2022 Summer School Homework:

Homework Problem 1: Read through the reading assignment and reproduce the following result for $e^+ + e^-$ annihilations at NLO with dim-reg. Final results for the NLO real and virtual contributions are

$$\sigma_r = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right],$$
$$\sigma_v = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right],$$

respectively. Therefore, summing over the LO and NLO contributions to the total cross section yields

$$\lim_{\epsilon\to 0} \sigma^{\rm tot}_{\gamma^*\to X} = \sigma_0 \left[1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2)\right]. \tag{1}$$

Homework Problem 2: Thrust:

(a) Consider the $2 \to 3$ ($e^+ + e^- \to q + \bar{q} + g$) process and derive the following thrust distribution for $T < 1$

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right]. \tag{2}$$

Hint: Use $2 \to 3$ cross section, the thrust distribution can be cast into

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{C_F \alpha_s}{2\pi} \int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T - \max[x_1, x_2, x_3]).$$

(b) After including the Born and virtual as well as real contributions, the cross section becomes

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right] + C\delta(1-T), \tag{3}$$

where C is a divergent constant, which can be determined by the following integral according to Eq. (1)

$$\int_{T_{\rm min}}^1 dT \frac{d\sigma}{\sigma_0 dT} = 1 + C_F \frac{3\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2), \quad \text{with} \quad T_{\rm min} = 2/3 \text{ for } \quad 2 \to 3 \quad \text{processes.} \tag{4}$$

From Eq. (2), show that the following one-loop expression for the thrust distribution satisfies Eq. (4)

$$\frac{d\sigma}{\sigma_0 dT} = \delta(1-T) + \frac{C_F \alpha_s}{2\pi} \left[\delta(1-T)\left(\frac{\pi^2}{3} - 1\right) - \frac{3(3T-2)(2-T)}{(1-T)_+}\right]$$
$$+ \frac{C_F \alpha_s}{2\pi} \frac{2(3T^2 - 3T + 2)}{T} \left[\frac{\ln(2T-1)}{(1-T)_+} - \left(\frac{\ln(1-T)}{1-T}\right)_+\right], \tag{5}$$

where the plus distribution is defined as $\int_a^1 dx g(x) (f(x))_+ = \int_a^1 dx g(x) f(x) - g(1) \int_0^1 dx f(x)$.

Homework Problem 3: EM fields of a massless particle and Weizsäcker-Williams Method

(a) Verify that the EM fields of the following shockwave solution

$$E^i = \frac{e}{2\pi} \frac{r_\perp^i}{r_\perp^2} \delta(t-z) \quad \text{and} \quad B^i = -\epsilon^{ij} \frac{e}{2\pi} \frac{r_\perp^j}{r_\perp^2} \delta(t-z) \tag{6}$$

satisfy the Maxwell equations with the source current $j^\mu = e n^\mu \delta^{(2)}(r_\perp) \delta(t-z)$ and $n^\mu = (1,0,0,1)$.

(b) Show that the covariant and light-cone gauge potentials below both give rise to the above EM fields

$$\text{Cov:}\quad A^0_{Cov} = A^z_{Cov} = -\frac{e}{4\pi}\ln\mu^2 r_\perp^2\,\delta(t-z),\quad A^\perp_{Cov} = 0; \tag{7}$$

$$\text{LC:}\quad A^0_{LC} = A^z_{LC} = 0,\quad A^\perp_{LC} = -\frac{e}{4\pi}\theta(t-z)\nabla\ln\mu^2 r_\perp^2; \tag{8}$$

(c) Show that these two gauge potentials are related by a gauge transformation $A^\mu_{LC} = A^\mu_{Cov} + \partial^\mu\Omega$.

(d) Show that the above EM fields lead to the photon distribution $xf_\gamma(k_\perp) = \alpha/(\pi^2 k_\perp^2)$.

