From Kadanoff-Baym to Spin Boltzmann Equation

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Outline

- Lecture 1: CTP formalism and KB equation
- Lecture 2: Wigner function and MVSD for fermions
- Lecture 3: SBE in terms of MVSD

Lecture 1: CTP formalism and KB equation

Introduction to Closed-Time-Path (CTP) or Schwinger-Keldysh (SK) formalism. For comparison we give a derivation of the correlation function in vacuum. Then we give a derivation of the correlation function in non-equilibrium in the CTP formalism.

In vacuum

Green functions in vacuum

Two-point Green function in vacuum

We define the two-point Green function in a full theory as

$$G(x,y) = \langle\Omega| T\phi(x)\phi(y) |\Omega\rangle \tag{1}$$

where $|\Omega\rangle$ is the ground state of the full Hamiltonian H and $\phi(x)$ is the scalar field operator in the Heisenberg picture. In Schroedinger picture, the full Hamitonian H is a sum of a free one and an interaction one,

$$H = H_0 + H_{\rm int} \tag{2}$$

Heisenberg and Schroedinger picture

At the initial time t_0, $\phi_{(S)}(\mathbf{x})$ is a field in Schroedinger picture and can be expanded as

$$\phi_{(S)}(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^{\dagger} e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{3}$$

We can obtain $\phi(x) = \phi_{(H)}(t, \mathbf{x})$ in the Heisenberg picture from $\phi_{(S)}(\mathbf{x})$ by

$$\phi(x) = e^{iH(t-t_0)} \phi_{(S)}(\mathbf{x}) e^{-iH(t-t_0)} \tag{4}$$

We see $H_{(H)} = H_{(S)} = H$.

Interaction picture

We can define the field operator in the interaction picture

$$\begin{aligned}\phi_{(I)}(x) &= e^{iH_0(t-t_0)}\phi_{(S)}(\mathbf{x})e^{-iH_0(t-t_0)} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x}+a_{\mathbf{p}}^{\dagger}e^{ip\cdot x}\right)\end{aligned} \tag{5}$$

where we have used $t_0 = 0$ and

$$\begin{aligned}a_{\mathbf{p}(I)} &= e^{iH_0t}a_{\mathbf{p}}e^{-iH_0t} = a_{\mathbf{p}}e^{-iE_pt} \\ a_{\mathbf{p}(I)}^{\dagger} &= e^{iH_0t}a_{\mathbf{p}}^{\dagger}e^{-iH_0t} = a_{\mathbf{p}}^{\dagger}e^{iE_pt}\end{aligned} \tag{6}$$

We note that $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^{\dagger}$ in Eq. (5) are in Schroedinger picture and independent of time. We have used $H_0 \equiv H_0^{(I)} = H_0^{(S)}$.

Evolution operator

Now we can express the Heisenberg picture field $\phi(x)$ in terms of $\phi_{(I)}(t, \mathbf{x})$

$$\begin{aligned}
\phi(x) &= e^{iH(t-t_0)}\phi_{(S)}(\mathbf{x})e^{-iH(t-t_0)} \\
&= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_{(I)}(x)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\
&= U^\dagger(t, t_0)\phi_{(I)}(x)U(t, t_0)
\end{aligned} \quad (7)$$

where $U(t, t_0)$ is the unitary operator connecting the interaction and Heisenberg picture,

$$\begin{aligned}
U(t, t_0) &= e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\
U(t_0, t) &= U^\dagger(t, t_0) = e^{iH(t-t_0)}e^{-iH_0(t-t_0)}
\end{aligned} \quad (8)$$

Field in Heisenberg picture

The field in Heisenberg picture is then

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[a_{\mathbf{p}}(t) e^{-ip\cdot x} + a_{\mathbf{p}}^{\dagger}(t) e^{ip\cdot x} \right]$$

where $a_{\mathbf{p}}(t)$ and $a_{\mathbf{p}}^{\dagger}(t)$ are operators in Heisenberg picture

$$\begin{aligned} a_{\mathbf{p}}(t) &= U^{\dagger}(t, t_0) a_{\mathbf{p}} U(t, t_0) \\ a_{\mathbf{p}}^{\dagger}(t) &= U^{\dagger}(t, t_0) a_{\mathbf{p}}^{\dagger} U(t, t_0) \end{aligned} \tag{9}$$

Evolution operator

The evolution operator $U(t, t_0)$ satisfies the Schroedinger equation

$$i\frac{d}{dt}U(t, t_0) = H_I U(t, t_0) \tag{10}$$

where $H_I \equiv H_{\rm int}^{(I)}$ is the interaction Hamiltonian in the interaction picture,

$$H_I = e^{iH_0(t-t_0)}H_{\rm int}^{(S)}e^{-iH_0(t-t_0)} \tag{11}$$

The derivation of (10) is

$$\begin{aligned} i\frac{d}{dt}U(t, t_0) &= U(t, t_0)H - H_0 U(t, t_0) \\ &= e^{iH_0(t-t_0)}H_{\rm int}^{(S)}e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\rm int}^{(S)}e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_I U(t, t_0) \end{aligned} \tag{12}$$

Evolution operator

The solution to $U(t, t_0)$ is

$$U(t, t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right)\right] \tag{13}$$

Note that $U(t, t_0)$ is a unitary operator which connects the interaction with Heisenberg picture.

Evolution operator for arbitary time

We can define an evolution operator in the interaction picture from t_1 to t_2

$$U(t_2, t_1) = U(t_2, t_0) U^{\dagger}(t_1, t_0) \\ = e^{iH_0(t_2 - t_0)} e^{-iH(t_2 - t_1)} e^{-iH_0(t_1 - t_0)} \tag{14}$$

We can prove that all elements of $U(t_2, t_1)$ consititute a U(1) group: (1) The inverse of $U(t_2, t_1)$ is $U(t_1, t_2)$: $U(t_2, t_1)U(t_1, t_2) = 1$. (2) Unitary condition $U^{\dagger}(t_2, t_1) = U(t_1, t_2)$. (3) The result of two continuous evolution is still an evolution: $U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1)$. One can verify $U(t, t_1)$ satisfies the same Schroedinger equation as in Eq. (10). So its solution is similar to (13),

$$U(t, t_1) = T \left[\exp \left(-i \int_{t_1}^{t} dt' H_I(t') \right) \right] \tag{15}$$

Vacuum states of H_0 and H

To deal with Eq. (1), we express the ground state of H_0 in terms of eigenstates of H,

$$\begin{aligned} |0\rangle &= \sum_n |n, H\rangle \langle n, H|0\rangle \\ \langle 0| &= \sum_n \langle 0|n, H\rangle \langle n, H| \end{aligned} \tag{16}$$

where $|n, H\rangle$ denote energy eigenstates of H with $|0, H\rangle = |\Omega\rangle$. We let $|0\rangle$ evolve in time with H

$$\begin{aligned} e^{-iH(t_\infty + t_0)} |0\rangle &= \sum_n e^{-iE_n(t_\infty + t_0)} |n, H\rangle \langle n, H|0\rangle \\ \langle 0| e^{-iH(t_\infty - t_0)} &= \sum_n e^{-iE_n(t_\infty - t_0)} \langle 0|n, H\rangle \langle n, H| \end{aligned} \tag{17}$$

where E_n denotes the energy of the state $|n, H\rangle$ with $E_n > E_{n-1}$.

Vacuum states of H_0 and H

In order to single out $|\Omega\rangle$ out of other states, we introduce a small imaginary part into t_∞ as $t_\infty \to t_\infty(1-i\epsilon)$, then we have

$$
\begin{aligned}
e^{-iH[t_\infty(1-i\epsilon)+t_0]}|0\rangle &= e^{-E_0\epsilon}\left\{\left(e^{-iE_0(t_\infty+t_0)}\langle\Omega|0\rangle\right)|\Omega\rangle\right.\\
&\left.+\sum_{n\neq 0} e^{-iE_n(t_\infty+t_0)}e^{-(E_n-E_0)t_\infty\epsilon}|n,H\rangle\langle n,H|0\rangle\right\}\\
\langle 0|e^{-iH[t_\infty(1-i\epsilon)-t_0]} &= e^{-E_0\epsilon}\left\{\left(e^{-iE_0(t_\infty-t_0)}\langle 0|\Omega\rangle\right)\langle\Omega|\right.\\
&\left.+\sum_{n\neq 0} e^{-iE_n(t_\infty-t_0)}e^{-(E_n-E_0)t_\infty\epsilon}\langle 0|n,H\rangle\langle n,H|\right\}
\end{aligned}
\qquad (18)
$$

Vacuum states of H_0 and H

Since all higher states are suppressed relative to the ground state by a factor $e^{-(E_n-E_0)t_\infty \epsilon}$, so we have

$$e^{-iH(t_\infty+t_0)}|0\rangle \approx \left(e^{-iE_0(t_\infty+t_0)}\langle\Omega|0\rangle\right)|\Omega\rangle$$
$$\langle 0| e^{-iH(t_\infty-t_0)} \approx \left(e^{-iE_0(t_\infty-t_0)}\langle 0|\Omega\rangle\right)\langle\Omega| \tag{19}$$

Vacuum states of H_0 and H

We can approximate

$$
\begin{aligned}
|\Omega\rangle &\approx \lim_{t_\infty \to \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty+t_0)} \langle \Omega|0\rangle^{-1} \right) e^{-iH(t_\infty+t_0)} |0\rangle \\
&= \lim_{t_\infty \to \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty+t_0)} \langle \Omega|0\rangle^{-1} \right) U(t_0, -t_\infty) |0\rangle \\
\langle \Omega| &\approx \lim_{t_\infty \to \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty-t_0)} \langle 0|\Omega\rangle^{-1} \right) \langle 0| e^{-iH(t_\infty+t_0)} \\
&\approx \lim_{t_\infty \to \infty(1-i\epsilon)} \left(e^{iE_0(t_\infty-t_0)} \langle 0|\Omega\rangle^{-1} \right) \langle 0| U(t_\infty, t_0)
\end{aligned} \tag{20}
$$

where we have used Eq. (8) and $H_0 |0\rangle = 0$.

