临界现象与泛函重整化群（三）

付伟杰

大连理工大学

粒子物理与核物理暑期学校，复旦大学，2022年8月13–21日

参考资料

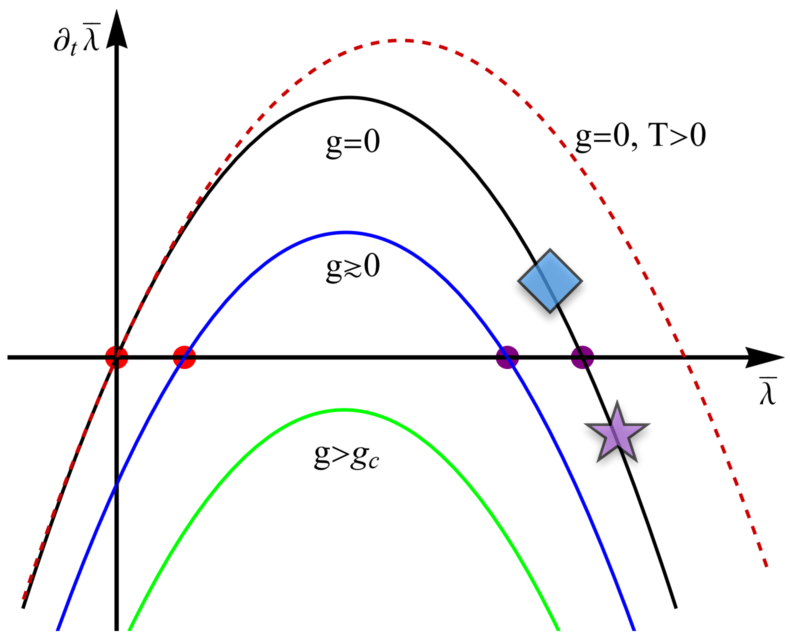
- W.-j. Fu, *QCD at finite temperature and density within the fRG approach: An overview*, (2022), arXiv:2205.00468 [hep-ph].
- Shang-Keng Ma, *Modern theory of critical phenomena*, (2000), Westview Press.
- 量子场论在线课程：*https://www.bilibili.com/video/BV11z411e7aU*

提纲

- 泛函重整化群与强子物理
- 泛函重整化群在低能有效模型中的应用
- 泛函重整化群在有限温有限密QCD中的应用
- 实时泛函重整化群

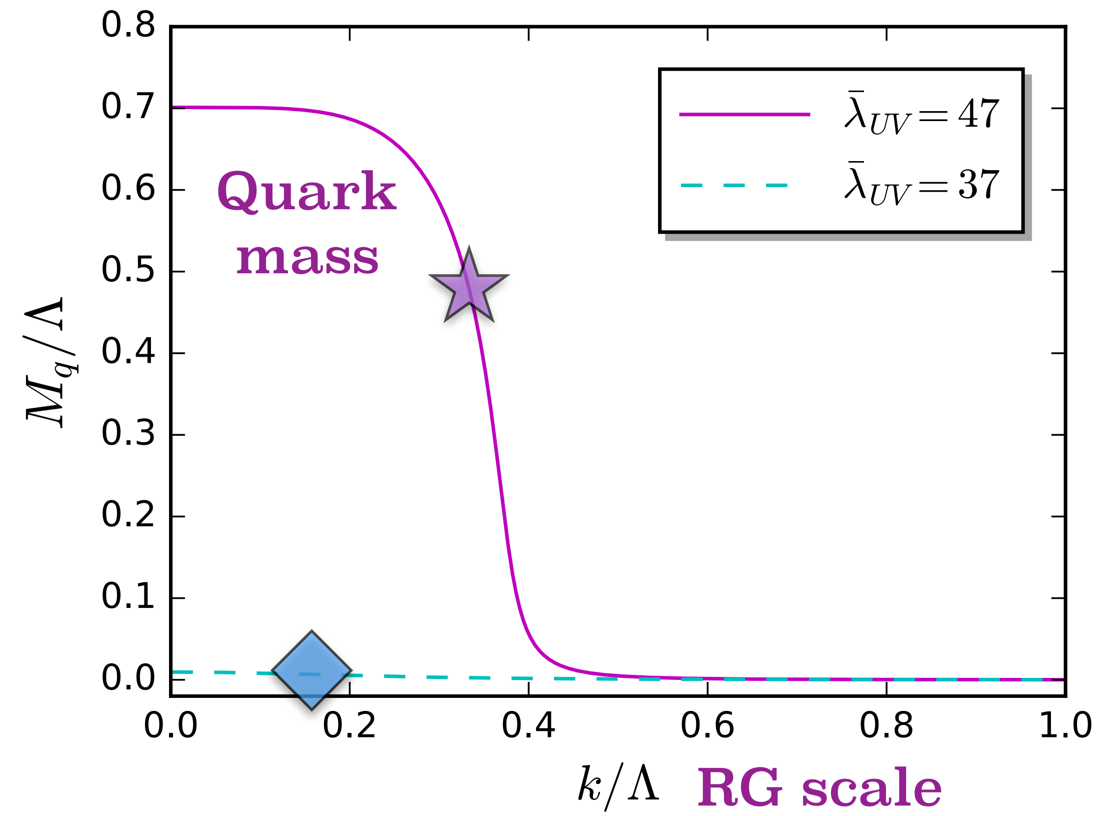
手征对称性自发破缺的重整化群描述

- 4-夸克耦合强度的 β 函数：

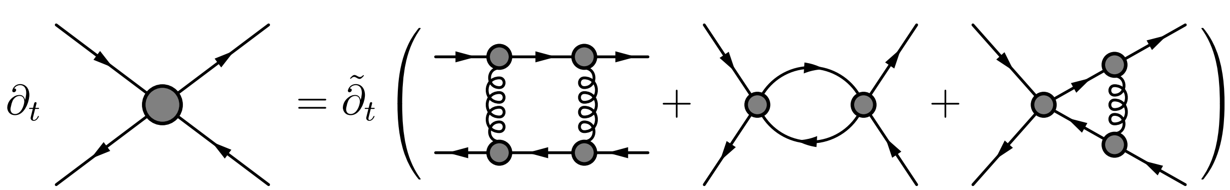


$$\partial_t\bar{\lambda} = (d-2)\bar{\lambda} - a\bar{\lambda}^2 - b\bar{\lambda}g^2 - cg^4\,,$$

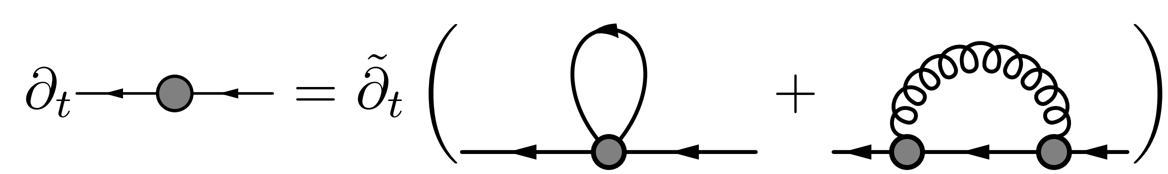
- 夸克动力学质量的产生：



4-夸克顶点的流方程



夸克传播子的流方程



Nambu—Jona-Lasinio 模型

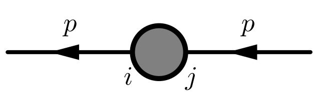
采用如下 non-local 的 NJL 有效作用量

$$\Gamma_k[\Phi] = \int_{x,y} \left[Z_{q,k}(x,y)\bar{q}(x)\gamma_\mu\partial_\mu q(y) + m_{q,k}(x,y)\bar{q}(x)q(y) \right]$$

$$- \int_{x,y,w,z} \sum_{\alpha\in\mathscr{B}} \lambda_{\alpha,k}(x,y,w,z)\mathcal{O}^{\alpha}_{ijlm}\bar{q}_i(x)q_j(y)\bar{q}_l(w)q_m(z)$$

这里 $\mathcal{O}^\alpha$ $(\alpha \in \mathscr{B})$ 是 Fierz 完备的 $N_f = 2$ 四夸克相互作用的基，共有10个相互作用道。两夸克关联函数：

$$\Gamma^{(2)\bar{q}q}_{k,ij}(p',p) \equiv - \frac{\delta^2\Gamma_k}{\delta\bar{q}_i(p')\delta q_j(p)}\bigg|_{\Phi=0}$$



$$\equiv -\Gamma^{(2)\bar{q}q}_{k,ij}(p',p)$$

$$= \left[Z_{q,k}(p)i(\gamma\cdot p)_{ij} + m_{q,k}(p)\delta_{ij} \right](2\pi)^4\delta^4(p'+p)$$

这样我们得到传播子

$$G^{q\bar{q}}_k(p,p') = \left[\Gamma^{(2)\bar{q}q}_k + R^{\bar{q}q}_k\right]^{-1}$$
$$= G^q_k(p)(2\pi)^4\delta^4(p'+p)$$

其中

$$G^q_k(p) = \frac{1}{Z_{q,k}(p)i\gamma\cdot p + Z_{q,k}r_F(p^2/k^2)i\gamma\cdot p + m_{q,k}(p)}$$

费米型的红外抑制 regulator

$$R^{\bar{q}q}_k = Z_{q,k}r_F(p^2/k^2)i\gamma\cdot p$$

Litim regulator

$$r_{F,\mathrm{opt}}(x) = \left(\frac{1}{\sqrt{x}} - 1\right)\Theta(1-x)$$

指数型 regulator

$$r_F(x) = r_{\exp,n}(x) = \frac{x^{n-1}}{e^{x^n}-1}$$

$$r_{F,\exp}(x) = \frac{1}{x}e^{-x}$$

Fierz 完备的 $N_f = 2$ 四夸克相互作用（1）

Fierz 完备的 $N_f = 2$ 四夸克相互作用的基包括10个相互作用道，这10个相互作用道又可以分为以下四组，第一组

$$\mathcal{O}^{(V-A)}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\gamma_\mu T^0 q)^2 - (\bar{q} i\gamma_\mu\gamma_5 T^0 q)^2$$

$$\mathcal{O}^{(V+A)}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\gamma_\mu T^0 q)^2 + (\bar{q} i\gamma_\mu\gamma_5 T^0 q)^2$$

$$\mathcal{O}^{(S-P)_+}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\, T^0 q)^2 - (\bar{q}\,\gamma_5 T^0 q)^2 + (\bar{q}\, T^a q)^2 - (\bar{q}\,\gamma_5 T^a q)^2$$

$$\mathcal{O}^{(V-A)^{\text{adj}}}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\gamma_\mu T^0 t^a q)^2 - (\bar{q} i\gamma_\mu\gamma_5 T^0 t^a q)^2$$

这四个相互作用道在 $SU_{\rm V}(N_f) \otimes SU_{\rm A}(N_f) \otimes U_{\rm V}(1) \otimes U_{\rm A}(1)$ 变换下是不变，其中 T^a 和 t^a 分别是 $SU(N_f)$ 群和 $SU(N_c)$ 群的生成元，$T^0 = 1/\sqrt{2N_f}\mathbf{1}_{N_f\times N_f}$。下面一组

$$\mathcal{O}^{(S+P)_-}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\, T^0 q)^2 + (\bar{q}\,\gamma_5 T^0 q)^2 - (\bar{q}\, T^a q)^2 - (\bar{q}\,\gamma_5 T^a q)^2,$$

$$\mathcal{O}^{(S+P)^{\text{adj}}_-}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\, T^0 t^a q)^2 + (\bar{q}\,\gamma_5 T^0 t^a q)^2 - (\bar{q}\, T^a t^b q)^2 - (\bar{q}\,\gamma_5 T^a t^b q)^2$$

在 $SU_{\rm V}(N_f) \otimes SU_{\rm A}(N_f) \otimes U_{\rm V}(1)$ 变换下是不变，但是破坏 $U_{\rm A}(1)$ 对称性

Fierz 完备的 $N_f = 2$ 四夸克相互作用（2）

第三组

$$\mathcal{O}^{(S-P)_-}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\,T^0 q)^2 - (\bar{q}\,\gamma_5 T^0 q)^2$$
$$-(\bar{q}\,T^a q)^2 + (\bar{q}\,\gamma_5 T^a q)^2$$
$$\mathcal{O}^{(S-P)^{\text{adj}}_-}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\,T^0 t^a q)^2 - (\bar{q}\,\gamma_5 T^0 t^a q)^2$$
$$-(\bar{q}\,T^a t^b q)^2 + (\bar{q}\,\gamma_5 T^a t^b q)^2$$

这两个相互作用道在 $SU_{\mathrm{V}}(N_f) \otimes U_{\mathrm{V}}(1) \otimes U_{\mathrm{A}}(1)$ 变换下是不变，但是破坏 $SU_{\mathrm{A}}(N_f)$ 对称性。最后一组

$$\mathcal{O}^{(S+P)_+}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\,T^0 q)^2 + (\bar{q}\,\gamma_5 T^0 q)^2$$
$$+(\bar{q}\,T^a q)^2 + (\bar{q}\,\gamma_5 T^a q)^2$$
$$\mathcal{O}^{(S+P)^{\text{adj}}_+}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\,T^0 t^a q)^2 + (\bar{q}\,\gamma_5 T^0 t^a q)^2$$
$$+(\bar{q}\,T^a t^b q)^2 + (\bar{q}\,\gamma_5 T^a t^b q)^2$$

在 $SU_{\mathrm{V}}(N_f) \otimes U_{\mathrm{V}}(1)$ 变换下是不变，但是破坏 $SU_{\mathrm{A}}(N_f)$ 和 $U_{\mathrm{A}}(1)$对称性。将$\mathcal{O}^{(S-P)_+}$， $\mathcal{O}^{(S+P)_-}$， $\mathcal{O}^{(S-P)_-}$， $\mathcal{O}^{(S+P)_+}$ 作适当的线性组合，可以得到和介子观测量直接联系的相互作用道

$$\mathcal{O}^{\sigma}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\,T^0 q)^2$$
$$\mathcal{O}^{\pi}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = -(\bar{q}\,\gamma_5 T^a q)^2$$
$$\mathcal{O}^{a}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = (\bar{q}\,T^a q)^2$$
$$\mathcal{O}^{\eta}_{ijlm}\bar{q}_i q_j \bar{q}_l q_m = -(\bar{q}\,\gamma_5 T^0 q)^2$$

四夸克顶点及其流方程

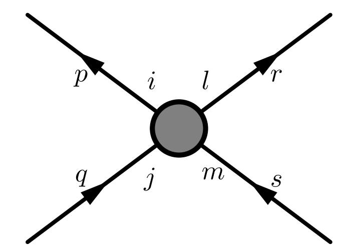
四夸克顶点

$$\Gamma^{(4)\bar{q}q\bar{q}q}_{k,ijlm}(p,q,r,s) \equiv \left.\frac{\delta^4\Gamma_k}{\delta\bar{q}_i(p)\delta q_j(q)\delta\bar{q}_l(r)\delta q_m(s)}\right|_{\Phi=0}$$

$$= 2\sum_{\alpha\in\mathscr{B}}\Big(\lambda^S_{\alpha,k}(p,q,r,s)(\mathscr{O}^\alpha_{ijlm}-\mathscr{O}^\alpha_{ljim})$$

$$+\lambda^A_{\alpha,k}(p,q,r,s)(\mathscr{O}^\alpha_{ijlm}+\mathscr{O}^\alpha_{ljim})\Big)$$

$$\times(2\pi)^4\delta^4(p+q+r+s)$$



$$\equiv -\Gamma^{(4)\bar{q}q\bar{q}q}_{k,ijlm}(p,q,r,s)$$

对称和反对称四夸克耦合分别为

$$\lambda^S_{\alpha,k}(p,q,r,s) \equiv \Big(\lambda_{\alpha,k}(p,q,r,s)+\lambda_{\alpha,k}(r,q,p,s)\Big)/2$$

$$\lambda^A_{\alpha,k}(p,q,r,s) \equiv \Big(\lambda_{\alpha,k}(p,q,r,s)-\lambda_{\alpha,k}(r,q,p,s)\Big)/2$$

若我们忽略反对称四夸克耦合的影响，即 $\lambda^A_{\alpha,k}=0$，那么

$$\lambda^S_{\alpha,k}(p,q,r,s)=\lambda_{\alpha,k}(p,q,r,s)=\lambda_{\alpha,k}(r,q,p,s)$$

其流方程为

$$\partial_t\lambda_{\alpha,k}(p_1,p_2,p_3,p_4)$$

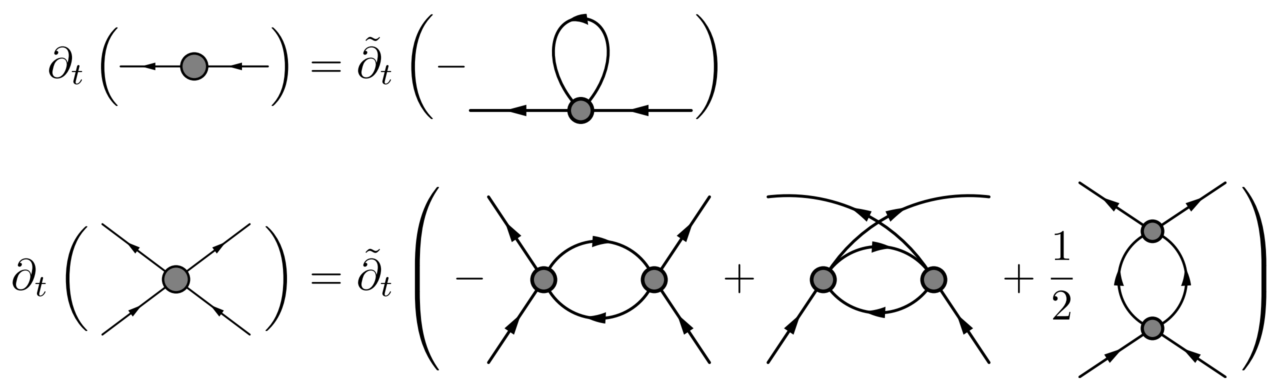
$$=\sum_{\alpha',\alpha''\in\mathscr{B}}\int\frac{d^4q}{(2\pi)^4}\Big[\lambda_{\alpha',k}(p_1,p_2,q+p_2-p_1,q)\lambda_{\alpha'',k}(p_3,p_4,q,q+p_2-p_1)\mathscr{F}^t_{\alpha'\alpha'',\alpha}$$

$$+\lambda_{\alpha',k}(p_3,p_2,q+p_2-p_3,q)\lambda_{\alpha'',k}(p_1,p_4,q,q+p_2-p_3)\mathscr{F}^u_{\alpha'\alpha'',\alpha}$$

$$+\lambda_{\alpha',k}(p_1,q,p_3,-q+p_1+p_3)\lambda_{\alpha'',k}(q,p_2,-q+p_1+p_3,p_4)\mathscr{F}^s_{\alpha'\alpha'',\alpha}\Big]$$

夸克两点关联函数的流方程

NJL模型两点和四点夸克关联函数的流方程



夸克质量的流方程

$$\partial_t m_{q,k}(p) = \int \frac{d^4q}{(2\pi)^4}\left(\tilde{\partial}_t \bar{G}_k^q(q)\right) m_{q,k}(q)\Big[\frac{3}{2}\lambda_{\pi,k}(p,p,q,q) \\ +\frac{23}{2}\lambda_{\sigma,k}(p,p,q,q) - \frac{3}{2}\lambda_{a,k}(p,p,q,q) \\ +\frac{1}{2}\lambda_{\eta,k}(p,p,q,q) + \frac{8}{3}\lambda_{(S+P)_-^{\mathrm{adj}},k}(p,p,q,q) \\ -\frac{16}{3}\lambda_{(S+P)_+^{\mathrm{adj}},k}(p,p,q,q) - 4\lambda_{(V+A),k}(p,p,q,q)\Big]$$

其中

$$\tilde{\partial}_t \bar{G}_k^q(q) = -2\left(\bar{G}_k^q(q)\right)^2 Z_{q,k}^R(q) q^2 \partial_t R_{F,k}(q)$$

我们有 $Z_{q,k}^R(q) = Z_{q,k}(q) + R_{F,k}(q)$， $R_{F,k}(q) = Z_{q,k} r_F(q^2/k^2)$，以及

$$\bar{G}_k^q(q) = \frac{1}{\left(Z_{q,k}^R(q)\right)^2 q^2 + m_{q,k}^2(q)}$$

夸克质量的产生

如果我们忽略四夸克耦合和夸克质量的动量依赖性，只保留标量和赝标的 σ 和 π 相互作用道，并且将四夸克耦合和夸克质量无量纲化，

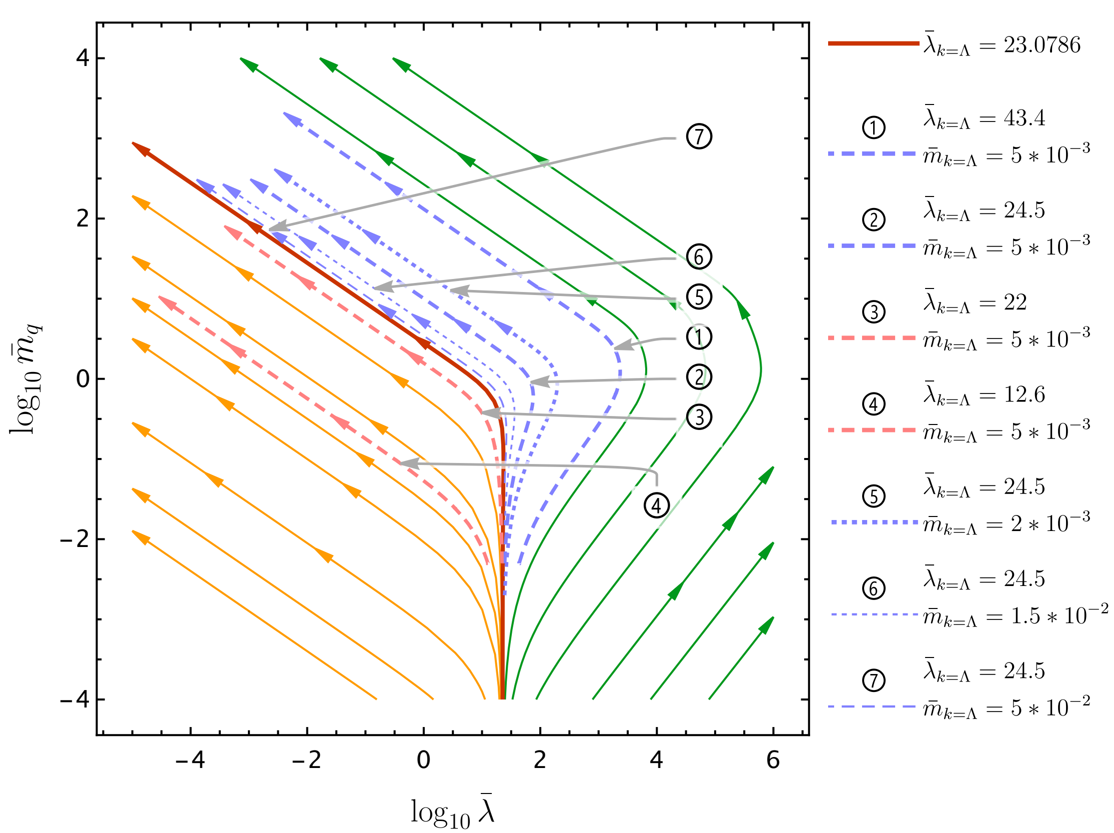
$$\bar{\lambda}_{\alpha,k} = \lambda_{\alpha,k} k^2, \qquad \bar{m}_{q,k} = m_{q,k}/k$$