Homework Problem 4: BFKL equation in the momentum and coordinate space

(a) As we mentioned in class, the BFKL equation in the dipole model can be written as

$$\partial_Y T(x,y;Y) = \frac{\bar{\alpha}_s}{2\pi}\int d^2z\frac{(x-y)^2}{(x-z)^2(z-y)^2}\left[T(x,z;Y)+T(z,y;Y)-T(x,y;Y)\right], \tag{9}$$

with $\bar{\alpha}_s = \dfrac{\alpha_s N_c}{\pi}$. Let us look for angular independent solution (the dominant one) and introduce the shorthand notation $x_{10} = x_1 - x_0$, where $x_{0,1}$ are 2-d vectors, thus we can cast the equation into

$$\partial_Y T(x_{10};Y) = \frac{\bar{\alpha}_s}{2\pi}\int d^2x_2\frac{x_{10}^2}{x_{12}^2 x_{20}^2}\left[T(x_{12};Y)+T(x_{20};Y)-T(x_{10};Y)\right]. \tag{10}$$

Suppose one can define

$$T(x;Y) = \int_{c-i\infty}^{c+i\infty}\frac{d\gamma}{2\pi i}\left(\frac{x^2}{x_{10}^2}\right)^\gamma T_\gamma(Y) \tag{11}$$

with x_{10} the initial dipole size, show that the BFKL equation can be converted into $dT_\gamma/dY = \bar{\alpha}_s\chi(\gamma)T_\gamma$, where the BFKL characteristic function $\chi(\gamma) = 2\psi(1)-\psi(1-\gamma)-\psi(\gamma)$ with $\psi(x)$ the digamma function. Hint: First show that

$$\chi(\gamma) = \frac{1}{2\pi}\int d^2x_2\frac{x_{10}^2}{x_{12}^2x_{20}^2}\left[\left(\frac{x_{12}^2}{x_{10}^2}\right)^\gamma + \left(\frac{x_{20}^2}{x_{10}^2}\right)^\gamma - 1\right] \tag{12}$$

and use the integral identity

$$\int_0^{2\pi}\frac{d\theta}{1-a\cos\theta} = \frac{1}{\sqrt{1-a^2}}\quad\text{with}\quad a<1$$

and the identity regarding the digamma function

$$\psi(\gamma) = -\gamma_E + \int_0^1 du\frac{1-u^{\gamma-1}}{1-u}. \tag{13}$$

with $\gamma_E \simeq 0.577$ the Euler constant.

(b) In the momentum space, the BFKL equation reads

$$\partial_Y G(l_\perp, l'_\perp;Y) = \frac{\bar{\alpha}_s}{\pi}\int\frac{d^2q_\perp}{(q_\perp - l_\perp)^2}\left[G(q_\perp,l'_\perp;Y) - \frac{l_\perp^2}{2q_\perp^2}G(l_\perp,l'_\perp;Y)\right], \tag{14}$$

where $G(l_\perp, l'_\perp;Y)$ is known as the BFKL propagator. In the Mellin space, show that the solution $G_\gamma(Y)$ has the same BFKL characteristic function, i.e., $G_\gamma(Y) = G_\gamma(0)\exp[\bar{\alpha}_s\chi(\gamma)Y]$.

Hint: Use the dimensional regularization ($\overline{MS}$ scheme with $S_\epsilon^{-1} = (4\pi e^{-\gamma_E})^{-\epsilon}$) and the following identity (see the appendix A in [$arXiv:1607.04726$])

$$J(\gamma) = S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{(k_\perp+q_\perp)^2}\left(\frac{k_\perp^2}{q_\perp^2}\right)^\gamma = \frac{1}{4\pi}\left(\frac{e^{\gamma_E}\mu^2}{k_\perp^2}\right)^\epsilon\frac{\Gamma(\epsilon+\gamma)}{\Gamma(\gamma)}\frac{\Gamma(-\epsilon)\Gamma(-\epsilon-\gamma+1)}{\Gamma(-2\epsilon-\gamma+1)}. \tag{15}$$