Normalization

The normalization condition for the ground state reads

$$1 = \langle \Omega | \Omega \rangle$$
$$= \lim_{t_\infty \to \infty(1-i\epsilon)} \left(e^{2iE_0 t_\infty} | \langle \Omega | 0 \rangle |^{-2} \right) \langle 0 | U(t_\infty, t_0) U(t_0, -t_\infty) | 0 \rangle \quad (21)$$

which determines the normalization constant

$$e^{2iE_0 t_\infty} | \langle \Omega | 0 \rangle |^{-2} = \lim_{t_\infty \to \infty(1-i\epsilon)} \frac{1}{\langle 0 | U(t_\infty, -t_\infty) | 0 \rangle} \quad (22)$$

Gell-Mann-Low theorem

Using Eqs. (7,20), Eq. (1) can be written as

$$
\begin{aligned}
G(x,y) &= \langle\Omega| T\phi(x)\phi(y) |\Omega\rangle \\
&= \lim_{t_\infty\to\infty(1-i\epsilon)} \frac{1}{\langle 0| U(t_\infty,-t_\infty) |0\rangle} \\
&\quad \times \langle 0| U(t_\infty,t_0) T U^\dagger(x_0,t_0)\phi_I(x) U(x_0,t_0) \\
&\quad \times U^\dagger(y_0,t_0)\phi_I(y) U(y_0,t_0) U(t_0,-t_\infty) |0\rangle \\
&= \lim_{t_\infty\to\infty(1-i\epsilon)} \frac{\langle 0| T[\phi_I(x)\phi_I(y) U(t_\infty,-t_\infty)] |0\rangle}{\langle 0| U(t_\infty,-t_\infty) |0\rangle}
\end{aligned}
\tag{23}
$$

Green functions in non-equilibrium

Two-point Green function in non-equilibrium

In non-equilibrium, the two point correlation function is defined through the density operator at the initial time

$$G(x,y) = \langle T\phi(x)\phi(y)\rangle \equiv \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)T\phi(x)\phi(y)] \qquad (24)$$

where the trace is taken over initial states which can be expanded in eigenstates of free Hamiltonian H_0.

Two-point function on CTP

The interaction H_I is is switched on in later time. Using Eq. (7), we can write Eq. (24) as

$$
\begin{aligned}
G(x,y) &= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)T\phi(x)\phi(y)] \\
&= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)TU^\dagger(x_0,t_0)\phi_I(x)U(x_0,t_0) \\
&\quad \times U^\dagger(y_0,t_0)\phi_I(y)U(y_0,t_0)] \\
&= \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)TU^\dagger(x_0,t_0)\phi_I(x)U(x_0,y_0)\phi_I(y)U(y_0,t_0)] \\
&\equiv \frac{1}{\mathrm{Tr}\rho(t_0)}\mathrm{Tr}[\rho(t_0)T_P\phi_I(x)\phi_I(y)U_{CTP}(t_0)] \qquad (25)
\end{aligned}
$$

Evolution operator on CTP

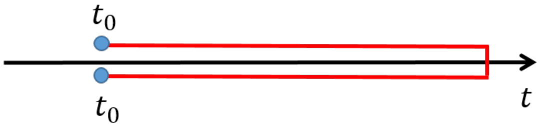
Here $U_{CTP}(t_0)$ is the unitary evolution operator defined on a closed-time path (CTP) starting from t_0 to $+\infty$ and back to t_0 (see Fig. 1),

$$\begin{aligned} U_{CTP}(t_0) &\equiv T_P \left[\exp \left(-i \int_{CTP} dt H_I(t) \right) \right] \\ &= T_P \left[\exp \left(-i \int_{t_0}^{\infty} dt_+ H_I(t_+) + i \int_{t_0}^{\infty} dt_- H_I(t_-) \right) \right] \end{aligned} \tag{26}$$

In Eqs. (25,26), T_P is the ordering operator on the CTP. From Eq. (26) we see that the interaction terms leading to vertices in Feynman rules on the negative time branch have got a minus sign relative to those on the positive time branch. The difference between $G(x, y)$ in vacuum and non-equilibrium is that there is no evolution of an in-state to a state at t_0 and a state at t_0 to an out-state. Only states at t_0 is involved which lead to the CTP.

CTP contour

Figure: The Closed Time Path.



Difference between vacuum and non-equilbrium

The difference between $G(x, y)$ in vacuum and non-equilibrium is that there is no evolution of an in-state to a state at t_0 and a state at t_0 to an out-state. Only states at t_0 is involved which lead to the CTP.

Two-point Green functions on CTP

There are two equivalent forms of Green functions on the CTP: (1) The original definition:

$$G(x_1, x_2) = \left\langle T_P \psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right\rangle \tag{27}$$

where x_{10} and x_{20} are on the CTP, and T_P denotes the time-order operator on the CTP. (2) Matrix form:

$$G = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = \begin{pmatrix} G^F & G^< \\ G^> & G^{\overline{F}} \end{pmatrix} \tag{28}$$

where we have used $G^{+-} = G^<$, $G^{-+} = G^>$, $G^{++} = G^F$, $G^{--} = G^{\overline{F}}$.

Two-point Green functions on CTP

The definition of Green functions are

$$\begin{aligned} G^{F}_{\alpha\beta}(x_1,x_2) &= \left\langle T\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right\rangle \\ G^{\overline{F}}_{\alpha\beta}(x_1,x_2) &= \left\langle T_A\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right\rangle \\ G^{<}_{\alpha\beta}(x_1,x_2) &= -\left\langle\overline{\psi}_\beta(x_2)\psi_\alpha(x_1)\right\rangle \\ G^{>}_{\alpha\beta}(x_1,x_2) &= \left\langle\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\right\rangle \end{aligned} \tag{29}$$

where T and T_A denote the time-order and time-antiorder operator respectively, and $\alpha,\beta=1,2,3,4$ denote Dirac indices.

Equivalent form of two-point function

We can use the physical representation by the unitary transformation

$$\begin{pmatrix} 0 & G^A \\ G^R & G^C \end{pmatrix} = U \begin{pmatrix} G^F & G^< \\ G^> & G^{\overline{F}} \end{pmatrix} U^{-1} \tag{30}$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \; U^{-1} = U^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{31}$$

Only three of them are independent due to the identity
$G^F + G^{\overline{F}} = G^> + G^<$.

The explicit form is

$$\begin{aligned} G^R &= \theta(t_1 - t_2)G^> + \theta(t_2 - t_1)G^< - G^< \\ &= \theta(t_1 - t_2)(G^> - G^<) \\ G^A &= \theta(t_1 - t_2)G^> + \theta(t_2 - t_1)G^< - G^> \\ &= \theta(t_2 - t_1)(G^< - G^>) \\ G^C &= G^F + G^{\overline{F}} = G^> + G^< \end{aligned} \tag{32}$$

Schematically, the Dyson-Schwinger (DS) equations on CTP read

$$\begin{aligned}
G^{-1} &= G_0^{-1} - \Sigma \\
G &= \frac{1}{G_0^{-1} - \Sigma} = \frac{1}{G_0^{-1}(1 - G_0\Sigma)} \\
&= \frac{1}{1 - G_0\Sigma} G_0 = \sum_{n=0} (G_0\Sigma)^n G_0 = G_0 + G\Sigma G_0 \\
G &= \frac{1}{(1 - \Sigma G_0) G_0^{-1}} = G_0 \frac{1}{1 - \Sigma G_0} \\
&= \sum_{n=0} G_0 (\Sigma G_0)^n = G_0 + G_0 \Sigma G \qquad (33) \\
G G_0^{-1} &= 1 + G\Sigma \\
G_0^{-1} G &= 1 + \Sigma G \qquad (34)
\end{aligned}$$

Dyson-Schwinger (DS) equation on CTP

The explicit form of the KB equation for two-point function on CTP is

$$\begin{aligned} & -i(i\gamma_\mu \partial^\mu_{x_1} - m)G(x_1, x_2) \\ = \quad & \delta^{(4)}_C(x_1 - x_2) + \int_C dx' \Sigma(x_1, x')G(x', x_2) \end{aligned} \tag{35}$$

where x_1, x_2 and x' are space-time points on the CTP, $dx \equiv d^4x$, and the integral is defined on the contour. The delta function on the CTP is expressed in terms of normal delta functions

$$\delta^{(4)}_C(x_1 - x_2) = \begin{cases} \delta^{(4)}(x_1 - x_2), & x_{1,2} \in t_+ \\ -\delta^{(4)}(x_1 - x_2), & x_{1,2} \in t_- \\ 0, & (x_1, x_2) \in (t_+, t_-) \text{ or } (t_-, t_+) \end{cases} \tag{36}$$

Equation (35) can also be put into an equivalent matrix form in normal coordinates.