得到

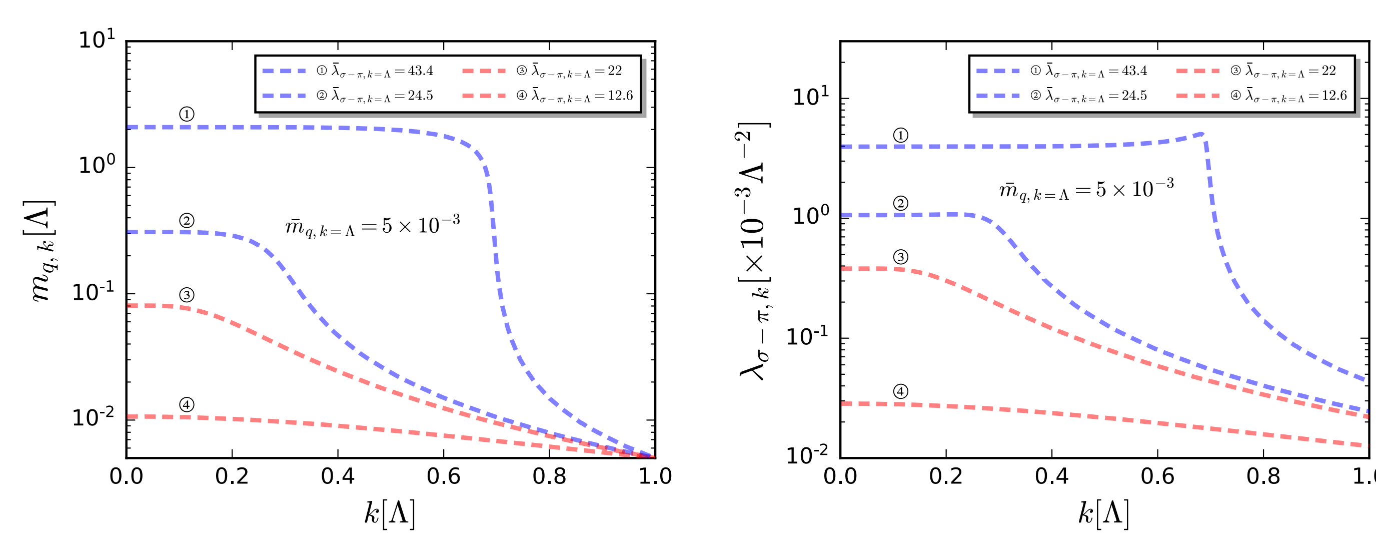
$$\partial_t \bar{\lambda}_{\sigma-\pi} = 2\bar{\lambda}_{\sigma-\pi} + \frac{\bar{\lambda}^2_{\sigma-\pi}}{2\pi^2}\int_0^\infty dx\, x^3 r_F{}'(x)\Big[-4\bar{m}_q^2 + 7x\big(1+r_F(x)\big)^2\Big]\frac{1+r_F(x)}{\Big[\big(1+r_F(x)\big)^2 x + \bar{m}_q^2\Big]^3}$$

$$\partial_t \bar{m}_q = -\bar{m}_q + \bar{m}_q\bar{\lambda}_{\sigma-\pi}\frac{13}{4\pi^2}\int_0^\infty dx\, x^3 r_F{}'(x)\frac{1+r_F(x)}{\Big[\big(1+r_F(x)\big)^2 x + \bar{m}_q^2\Big]^2}$$

- 质量和耦合平面的流图

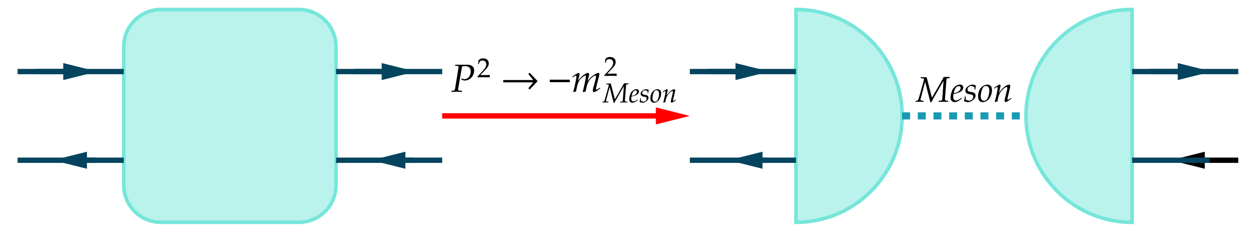


- 质量和耦合随RG能标的依赖



束缚态的重整化群描述（1）

束缚态的信息包含在相应相互作用道的四夸克顶点，如右图所示，当外动量接近于某一介子的在壳质量，导致共振的发生，这样四夸克顶点可以近似为两个夸克介子顶点通过一个介子传播子相联系。

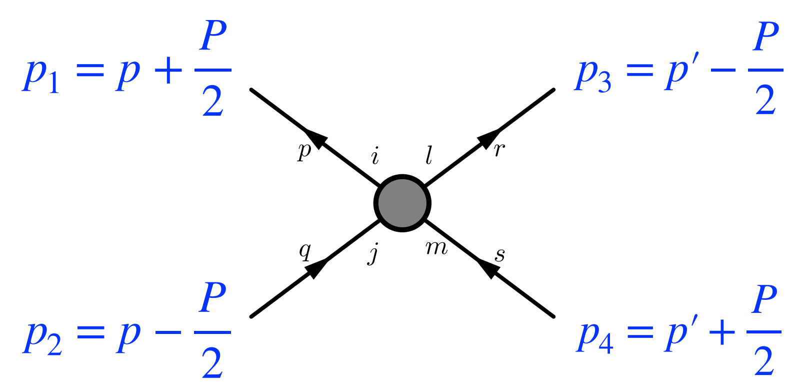


四夸克顶点的 Mandelstam 变量分别为

$$s = (p_1 + p_3)^2 = (p + p')^2$$

$$t = (p_1 - p_2)^2 = P^2$$

$$u = (p_1 - p_4)^2 = (p - p')^2$$



下面以 π 介子为例，当发生共振，其相互作用道的四夸克耦合强度的流方程可以作单动量道近似

$$\partial_t \lambda_{\pi,k}(P^2) = \mathscr{C}_k(P^2)\lambda^2_{\pi,k}(P^2) + \mathscr{A}_k(t,u,s)$$

其中

$$\mathscr{C}_k(P^2) = \int \frac{d^4q}{(2\pi)^4} \mathscr{F}^t{}_{\pi\pi,\pi}$$

以及

$$\mathscr{A}_k(t,u,s) = \int \frac{d^4q}{(2\pi)^4} \Bigg\{ \sum_{\alpha',\alpha''\in\mathscr{B}} \Big[\lambda_{\alpha',k}\lambda_{\alpha'',k}\Big(\mathscr{F}^t{}_{\alpha'\alpha'',\pi} + \mathscr{F}^u{}_{\alpha'\alpha'',\pi} + \mathscr{F}^s{}_{\alpha'\alpha'',\pi}\Big)\Big] - \lambda^2_{\pi,k}\mathscr{F}^t{}_{\pi\pi,\pi} \Bigg\}$$

进一步我们令 $p = p' = 0$，即 $s = u = 0$，那么

$$\mathscr{A}_k(t,u,s) \to \mathscr{A}_k(P^2)$$

在忽略 $\mathscr{A}_k(P^2)$的情况下，我们可以解析求解 $\lambda_{\pi,k}$ 的方程，得到

$$\lambda_{\pi,k=0}(P^2) = \frac{\lambda_{\pi,k=\Lambda}}{1 - \lambda_{\pi,k=\Lambda}\int_\Lambda^0 \mathscr{C}_k(P^2)\frac{dk}{k}}$$

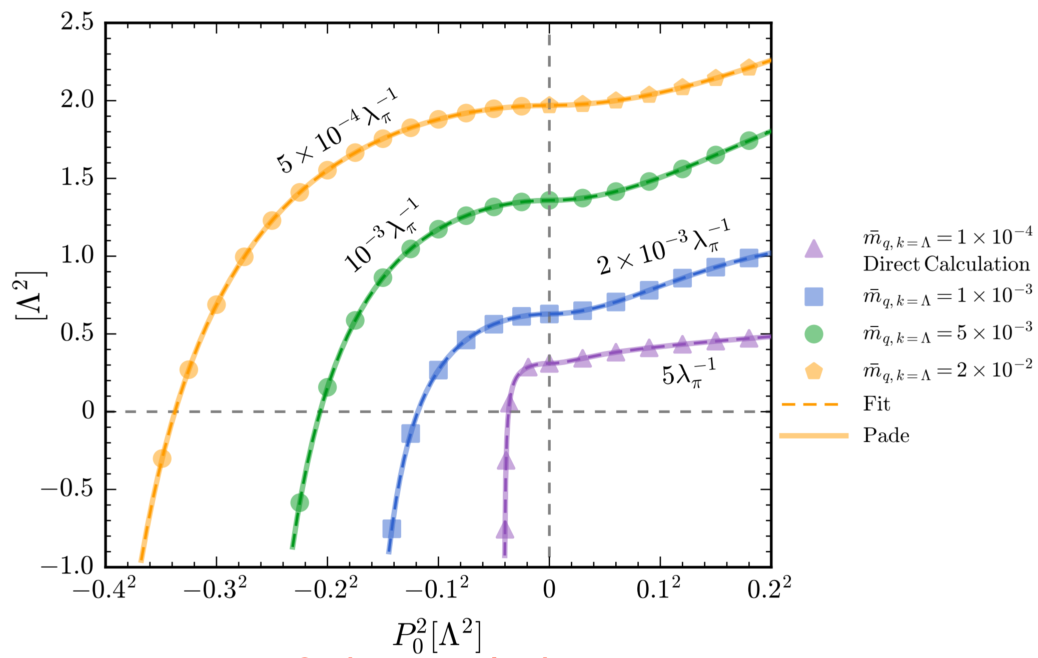
上式 pole 的位置确定了介子的质量

$$1 - \lambda_{\pi,k=\Lambda}\int_\Lambda^0 \mathscr{C}_k(P^2 = -m^2_\pi)\frac{dk}{k} = 0$$

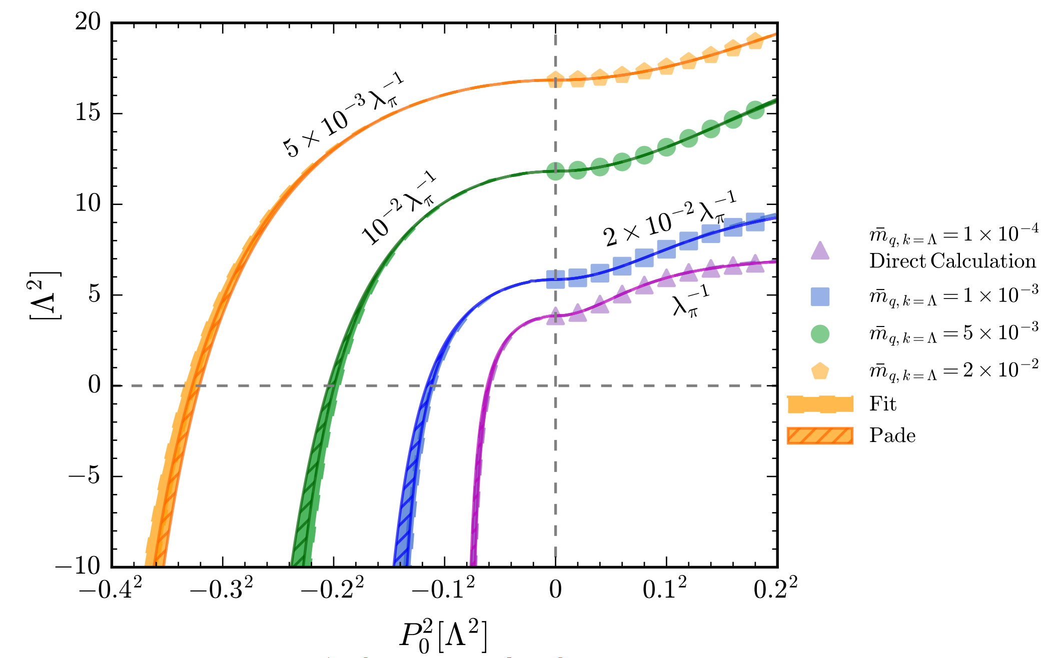
束缚态的重整化群描述（2）

- 四夸克耦合的倒数

$$\frac{1}{\lambda_{\pi,k=0}}$$

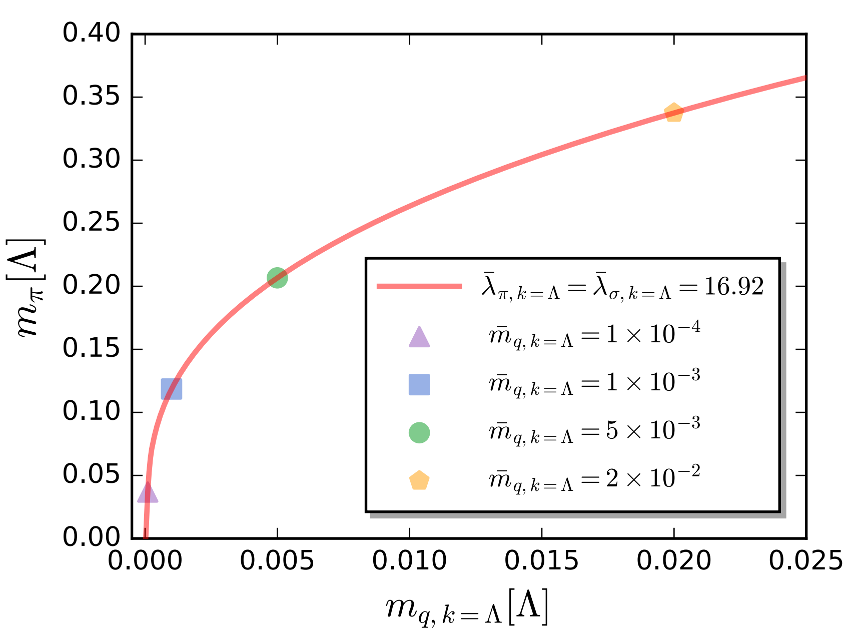


3d regulator



4d regulator

- π 介子质量对流夸克质量的依赖



满足 Goldstone 定理

解析延拓

$$\lambda_{\pi,k=0}(P^2) \approx \frac{a_0 + a_2P^2 + a_4P^4}{c_0 + P^2 + c_4P^4}$$

Pade 近似

$$\lambda_{\pi,k=0}[n,n](P^2) \approx \lambda_{\pi,k=0}(P^2)\,, \qquad P^2 > 0$$

夸克-介子 (QM) 模型

$N_f = 2$ 的夸克-介子模型的有效作用量:

$$\Gamma_k = \int_x \left\{ Z_{q,k}\bar{q}\left[\gamma_\mu \partial_\mu - \gamma_0(\hat{\mu} + igA_0)\right]q + \frac{1}{2}Z_{\phi,k}(\partial_\mu \phi)^2 \right.$$
$$\left. + h_k \bar{q}\left(T^0\sigma + i\gamma_5 \vec{T}\cdot\vec{\pi}\right)q + V_k(\rho, A_0) - c\sigma \right\}$$

有效势可以分解成两部分之和

$$V_k(\rho, A_0) = V_{\text{glue},k}(A_0) + V_{\text{mat},k}(\rho, A_0)$$

第一部分是胶子势，也就是 Polyakov loop 势的贡献，第二部分是物质场的贡献，下面我们将 $V_{\text{mat},k}$ 直接记为 V_k，容易得到其流方程为

$$\partial_t V_k(\rho) = \frac{k^4}{4\pi^2}\left[\left(N_f^2 - 1\right) l_0^{(B,4)}(\tilde{m}_{\pi,k}^2, \eta_{\phi,k}; T) + l_0^{(B,4)}(\tilde{m}_{\sigma,k}^2, \eta_{\phi,k}; T)\right.$$
$$\left. - 4N_c N_f l_0^{(F,4)}(\tilde{m}_{q,k}^2, \eta_{q,k}; T, \mu)\right]$$

这里阈值函数分别为

$$l_0^{(B,4)}(\tilde{m}_{\phi,k}^2, \eta_{\phi,k}; T) = \frac{2}{3}\left(1 - \frac{\eta_{\phi,k}}{5}\right)\frac{1}{\sqrt{1 + \tilde{m}_{\phi,k}^2}}\left(\frac{1}{2} + n_B(\tilde{m}_{\phi,k}^2; T)\right)$$

以及

$$l_0^{(F,4)}(\tilde{m}_{q,k}^2, \eta_{q,k}; T, \mu)$$
$$= \frac{2}{3}\left(1 - \frac{\eta_{q,k}}{4}\right)\frac{1}{2\sqrt{1 + \tilde{m}_{q,k}^2}}\left(1 - n_F(\tilde{m}_{q,k}^2; T, \mu, L, \bar{L}) - n_F(\tilde{m}_{q,k}^2; T, -\mu, \bar{L}, L)\right)$$

有效势流方程的求解

有效势流方程中介子和夸克质量：

$$\tilde{m}_{\pi,k}^2 = \frac{V_k'(\rho)}{k^2 Z_{\phi,k}}, \qquad \tilde{m}_{\sigma,k}^2 = \frac{V_k'(\rho) + 2\rho V_k''(\rho)}{k^2 Z_{\phi,k}}, \qquad \tilde{m}_{q,k}^2 = \frac{h_k^2 \rho}{2k^2 Z_{q,k}^2}$$

夸克和介子的反常量纲

$$\eta_{q,k} = -\frac{\partial_t Z_{q,k}}{Z_{q,k}}, \qquad \eta_{\phi,k} = -\frac{\partial_t Z_{\phi,k}}{Z_{\phi,k}}$$

有效势的求解常用的方法包括：Taylor 展开，有效势格点 (Schaefer, Wambach, arXiv: nucl-th/0403039)，Chebyshev 展开 (Borchardt, Knorr, arXiv:1502.07511)，Galerkin 方法 (Grossi and Wink, arXiv:1903.09503) 等等。前面我们介绍过在零点的 Taylor 展开，下面我们再来讨论一下同样很常用的在物理点的 Taylor 展开：

$$\bar{V}_k(\bar{\rho}) = \sum_{n=0}^{N_v} \frac{\bar{\lambda}_{n,k}}{n!}(\bar{\rho} - \bar{\kappa}_k)^n$$

其中我们使用了 RG 不变的物理量

$$\bar{V}_k(\bar{\rho}) = V_k(\rho), \qquad \bar{\rho} = Z_{\phi,k}\rho, \qquad \bar{\kappa}_k = Z_{\phi,k}\kappa_k, \qquad \bar{\lambda}_{n,k} = \frac{\lambda_{n,k}}{(Z_{\phi,k})^n}$$

这样得到

$$\partial_{\bar{\rho}}^n \left(\partial_t \Big|_\rho \bar{V}_k(\bar{\rho}) \right) \Big|_{\bar{\rho}=\bar{\kappa}_k} = (\partial_t - n\eta_{\phi,k})\bar{\lambda}_{n,k} - (\partial_t \bar{\kappa}_k + \eta_{\phi,k}\bar{\kappa}_k)\bar{\lambda}_{n+1,k}$$

由运动方程

$$\frac{\partial}{\partial \bar{\rho}} \left(\bar{V}_k(\bar{\rho}) - \bar{c}_k \bar{\sigma} \right) \Big|_{\bar{\rho}=\bar{\kappa}_k} = 0$$

得到展开点的流方程

$$\partial_t \bar{\kappa}_k = -\frac{\bar{c}_k^2}{\bar{\lambda}_{1,k}^3 + \bar{c}_k^2 \bar{\lambda}_{2,k}} \left[\partial_{\bar{\rho}} \left(\partial_t \Big|_\rho \bar{V}_k(\bar{\rho}) \right) \Big|_{\bar{\rho}=\bar{\kappa}_k} + \eta_{\phi,k} \left(\frac{\bar{\lambda}_{1,k}}{2} + \bar{\kappa}_k \bar{\lambda}_{2,k} \right) \right]$$

有效势的 Chebyshev 展开

我们也可以利用 Chebyshev 多项式展开有效势

$$\bar{V}_k(\bar{\rho}) = \sum_{n=1}^{N_v} c_{n,k} T_n(\bar{\rho}) + \frac{1}{2} c_{0,k}$$

T_n 是 n 阶 Chebyshev 多项式，$c_{n,k}$ 是其相应的展开系数，容易得到

$$\partial_t \Big|_{\rho} \bar{V}_k(\bar{\rho}) = \sum_{n=1}^{N_v} \Big(\partial_t c_{n,k} - d_{n,k} \eta_{\phi,k}(\bar{\rho}) \bar{\rho} \Big) T_n(\bar{\rho}) + \frac{1}{2} \Big(\partial_t c_{0,k} - d_{0,k} \eta_{\phi,k}(\bar{\rho}) \bar{\rho} \Big)$$

系数 $d_{n,k}$ 通过递推关系式与 $c_{n,k}$ 联系，展开系数的流方程为

$$\begin{aligned} \partial_t c_{m,k} = & \frac{2}{N+1} \sum_{i=0}^{N} \Big(\partial_t \Big|_{\rho} \bar{V}_k(\bar{\rho}_i) \Big) T_m(\bar{\rho}_i) \\ & + \frac{2}{N+1} \sum_{n=1}^{N_v} \sum_{i=0}^{N} d_{n,k} T_m(\bar{\rho}_i) T_n(\bar{\rho}_i) \eta_{\phi,k}(\bar{\rho}_i) \bar{\rho}_i \\ & + \frac{1}{N+1} d_{0,k} \sum_{i=0}^{N} T_m(\bar{\rho}_i) \eta_{\phi,k}(\bar{\rho}_i) \bar{\rho}_i \end{aligned}$$

其中 i 代表对 $T_{N+1}(\bar{\rho})$ 的 $N+1$ 个零点位置 $\bar{\rho}_i$ 的求和

$N_f = 2+1$ 味QM模型（1）

$N_f = 2+1$ 味 QM 模型的有效作用量：

$$\Gamma_k = \int_x \Big\{ Z_{q,k}\bar{q}\Big[\gamma_\mu\partial_\mu - \gamma_0(\hat{\mu} + igA_0)\Big]q + h_k\,\bar{q}\,\Sigma_5 q + Z_{\phi,k}\mathrm{tr}(\bar{D}_\mu\Sigma\cdot\bar{D}_\mu\Sigma^\dagger)$$

$$+V_{\rm glue}(L,\bar{L}) + V_k(\rho_1,\rho_2) - c_A\xi - c_l\sigma_l - \frac{1}{\sqrt{2}}\,c_s\sigma_s \Big\}$$

标量和赝标介子八重态和单态处于 $U(N_f=3)$ 群的伴随表示

$$\Sigma = T^a(\sigma^a + i\pi^a), \quad a = 0,1,\ldots,8$$

这里 $T^0 = 1/\sqrt{2N_f}\mathbf{1}_{N_f\times N_f}$，$T^a = \lambda^a/2$ 当 $a\neq 0$，λ^a 是 Gell-Mann 矩阵。介子场的协变导数：

$$\bar{D}_\mu\Sigma = \partial_\mu + \delta_{\mu 0}\big[\hat{\mu},\Sigma\big]$$

$\big[\hat{\mu},\Sigma\big]$ 是化学势矩阵和介子矩阵的对易，介子虽然不具有重子数化学势，但有可能携带电荷或者奇异数化学势，夸克化学势和守恒荷化学势的关系：

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q, \qquad \mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q, \qquad \mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S$$

Yukawa 耦合项

$$\Sigma_5 = T^a(\sigma^a + i\gamma_5\pi^a)$$

$N_f = 2 + 1$ 味QM模型（2）

有效势 $V_k(\rho_1, \rho_2)$ 的变量是 ρ_1 和 ρ_2，其分别为

$$\rho_1 = \mathrm{tr}(\Sigma \cdot \Sigma^\dagger)$$

$$\rho_2 = \mathrm{tr}\left(\Sigma \cdot \Sigma^\dagger - \frac{1}{3}\rho_1 \mathbf{1}_{3\times 3}\right)^2$$