Kadanoff-Baym (KB) equation

The equation for $G^<$ for $(x_1, x_2) \in (t_+, t_-)$,

$$\begin{aligned} -i(i\hbar\gamma_\mu\partial^\mu_{x_1} - m)G^<(x_1, x_2) &= \hbar\int_{-\infty}^{\infty} dx' \Big[\Sigma^F(x_1, x')G^<(x', x_2) \\ &\quad -\Sigma^<(x_1, x')G^{\overline{F}}(x', x_2)\Big] \\ &= \hbar\int_{-\infty}^{\infty} dx' \Big[\Sigma^R(x_1, x')G^<(x', x_2) \\ &\quad +\Sigma^<(x_1, x')G^A(x', x_2)\Big] \end{aligned} \quad (37)$$

Note that all x_1, x_2 and x' are ordinary coordinates (not on CTP).

Proof of Eq. (37)

For $(x_1, x_2) \in (t_+, t_-)$, the CTP integral in r.h.s. of Eq. (35) can be rewritten into a normal integral

$$
\begin{aligned}
I_{+-} &= \int_C dx' \Sigma(x_1, x') G(x', x_2) \\
&= \int_{t_0}^{\infty} dx' \left[\Sigma^F(x_1, x') G^<(x', x_2) - \Sigma^<(x_1, x') G^{\overline{F}}(x', x_2) \right] \quad (38)
\end{aligned}
$$

Then we can express Σ^F and $G^{\overline{F}}$ in terms of $\Sigma^{\lessgtr}$ and $G^{\lessgtr}$,

$$
\begin{aligned}
\Sigma^F(x_1, x') &= \theta(t_1 - t')\Sigma^>(x_1, x') + \theta(t' - t_1)\Sigma^<(x_1, x') \\
G^{\overline{F}}(x', x_2) &= \theta(t' - t_2)G^<(x', x_2) + \theta(t_2 - t')G^>(x', x_2) \quad (39)
\end{aligned}
$$

Proof of Eq. (37)

Then we have

$$\begin{aligned} I_{+-} &= \int_{t_0}^{\infty} dx' \left\{ \left[\theta(t_1 - t')\Sigma^{>}(x_1, x') + \theta(t' - t_1)\Sigma^{<}(x_1, x') \right] G^{<}(x', x_2) \right. \\ &\quad \left. - \Sigma^{<}(x_1, x') \left[\theta(t' - t_2) G^{<}(x', x_2) + \theta(t_2 - t') G^{>}(x', x_2) \right] \right\} \\ &= \int_{t_0}^{t_1} dx' \Sigma^{>}(x_1, x') G^{<}(x', x_2) + \int_{t_1}^{\infty} dx' \Sigma^{<}(x_1, x') G^{<}(x', x_2) \\ &\quad - \int_{t_2}^{\infty} dx' \Sigma^{<}(x_1, x') G^{<}(x', x_2) - \int_{t_0}^{t_2} dx' \Sigma^{<}(x_1, x') G^{>}(x', x_2) \end{aligned} \tag{40}$$

Proof of Eq. (37)

(Continued)

$$
\begin{aligned}
I_{+-} &= \int_{t_0}^{t_1} dx' \left[\Sigma^{>}(x_1, x') - \Sigma^{<}(x_1, x')\right] G^{<}(x', x_2) \\
&\quad - \int_{t_0}^{t_2} dx' \Sigma^{<}(x_1, x') \left[G^{>}(x', x_2) - G^{<}(x', x_2)\right] \\
&\quad + \left[\int_{t_1}^{\infty} - \int_{t_2}^{\infty} + \int_{t_0}^{t_1} - \int_{t_0}^{t_2}\right] \Sigma^{<}(x_1, x') G^{<}(x', x_2) \\
&= \int_{t_0}^{t_1} dx' \left[\Sigma^{>}(x_1, x') - \Sigma^{<}(x_1, x')\right] G^{<}(x', x_2) \\
&\quad - \int_{t_0}^{t_2} dx' \Sigma^{<}(x_1, x') \left[G^{>}(x', x_2) - G^{<}(x', x_2)\right] \\
&= \int_{t_0}^{\infty} dx' \left[\Sigma^{R}(x_1, x') G^{<}(x', x_2) + \Sigma^{<}(x_1, x') G^{A}(x', x_2)\right] \quad (41)
\end{aligned}
$$

Proof of Eq. (37)

Here we have used

$$\Sigma^R(x_1, x') = \theta(t_1 - t') \left[\Sigma^>(x_1, x') - \Sigma^<(x_1, x')\right]$$
$$G^A(x', x_2) = -\theta(t_2 - t') \left[G^>(x', x_2) - G^<(x', x_2)\right]$$

KB equation in matrix form

Equivalently one can write Eqs. (35) in the matrix form corresponding to DS equation $G_0^{-1} G = 1 + \Sigma G$,

$$
\begin{aligned}
& -i(i\gamma_\mu \partial_{x_1}^\mu - m) \left(\begin{array}{cc} G^F & G^< \\ G^> & G^{\overline{F}} \end{array} \right) (x_1, x_2) \\
= \; & \delta^{(4)}(x_1 - x_2) \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \\
& + \int_{-\infty}^{\infty} dx' \left(\begin{array}{cc} \Sigma^F & -\Sigma^< \\ \Sigma^> & -\Sigma^{\overline{F}} \end{array} \right) (x_1, x') \left(\begin{array}{cc} G^F & G^< \\ G^> & G^{\overline{F}} \end{array} \right) (x', x_2) \quad (42)
\end{aligned}
$$

Note that all x_1, x_2 and x' are ordinary coordinates (not on CTP).

KB equation in matrix form

Corresponding to $GG_0^{-1} = 1 + G\Sigma$, we can also derive the KB equation with the differential operators acting on Green's functions in the left

$$
\begin{aligned}
& i\begin{pmatrix} G^F & G^< \\ G^> & G^{\overline{F}} \end{pmatrix}(x_1,x_2)(i\gamma_\mu \overleftarrow{\partial}{}^\mu_{x_2} + m) \\
= \;& \delta^{(4)}(x_1 - x_2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
& + \int_{-\infty}^{\infty} dx' \begin{pmatrix} G^F & -G^< \\ G^> & -G^{\overline{F}} \end{pmatrix}(x_1,x')\begin{pmatrix} \Sigma^F & \Sigma^< \\ \Sigma^> & \Sigma^{\overline{F}} \end{pmatrix}(x',x_2) \quad (43)
\end{aligned}
$$

Wigner transformation (WT)

We rewrite x_1 and x_2 by

$$X = \frac{1}{2}(x_1 + x_2)$$
$$y = x_1 - x_2 \tag{44}$$

and then make Fourier transformation with respect to y of Eq. (37). The Fourier transform is defined as

$$\begin{aligned}\widetilde{A}(p) &= \int dx e^{ip\cdot x} A(x) \\ A(x) &= \int [dp] e^{-ip\cdot x} \widetilde{A}(p)\end{aligned} \tag{45}$$

where $[dp] \equiv d^4 p/(2\pi)^4$.

WT for Kinetic terms

We take WT for Eq. (37). The kinetic term becomes

$$
\begin{aligned}
I_{\mathrm{kin}} &= \int dy e^{ip\cdot y}\left[-i(i\gamma_\mu \partial^\mu_{x_1} - m)G(x_1,x_2)\right] \\
&= -i\int dy e^{ip\cdot y}\left(i\gamma_\mu \frac{1}{2}\partial^\mu_X + i\gamma_\mu \partial^\mu_y - m\right)G^<(y,X) \\
&= -i\int dy e^{ip\cdot y}\left(i\gamma_\mu \frac{1}{2}\partial^\mu_X - i\gamma_\mu \overleftarrow{\partial}^\mu_y - m\right)G^<(y,X) \\
&= -i\left(i\gamma_\mu \frac{1}{2}\partial^\mu_X + \gamma_\mu p^\mu - m\right)G^<(p,X)
\end{aligned}
\tag{46}
$$

We now make a gradient expansion of l.h.s. of Eq. (37). First we should define coordinate variables

$$
\begin{aligned}
(x_1, x_2) &\rightarrow (X, y) \\
(x_1, x') &\rightarrow \left(X + \frac{1}{2}y', y - y'\right) \\
(x', x_2) &\rightarrow \left(X - \frac{1}{2}(y - y'), y'\right) \\
y' &= x' - x_2 \\
y - y' &= x_1 - x_2 - (x' - x_2) \\
&= x_1 - x'
\end{aligned} \tag{47}
$$

WT and gradient expansion for collision terms

The collision term in Eq. (37) can be rewritten as Eq. (41) ($\hbar$ is suppressed). By setting $t_0 = -\infty$ we obtain at the leading order of gradient expansion

$$
\begin{aligned}
I_{+-}(x_1, x_2) &= \int_{-\infty}^{\infty} dx' \left[\Sigma^R(x_1, x')G^<(x', x_2) + \Sigma^<(x_1, x')G^A(x', x_2)\right] \\
&= \int_{-\infty}^{\infty} dy' \Sigma^R \left(y - y', X + \frac{1}{2}y'\right) G^< \left(y', X - \frac{1}{2}(y - y')\right) \\
&\quad + \int_{-\infty}^{\infty} dy' \Sigma^< \left(y - y', X + \frac{1}{2}y'\right) G^A \left(y', X - \frac{1}{2}(y - y')\right) \\
&\approx I_{+-}^{(0)} + I_{+-}^{(1)} \qquad (48)
\end{aligned}
$$

where $I_{+-}^{(0)}$ and $I_{+-}^{(1)}$ are the contribution of the leading and next-to-leading order in space-time derivatives, respectively.

WT and gradient expansion for collision terms

They are given by

$$
\begin{aligned}
I_{+-}^{(0)}(y,X) &= \int_{-\infty}^{\infty} dy'\Sigma^R(y-y',X)G^<(y',X) \\
&\quad + \int_{-\infty}^{\infty} dy'\Sigma^<(y-y',X)G^A(y',X) \\
I_{+-}^{(1)}(y,X) &= \frac{1}{2}\int_{-\infty}^{\infty} dy' y'_\mu \partial_X^\mu \Sigma^R(y-y',X)G^<(y',X) \\
&\quad -\frac{1}{2}\int_{-\infty}^{\infty} dy'\Sigma^R(y-y',X)(y_\mu - y'_\mu)\partial_X^\mu G^<(y',X) \\
&\quad +\frac{1}{2}\int_{-\infty}^{\infty} dy' y'_\mu \partial_X^\mu \Sigma^<(y-y',X)G^A(y',X) \\
&\quad -\frac{1}{2}\int_{-\infty}^{\infty} dy'\Sigma^<(y-y',X)(y_\mu - y'_\mu)\partial_X^\mu G^A(y',X) \qquad (49)
\end{aligned}
$$

WT for collision terms

The Wigner transformation of $I_{+-}^{(0)}$ and $I_{+-}^{(1)}$ reads

$$
\begin{aligned}
I_{+-}^{(0)}(p,X) &= \int dye^{ip\cdot y/\hbar} I_{+-}^{(0)}(y,X) \\
&= \Sigma^R(p,X)G^<(p,X) + \Sigma^<(p,X)G^A(p,X) \qquad (50) \\
I_{+-}^{(1)}(p,X) &= \int dye^{ip\cdot y/\hbar} I_{+-}^{(1)}(y,X) \\
&= -\frac{1}{2}i\hbar\left[\partial_X^\mu \Sigma^R(p,X)\partial_\mu^p G^<(p,X)\right. \\
&\left. -\partial_\mu^p \Sigma^R(p,X)\partial_X^\mu G^<(p,X)\right] \\
&- \frac{1}{2}i\hbar\left[\partial_X^\mu \Sigma^<(p,X)\partial_\mu^p G^A(p,X)\right. \\
&\left. -\partial_\mu^p \Sigma^<(p,X)\partial_X^\mu G^A(p,X)\right] \qquad (51)
\end{aligned}
$$

where we have replaced $y \to -i\hbar\partial_p$.

We can use Poisson bracket

$$\{A, B\}_{P.B.} \equiv \partial_X^\mu A \partial_\mu^p B - \partial_\mu^p A \partial_X^\mu B \tag{52}$$

to rewrite (51) as

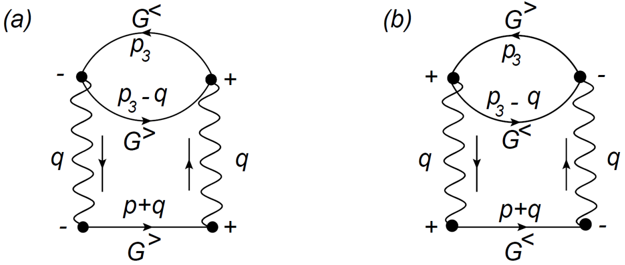
$$\begin{aligned} I_{+-}^{(1)}(p, X) &= \int dy e^{ip\cdot y/\hbar} I_{+-}^{(1)}(y, X) \\ &= -\frac{1}{2} i\hbar \left\{ \Sigma^R(p, X), G^<(p, X) \right\}_{P.B.} \\ &\quad -\frac{1}{2} i\hbar \left\{ \Sigma^<(p, X), G^A(p, X) \right\}_{P.B.} \end{aligned} \tag{53}$$

Then the KB equation for $G^<$ is then

$$
\begin{aligned}
& \left(i\gamma_\mu \frac{1}{2}\hbar\partial_X^\mu + \gamma_\mu p^\mu - m\right) G^<(p,X) \\
= \;& i\hbar\left[\Sigma^R(p,X)G^<(p,X) + \Sigma^<(p,X)G^A(p,X)\right] \\
& +\frac{1}{2}\hbar^2\left[\left\{\Sigma^R(p,X), G^<(p,X)\right\}_{P.B.}\right. \\
& \left. + \left\{\Sigma^<(p,X), G^A(p,X)\right\}_{P.B.}\right]
\end{aligned} \tag{54}
$$

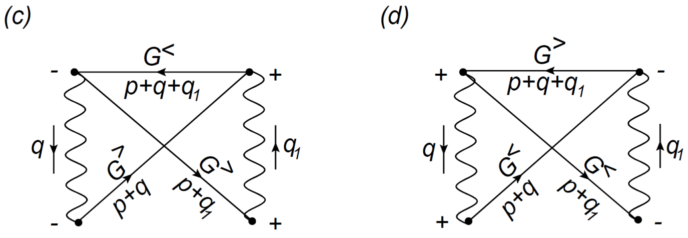
where we have recovered $\hbar$. Note that every propagator contributes $\hbar$, every vertex contributes $\hbar^{-1}$, so for Feynman diagram in Fig. (2) is in the order $\hbar$. So the first term in the right-hand side is in the order $\hbar$, while the second term is in the order $\hbar^2$ due to spatial derivative.

Figure: Feynman diagrams in QED for (a,c) $\Sigma^{>}(p)$ and (b,d) $\Sigma^{<}(p)$. Solid lines represent fermion propagators, wavy lines represent photon propagators.



Feynmann diagram for selfenergy

Figure: Feynman diagrams in QED for (a,c) $\Sigma^{>}(p)$ and (b,d) $\Sigma^{<}(p)$. Solid lines represent fermion propagators, wavy lines represent photon propagators.



Retarded and Advanced Green functions

The retarded Green function is defined as

$$G^R(y,X) = \theta(y_0)\left[G^>(y,X) - G^<(y,X)\right] \tag{55}$$

We take Wigner transformation

$$\begin{aligned}
G^R(p,X) &= \int d^4y e^{ip\cdot y} G^R(y,X) \\
&= \int d^4y e^{ip\cdot y}\theta(y_0)\left[G^>(y,X) - G^<(y,X)\right] \\
&= \int d^4y e^{ip\cdot y}\frac{1}{2\pi i}\int_{-\infty}^{\infty} dq_0 \frac{1}{q_0 - i\epsilon} e^{iq_0 y_0} \\
&\times \int [d^4k] e^{-ik\cdot y}[G^>(k,X) - G^<(k,X)]
\end{aligned} \tag{56}$$

where we have used

$$\theta(y_0) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dq_0 \frac{1}{q_0 - i\epsilon} e^{iq_0 y_0} \tag{57}$$

We first complete the integral over y_0 and then over q_0

$$
\begin{aligned}
G^R(p,X) &= \frac{1}{2\pi i}\int[d^4k]\int dq_0\int dy_0 d^3y e^{-i(\mathbf{p}-\mathbf{k})\cdot\mathbf{y}} e^{i(p_0+q_0-k_0)y_0} \\
&\times \frac{1}{q_0-i\epsilon}[G^>(k,X)-G^<(k,X)] \\
&= \frac{1}{2\pi i}\int[d^4k]\int dq_0(2\pi)^4\delta(p_0+q_0-k_0)\delta^{(3)}(\mathbf{p}-\mathbf{k}) \\
&\times \frac{1}{q_0-i\epsilon}[G^>(k,X)-G^<(k,X)] \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dk_0 \frac{1}{k_0-p_0-i\epsilon} \\
&\times [G^>(k_0,\mathbf{p},X)-G^<(k_0,\mathbf{p},X)]
\end{aligned} \tag{58}
$$

Retarded and Advanced Green functions

For the advanced two-point function we can derive the similar formula

$$G^A(p,X) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{k_0 - p_0 + i\epsilon}[G^>(k_0,\mathbf{p},X) - G^<(k_0,\mathbf{p},X)] \quad (59)$$

Similar to Eq. (58,59), we have

$$\begin{aligned} \Sigma^R(p,X) &= \frac{1}{2\pi i}\int dk_0 \frac{1}{k_0 - p_0 - i\epsilon}\left[\Sigma^>(k_0,\mathbf{p},X) - \Sigma^<(k_0,\mathbf{p},X)\right] \\ \Sigma^A(p,X) &= \frac{1}{2\pi i}\int dk_0 \frac{1}{k_0 - p_0 + i\epsilon}\left[\Sigma^>(k_0,\mathbf{p},X) - \Sigma^<(k_0,\mathbf{p},X)\right] \end{aligned} \quad (60)$$

On-shell approximation

If we only consider the scattering effect which corresponds to taking the imaginary part of $(k_0 - p_0 \pm i\epsilon)^{-1}$, then we have

$$\frac{1}{k_0 - p_0 \pm i\epsilon} \approx \mp i\pi\delta(k_0 - p_0) \tag{61}$$

So we obtain

$$\begin{aligned} O^R(p) &\approx \frac{1}{2}\left[O^>(p) - O^<(p)\right] \\ O^A(p) &\approx -\frac{1}{2}\left[O^>(p) - O^<(p)\right] \end{aligned} \tag{62}$$

where $O = \Sigma, G$.