ρ_1 和 ρ_2在 $SU_\mathrm{A}(N_f) \otimes U_\mathrm{A}(1)$ 变换下是不变，当满足运动方程，有

$$\rho_1\Big|_{\mathrm{EoM}} = \frac{1}{2}(\sigma_l^2 + \sigma_s^2)$$

$$\rho_2\Big|_{\mathrm{EoM}} = \frac{1}{24}(\sigma_l^2 - 2\sigma_s^2)^2$$

味道空间的场通过下面的转动与八重态和单态的场相联系：

$$\begin{pmatrix} \phi_l \\ \phi_s \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \phi_8 \\ \phi_0 \end{pmatrix}$$

此外，Kobayashi-Maskawa-'t Hooft determinant 项：

$$\xi = \det(\Sigma) + \det(\Sigma^\dagger)$$

具有 $SU_\mathrm{A}(N_f)$ 对称性，但是破坏 $U_\mathrm{A}(1)$，轻、奇异夸克的质量分别为

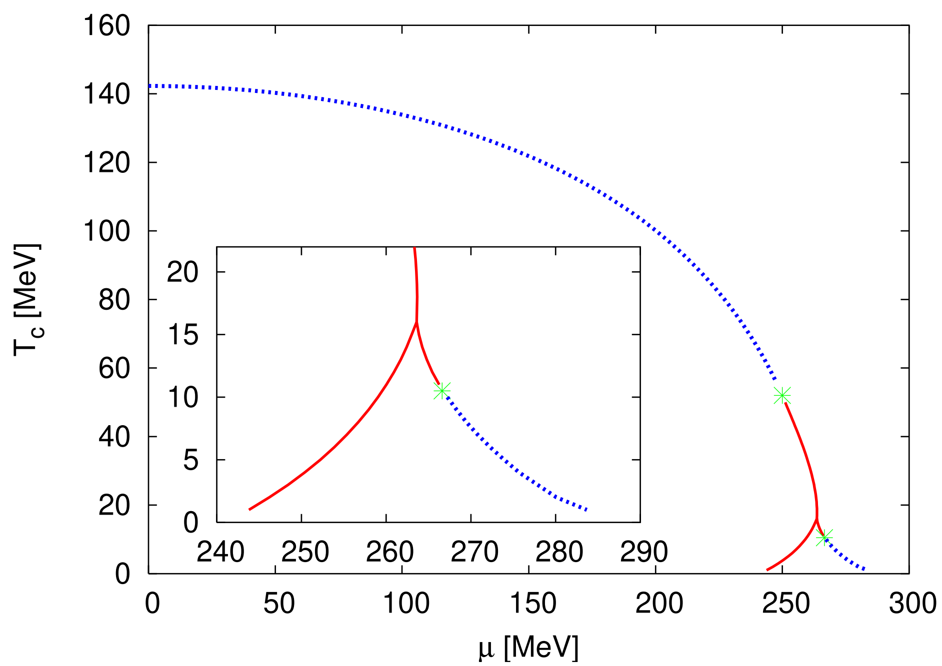
$$m_{l,k} = \frac{h_k}{2}\sigma_l, \qquad m_{s,k} = \frac{h_k}{\sqrt{2}}\sigma_s$$

π 和 K 介子弱衰变常数分别为

$$f_\pi = \sigma_l, \qquad f_K = \frac{\sigma_l + \sqrt{2}\,\sigma_s}{2}$$

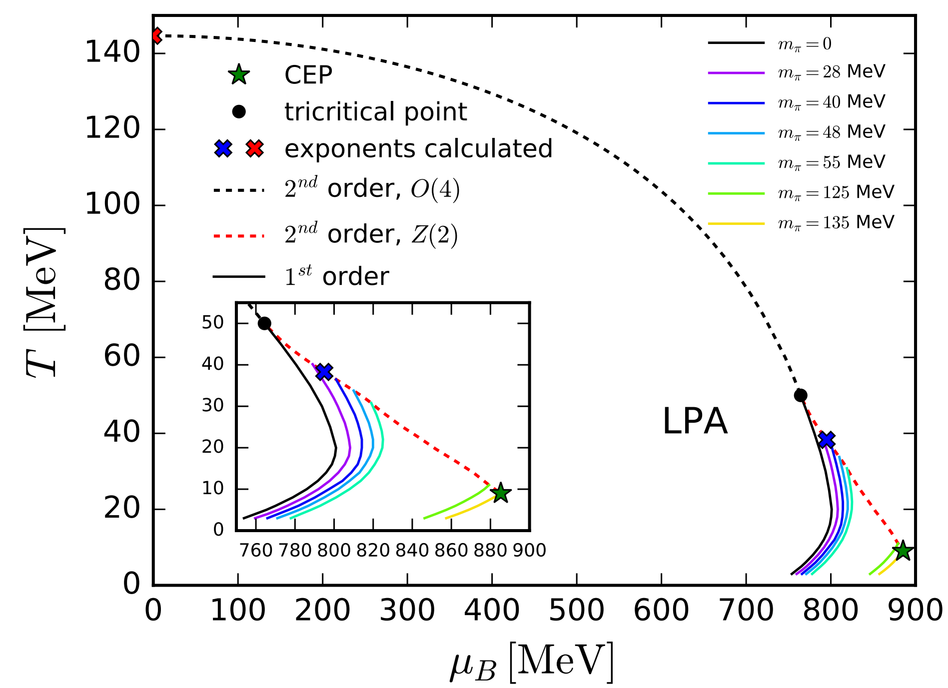
QM 模型的相图（1）

- 有效势 grid



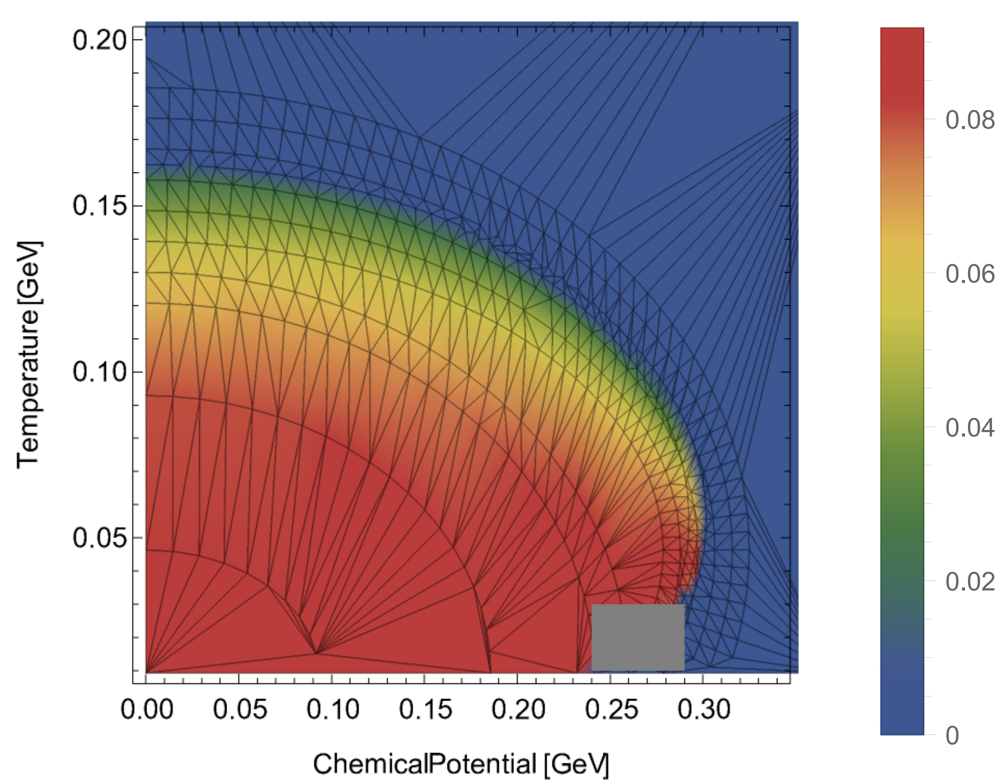
Schaefer, Wambach, arXiv: nucl-th/0403039

- Chebyshev 展开



Chen, *et al.* , arXiv: 2101.08484

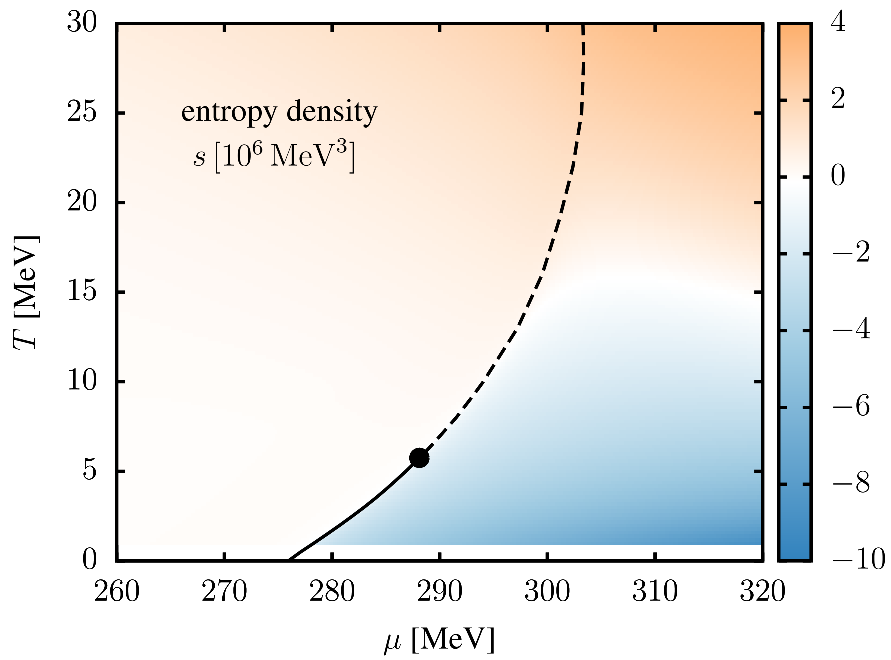
- Galerkin 方法



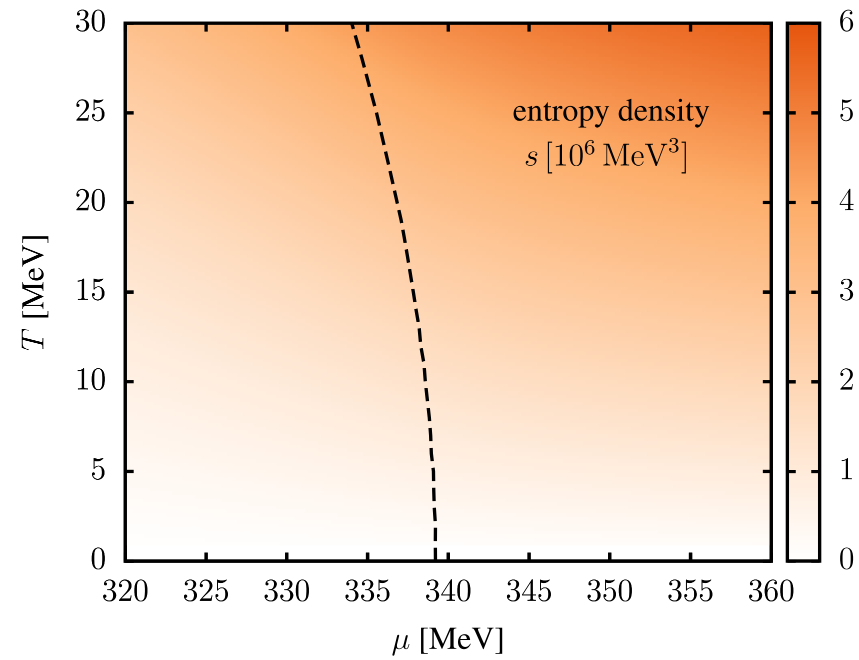
Grossi, *et al.* , arXiv: 2102.01602

QM 模型的相图（2）

- 相边界回弯与 regulator 的依赖性　Otto, *et al.* , arXiv: 2206.13067

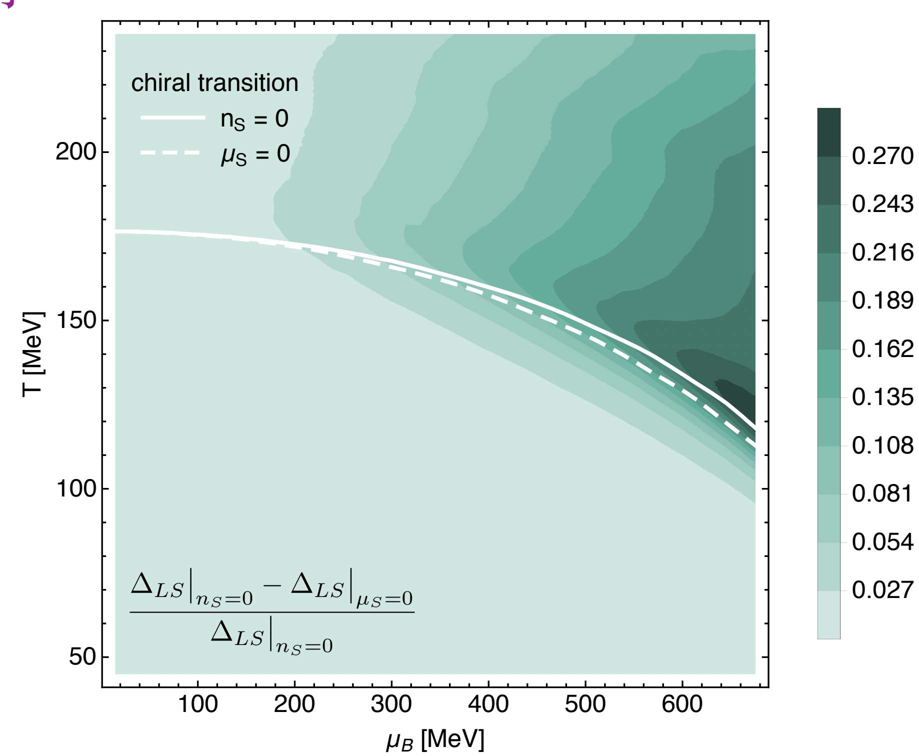


$R_k^{\mathrm{flat,3d}}(\mathbf{p}) = (k^2 - \mathbf{p}^2)\Theta(k^2 - \mathbf{p}^2)$



$R_k^{\mathrm{mass,3d}}(\mathbf{p}) = k^2\Theta(k_\phi^2 - \mathbf{p}^2)$

- 奇异数中性对相边界影响



Fu, *et al.*, arXiv: 1808.00410

热力学和状态方程

约化手征凝聚:

$$\Delta_{l,s}(T,\mu_q)=\frac{\left(\sigma_l-\sqrt{2}\frac{c_l}{c_s}\sigma_s\right)_{T,\mu_q}}{\left(\sigma_l-\sqrt{2}\frac{c_l}{c_s}\sigma_s\right)_{0,0}}$$

热力学势密度:

$$\Omega[T,\mu]=\frac{T}{V}\left(\Gamma_{k=0}[\Phi_{\rm EoM}]\Big|_{T,\mu}-\Gamma_{k=0}[\Phi_{\rm EoM}]\Big|_{T=\mu=0}\right)$$

压强:

$$p=-\Omega[T,\mu]$$

熵密度:

$$s=\frac{\partial p}{\partial T}$$

Trace anomaly:

$$\Delta=\epsilon-3p$$

能量密度:

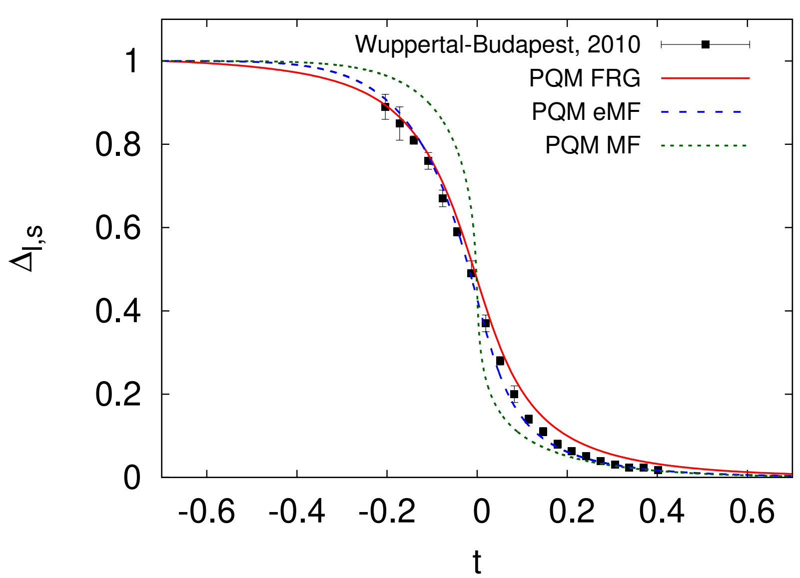
$$\epsilon=-p+Ts+\sum_{f=u,d,s}\mu_f n_f$$

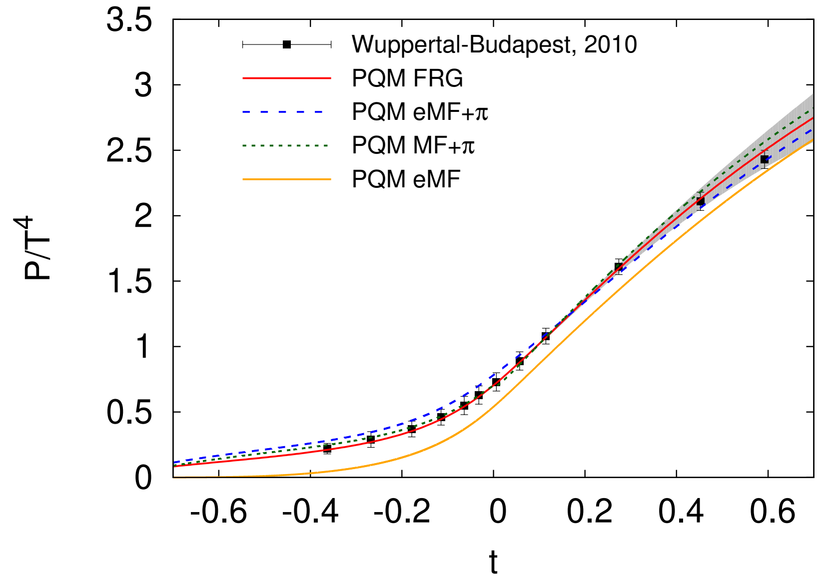
夸克数密度:

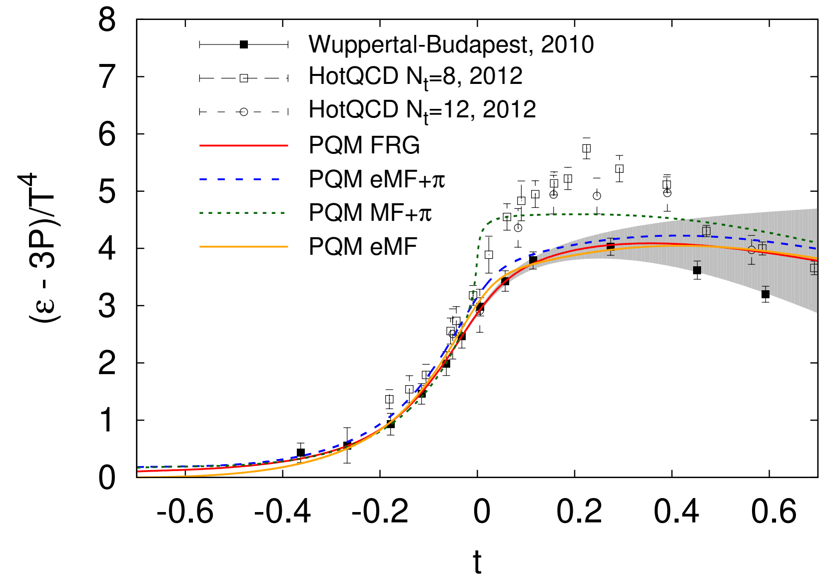
$$n_f=\frac{\partial p}{\partial \mu_f}$$

QM 模型的状态方程

- 约化手征凝聚
- 压强
- Trace anomaly

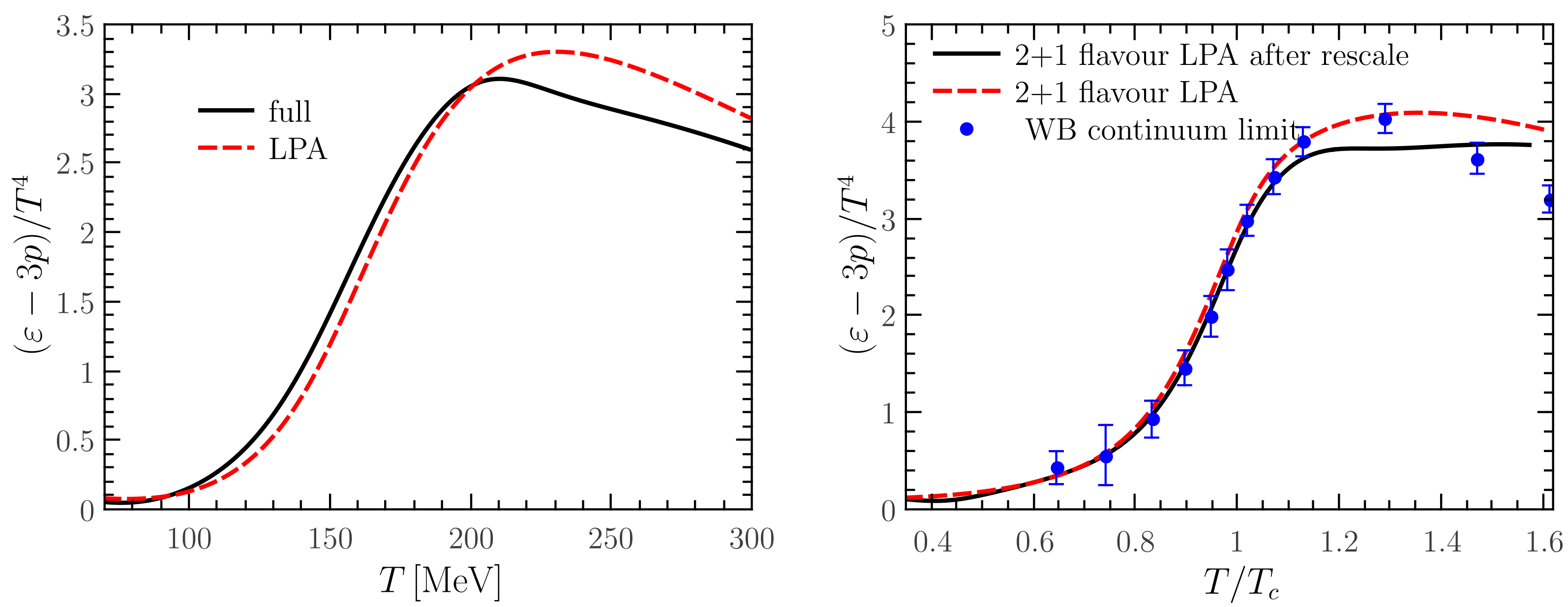






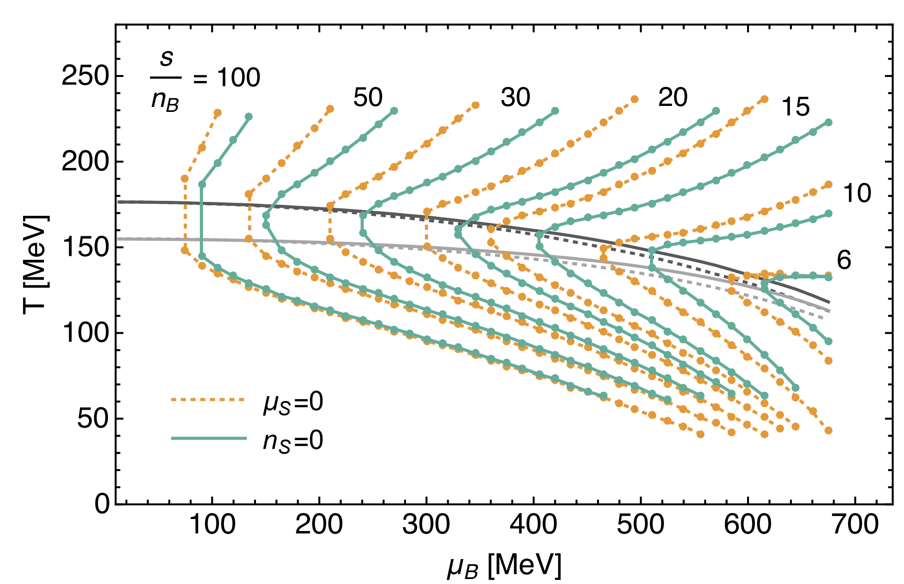
Herbst, *et al.* , arXiv: 1308.3621

- Trace anomaly beyond LPA



Fu, *et al.*, arXiv: 1508.06504

- 等熵线



Fu, *et al.*, arXiv: 1808.00410

重子数涨落

由热力学势，可以得到净重子数的 n 阶广义磁化率：

$$\chi_n^B = \frac{\partial^n}{\partial(\mu_B/T)^n}\frac{p}{T^4}$$

净重子数分布的 n 阶矩定义为

$$\langle(\delta N_B)^n\rangle = \sum_{N_B=-\infty}^{\infty}(\delta N_B)^n P(N_B)$$