Then the KB equation for $G^{<}$ becomes

$$
\begin{aligned}
& \left(i\gamma_{\mu}\frac{1}{2}\hbar\partial_{X}^{\mu}+\gamma_{\mu}p^{\mu}-m\right)G^{<}(p,X) \\
= & -i\frac{1}{2}\hbar\left[\Sigma^{<}(p,X)G^{>}(p,X)-\Sigma^{>}(p,X)G^{<}(p,X)\right] \\
& -\frac{1}{4}\hbar^{2}\left[\left\{\Sigma^{<}(p,X),G^{>}(p,X)\right\}_{P.B.}\right. \\
& \left.-\left\{\Sigma^{>}(p,X),G^{<}(p,X)\right\}_{P.B.}\right] \qquad (63)
\end{aligned}
$$

Similarly one can derive the KB equation for $G^{>}$ from Eq. (63) by changing $G^{<}\rightarrow G^{>}$ in the left-hand side. The collision term in the r.h.s. is denoted as I_{coll}.

Introduction to Matrix Valued Spin dependent Distribution (MVSD) for fermions.

Fermion fields

Quantized free Fermion fields are

$$\begin{aligned}\psi(x) &= \sum_s \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \frac{1}{2E_k} \\ &\quad \times \left[a(s,\mathbf{k}) u(s,\mathbf{k}) e^{-ik\cdot x/\hbar} + b^\dagger(s,\mathbf{k}) v(s,\mathbf{k}) e^{ik\cdot x/\hbar} \right] \\ \overline{\psi}(x) &= \sum_s \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \frac{1}{2E_k} \\ &\quad \times \left[a^\dagger(s,\mathbf{k}) \overline{u}(s,\mathbf{k}) e^{ik\cdot x/\hbar} + b(s,\mathbf{k}) \overline{v}(s,\mathbf{k}) e^{-ik\cdot x/\hbar} \right] \end{aligned} \tag{64}$$

where $k^0 \equiv E_k \equiv E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, the index $s = \pm 1$ denotes the spin state parallel or anti-parallel to the quantization direction $\mathbf{n}$, and $a(s,\mathbf{k})$, $a^\dagger(s,\mathbf{k})$, $b(s,\mathbf{k})$, and $b^\dagger(s,\mathbf{k})$ are creation and annihilation operators for fermions and antifermions, respectively, and $u(\mathbf{k},s)$ and $v(\mathbf{k},s)$ are spinors for fermions and antifermions, respectively.

These operators satisfy anticommutation relations

$$\left\{a(r,\mathbf{p},t),a^{\dagger}(s,\mathbf{q},t)\right\}=\left\{b(r,\mathbf{p},t),b^{\dagger}(s,\mathbf{q},t)\right\}$$
$$=2E_{\mathbf{p}}(2\pi\hbar)^{3}\delta_{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q}) \quad (65)$$

and all other anticommutators are vanishing. We have equal time anti-commutation relations

$$\begin{aligned}\{\psi_{\alpha}(t,\mathbf{x}),\psi_{\beta}(t,\mathbf{y})\} &= \{\psi_{\alpha}^{\dagger}(t,\mathbf{x}),\psi_{\beta}^{\dagger}(t,\mathbf{y})\}\\ \{\psi_{\alpha}(t,\mathbf{x}),\psi_{\beta}^{\dagger}(t,\mathbf{y})\} &= \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})\end{aligned} \quad (66)$$

Fermion spinors

In Dirac representation, the spinors take the explicit form

$$\begin{aligned} u(s,\mathbf{p}) &= \sqrt{E_p+m}\left(\begin{array}{c} \chi_s \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+m}\chi_s \end{array}\right) \\ v(s,\mathbf{p}) &= -i\sqrt{E_p+m}\left(\begin{array}{c} \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E_p+m}\sigma_2\chi_s^* \\ \sigma_2\chi_s^* \end{array}\right) \end{aligned} \quad (67)$$

They satisfy the relation $v(s,\mathbf{p}) = i\gamma^2 u^*(s,\mathbf{p})$, which one readily proves using $\sigma_2\boldsymbol{\sigma}\sigma_2 = -\boldsymbol{\sigma}^*$.

The spin eigenstates of χ_s along $\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ are

$$\chi_+ = \left(\begin{array}{c} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{array} \right), \quad \chi_- = \left(\begin{array}{c} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{array} \right) \tag{68}$$

which satisfy

$$\begin{aligned} \sigma\cdot\mathbf{n} &= \left(\begin{array}{cc} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{array} \right) \\ (\sigma\cdot\mathbf{n})\chi_s &= s\chi_s \end{aligned} \tag{69}$$

One can check

$$(\sigma_2\chi_s^*)^\dagger \boldsymbol{\sigma}\sigma_2\chi_s^* = -\chi_s^\dagger\boldsymbol{\sigma}\chi_s \tag{70}$$

MVSD for fermions

The Wigner function can be defined from the two-point function $G^{<}(x_1, x_2)$ in Eq. (29)

$$
\begin{aligned}
G^{<}_{\alpha\beta}(x,p) &= \int d^4y e^{ip\cdot y/\hbar} G^{<}(x_1,x_2) \\
&= -\int d^4y e^{ip\cdot y/\hbar} \left\langle \overline{\psi}_\beta \left(x - \frac{y}{2}\right) \psi_\alpha \left(x + \frac{y}{2}\right) \right\rangle \\
&= -\int d^4y e^{ip\cdot y/\hbar} \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3} \frac{d^3\mathbf{q}}{(2\pi\hbar)^3} \sum_{s,r} \frac{1}{2E_k} \frac{1}{2E_q} \\
&\times \left\{ \left\langle a^\dagger(\mathbf{k},s) a(\mathbf{q},r) \right\rangle \overline{u}_\beta(\mathbf{k},s) u_\alpha(\mathbf{q},r) e^{i(k-q)\cdot x/\hbar - i(k+q)\cdot y/(2\hbar)} \right. \\
&+ \left. \left\langle b(\mathbf{k},s) b^\dagger(\mathbf{q},r) \right\rangle \overline{v}_\beta(\mathbf{k},s) v_\alpha(\mathbf{q},r) e^{-i(k-q)\cdot x/\hbar + i(k+q)\cdot y/(2\hbar)} \right\}
\end{aligned}
\tag{71}
$$

where we have used the field quantization form of (64) and neglected $\left\langle a^\dagger(\mathbf{k},s) b^\dagger(\mathbf{q},r) \right\rangle$ and $\langle b(\mathbf{k},s) a(\mathbf{q},r) \rangle$.

MVSD for fermions

We can carry out integral over y and obtain

$$
\begin{aligned}
G^{<}_{\alpha\beta}(x,p) = -(2\pi\hbar)^4 \int \frac{d^3\mathbf{k}}{(2\pi\hbar)^3}\frac{d^3\mathbf{q}}{(2\pi\hbar)^3}\sum_{s,r}\frac{1}{2E_k}\frac{1}{2E_q} \\
\times \left\{ \left\langle a^\dagger(\mathbf{k},s)a(\mathbf{q},r)\right\rangle \overline{u}_\beta(\mathbf{k},s)u_\alpha(\mathbf{q},r)e^{i(k-q)\cdot x/\hbar} \right. \\
\times\, \delta^{(4)}[p-(k+q)/2] \\
+ \left\langle b(\mathbf{k},s)b^\dagger(\mathbf{q},r)\right\rangle \overline{v}_\beta(\mathbf{k},s)v_\alpha(\mathbf{q},r) \\
\left. \times\, e^{-i(k-q)\cdot x/\hbar}\delta^{(4)}[p+(k+q)/2] \right\} \qquad (72)
\end{aligned}
$$

where we have used the field quantization form of (64) and neglected $\left\langle a^\dagger(\mathbf{k},s)b^\dagger(\mathbf{q},r)\right\rangle$ and $\langle b(\mathbf{k},s)a(\mathbf{q},r)\rangle$.

MVSD for fermions

We change integral variables from q and k to p' and u ($q_0 = E_{\mathbf{p}'+\mathbf{u}/2}$ and $k_0 = E_{\mathbf{p}'-\mathbf{u}/2}$)

$$
\begin{aligned}
p' &= \frac{1}{2}(k+q) \\
p'_0 &= \frac{1}{2}(E_{\mathbf{p}'+\mathbf{u}/2} + E_{\mathbf{p}'-\mathbf{u}/2}) \\
u &= q-k \\
u_0 &= E_{\mathbf{p}'+\mathbf{u}/2} - E_{\mathbf{p}'-\mathbf{u}/2}
\end{aligned}
\tag{73}
$$

We have

$$
\begin{aligned}
G^{<}_{\alpha\beta}(x,p) = &-(2\pi\hbar)^4 \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3}\frac{d^3\mathbf{u}}{(2\pi\hbar)^3}\sum_{s,r}\frac{1}{2E_{\mathbf{p}+\mathbf{u}/2}}\frac{1}{2E_{\mathbf{p}-\mathbf{u}/2}} \\
&\times\left\{\left\langle a^{\dagger}\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)a\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\right\rangle\right. \\
&\times u_{\alpha}\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\overline{u}_{\beta}\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)e^{-iu\cdot x/\hbar}\delta^{(4)}(p-p') \\
&+\left[2E_{\mathbf{p}}(2\pi\hbar)^3\delta_{rs}\delta^{(3)}(\mathbf{u})\right. \\
&\left.-\left\langle b^{\dagger}\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)b\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)\right\rangle\right] \\
&\left.\times v_{\alpha}\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\overline{v}_{\beta}\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)e^{iu\cdot x/\hbar}\delta^{(4)}(p+p')\right\}
\end{aligned}
\tag{74}
$$