其中 $\delta N_B = N_B - \langle N_B\rangle$，$P(N_B)$ 表示净重子数的几率分布，容易得到 n 阶广义磁化率和相应的 n 阶矩的关系

$$\chi_1^B = \frac{1}{VT^3}\langle N_B\rangle, \qquad \chi_2^B = \frac{1}{VT^3}\langle(\delta N_B)^2\rangle$$

$$\chi_3^B = \frac{1}{VT^3}\langle(\delta N_B)^3\rangle, \qquad \chi_4^B = \frac{1}{VT^3}\Big(\langle(\delta N_B)^4\rangle - 3\langle(\delta N_B)^2\rangle^2\Big)$$

前四阶分别对应净重子数分布的平均值，方差，偏斜度，峰度：

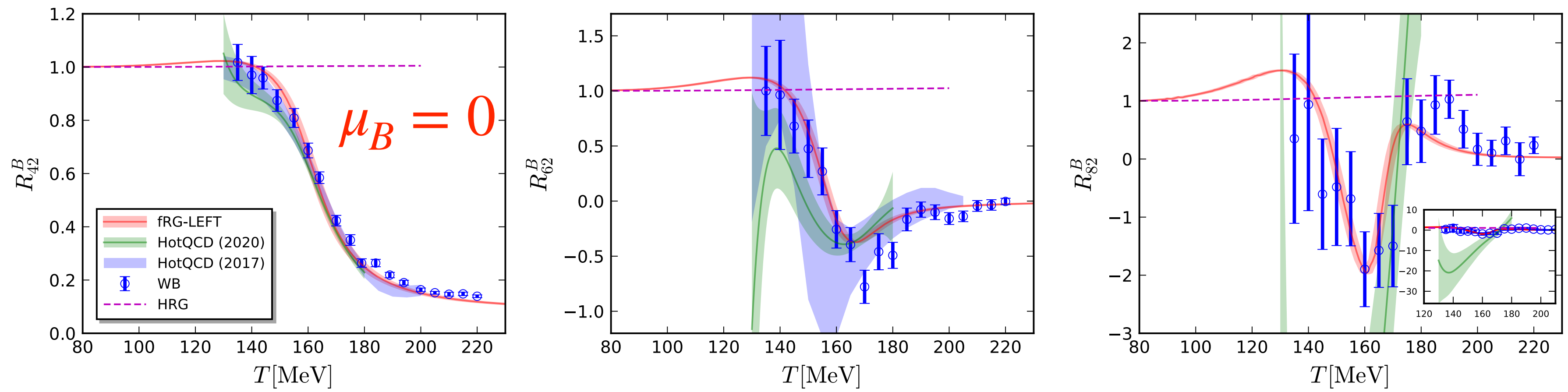
$$M = VT^3\chi_1^B, \qquad \sigma^2 = VT^3\chi_2^B, \qquad S = \frac{\chi_3^B}{\chi_2^B\sigma}, \qquad \kappa = \frac{\chi_4^B}{\chi_2^B\sigma^2}$$

为了消除对体积的依赖性，通常我们也使用涨落的比值：

$$R_{nm}^B = \frac{\chi_n^B}{\chi_m^B}$$

QM 模型的重子数涨落

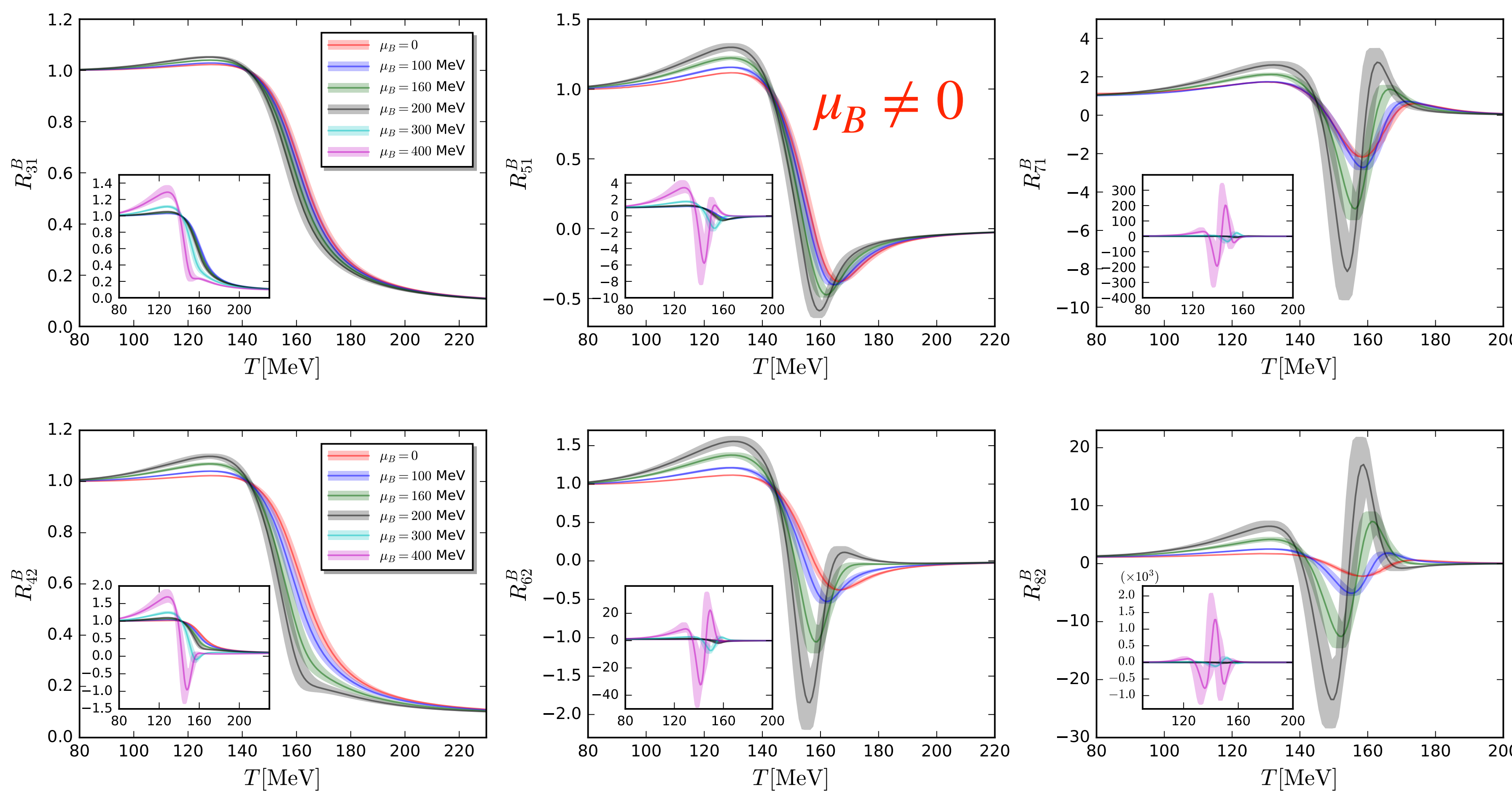
- fRG 和格点QCD以及强子共振气体(HRG)的比较



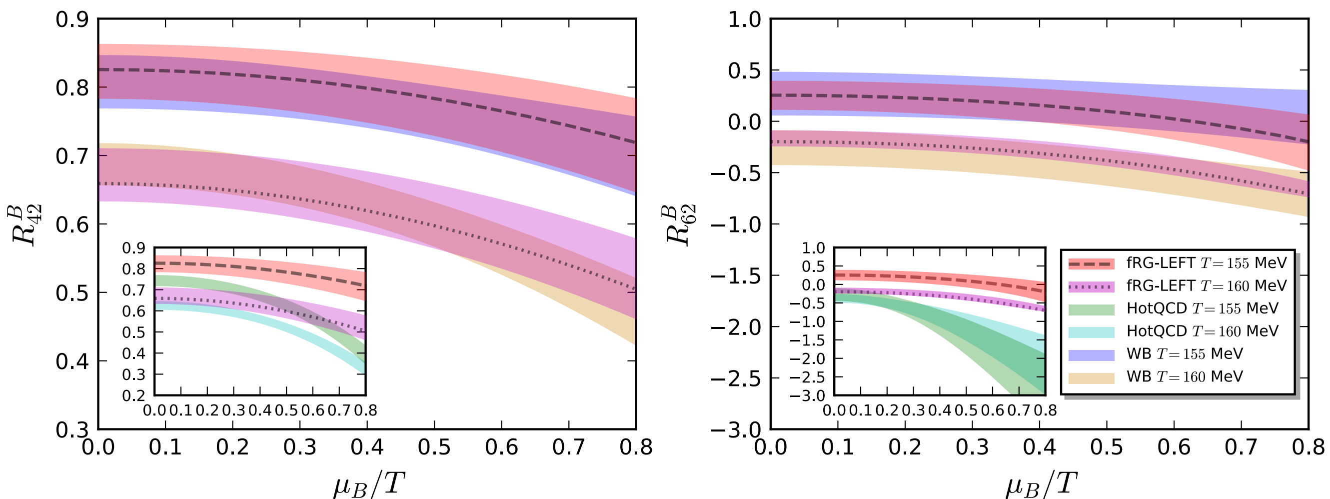
HotQCD: A. Bazavov *et al.*, *PRD* 95 (2017) 054504; *PRD* 101 (2020) 074502 fRG: WF, Luo, Pawlowski, Rennecke, Wen, Yin, *PRD* 104 (2021) 094047

WB: S. Borsanyi *et al.*, *JHEP* 10 (2018) 205

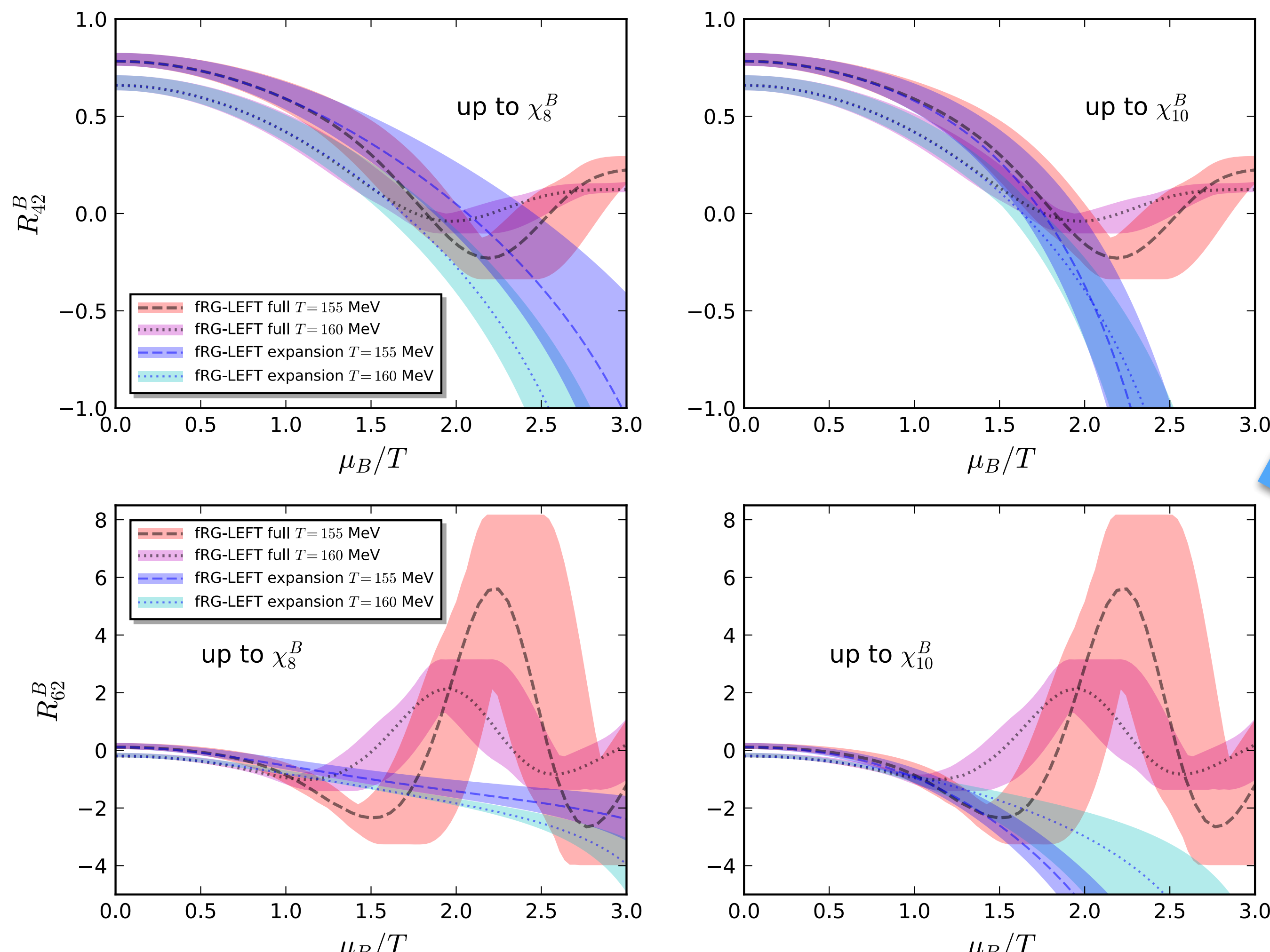
- 有限密度



化学势泰勒展开的收敛半径



$\mu_B \neq 0$ **fRG vs Lattice**



full vs expanded

将压强在 $\mu_B = 0$ 作 Taylor 展开

$$\frac{p(\mu_B)}{T^4} = \frac{p(0)}{T^4} + \sum_{i=1}^{\infty} \frac{\chi^B_{2i}(0)}{(2i)!}\hat{\mu}_B^{2i}$$

同样也可以得到有限化学势下的涨落：

$$\chi^B_2(\mu_B) \simeq \chi^B_2(0) + \frac{\chi^B_4(0)}{2!}\hat{\mu}_B^2 + \frac{\chi^B_6(0)}{4!}\hat{\mu}_B^4 + \frac{\chi^B_8(0)}{6!}\hat{\mu}_B^6$$

$$\chi^B_4(\mu_B) \simeq \chi^B_4(0) + \frac{\chi^B_6(0)}{2!}\hat{\mu}_B^2 + \frac{\chi^B_8(0)}{4!}\hat{\mu}_B^4$$

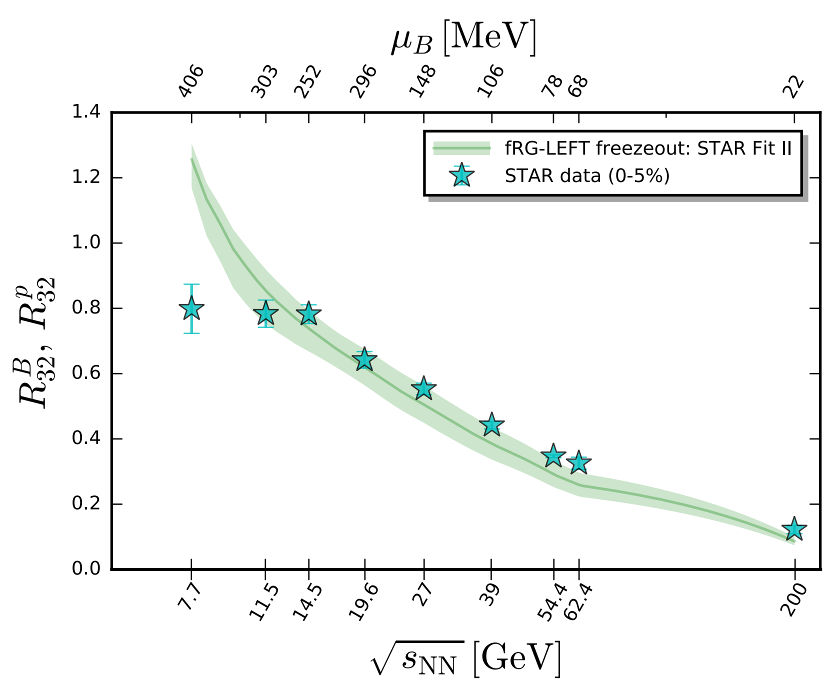
$$\chi^B_6(\mu_B) \simeq \chi^B_6(0) + \frac{\chi^B_8(0)}{2!}\hat{\mu}_B^2$$

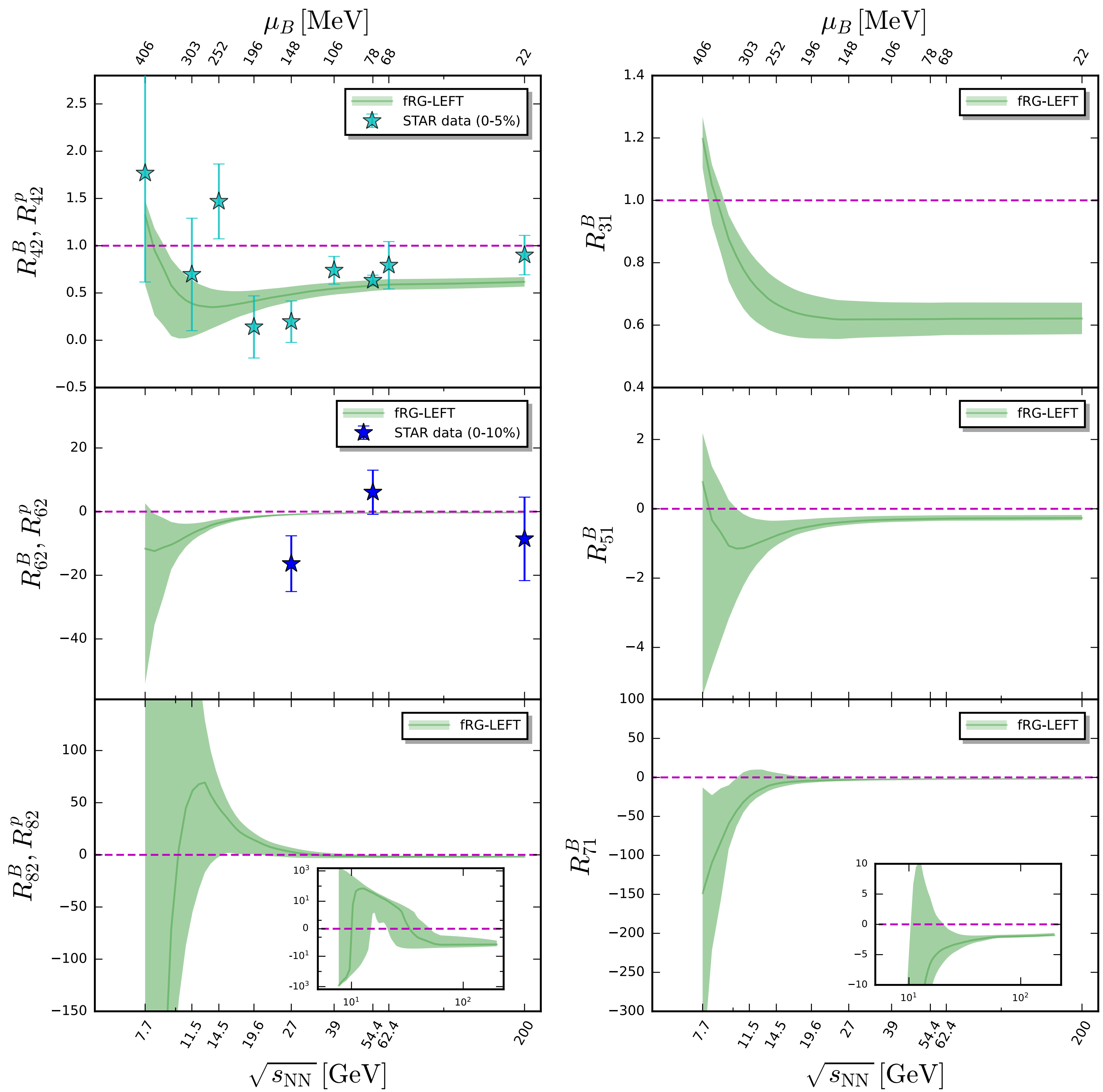
$T = 155\,\mathrm{MeV}: \quad |\mu_B/T|_{\mathrm{Max}} \approx 1.5\,,$

$T = 160\,\mathrm{MeV}: \quad |\mu_B/T|_{\mathrm{Max}} \approx 1.2$

Yang-Lee edge singularity?

理论与实验涨落观测量的比较





STAR (R^p_{32}, R^p_{42}):

J. Adam *et al.* (STAR), *PRL* 126 (2021) 092301

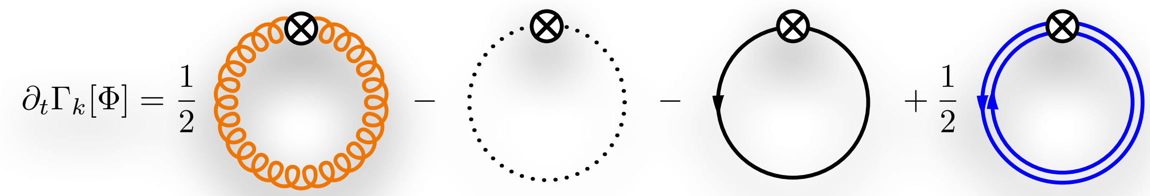
STAR (R^p_{62}):

M. Abdallah *et al.* (STAR), arXiv:2105.14698

fRG:

WF, Luo, Pawlowski, Rennecke, Wen, Yin,
PRD 104 (2021) 094047

有限温有限密 QCD



QCD 的有效作用量:

$$
\begin{aligned}
&\Gamma_k[\Phi] \\
&= \int_x \Big\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + Z_{c,k} (\partial_\mu \bar{c}^a) D^{ab}_\mu c^b + \frac{1}{2\xi} \left(\partial_\mu A^a_\mu\right)^2 + Z_{q,k} \bar{q} \left(\gamma_\mu D_\mu - \gamma_0 \hat{\mu}\right) q \\
&\quad + m_s(\sigma_s) \bar{q}_s q_s - \lambda_{q,k} \left[\left(\bar{q}_l T^0 q_l\right)^2 + \left(\bar{q}_l i\gamma_5 \vec{T} q_l\right)^2 \right] + h_k \bar{q}_l \left(T^0 \sigma + i\gamma_5 \vec{T} \cdot \vec{\pi}\right) q_l \\
&\quad + \frac{1}{2} Z_{\phi,k} (\partial_\mu \phi)^2 + V_k(\rho, A_0) - c_\sigma \sigma - \frac{1}{\sqrt{2}} c_{\sigma_s} \sigma_s \Big\} + \Delta\Gamma_{\text{glue}}
\end{aligned}
$$

其中 $\Phi = (A, c, \bar{c}, q, \bar{q}, \sigma, \pi)$，$\phi = (\sigma, \pi)$，各个场的波函数重整化由下式给出

$$\bar{\Phi} = Z^{1/2}_{\Phi,k} \Phi$$

胶子场强张量:

$$F^a_{\mu\nu} = Z^{1/2}_{A,k} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + Z^{1/2}_{A,k} \bar{g}_{\text{glue},k} f^{abc} A^b_\mu A^c_\nu\right)$$

强耦合:

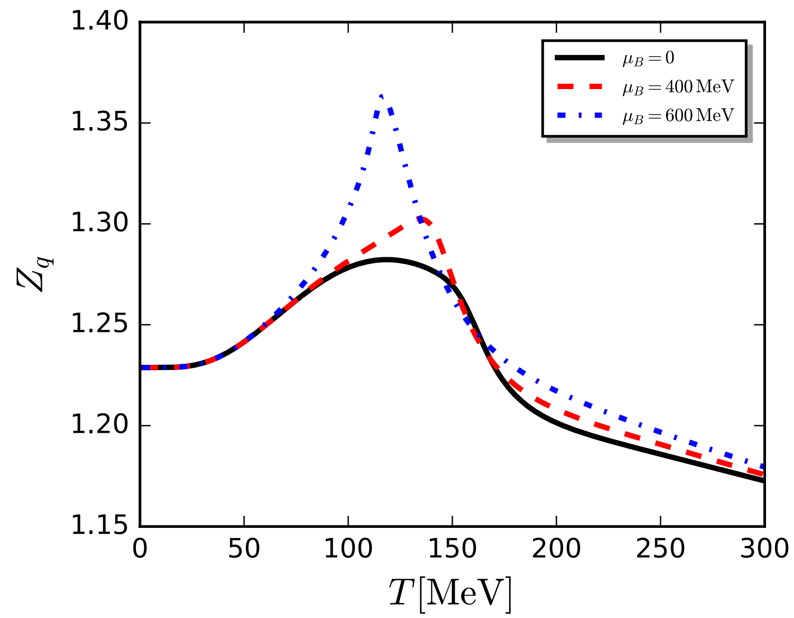
$$\bar{g}_{A^3,k} = \frac{\lambda_{A^3,k}}{Z^{3/2}_{A,k}}, \quad \bar{g}_{A^4,k} = \frac{\lambda^{1/2}_{A^4,k}}{Z_{A,k}}, \quad \bar{g}_{\bar{c}cA,k} = \frac{\lambda_{\bar{c}cA,k}}{Z^{1/2}_{A,k} Z_{c,k}}, \quad \bar{g}_{\bar{q}qA,k} = \frac{\lambda_{\bar{q}qA,k}}{Z^{1/2}_{A,k} Z_{q,k}}$$

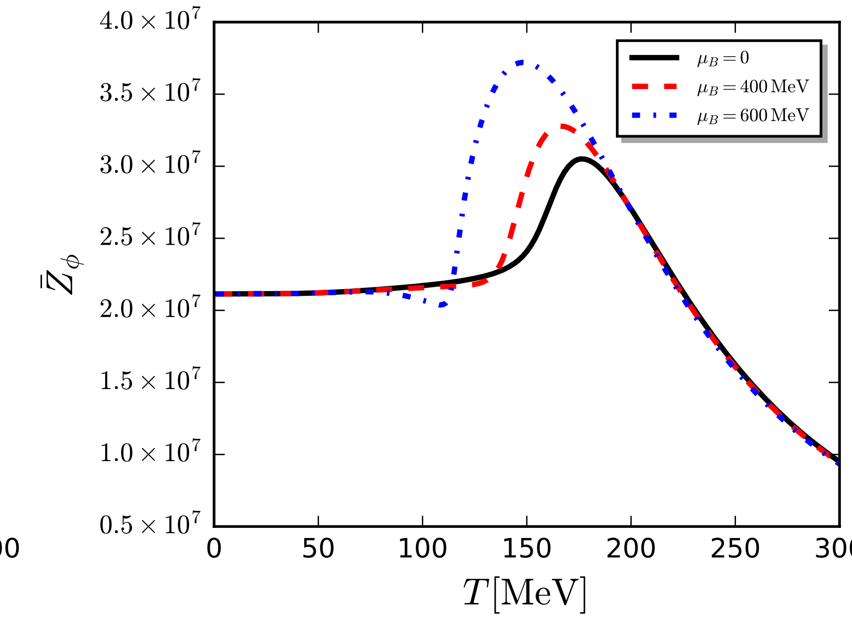
基本和伴随表示的协变导数分别为

$$D_\mu = \partial_\mu - i Z^{1/2}_{A,k} \bar{g}_{\bar{q}qA,k} A^a_\mu t^a$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} - Z^{1/2}_{A,k} \bar{g}_{\bar{c}cA,k} f^{abc} A^c_\mu$$

夸克、介子传播子及其反常量纲





WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

场 Φ 的反常量纲由下面的定义给出：

$$\eta_{\Phi,k} = -\frac{\partial_t Z_{\Phi,k}}{Z_{\Phi,k}}$$

其中 $Z_{\Phi,k}$ 是场 Φ 的波函数重整化，夸克两点关联函数：

$$\Gamma_k^{(2)\bar{q}q}(p) = -\frac{\delta^2 \Gamma_k[\Phi]}{\delta \bar{q}(-p)\delta q(p)}\bigg|_{\Phi=\Phi_{\mathrm{EoM}}}$$

$$= Z_{q,k}(p) i\gamma\cdot p + m_{q,k}(p)$$

投影到矢量和标量道，得到

$$Z_{q,k}(p) = \frac{1}{4i}\frac{1}{p^2}\mathrm{tr}\Big(\gamma\cdot p\,\Gamma_k^{(2)\bar{q}q}(p)\Big)$$

$$m_{q,k}(p) = \frac{1}{4}\mathrm{tr}\Big(\Gamma_k^{(2)\bar{q}q}(p)\Big)$$

夸克的反常量纲：

$$\eta_{q,k}(p_0) = \frac{1}{4Z_{q,k}(p_0)}\mathrm{Re}\left[\frac{\partial}{\partial(|\vec{p}\,|^2)}\mathrm{tr}\left(i\vec{\gamma}\cdot\vec{p}\left(\partial_t \Gamma_k^{(2)\bar{q}q}(p)\right)\right)\right]_{\vec{p}=0}$$

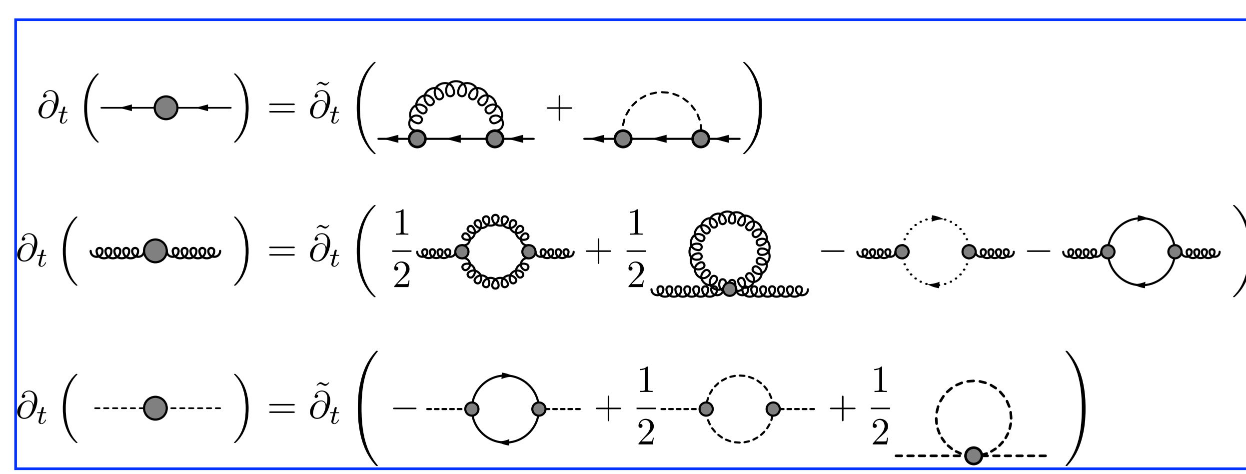
介子两点关联函数：

$$\Gamma_k^{(2)\pi\pi}(p) = \frac{\delta^2\Gamma_k[\Phi]}{\delta\pi_i(-p)\delta\pi_j(p)}\bigg|_{\Phi=\Phi_{\mathrm{EoM}}}$$

$$= \left(Z_{\pi,k}(p)p^2 + m_{\pi,k}^2\right)\delta_{ij}$$

介子的反常量纲：

$$\eta_\phi(0,\vec{p}\,) = -\frac{\delta_{ij}}{3Z_\phi(0,\vec{p}\,)}\frac{\partial_t\Gamma^{(2)}_{\pi_i\pi_j}(0,\vec{p}\,) - \partial_t\Gamma^{(2)}_{\pi_i\pi_j}(0,0)}{\vec{p}^{\,2}}$$



胶子传播子及其反常量纲

胶子反常量纲可以分为三个部分的贡献:

$$\eta_A = \eta_{A,\mathrm{vac}}^{\mathrm{QCD}} + \Delta\eta_A^{\mathrm{glue}} + \Delta\eta_A^{q}$$

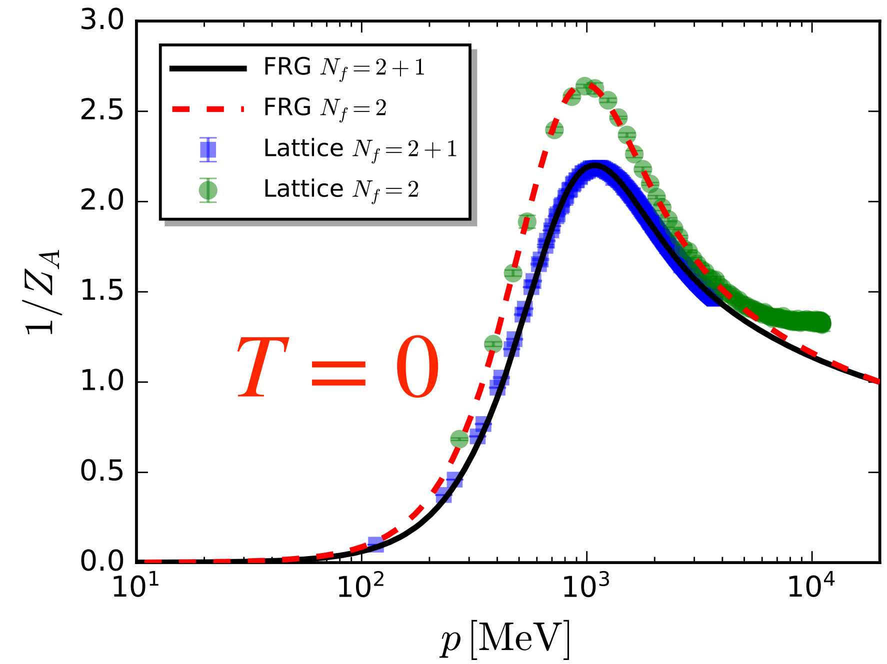
其中 $\eta_{A,\mathrm{vac}}^{\mathrm{QCD}}$ 代表真空的贡献，后两项代表有限温度有限密度介质的贡献，分别来自于胶子部分和夸克部分，真空部分又可以写成

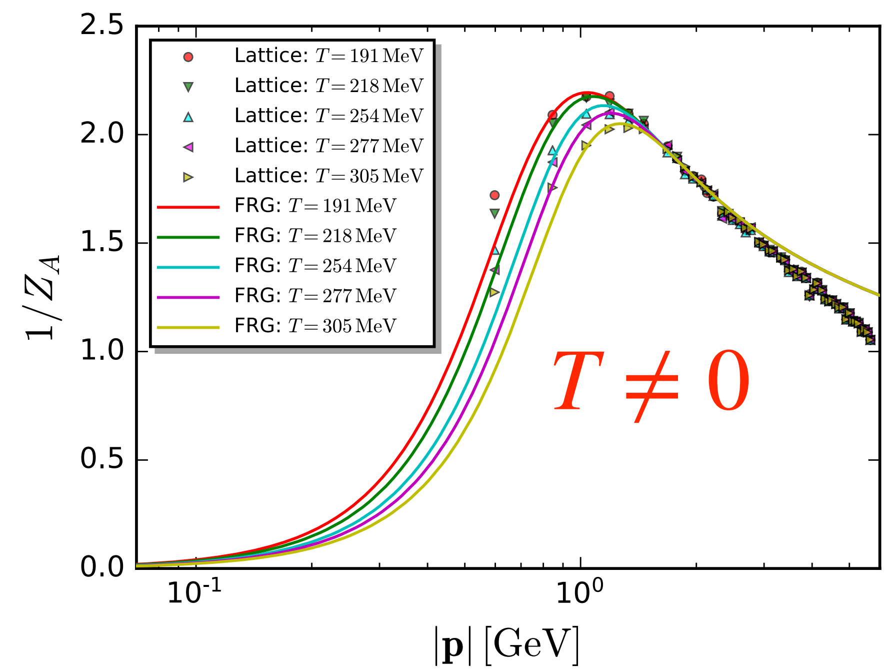
$$\eta_{A,\mathrm{vac}}^{\mathrm{QCD}} = \eta_{A,\mathrm{vac}}^{\mathrm{QCD}}\Big|_{N_f=2} + \eta_{A,\mathrm{vac}}^{s}$$

等号右边分别代表两味轻夸克和奇异夸克的贡献，前者可以从更完善的真空 fRG 计算或者格点 QCD 计算中得到，即

$$\eta_{A,\mathrm{vac}}^{\mathrm{QCD}}\Big|_{N_f=2} = -\left.\frac{p\partial_p Z_{A,k=0}^{\mathrm{QCD}}(p)}{Z_{A,k=0}^{\mathrm{QCD}}(p)}\right|_{p=k}$$

而后者通过自洽的计算得到，这样得到的 $N_f = 2+1$ 味的胶子传播子可与其他的计算结果进行比较





Lattice $N_f = 2$: Sternbeck *et al.*, *PoS* (2012) LATTICE2012, 243
Lattice $N_f = 2+1$: Boucaud *et al.*, *PRD* 98 (2018) 114515
fRG $N_f = 2$: Cyrol, Mitter, Pawlowski,Strodthoff, *PRD* 97 (2018) 054006
fRG: WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

强耦合常数

在非微扰区域，由于胶子质量 gap 的产生，我们需要区分不同的强耦合常数，比如规范场部分：

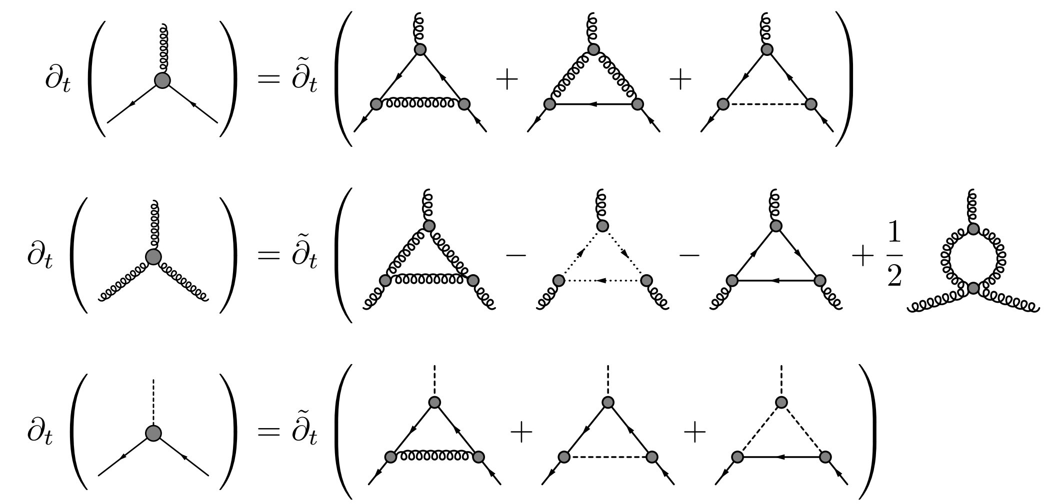
$$\alpha_{A^3,k} = \frac{1}{4\pi}\frac{\lambda^2_{A^3,k}}{Z^3_{A,k}}, \qquad \alpha_{A^4,k} = \frac{1}{4\pi}\frac{\lambda_{A^4,k}}{Z^2_{A,k}}, \qquad \alpha_{\bar{c}cA,k} = \frac{1}{4\pi}\frac{\lambda^2_{\bar{c}cA,k}}{Z_{A,k}\, Z^2_{c,k}}$$

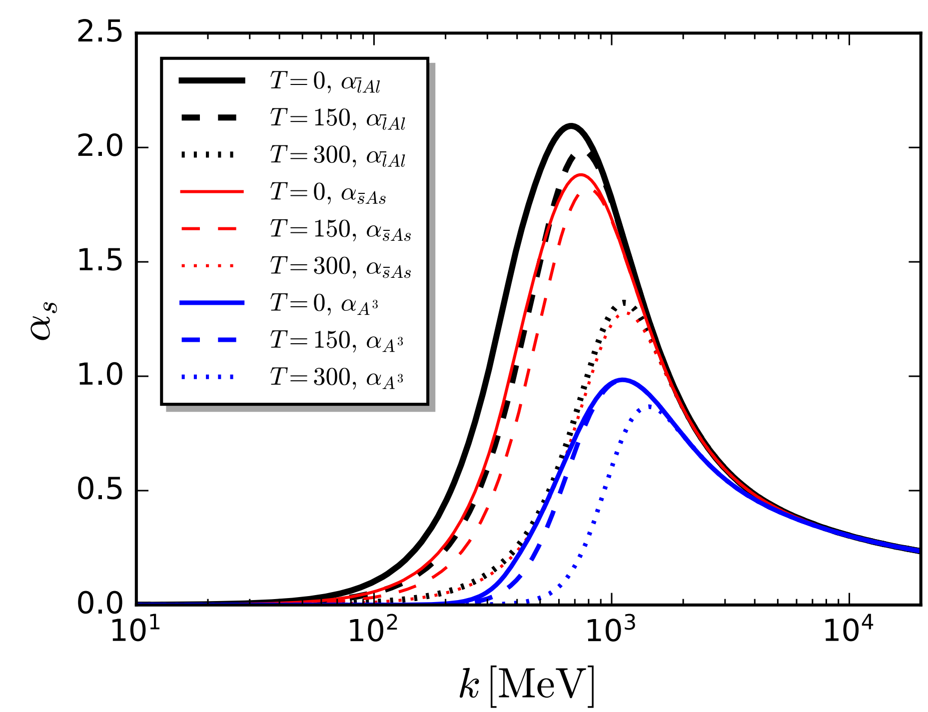
物质场部分

$$\alpha_{\bar{l}lA,k} = \frac{1}{4\pi}\frac{\lambda^2_{\bar{l}lA,k}}{Z_{A,k}\, Z^2_{q,k}}, \qquad \alpha_{\bar{s}sA,k} = \frac{1}{4\pi}\frac{\lambda^2_{\bar{s}sA,k}}{Z_{A,k}\, Z^2_{q,k}}$$

我们以夸克胶子顶点为例，其流方程为

$$\partial_t \bar{g}_{\bar{q}qA,k} = \left(\frac{1}{2}\eta_A + \eta_q\right)\bar{g}_{\bar{q}qA,k} + \frac{1}{8(N_c^2-1)} \times \mathrm{tr}\left[\left(\overline{\mathrm{Flow}}^{(3)}_{\bar{q}qA}\right)^a_\mu \left(S^{(3)}_{\bar{q}qA}\right)^a_\mu\right](\{p\})$$





WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

QCD 的动力学强子化

引入一个 RG 能标依赖的复合场

$$\hat{\phi}_k(\hat{\varphi})，其中基本场为\ \hat{\varphi} = (\hat{A}, \hat{c}, \hat{\bar{c}}, \hat{q}, \hat{\bar{q}})$$

使得对于比如 $\sigma - \pi$ 道，有

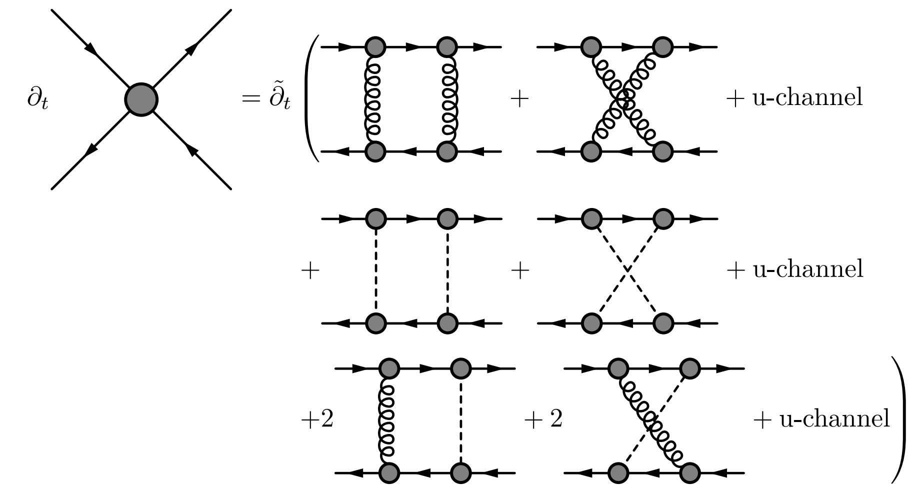
$$\langle \partial_t \hat{\phi}_k \rangle = \dot{A}_k\, \bar{q}\tau q，其中\ \tau = (T^0, i\gamma_5 \vec{T})$$

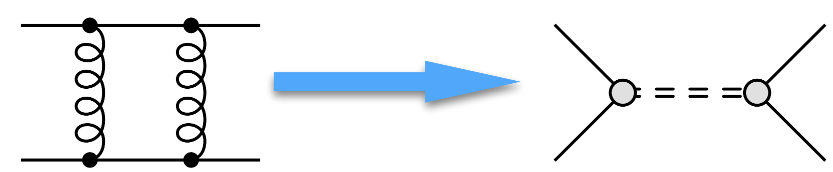
这样有效作用量流方程被修正为

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2}\mathrm{STr}\big(G_k[\Phi]\,\partial_t R_k\big) + \mathrm{Tr}\left(G_{\phi\Phi_a}[\Phi]\frac{\delta\langle\partial_t\hat{\phi}_k\rangle}{\delta\Phi_a} R_\phi\right)$$

$$-\int \langle \partial_t \hat{\phi}_{k,i}\rangle \left(\frac{\delta\Gamma_k[\Phi]}{\delta\phi_i} + c_\sigma \delta_{i\sigma}\right)$$

WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032





对每一个RG 能标，作Hubbard-Stratonovich 变换

利用该修正后的 Wetterich 方程，我们可以得到四夸克耦合的流方程：

$$\partial_t \tilde{\lambda}_{q,k} = 2(1+\eta_{q,k})\tilde{\lambda}_{q,k} + \overline{\mathrm{Flow}}^{(4)}_{(\bar{q}\tau q)^2} + \dot{\bar{A}}\,\bar{h}_k$$

要求

$$\tilde{\lambda}_{q,k} = 0\,, \qquad \forall k$$

得到强子化函数

$$\dot{\bar{A}} = -\frac{1}{\bar{h}_k}\overline{\mathrm{Flow}}^{(4)}_{(\bar{q}\tau q)^2}$$