MVSD for fermions

Then complete the integral over **p**

$$
\begin{aligned}
G^{<}_{\alpha\beta}(x,p) = & -(2\pi\hbar)\int\frac{d^3\mathbf{u}}{(2\pi\hbar)^3}\sum_{s,r}\frac{1}{2E_{\mathbf{p}+\mathbf{u}/2}}\frac{1}{2E_{\mathbf{p}-\mathbf{u}/2}} \\
& \times\left\{\left\langle a^{\dagger}\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)a\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\right\rangle\right. \\
& \times u_{\alpha}\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\overline{u}_{\beta}\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)e^{-iu\cdot x/\hbar}\delta(p_0-p_0') \\
& +\left[2E_{\mathbf{p}}(2\pi\hbar)^3\delta_{rs}\delta^{(3)}(\mathbf{u})\right. \\
& \left.-\left\langle b^{\dagger}\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)b\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)\right\rangle\right] \\
& \left.\times v_{\alpha}\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\overline{v}_{\beta}\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)e^{iu\cdot x/\hbar}\delta(p_0+p_0')\right\} \quad (75)
\end{aligned}
$$

MVSD: leading order result

For the leading order resuld of MVSD, we make approximation

$$
\begin{aligned}
u_\alpha\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\overline{u}_\beta\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right) &\approx u_\alpha(\mathbf{p},r)\,\overline{u}_\beta(\mathbf{p},s) \\
v_\alpha\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\overline{v}_\beta\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},s\right) &\approx v_\alpha(-\mathbf{p},r)\,\overline{v}_\beta(-\mathbf{p},s) \\
\delta\left(p_0-\frac{E_{\mathbf{p}+\mathbf{u}/2}+E_{\mathbf{p}-\mathbf{u}/2}}{2}\right) &\approx \delta(p_0-E_p) \\
\delta\left(p_0+\frac{E_{\mathbf{p}+\mathbf{u}/2}+E_{\mathbf{p}-\mathbf{u}/2}}{2}\right) &\approx \delta(p_0+E_p) \\
\frac{1}{E_{\mathbf{p}+\mathbf{u}/2}E_{\mathbf{p}-\mathbf{u}/2}} &\approx \frac{1}{E_p^2}
\end{aligned}
\tag{76}
$$

We define MVSD (spin density matrix) at the leading order

$$f_{rs}^{(+,0)}(x,\mathbf{p}) = \frac{1}{2E_p}\int \frac{d^3\mathbf{u}}{(2\pi\hbar)^3} e^{-iu\cdot x/\hbar} \left\langle a^\dagger\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right) a\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\right\rangle$$

$$= \int \frac{d^4 u}{2(2\pi\hbar)^3} e^{-iu\cdot x/\hbar}\delta(p\cdot u) \left\langle a^\dagger\left(\mathbf{p}-\frac{1}{2}\mathbf{u},s\right) a\left(\mathbf{p}+\frac{1}{2}\mathbf{u},r\right)\right\rangle$$

$$f_{sr}^{(-,0)}(x,-\mathbf{p}) = \frac{1}{2E_p}\int \frac{d^3\mathbf{u}}{(2\pi\hbar)^3} e^{iu\cdot x/\hbar} \left\langle b^\dagger\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},r\right) b\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},s\right)\right\rangle$$

$$= \int \frac{d^4 u}{2(2\pi\hbar)^3} e^{-iu\cdot x/\hbar}\delta(\overline{p}\cdot u) \times \left\langle b^\dagger\left(-\mathbf{p}-\frac{1}{2}\mathbf{u},r\right) b\left(-\mathbf{p}+\frac{1}{2}\mathbf{u},s\right)\right\rangle \quad (77)$$

The MVSD $f_{sr}^{(0)}$ can be written as (scalar + polarization part)

$$f^{(0)} = \frac{1}{2}\mathrm{Tr}\left(f^{(0)}\right) + \frac{1}{2}\boldsymbol{\tau}\cdot\mathrm{Tr}\left(\boldsymbol{\tau} f^{(0)}\right) \quad (78)$$

In Eq. (77) we used $\overline{p}^{\mu} = (E_p, -\mathbf{p})$, and we have flipped the sign of $\mathbf{u}$ in the second equality of $f_{sr}^{(-,0)}(x, \mathbf{p})$ and reused $u_0 = E_{-\mathbf{p}+\mathbf{u}/2} - E_{-\mathbf{p}-\mathbf{u}/2}$, so that the definition of u^{μ} is the same as the fermion.

WF in MVSD: leading order result

Then the Wigner function in terms of MVSD at the leading order reads

$$
\begin{aligned}
G_{\alpha\beta}^{<(0)}(x,p) &= -(2\pi\hbar)\theta(p_0)\delta\left(p^2-m^2\right) \\
&\times \sum_{s,r} u_\alpha(\mathbf{p},r)\,\overline{u}_\beta(\mathbf{p},s)\, f_{rs}^{(+,0)}(x,\mathbf{p}) \\
&- (2\pi\hbar)\theta(-p_0)\delta\left(p^2-m^2\right) \\
&\times \sum_{s,r} v_\alpha(-\mathbf{p},r)\,\overline{v}_\beta(-\mathbf{p},s)\left[\delta_{sr} - f_{sr}^{(-,0)}(x,-\mathbf{p})\right] \\
G_{\alpha\beta}^{>(0)}(x,p) &= (2\pi\hbar)\theta(p_0)\delta\left(p^2-m^2\right) \\
&\times \sum_{s,r} u_\alpha(\mathbf{p},r)\,\overline{u}_\beta(\mathbf{p},s)\left[\delta_{rs} - f_{rs}^{(+,0)}(x,\mathbf{p})\right] \\
&+ (2\pi\hbar)\theta(-p_0)\delta\left(p^2-m^2\right) \\
&\times \sum_{s,r} v_\alpha(-\mathbf{p},r)\,\overline{v}_\beta(-\mathbf{p},s)\, f_{sr}^{(-,0)}(x,-\mathbf{p})
\end{aligned}
\tag{79}
$$

Components of WF

WF can be decomposed in terms of Clifford algebra basis

$$\Gamma_a = \{1, \gamma^\mu, i\gamma_5, \gamma_5\gamma^\mu, \sigma^{\mu\nu}\} \tag{80}$$

with $\gamma_5 = \gamma^5 = i\gamma^1\gamma^2\gamma^3\gamma^4$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Then $G^{<(0)}$ can be decomposed as

$$G^{<(0)} = \frac{1}{4}\left(\mathcal{F}^{(0)} + i\gamma_5\mathcal{P}^{(0)} + \gamma^\mu\mathcal{V}_\mu^{(0)} + \gamma_5\gamma^\mu\mathcal{A}_\mu^{(0)} + \frac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}^{(0)}\right) \tag{81}$$

The components can be extracted as

$$\begin{aligned} \mathcal{F}^{(0)} &= \mathrm{Tr}\left[G^{<(0)}\right], \quad \mathcal{P}^{(0)} = -i\mathrm{Tr}\left[\gamma_5 G^{<(0)}\right] \\ \mathcal{V}^\mu_{(0)} &= \mathrm{Tr}\left[\gamma^\mu G^{<(0)}\right], \quad \mathcal{A}^\mu_{(0)} = \mathrm{Tr}\left[\gamma^\mu\gamma_5 G^{<(0)}\right] \\ \mathcal{S}^{\mu\nu}_{(0)} &= \mathrm{Tr}\left[\sigma^{\mu\nu} G^{<(0)}\right] \end{aligned} \tag{82}$$

Clifford Components of WF at 0-th order

The Clifford component are

$$
\begin{aligned}
\mathcal{P}^{(0)} &= 0 \\
\mathcal{F}^{(0)} &= -(2\pi\hbar)2m\delta\left(p^2-m^2\right) \\
&\quad \times\left\{\theta(p_0)\mathrm{Tr}\left[f^{(+,0)}\right]-\theta(-p_0)\mathrm{Tr}\left[1-f^{(-,0)}(-\mathbf{p})\right]\right\} \\
\mathcal{V}^{\mu}_{(0)} &= \frac{p^{\mu}}{m}\mathcal{F}^{(0)} \\
\mathcal{A}^{\mu}_{(0)} &= -(2\pi\hbar)2m\delta\left(p^2-m^2\right)\left\{\theta(p_0)n^{\mu}(\mathbf{p},\mathbf{n}_j)\mathrm{Tr}\left(\tau_j f^{(+,0)}\right)\right. \\
&\quad \left.+\theta(-p_0)n^{\mu}(-\mathbf{p},\mathbf{n}_j)\mathrm{Tr}\left[\tau_j f^{(-,0)}(-\mathbf{p})\right]\right\} \\
\mathcal{S}^{\mu\nu}_{(0)} &= -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}\mathcal{A}^{(0)}_{\beta}
\end{aligned}
\tag{83}
$$