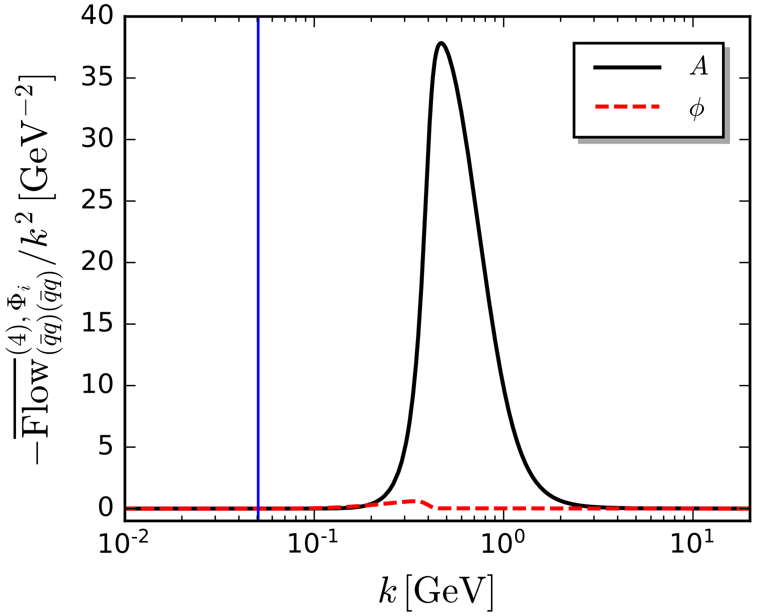
代入下面的方程，可以得到 Yukawa 耦合的流方程

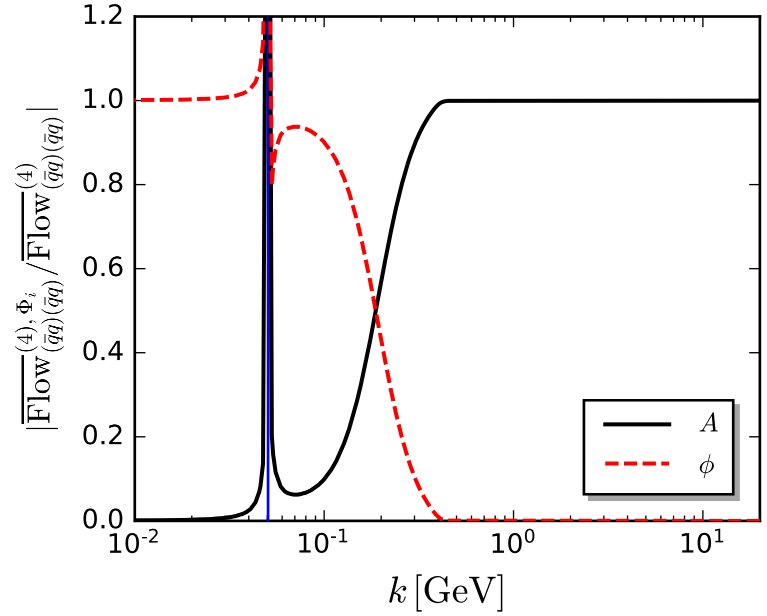
$$\partial_t \bar{h}_k = \left(\frac{1}{2}\eta_{\phi,k} + \eta_{q,k}\right)\bar{h}_k + \overline{\mathrm{Flow}}^{(3)}_{(\bar{q}\vec{\tau}q)\vec{\pi}} - \tilde{m}^2_{\pi,k}\dot{\bar{A}}$$

Gies, Wetterich , *PRD* 65 (2002) 065001; 69 (2004) 025001
Pawlowski, *AP* 322 (2007) 2831
Flörchinger, Wetterich, *PLB* 680 (2009) 371

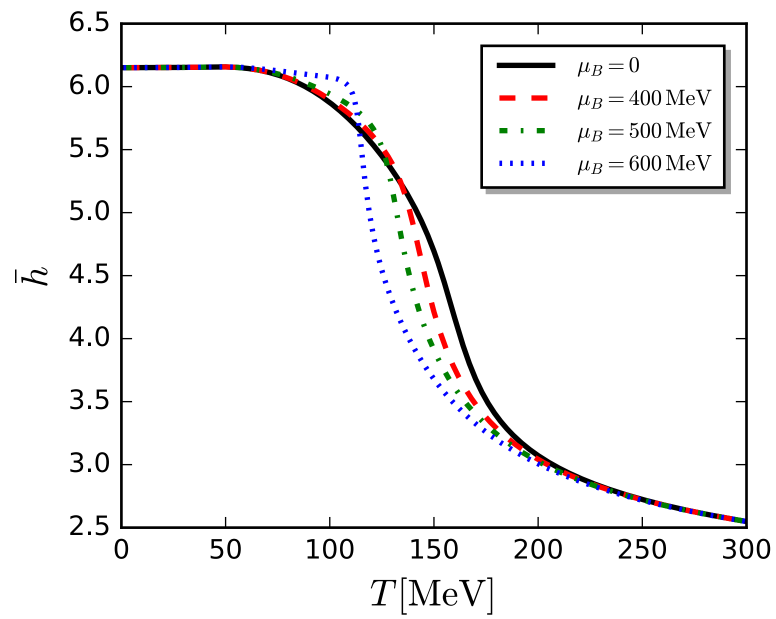
四夸克耦合和 Yukawa 耦合

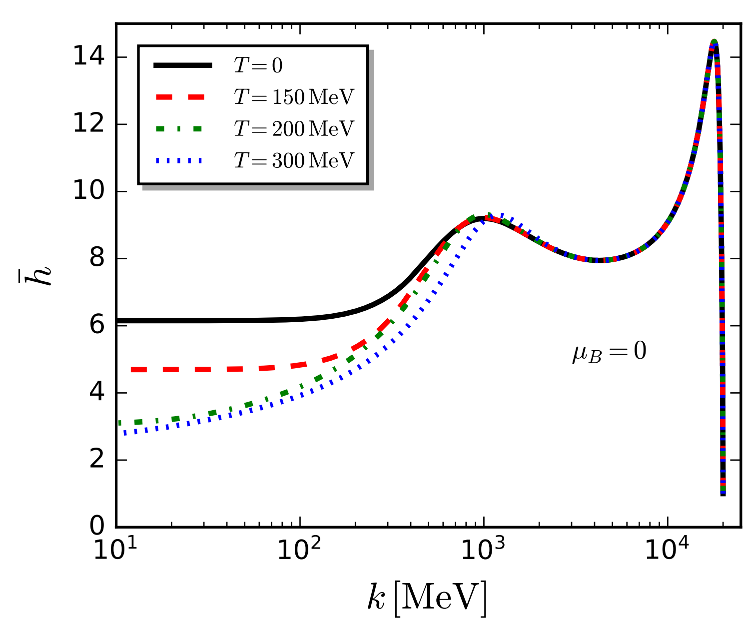
- 四夸克耦合





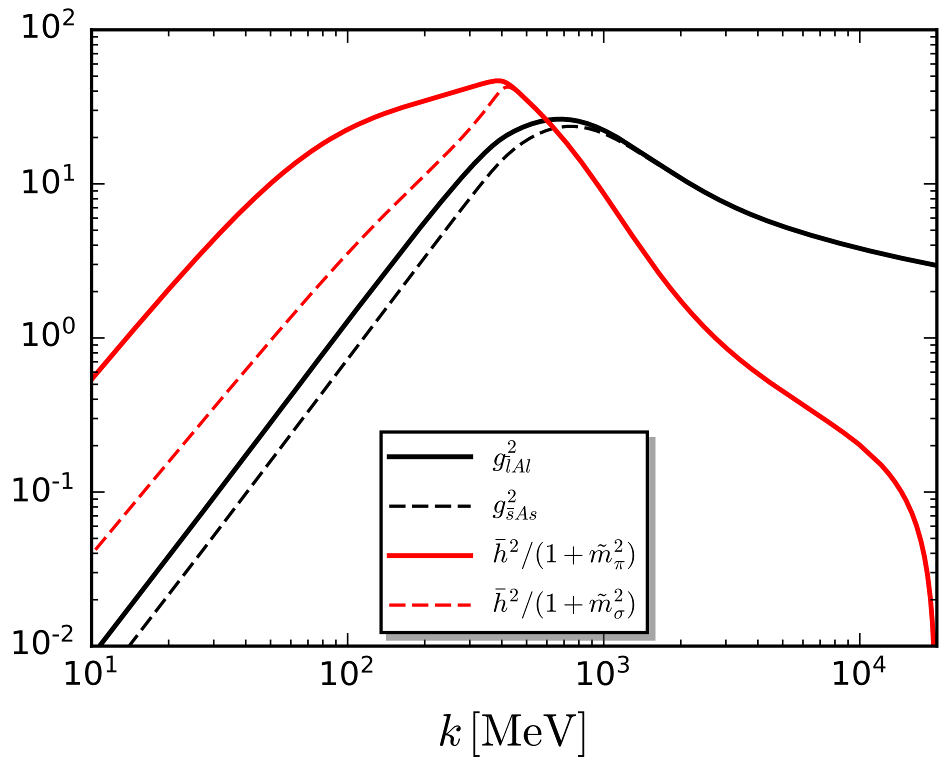
- Yukawa 耦合



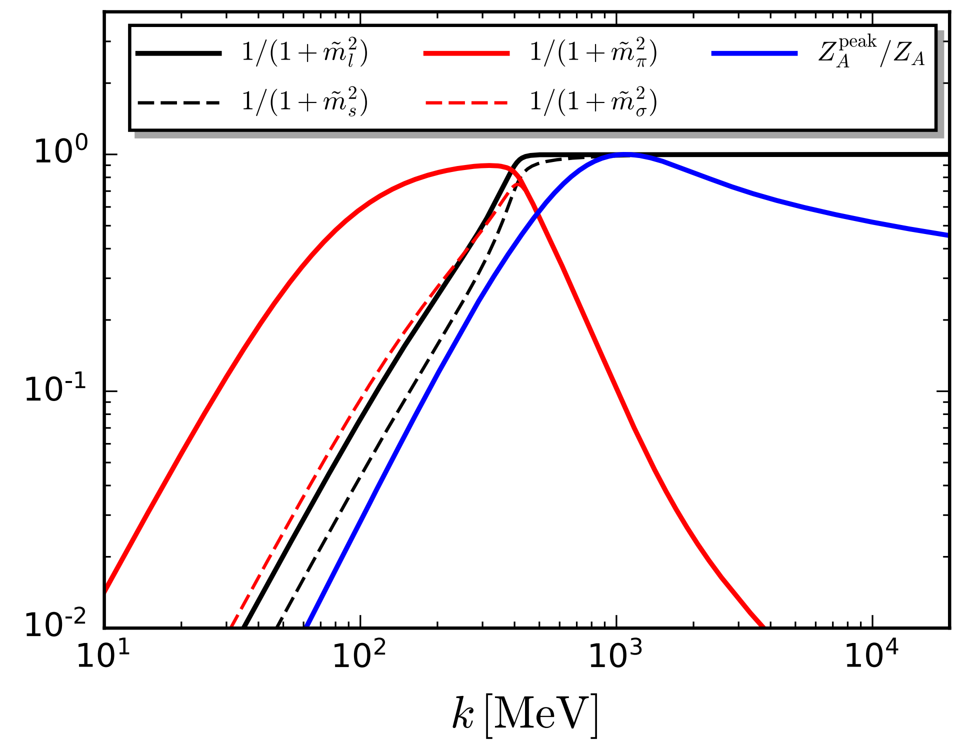


从 QCD 到低能有效模型的自然呈现

- 交换耦合强度



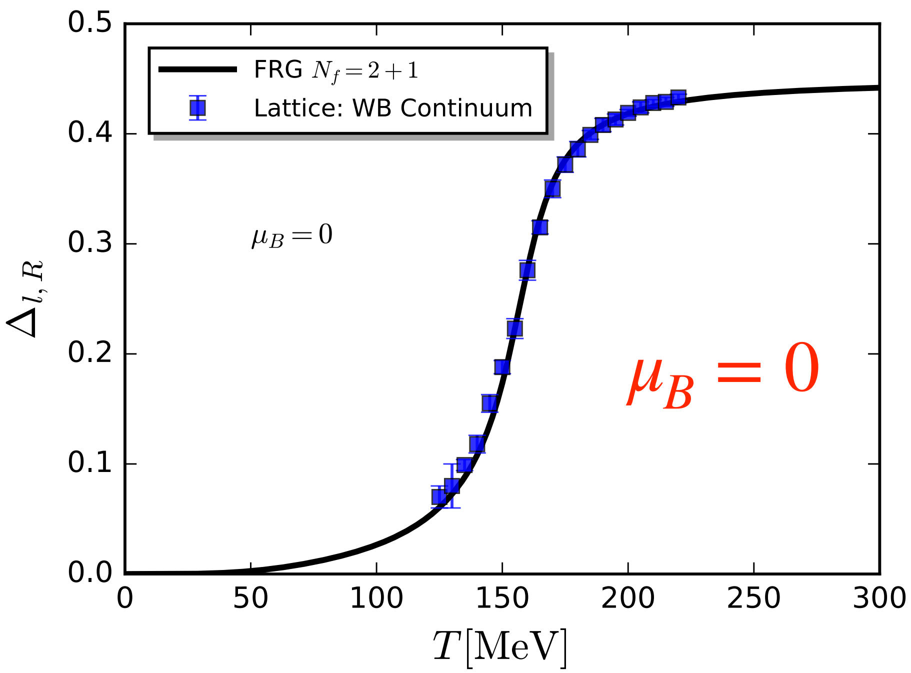
- 传播子 gapping



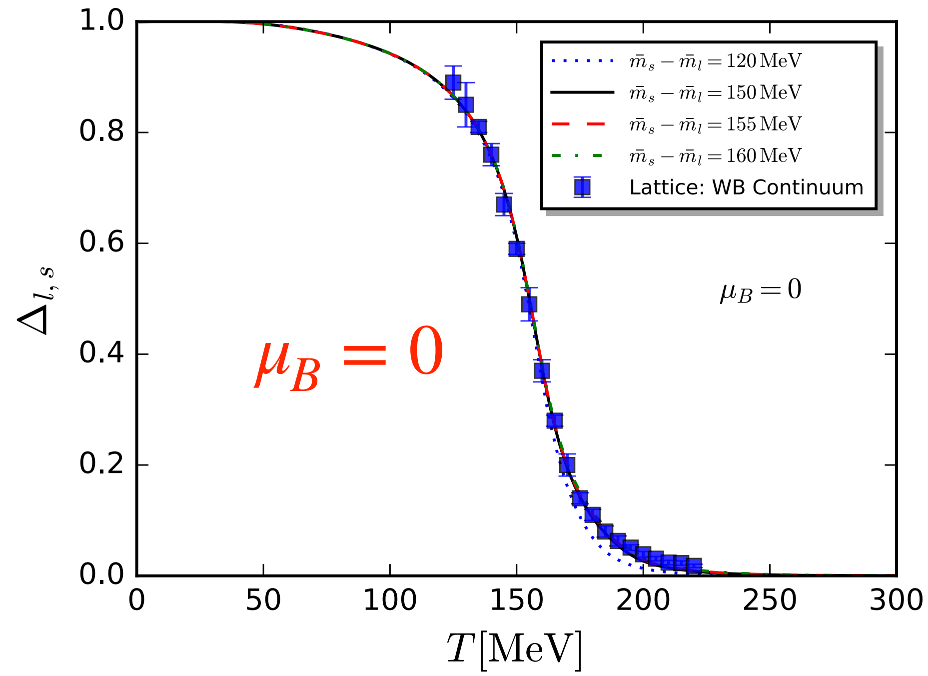
- Composite (mesonic) degrees of freedom take over active dynamics from partonic ones when the RG scale is lowered down $k \lesssim 600 \sim 800$ MeV.
- LEFTs emerge naturally from fundamental theory in the regime of low energy, in agreement with the viewpoint of RG.

夸克凝聚

- 重整化轻夸克凝聚



- 约化凝聚



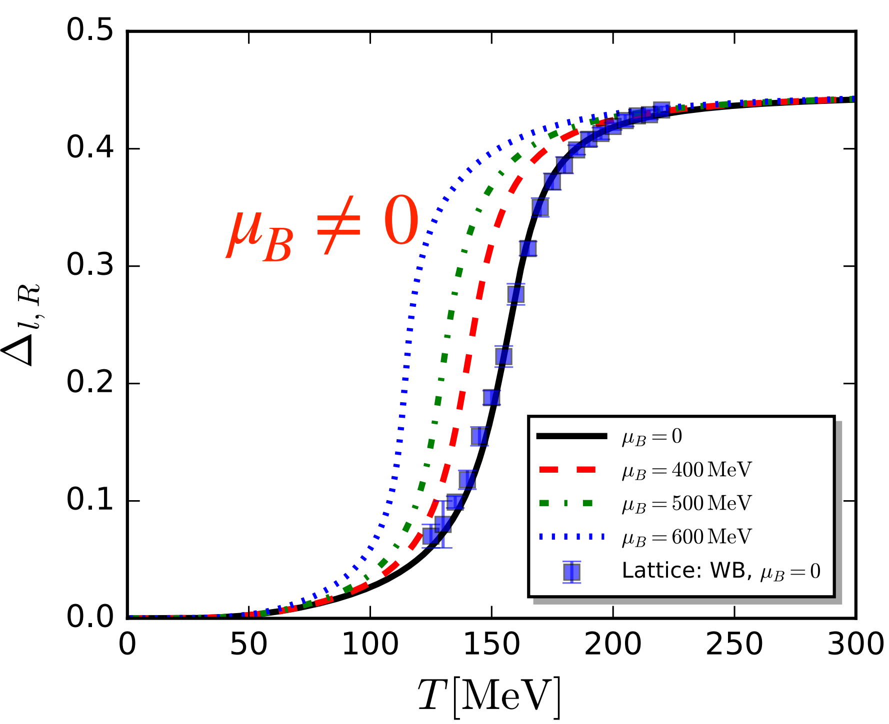
夸克凝聚

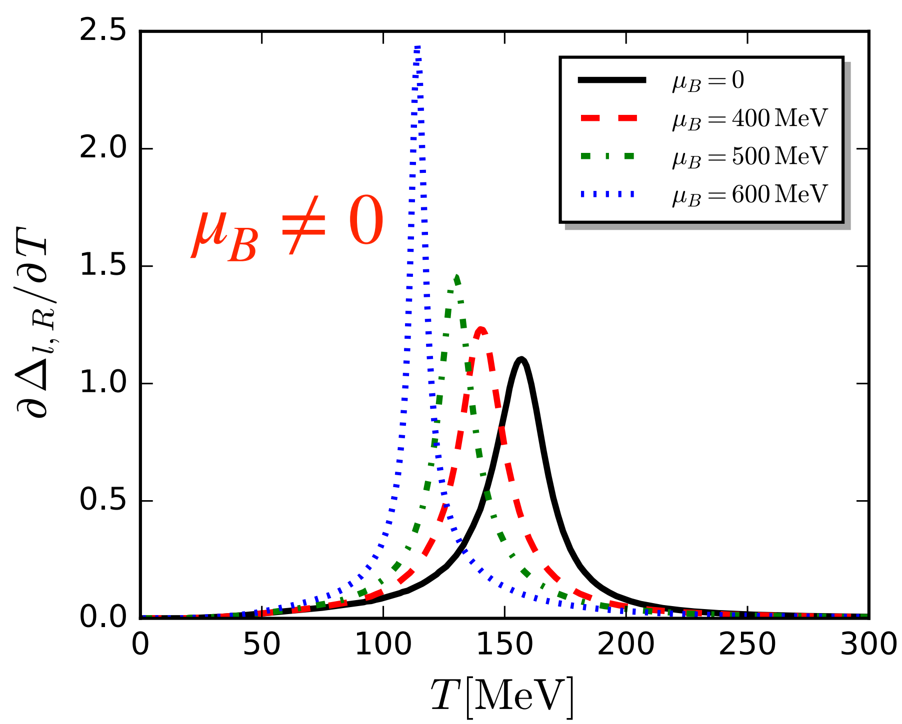
$$\Delta_{q_i} \simeq -m^0_{q_i} T \sum_{n\in\mathbb{Z}} \int \frac{d^3q}{(2\pi)^3} \mathrm{tr}\, G_{q_i\bar{q}_i}(q)\,,$$

$$\Delta_{q_i,R} = \frac{1}{\mathcal{N}_R}\Big[\Delta_{q_i}(T,\mu_q) - \Delta_{q_i}(0,0)\Big]\,.$$

约化凝聚

$$\Delta_{l,s}(T,\mu_q) = \frac{\Delta_l(T,\mu_q) - \left(\frac{m^0_l}{m^0_s}\right)^2 \Delta_s(T,\mu_q)}{\Delta_l(0,0) - \left(\frac{m^0_l}{m^0_s}\right)^2 \Delta_s(0,0)}$$

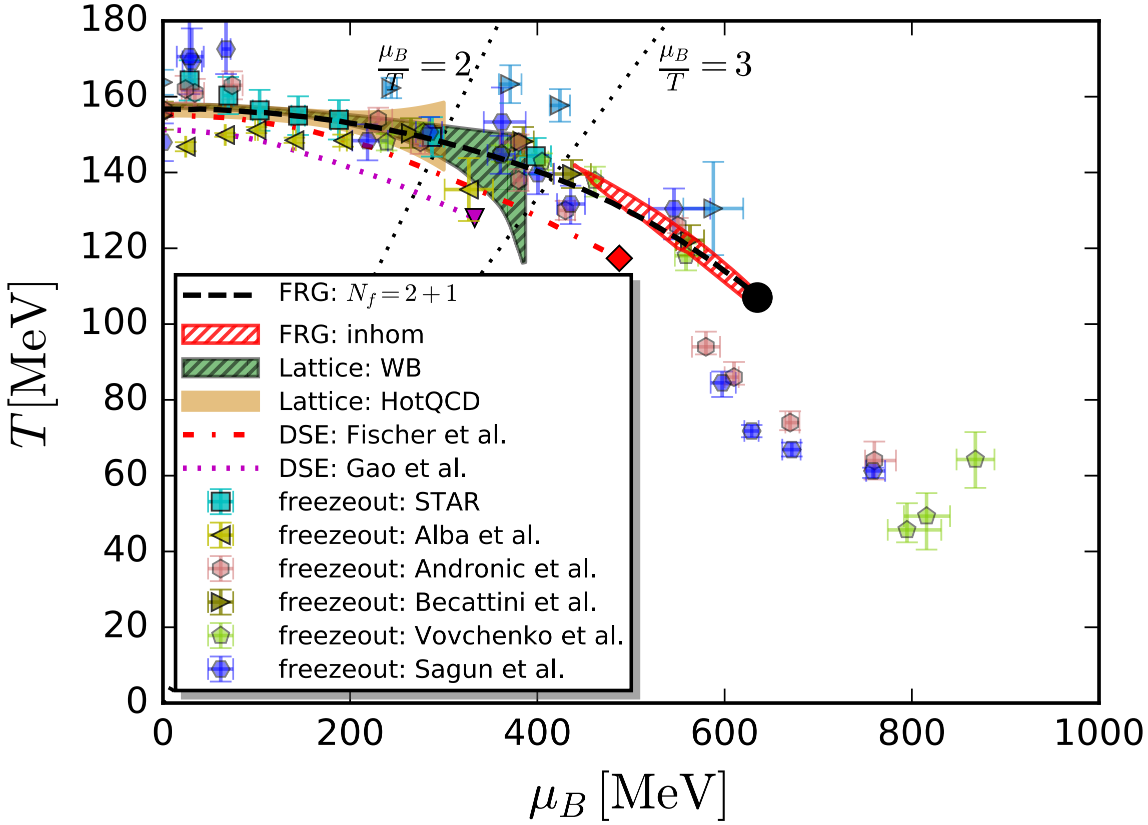




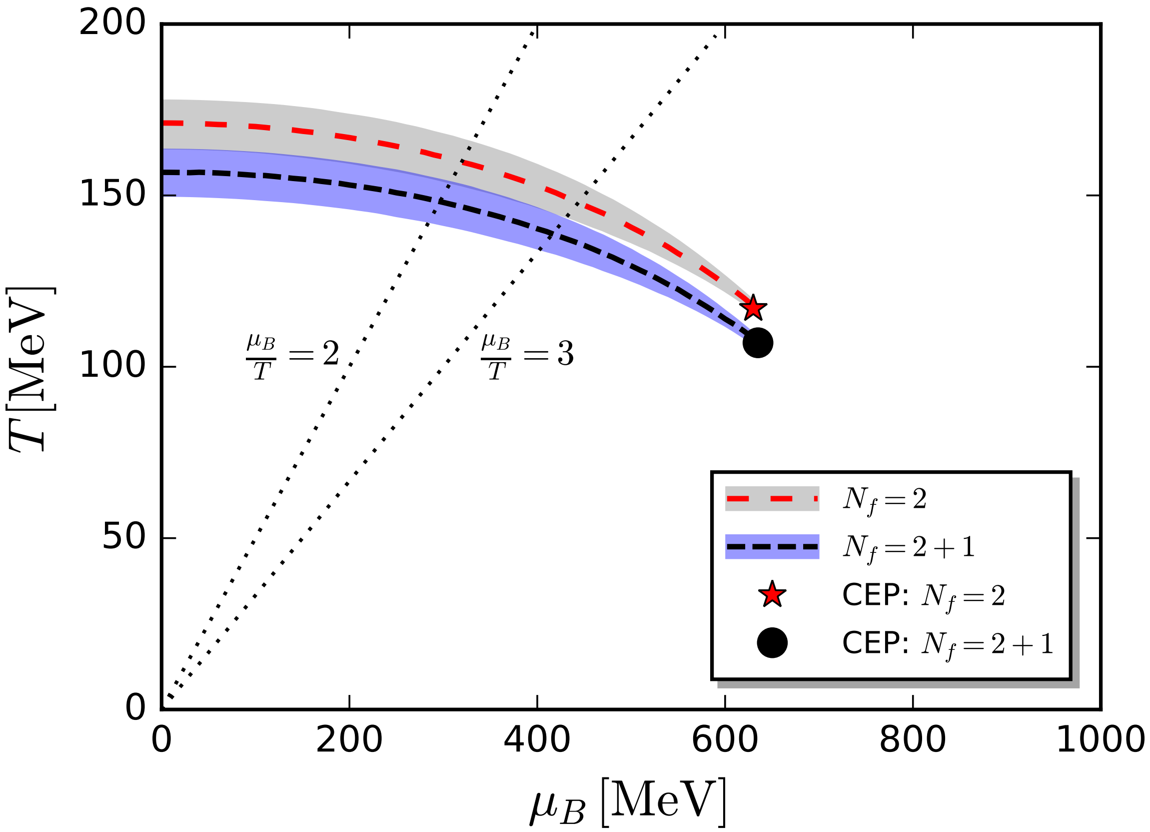
Lattice: Borsanyi *et al.* (WB), *JHEP* 09 (2010) 073

fRG: WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

相边界和相边界曲率



fRG: WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032



CEP 位置:

$$(T_{\mathrm{CEP}}, \mu_{B_{\mathrm{CEP}}})_{N_f=2+1} = (107, 635)\mathrm{MeV}$$
$$(T_{\mathrm{CEP}}, \mu_{B_{\mathrm{CEP}}})_{N_f=2} = (117, 630)\mathrm{MeV}$$

相边界曲率:

$$\frac{T_c(\mu_B)}{T_c} = 1 - \kappa\left(\frac{\mu_B}{T_c}\right)^2 + \lambda\left(\frac{\mu_B}{T_c}\right)^4 + \cdots$$

$$\kappa_{N_f=2+1} = 0.0142(2)$$
$$\kappa_{N_f=2} = 0.0176(1)$$

格点 QCD 结果:

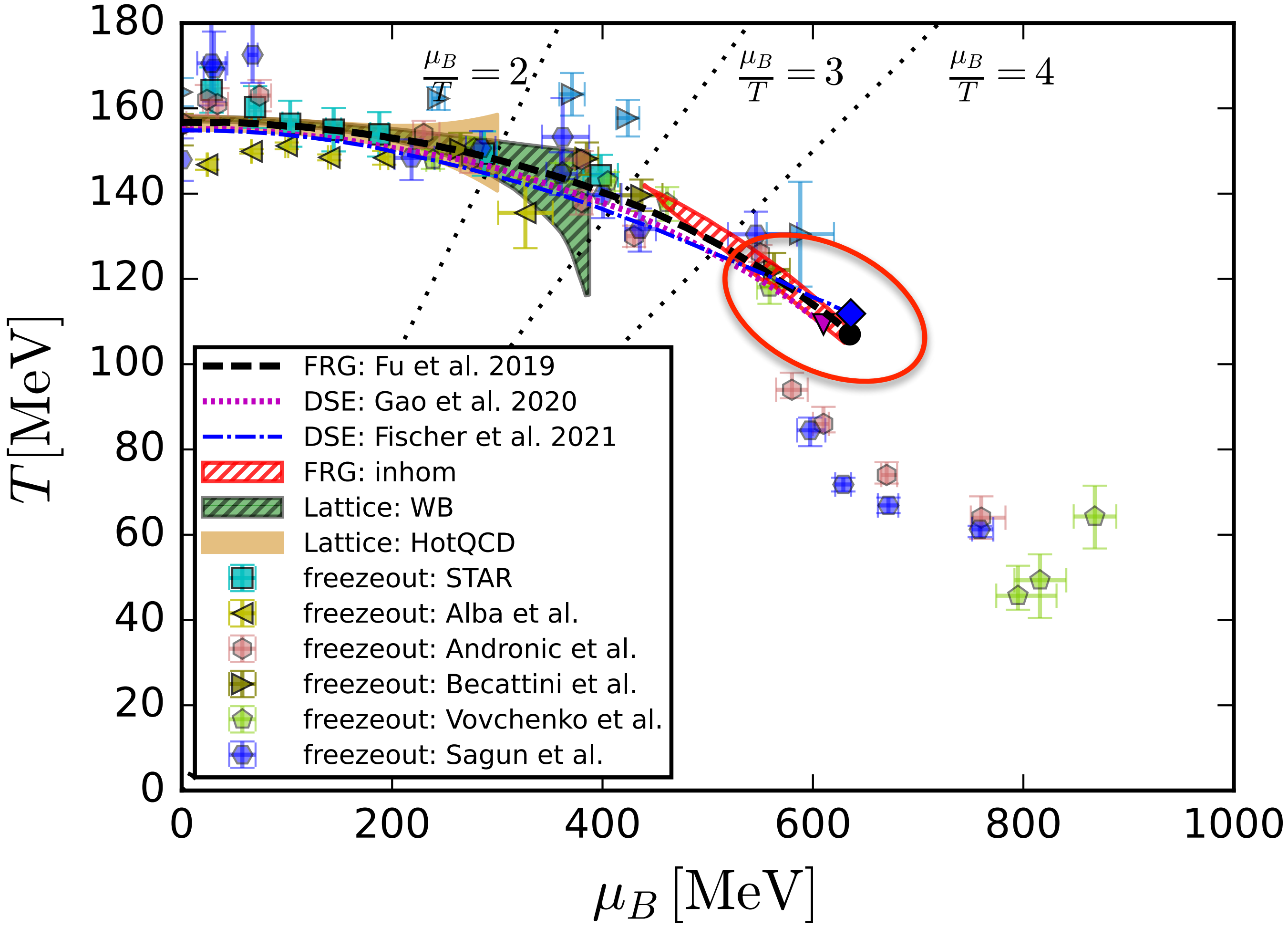
$$\kappa = 0.0149 \pm 0.0021$$

Lattice: Bellwied *et al.* (WB), *PLB* 751 (2015) 559

$$\kappa = 0.015 \pm 0.004$$

Lattice: Bazavov *et al.* (HotQCD), *PLB* 795 (2019) 15

泛函 QCD 关于 CEP 位置的最新估计



文献中关于 CEP 位置的最新估计

fRG:

● $(T, \mu_B)_{\text{CEP}} = (107, 635)$**MeV**

fRG: WF, Pawlowski, Rennecke, *PRD* 101 (2020), 054032

DSE:

▼ $(T, \mu_B)_{\text{CEP}} = (109, 610)$**MeV**

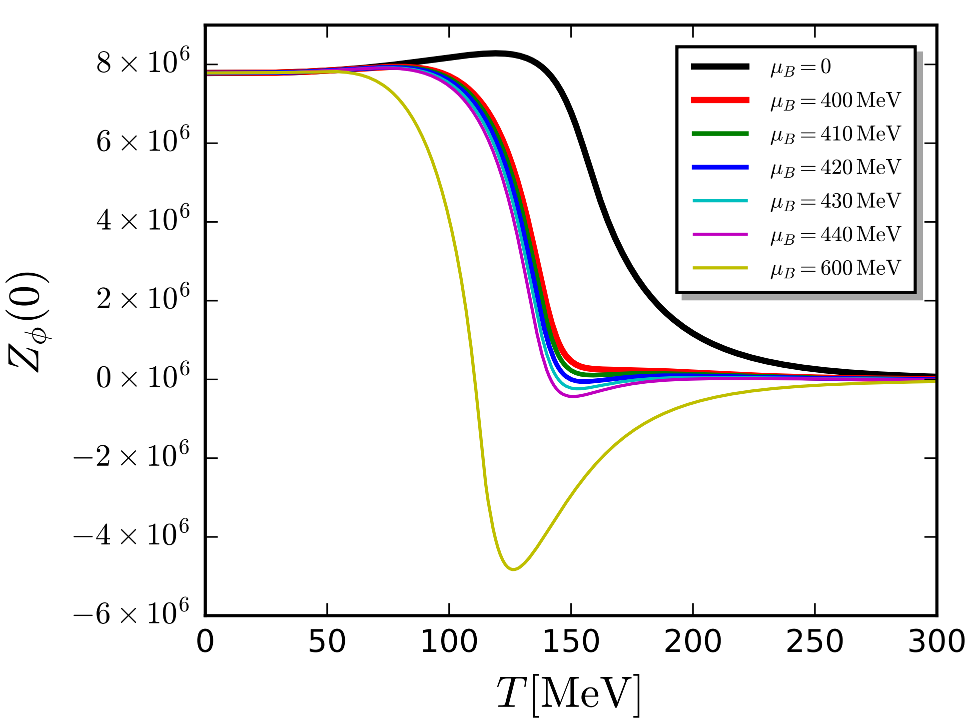
DSE (fRG): Gao, Pawlowski, *PLB* 820 (2021) 136584

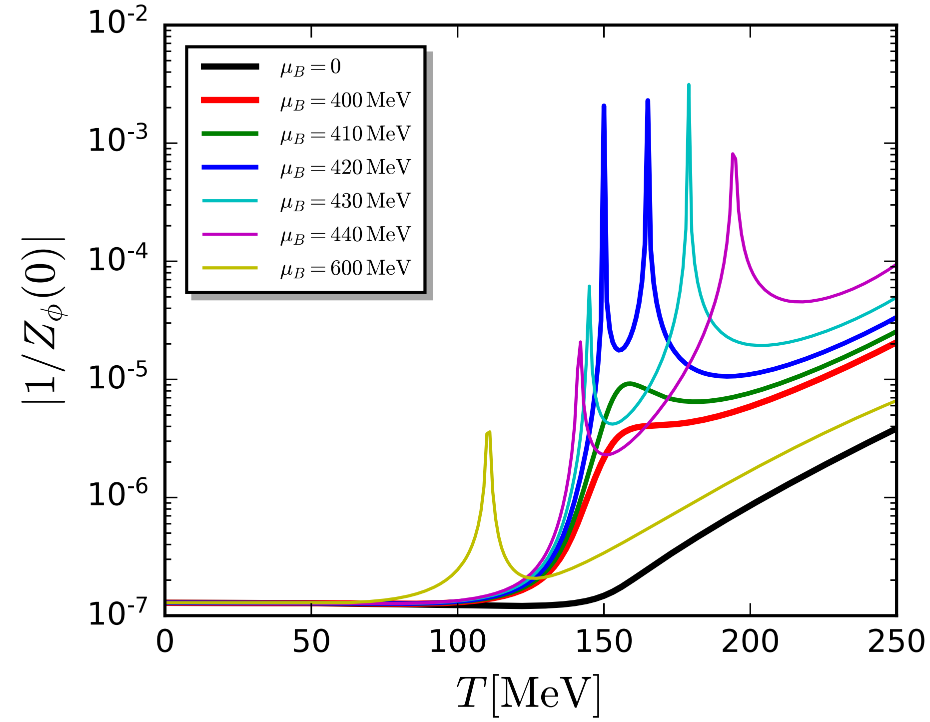
◆ $(T, \mu_B)_{\text{CEP}} = (112, 636)$**MeV**

DSE: Gunkel, Fischer, *PRD* 104 (2021) 5, 054022

- No CEP observed in $\mu_B/T \lesssim 2 \sim 3$ from lattice QCD. Karsch, *PoS* CORFU2018 (2019)163
- Recent studies of QCD phase structure from both fRG and DSE have shown convergent estimate for the location of CEP.
- Considering relatively larger errors when $\mu_B/T \gtrsim 4$, one arrives at a reasonable estimation : 450 MeV $\lesssim \mu_{B_{\text{CEP}}} \lesssim 650$ MeV.

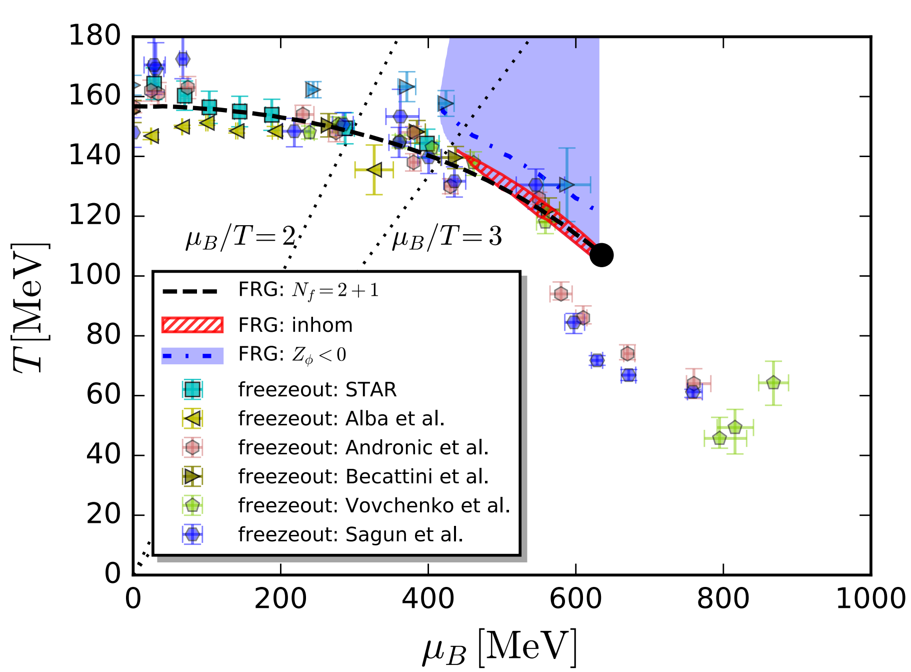
非均匀不稳定性

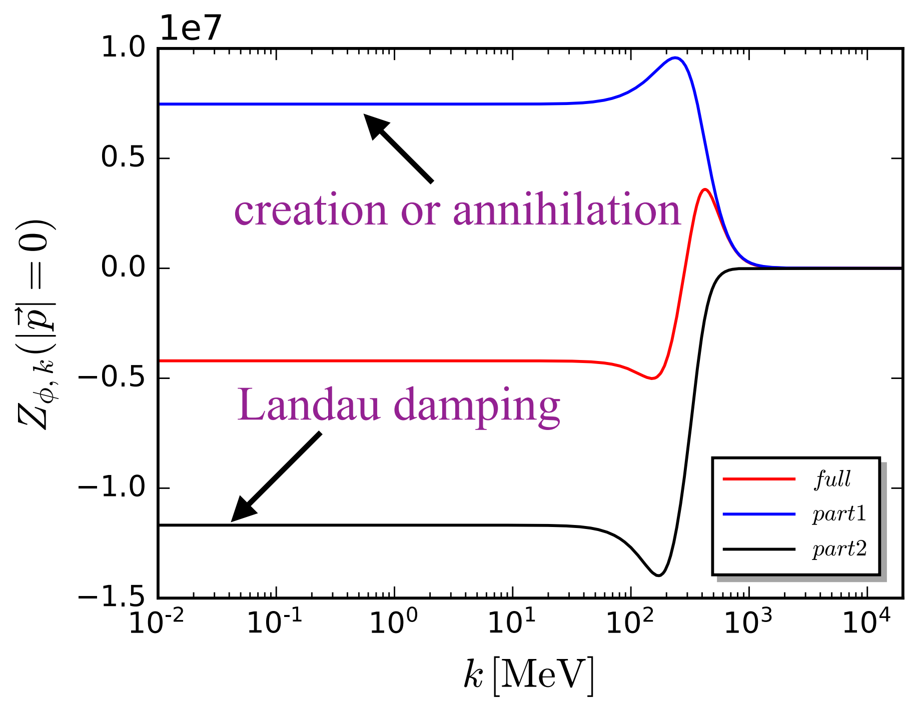




Two point function for the meson:

$$\Gamma^{(2)}_{\phi\phi}(p) = Z_\phi(p^2)\,p^2 + m_\phi^2\,,$$





- Inhomogeneous instability is resulted from Landau damping of two quarks in thermal bath in the regime of large baryon chemical potential.

WF, Pawlowski, Rennecke, *PRD* 101 (2020) 054032

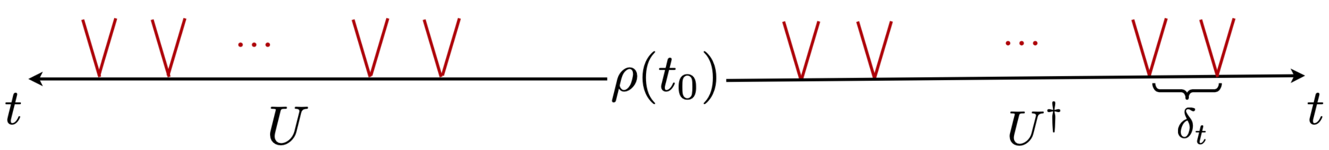
Schwinger-Keldysh 路径积分

- Schrödinger 方程:

$$U(t,t_0) = e^{-iH(t-t_0)}$$

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \quad \longrightarrow \quad |\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle,$$

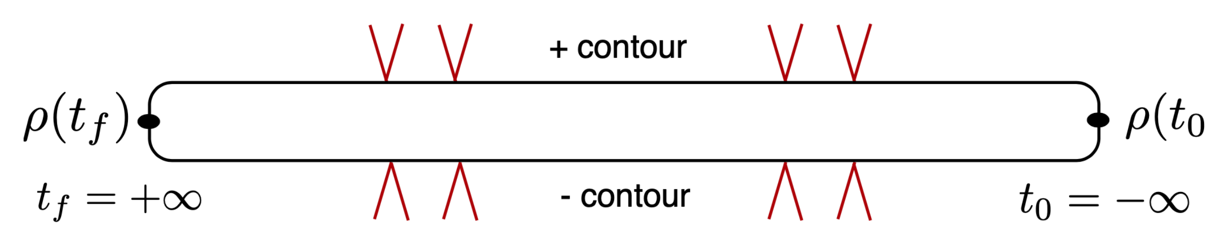
- von Neumann 方程:



$$\partial_t\rho(t) = -i[H,\rho(t)] \quad \longrightarrow \quad \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0),$$

- Keldysh 配分函数:

$$Z = \mathrm{tr}\,\rho(t),$$



- 两点闭时路径 (CTP) 格林函数:

$$G(x,y) \equiv -i\mathrm{tr}\{T_p(\phi(x)\phi^\dagger(y)\rho)\}$$
$$\equiv -i\langle T_p(\phi(x)\phi^\dagger(y))\rangle\,,$$

$$\Rightarrow$$

$$G(x,y) = \begin{pmatrix} G_{++} & G_{+-} \\ G_{-+} & G_{--} \end{pmatrix}$$
$$\equiv \begin{pmatrix} G_F & G_+ \\ G_- & G_{\tilde{F}} \end{pmatrix}\,,$$

$$G_F(x,y) \equiv -i\langle T(\phi(x)\phi^\dagger(y))\rangle\,,$$
$$G_+(x,y) \equiv -i\langle \phi^\dagger(y)\phi(x)\rangle\,,$$
$$G_-(x,y) \equiv -i\langle \phi(x)\phi^\dagger(y)\rangle\,,$$
$$G_{\tilde{F}}(x,y) \equiv -i\langle \tilde{T}(\phi(x)\phi^\dagger(y))\rangle\,,$$

Schwinger, J. Math. Phys. 2 (1961) 407;
Keldysh, Zh. Eksp. Teor. Fiz. 47 (1964) 1515;
Chou, Su, Hao, Yu, Phys. Rept. 118 (1985) 1

Keldysh 路径积分框架下的泛函重整化群

- 在 fRG 的路径积分中考虑两个时间分支:

$$Z_k[J_c, J_q] = \int (\mathscr{D}\varphi_c \mathscr{D}\varphi_q) \exp\left\{ i\left(S[\varphi] + \Delta S_k[\varphi] + (J_q^i \varphi_{i,c} + J_c^i \varphi_{i,q}) \right) \right\},$$

其中

$$\Delta S_k[\varphi] = \frac{1}{2}(\varphi_{i,c}, \varphi_{i,q}) \begin{pmatrix} 0 & R_k^{ij} \\ (R_k^{ij})^* & 0 \end{pmatrix} \begin{pmatrix} \varphi_{j,c} \\ \varphi_{j,q} \end{pmatrix}$$

$$= \frac{1}{2}\left(\varphi_{i,c} R_k^{ij} \varphi_{j,q} + \varphi_{i,q} (R_k^{ij})^* \varphi_{j,c} \right),$$

Keldysh 转动:

$$\begin{cases} \varphi_{i,+} = \frac{1}{\sqrt{2}}(\varphi_{i,c} + \varphi_{i,q}), \\ \varphi_{i,-} = \frac{1}{\sqrt{2}}(\varphi_{i,c} - \varphi_{i,q}), \end{cases}$$

- 这样可以得到闭时路径的流方程:

$$\partial_\tau \Gamma_k[\Phi] = \frac{i}{2} \mathrm{STr}\left[\left(\partial_\tau R_k^* \right) G_k \right],$$

$$R_k^{ab} \equiv \begin{pmatrix} 0 & R_k^{ij} \\ (R_k^{ij})^* & 0 \end{pmatrix},$$

$$iG(x,y) = \begin{pmatrix} iG^K(x,y) & iG^R(x,y) \\ iG^A(x,y) & 0 \end{pmatrix},$$

$$iG^R(x,y) = \theta(x^0 - y^0)\langle [\phi(x), \phi^*(y)] \rangle,$$
$$iG^A(x,y) = \theta(y^0 - x^0)\langle [\phi^*(y), \phi(x)] \rangle,$$
$$iG^K(x,y) = \langle \{ \phi(x), \phi^*(y) \} \rangle,$$

实时 fRG 框架下的 $O(N)$ 标量理论

- 同样我们可以将实时的流方程改写为:

$$\partial_\tau \Gamma_k[\Phi] = \frac{i}{2}\mathrm{STr}\Big[\tilde{\partial}_\tau \ln\big(\Gamma_k^{(2)}[\Phi] + R_k\big)\Big],$$