Polarization direction

Note that $p^\mu = (p_0, \mathbf{p})$, and constraint $p_\mu \mathcal{A}^\mu_{(0)} = 0$ since $\theta(p_0) p \cdot n(\mathbf{p}, \mathbf{n}_j) = 0$ and $\theta(-p_0) p \cdot n(-\mathbf{p}, \mathbf{n}_j) = 0$. Here we have [$\mathbf{n}_3$ is the spin quantization direction in the particle's rest frame, $n^\mu(\mathbf{p}, \mathbf{n}_j)$ is the 4-vector of the direction $\mathbf{n}_j$ in the rest frame]

$$\begin{aligned}
n^\mu(\mathbf{p}, \mathbf{n}_j) &= \left(\frac{\mathbf{n}_j \cdot \mathbf{p}}{m}, \mathbf{n}_j + \frac{(\mathbf{n}_j \cdot \mathbf{p})\mathbf{p}}{m(E_p + m)} \right), \\
\chi_s^\dagger \boldsymbol{\sigma} \chi_r &= \begin{pmatrix} \mathbf{n} & \mathbf{n}_1 - i\mathbf{n}_2 \\ \mathbf{n}_1 + i\mathbf{n}_2 & -\mathbf{n} \end{pmatrix}_{sr} = \mathbf{n}_j (\tau_j)_{sr} \\
\mathbf{n}_1 &= (\cos\phi\cos\theta, \sin\phi\cos\theta, -\sin\theta) \\
\mathbf{n}_2 &= (-\sin\phi, \cos\phi, 0) \\
\mathbf{n}_3 &\equiv \mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad (84)
\end{aligned}$$

Formulas for spinors

In deriving Eq. (83) we used following formulas with $p^\mu = (E_p, \mathbf{p})$

$$
\begin{aligned}
u(r,\mathbf{p})\overline{u}(s,\mathbf{p}) &= \frac{1}{2}\left(m+\gamma^\mu p_\mu\right)\delta_{rs} + \frac{1}{2} m\gamma^5\gamma^\mu n_\mu(\mathbf{p},\mathbf{n}_j)(\tau_j)_{sr} \\
&\quad -\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\sigma^{\mu\nu}p^\alpha n^\beta(\mathbf{p},\mathbf{n}_j)(\tau_j)_{sr} \\
v(r,\mathbf{p})\overline{v}(s,\mathbf{p}) &= i\gamma^2 u^*(r,\mathbf{p})[i\gamma^2 u^*(s,\mathbf{p})]^\dagger \gamma^0 \\
&= \gamma^2\left[u(r,\mathbf{p})\overline{u}(s,\mathbf{p})\right]^*\gamma^2 \\
&= \frac{1}{2}\gamma^2\left(m+\gamma^{\mu *} p_\mu\right)\gamma^2\delta_{rs} + \frac{1}{2} m\gamma^2\gamma^5\gamma^{\mu *}\gamma^2 n_\mu(\mathbf{p},\mathbf{n}_j)(\tau_j^*)_{sr} \\
&\quad -\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\gamma^2\sigma^{\mu\nu *}\gamma^2 p^\alpha n^\beta(\mathbf{p},\mathbf{n}_j)(\tau_j^*)_{sr} \\
&= \frac{1}{2}\left(-m+\gamma^\mu p_\mu\right)\delta_{rs} - \frac{1}{2} m\gamma^5\gamma^\mu n_\mu(\mathbf{p},\mathbf{n}_j)(\tau_j)_{rs} \\
&\quad -\frac{1}{4}\epsilon_{\mu\nu\alpha\beta}\sigma^{\mu\nu}p^\alpha n^\beta(\mathbf{p},\mathbf{n}_j)(\tau_j)_{rs}
\end{aligned} \tag{85}
$$

Lecture 3: SBE in terms MVSD

Introduction to SBE in terms of MVSD.

KBE: Clifford components

The KB equation (63) is a Dirac-type equation

$$(\gamma_\mu K^\mu - m)\, G^<(x,p) = I_{\mathrm{coll}} \tag{86}$$

where the operator K^μ is defined as

$$K^\mu \equiv p^\mu + \frac{i\hbar}{2}\partial_x^\mu \tag{87}$$

Acting the operator $\gamma_\nu K^\nu + m$ on the KB equation (86) we can obtain a Klein-Gordon-type equation

$$\left(K^2 - m^2\right) G^<(x,p) = (\gamma_\nu K^\nu + m)\, I_{\mathrm{coll}} \tag{88}$$

where we have used the property $[K_\mu, K_\nu] = 0$.

KBE: Clifford components

Taking the Hermitian conjugate of Eq. (86), we obtain a conjugate equation

$$\left(K^{2*} - m^2\right) G^<(x, p) = \gamma^0 \left[\left(\gamma_\nu K^\nu + m\right) I_{\mathrm{coll}}\right]^\dagger \gamma^0 \tag{89}$$

where we have used the Hermitian property of the Wigner function

$$[G^<(x, p)]^\dagger = \gamma^0 G^<(x, p)\gamma^0 \tag{90}$$

In Eqs. (88) and (89) we have

$$\begin{aligned} K^2 &= p^2 - \frac{\hbar^2}{4}\partial_x^2 + i\hbar p \cdot \partial_x \\ K^{2*} &= p^2 - \frac{\hbar^2}{4}\partial_x^2 - i\hbar p \cdot \partial_x \end{aligned} \tag{91}$$

By taking the sum and difference of Eq. (88) and (89) we obtain the modified on-shell condition and Boltzmann equation for the Wigner function

$$\begin{aligned}\left(p^2-\frac{\hbar^2}{4}\partial_x^2-m^2\right)G^<(x,p) &= \frac{1}{2}(\gamma^\mu K_\mu+m)I_{\mathrm{coll}} \\ &+\frac{1}{2}\gamma^0\left[(\gamma^\mu K_\mu+m)I_{\mathrm{coll}}\right]^\dagger\gamma^0 \\ \hbar p\cdot\partial_x G^<(x,p) &= -\frac{i}{2}(\gamma^\mu K_\mu+m)I_{\mathrm{coll}} \\ &+\frac{i}{2}\gamma^0\left[(\gamma^\mu K_\mu+m)I_{\mathrm{coll}}\right]^\dagger\gamma^0 \end{aligned} \tag{92}$$

Then we obtain equations for components

$$\begin{aligned}\left(p^2-\frac{\hbar^2}{4}\partial_x^2-m^2\right)\mathrm{Tr}\left(\Gamma_a G^<\right) &= \mathrm{ReTr}\left[\Gamma_a(\gamma\cdot K+m)I_{\mathrm{coll}}\right], \\ \hbar p\cdot\partial_x\mathrm{Tr}\left(\Gamma_a G^<\right) &= \mathrm{ImTr}\left[\Gamma_a(\gamma\cdot K+m)I_{\mathrm{coll}}\right], \quad (93)\end{aligned}$$

where we have used $\Gamma_a^\dagger=\gamma_0\Gamma_a\gamma_0$.

We can solve KB equation (63) order by order in powers of $\hbar$. To this end, we expand every quantity (functions as well as operators) as

$$F = \sum_{n=0}^{\infty} \hbar^n F^{(n)} \tag{94}$$

and truncate this expansion at a given order n.

Zeroth order in $\hbar$

In the following, we choose the scalar and axial-vector components $\mathcal{F}$ and $\mathcal{A}_\mu$ as independent ones. Then we can express the other components in terms of $\mathcal{F}$ and $\mathcal{A}_\mu$. At zeroth order we have

$$\begin{aligned} \mathcal{P}^{(0)} &= 0 \\ \mathcal{V}^{\mu}_{(0)} &= \frac{p^\mu}{m}\mathcal{F}^{(0)} \\ \mathcal{S}^{\mu\nu}_{(0)} &= -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_\alpha \mathcal{A}^{(0)}_\beta \end{aligned} \tag{95}$$

Note that $\mathcal{A}^{\mu}_{(0)}$ is constrained by $p \cdot \mathcal{A}_{(0)} = 0$.

The Boltzmann equations for $\mathcal{F}$ and $\mathcal{A}_\mu$

$$\begin{aligned} p\cdot\partial_x\mathcal{F}^{(0)} &= 2m\,\mathrm{Im}\,\mathrm{Tr}\left(I_{\mathrm{coll}}^{(1)}\right) \\ p\cdot\partial_x\mathcal{A}^{(0)\mu} &= -\epsilon^{\mu\nu\alpha\beta}p_\nu\,\mathrm{Im}\,\mathrm{Tr}\left(\sigma_{\alpha\beta}I_{\mathrm{coll}}^{(1)}\right) \end{aligned} \tag{96}$$

Other components can be obtained from $\mathcal{F}$ and $\mathcal{A}_\mu$

$$\begin{aligned} \mathcal{P}^{(1)} &= -\frac{1}{2m}\partial_x^\mu\mathcal{A}_\mu^{(0)} + \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(i\gamma^5 I_{\mathrm{coll}}^{(1)}\right) \\ \mathcal{V}_\mu^{(1)} &= \frac{1}{m}p_\mu\mathcal{F}^{(1)} - \frac{1}{2m}\partial_x^\nu\mathcal{S}_{\nu\mu}^{(0)} - \frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(\gamma_\mu I_{\mathrm{coll}}^{(1)}\right) \\ \mathcal{S}_{\mu\nu}^{(1)} &= -\frac{1}{m}\epsilon_{\mu\nu\alpha\beta}p^\alpha\mathcal{A}^{(1)\beta} + \frac{1}{2m}\partial_{x[\mu}\mathcal{V}_{\nu]}^{(0)} \\ &\quad -\frac{1}{m}\mathrm{Re}\,\mathrm{Tr}\left(\sigma_{\mu\nu}I_{\mathrm{coll}}^{(1)}\right) \end{aligned} \tag{97}$$