其中

$$\Gamma_k^{(2)} + R_k = \mathscr{P}_k + \mathscr{F}_k, \qquad \mathscr{P}_k = \begin{pmatrix} 0 & \mathscr{P}_k^A \\ \mathscr{P}_k^R & \mathscr{P}_k^K \end{pmatrix}, \qquad \mathscr{P}_k^K = \begin{pmatrix} \mathscr{P}_{\sigma,k}^K & 0 \\ 0 & \mathscr{P}_{\pi,k}^K \end{pmatrix},$$

$$\mathscr{P}_{\sigma,k}^K = 2i\epsilon\,\mathrm{sgn}(q_0)\coth\Big(\frac{q_0}{2T}\Big),$$

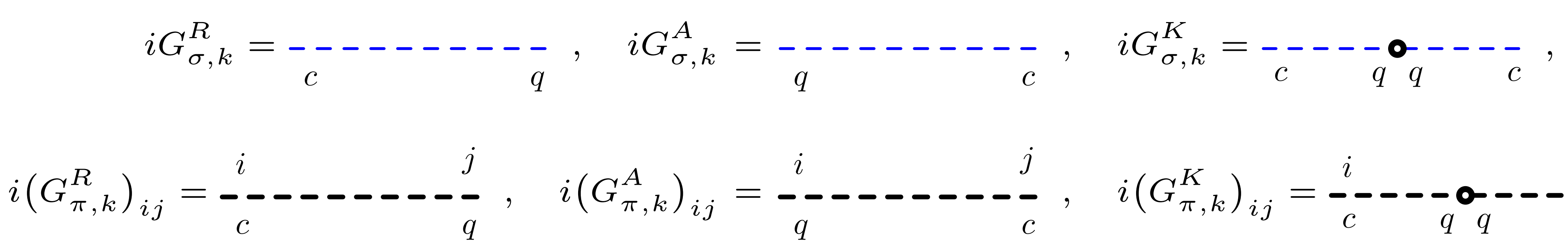
传播子:

$$G_k = \big(\mathscr{P}_k\big)^{-1} = \begin{pmatrix} G_k^K & G_k^R \\ G_k^A & 0 \end{pmatrix}, \qquad iG_k^K = (iG_k^R)(i\mathscr{P}_k^K)(iG_k^A),$$

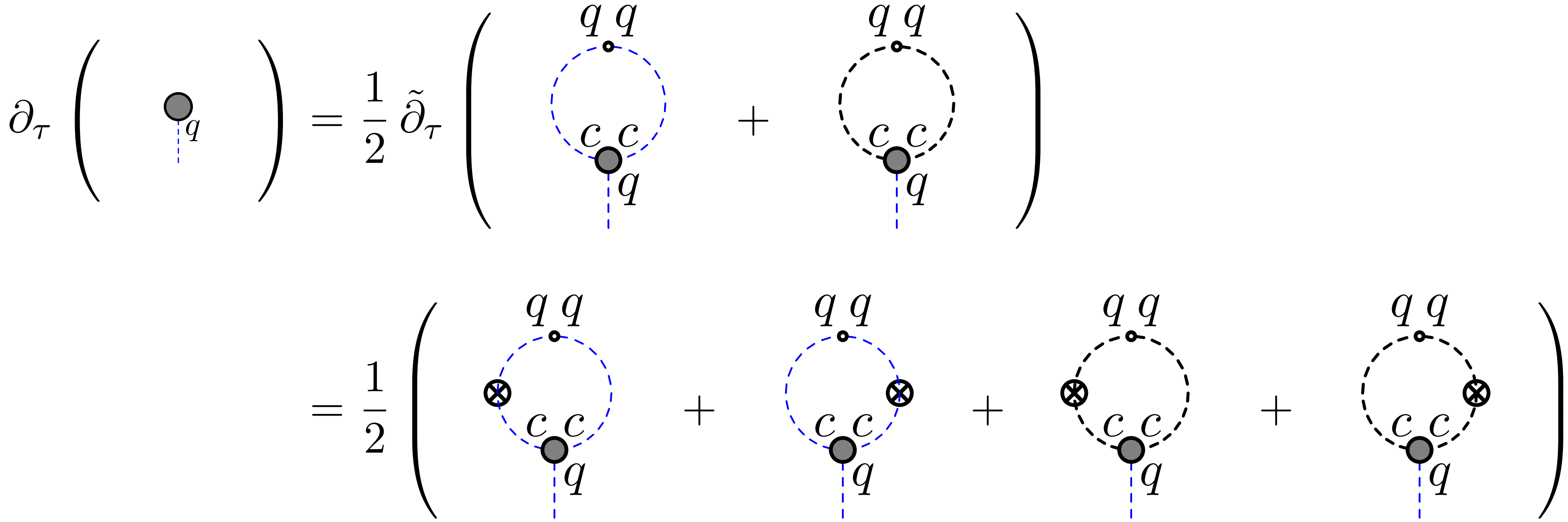
涨落耗散定理:

$$G_k^K = \big(G_k^R - G_k^A\big)\coth\Big(\frac{q_0}{2T}\Big),$$

- 各种传播子的图形表示:



有效势的流方程



- 对 σ_q 求导作相应的投影:

$$\partial_\tau \left(\frac{i\delta\Gamma_k[\Phi]}{\delta\sigma_q} \right)\Bigg|_{\Phi=0} = \frac{i}{2}\tilde{\partial}_\tau \left(\frac{i\delta\,\mathrm{STr}\big(G_k \mathscr{F}_k\big)}{\delta\sigma_q} \right)\Bigg|_{\Phi=0} + \cdots ,$$

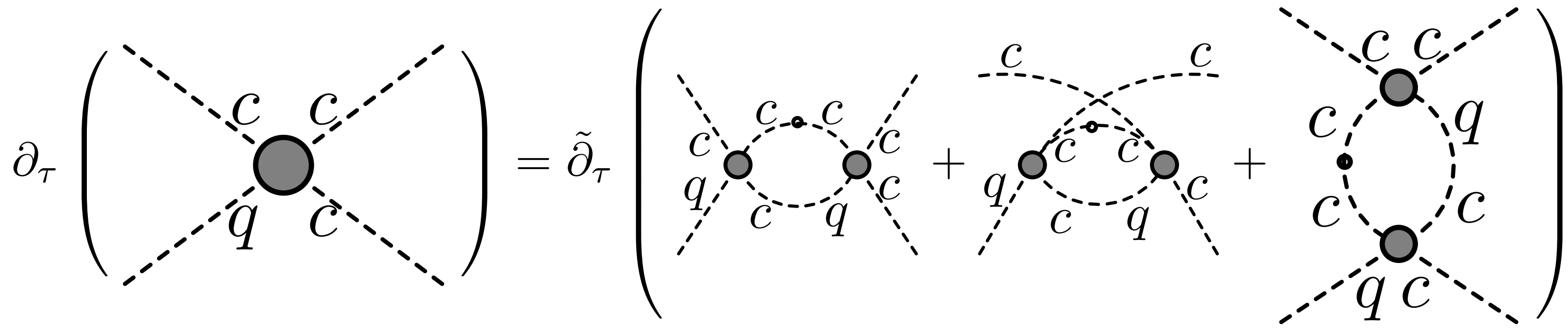
得到

$$\partial_\tau V_k'(\bar\rho_c) = \frac{\partial}{\partial\bar\rho_c}\left\{ -\frac{i}{4}\int \frac{d^4q}{(2\pi)^4}\big(\partial_\tau R_{\phi,k}(q)\big)\Big[G^K_{\sigma,k}(q) + \big(G^K_{\pi,k}\big)_{ii}(q)\Big] \right\},$$

对 $\bar\rho_c$ 积分

$$\partial_\tau V_k(\bar\rho_c) = -\frac{i}{4}\int \frac{d^4q}{(2\pi)^4}\big(\partial_\tau R_{\phi,k}(q)\big)\Big[G^K_{\sigma,k}(q) + \big(G^K_{\pi,k}\big)_{ii}(q)\Big] .$$

对称相两点和四点函数的流方程



- 四点顶点的一般形式：

$$i\Gamma^{(4)}_{k,\phi_{i,q}\phi_{j,c}\phi_{k,c}\phi_{l,c}}(p_i,p_j,p_k,p_l)$$

$$= -\frac{i}{3}\Big[\lambda^{\text{eff}}_{4\pi,k}(p_i,p_j,p_k,p_l)\delta_{il}\delta_{jk} + \lambda^{\text{eff}}_{4\pi,k}(p_i,p_k,p_l,p_j)\delta_{ij}\delta_{kl}$$

$$+ \lambda^{\text{eff}}_{4\pi,k}(p_i,p_l,p_j,p_k)\delta_{ik}\delta_{jl}\Big],$$

四点顶点的流：

$$\partial_\tau \lambda^{\text{eff}}_{4\pi,k}(p_i,p_j,p_k,p_l)$$

$$=\frac{\lambda^2_{4\pi,k}}{3}\Big[(N+4)\tilde{\partial}_\tau I_k(-p_i-p_l) + 2\,\tilde{\partial}_\tau I_k(-p_i-p_k)$$

$$+ 2\,\tilde{\partial}_\tau I_k(-p_i-p_j)\Big],$$

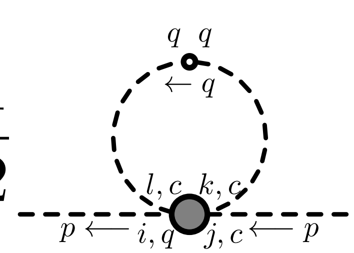
其中

$$I_k(p) \equiv i\int \frac{d^4q}{(2\pi)^4} G^K_{\pi,k}(q) G^A_{\pi,k}(q-p)\,.$$

- 两点函数：

$$i\Gamma^{(2)}_{k,\phi_{i,q}\phi_{j,c}}(p) = i\delta_{ij}\Big(Z_{\phi,k}(p^2)p^2 - m^2_{\pi,k}\Big),$$

$$-i\Sigma_{k,ij}(p) \equiv \frac{1}{2}$$



,

自能的流：

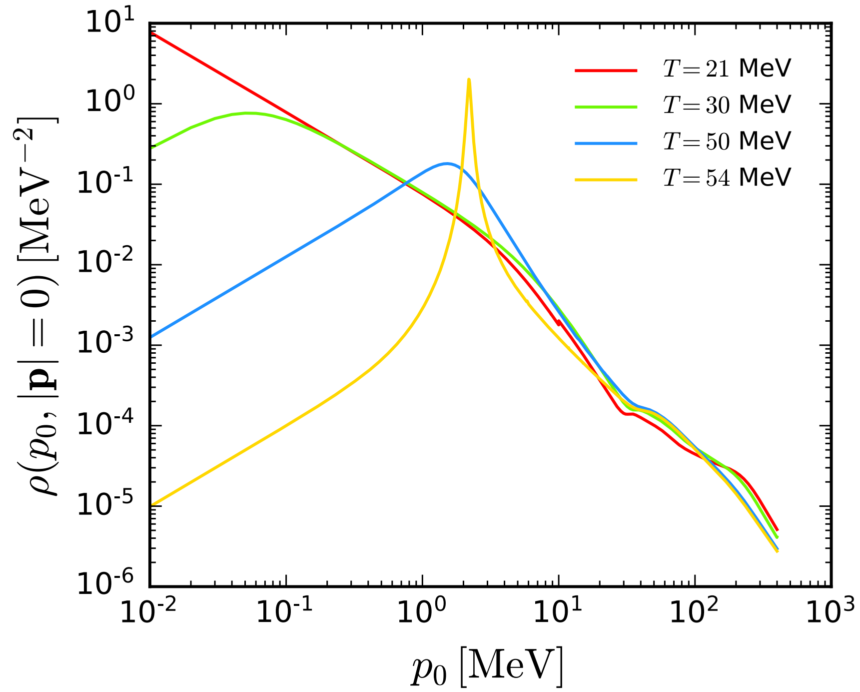
$$\partial_\tau \Gamma^{(2)}_{k,\phi_q\phi_c}(p) = -\tilde{\partial}_\tau \Sigma_k(p)$$

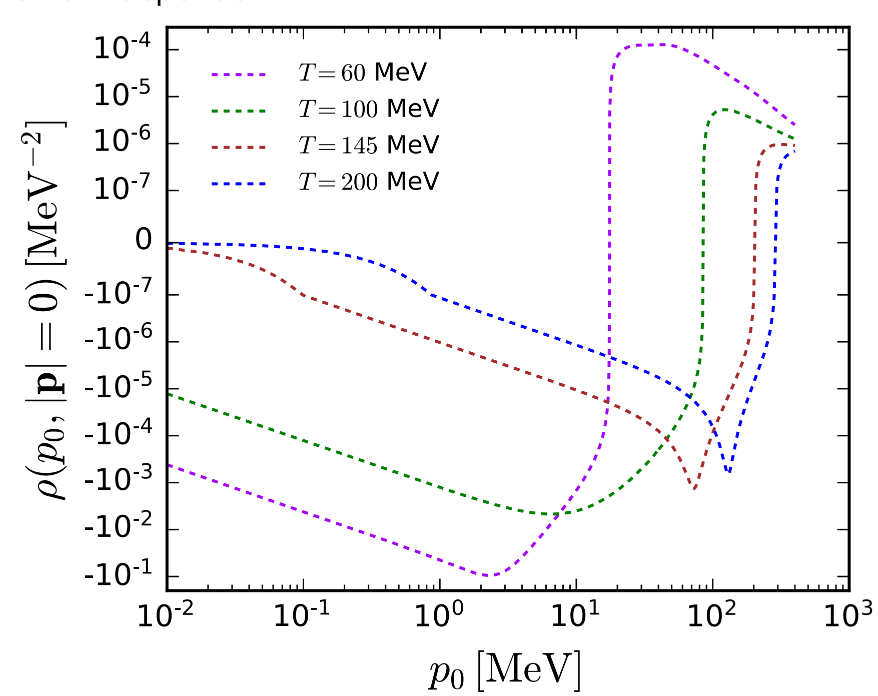
$$=(-\frac{i}{6})(N+2)\int \frac{d^4q}{(2\pi)^4}\tilde{\partial}_\tau\Big(G^K_{\pi,k}(q)\Big)$$

$$\times\, \bar{\lambda}^{\text{eff}}_{4\pi,k}(p_0,|\vec{p}|,q_0,|\vec{q}|,\cos\theta)\,.$$

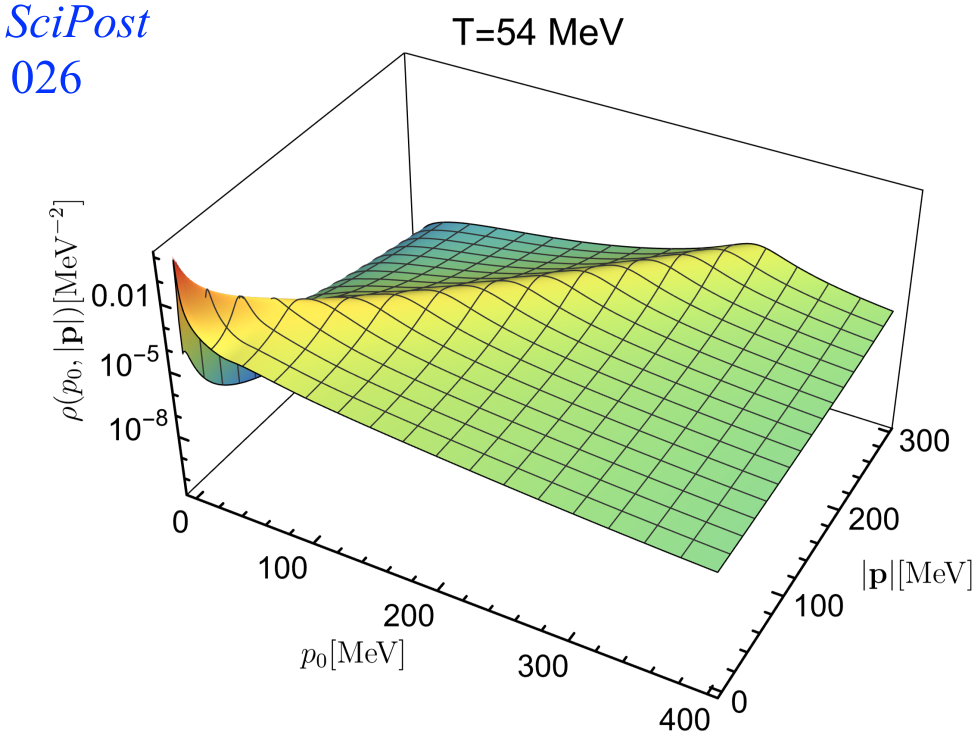
谱函数

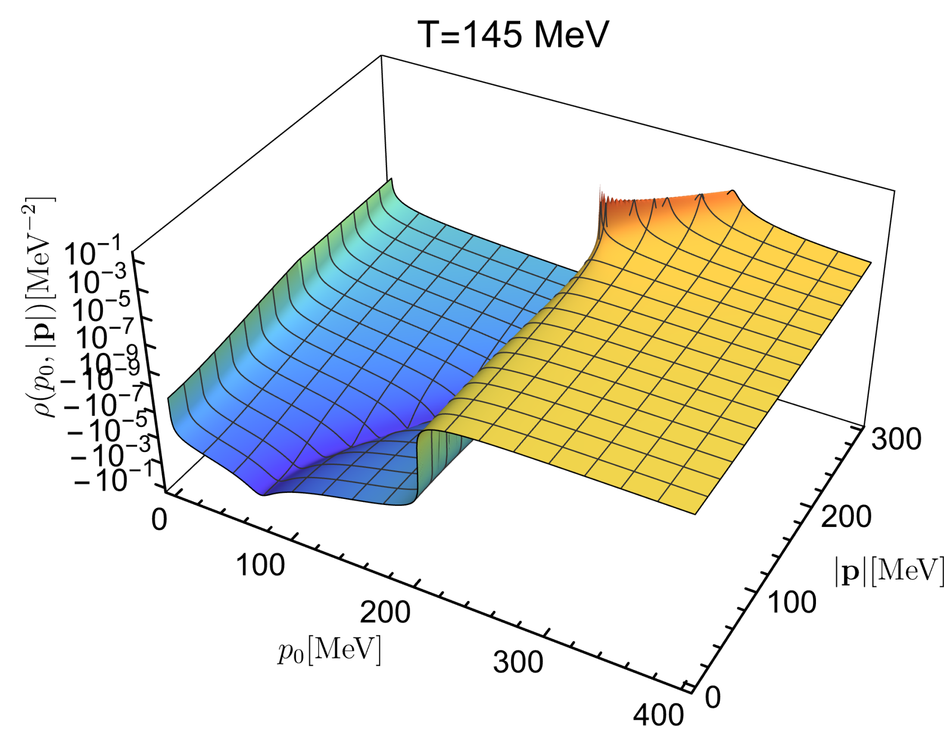
real time:momentum dependent





Tan, Chen, WF, *SciPost Phys.* 12 (2022) 026





$$G_R(p_0, |\vec{p}\,|) = -\int_{-\infty}^{\infty} \frac{dp_0'}{2\pi} \frac{\rho(p_0', |\vec{p}\,|)}{p_0' - (p_0 + i\epsilon)},$$

谱函数：

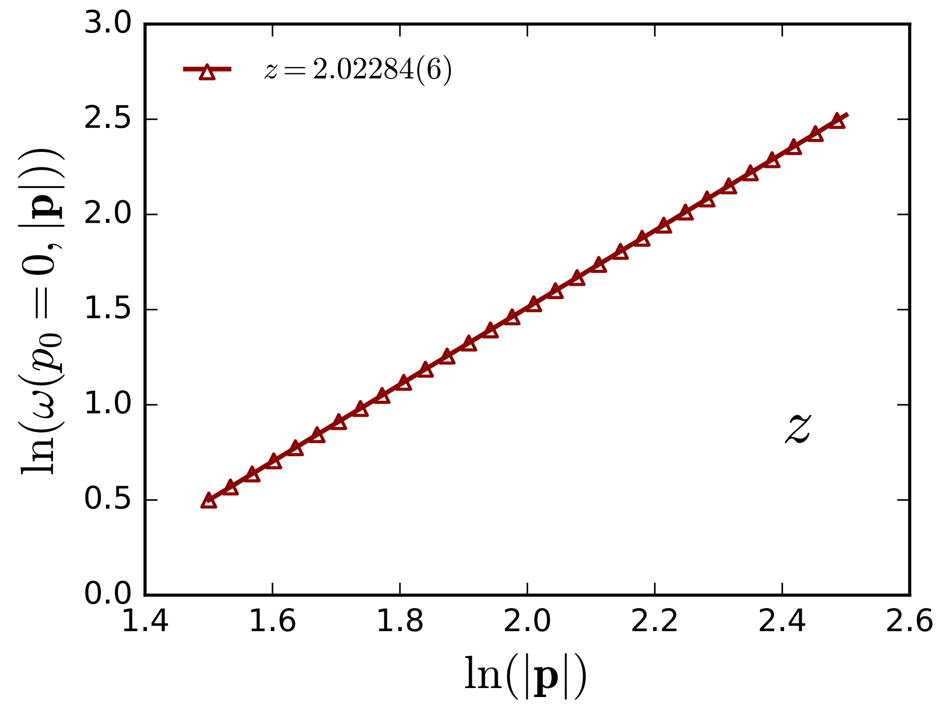
$$\rho(p_0, |\vec{p}\,|) = \frac{2\Im\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0, |\vec{p}\,|)}{\left[\Re\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0, |\vec{p}\,|)\right]^2 + \left[\Im\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0, |\vec{p}\,|)\right]^2}.$$

动力学临界指数

● 运动学系数:

$$\frac{1}{\Gamma(|\vec{p}\,|)} = -i\frac{\partial\Gamma^{(2)}_{\phi_q\phi_c}(p_0,|\vec{p}\,|)}{\partial p_0}\bigg|_{p_0=0} = \frac{\partial\Im\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0,|\vec{p}\,|)}{\partial p_0}\bigg|_{p_0=0},$$

● 耗散特征频率:



$$\omega(|\vec{p}\,|) = \Gamma(|\vec{p}\,|)\left(-\Gamma^{(2)}_{\phi_q\phi_c}(p_0=0,|\vec{p}\,|)\right) = -\Gamma(|\vec{p}\,|)\Re\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0=0,|\vec{p}\,|)\,.$$

$\omega(|\vec{p}\,|) \propto |\vec{p}\,|^z$ $\quad z \simeq 2.023$

Tan, Chen, WF, *SciPost Phys.* 12 (2022) 026

这里使用了

$$\Re\,\Gamma^{(2)}_{\phi_q\phi_c}(-p_0,|\vec{p}|) = \Re\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0,|\vec{p}|)\,,$$

$$\Im\,\Gamma^{(2)}_{\phi_q\phi_c}(-p_0,|\vec{p}|) = -\Im\,\Gamma^{(2)}_{\phi_q\phi_c}(p_0,|\vec{p}|)\,.$$

O(3):

$z \simeq 2.025\,,$

Duclut, Delamotte, *PRE* 95 (2017) 1, 012107.

Model A:

real-time classical-statistical lattice simulations

$z = 1.92(11)\,,$ Schweitzer, Schlichting, von Smekal, *NPB* 960 (2020) 115165.

Model G:

Relativistic O(4) should belong to Model G

$z = 3/2\,,$ Rajagopal and Wilczek, *NPB* 399 (1993) 395.

O(4):

real-time classical-statistical lattice simulations

$z \sim 2\,,$ Schlichting, Smith, L. von Smekal , *NPB* 950 (2020) 114868.

总结

- ★ 泛函重整化群除了能很好地研究临界现象，它还是一种目前被广泛使用的非微扰连续场论的理论方法

- ★ 我们简要地介绍了泛函重整化群在强子的性质、低能有效模型、有限温有限密 QCD 的相结构、非微扰实时场论等方面的应用和研究

谢谢！