The Boltzmann equations for $\mathcal{F}$ and $\mathcal{A}_\mu$

$$
\begin{aligned}
p\cdot\partial_x\mathcal{F}^{(1)} &= 2m\,\mathrm{Im}\,\mathrm{Tr}\left(I_{\mathrm{coll}}^{(2)}\right)+\mathrm{Re}\,\mathrm{Tr}\left(\gamma\cdot\partial_x I_{\mathrm{coll}}^{(1)}\right)\\
p\cdot\partial_x\mathcal{A}^{(1)\mu} &= -2p^\mu\,\mathrm{Im}\,\mathrm{Tr}\left(\gamma^5 I_{\mathrm{coll}}^{(2)}\right)-2\,\mathrm{Im}\,\mathrm{Tr}\left(\gamma\cdot p\gamma^5\gamma^\mu I_{\mathrm{coll}}^{(2)}\right)\\
&\quad-\mathrm{Re}\,\mathrm{Tr}\left(\gamma^5\partial_x^\mu I_{\mathrm{coll}}^{(1)}\right)
\end{aligned}
\tag{98}
$$

At order $\mathcal{O}(\hbar^2)$ the collision term reads

$$
I_{\mathrm{coll}}^{(2)}\equiv\Delta I_{\mathrm{coll}}^{(1)}+I_{\mathrm{coll,PB}}^{(0)}
\tag{99}
$$

Here we have defined

$$\Delta I_{\mathrm{coll}}^{(1)} = -\frac{i}{2}\Big[\Sigma^{<(1)}(x,p)G^{>(0)}(x,p) - \Sigma^{>(1)}(x,p)G^{<(0)}(x,p) + \Sigma^{<(0)}(x,p)G^{>(1)}(x,p) - \Sigma^{>(0)}(x,p)G^{<(1)}(x,p)\Big]$$

arises from an expansion to first order in $\hbar$ of the first collision term in Eq. (63), while

$$I_{\mathrm{coll,PB}}^{(0)} = -\frac{1}{4}\Big[\left\{\Sigma^{<(0)}(x,p), G^{>(0)}(x,p)\right\}_{\mathrm{PB}} - \left\{\Sigma^{>(0)}(x,p), G^{<(0)}(x,p)\right\}_{\mathrm{PB}}\Big] \tag{100}$$

is the leading-order contribution from the second collision term in Eq. (63).

SBE in MVSD at leading order

To extract the fermion sector in the left-hand side, we take an integration over $p_0 = [0, +\infty]$ of Eq. (96) for the scalar and axial vector components to obtain the Boltzmann equation for the scalar and polarization part of MVSD

$$\frac{1}{E_p} p \cdot \partial_x \mathrm{tr}\left[f^{(0)}(x,p)\right] = -\frac{1}{\pi\hbar}\int_0^\infty dp_0 \mathrm{Im}\mathrm{Tr}\left(I_{\mathrm{coll}}^{(1)}\right) \tag{101}$$

$$\begin{aligned} & \frac{1}{E_p} p \cdot \partial_x \mathrm{tr}\left[n_j^{(+)\mu} \tau_j^T f^{(0)}(x,p)\right] \\ & = \frac{1}{2\pi\hbar m}\epsilon^{\mu\nu\alpha\beta}\int_0^\infty dp_0 p_\nu \mathrm{Im}\mathrm{Tr}\left(\sigma_{\alpha\beta} I_{\mathrm{coll}}^{(1)}\right) \end{aligned} \tag{102}$$

From the scalar and polarization part of MVSD, one can rebuild $f_{sr}^{(0)}$ as

$$f^{(0)} = \frac{1}{2}\mathrm{Tr}\left(f^{(0)}\right) + \frac{1}{2}\boldsymbol{\tau} \cdot \mathrm{Tr}\left(\boldsymbol{\tau} f^{(0)}\right). \tag{103}$$

SBE in MVSD at leading order (scalar part)

$$
\begin{aligned}
&\frac{1}{E_p} p \cdot \partial_x \mathrm{tr}\left[f^{(0)}(x,p)\right] \\
=&\int_0^\infty \frac{dp_0}{2\pi\hbar} \mathrm{ReTr}\left[\Sigma^{<(0)} G^{>(0)} - \Sigma^{>(0)} G^{<(0)}\right] \\
=&\frac{1}{2E_p} \int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi\hbar)^3 2E_2} \frac{d^3\mathbf{p}_3}{(2\pi\hbar)^3 2E_3} \\
&\times (2\pi\hbar)^4 \delta^{(4)}(p + p_3 - p_1 - p_2) \\
&\times \left\{ f^{(0)}_{s_1 r_1}(p_1) f^{(0)}_{s_2 r_2}(p_2) \left[1 - f^{(0)}_{s_3 r_3}(p_3)\right] \left[1 - f^{(0)}_{sr}(p)\right] \right. \\
&\left. - f^{(0)}_{s_3 r_3}(p_3) f^{(0)}_{sr}(p) \left[1 - f^{(0)}_{s_1 r_1}(p_1)\right] \left[1 - f^{(0)}_{s_2 r_2}(p_2)\right] \right\} \\
&\times \mathrm{Re}\left(M_a^{\mathrm{scalar}} + M_b^{\mathrm{scalar}}\right)
\end{aligned}
\tag{104}
$$

$$
\begin{aligned}
&\frac{1}{E_p} p \cdot \partial_x \mathrm{tr}\left[n_j^{(+)\mu} \tau_j^T f^{(0)}(x,p) \right] \\
&= -\frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} \int_0^\infty \frac{dp_0}{2\pi\hbar} p_\nu \mathrm{ReTr}\left[\sigma_{\alpha\beta} \left(\Sigma^{<(0)} G^{>(0)} - \Sigma^{>(0)} G^{<(0)} \right) \right] \\
&= -\frac{1}{2E_p} \cdot \frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} p_\nu \int \frac{d^3\mathbf{p}_1}{(2\pi\hbar)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi\hbar)^3 2E_2} \frac{d^3\mathbf{p}_3}{(2\pi\hbar)^3 2E_3} \\
&\times (2\pi\hbar)^4 \delta^{(4)}(p + p_3 - p_1 - p_2) \\
&\times \left\{ f_{s_1 r_1}^{(0)}(p_1) f_{s_2 r_2}^{(0)}(p_2) \left[1 - f_{s_3 r_3}^{(0)}(p_3) \right] \left[1 - f_{sr}^{(0)}(p) \right] \right. \\
&\left. - f_{s_3 r_3}^{(0)}(p_3) f_{sr}^{(0)}(p) \left[1 - f_{s_1 r_1}^{(0)}(p_1) \right] \left[1 - f_{s_2 r_2}^{(0)}(p_2) \right] \right\} \\
&\times \mathrm{Re}\left(M_{a,\alpha\beta}^{\mathrm{pol}} + M_{b,\alpha\beta}^{\mathrm{pol}} \right) \qquad (105)
\end{aligned}
$$

Two-point functions at $O(\hbar)$

At $O(\hbar)$, two-point Green functions have three contributions

$$G^{\lessgtr(1)} = G_{\mathrm{qc}}^{\lessgtr(1)} + G_{\nabla}^{\lessgtr(1)} + G_{\mathrm{off}}^{\lessgtr(1)} \tag{106}$$

where $G_{\mathrm{qc}}^{\lessgtr(1)}$ is the quasi-classical part (due to corrections from MVSD), $G_{\nabla}^{\lessgtr(1)}$ is the gradient and collision part, and $G_{\mathrm{off}}^{\lessgtr(1)}$ is the off-shell part. Here we neglect $G_{\mathrm{off}}^{\lessgtr(1)}$ since its contribution is of higher order in the coupling constant. Note that $G^{\lessgtr(0)}$ is quasi-classical automatically. By definition, $G_{\mathrm{qc}}^{\lessgtr(1)}$ satisfies

$$\begin{aligned}
\mathcal{P}_{\mathrm{qc}}^{(1)} &= 0 \\
\mathcal{V}_{\mathrm{qc}}^{(1)\mu} &= \frac{p^{\mu}}{m}\mathcal{F}_{\mathrm{qc}}^{(1)} \\
\mathcal{S}_{\mathrm{qc}}^{(1)\mu\nu} &= -\frac{1}{m}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}\mathcal{A}_{\beta(1)}^{\mathrm{qc}}
\end{aligned} \tag{107}$$

Quasi-classical part of two-point functions at $O(\hbar)$

Here $\mathcal{F}_{\mathrm{qc}}^{(1)}$ and $\mathcal{A}_{\mathrm{qc}}^{(1)\beta}$ are given by

$$\begin{aligned}
\mathcal{F}_{\mathrm{qc}}^{(1)} &= -(2\pi\hbar)2m\delta\left(p^2-m^2\right) \\
&\quad \times\left\{\theta(p_0)\mathrm{Tr}\left[f^{(+,1)}\right]+\theta(-p_0)\mathrm{Tr}\left[f^{(-,1)}(-\mathbf{p})\right]\right\} \\
\mathcal{A}_{\mathrm{qc}}^{(1)\mu} &= -(2\pi\hbar)2m\delta\left(p^2-m^2\right)\left\{\theta(p_0)n^{\mu}(\mathbf{p},\mathbf{n}_j)\mathrm{Tr}\left(\tau_j f^{(+,1)}\right)\right. \\
&\quad \left.+\theta(-p_0)n^{\mu}(-\mathbf{p},\mathbf{n}_j)\mathrm{Tr}\left[\tau_j f^{(-,1)}(-\mathbf{p})\right]\right\} \qquad (108)
\end{aligned}$$

We see the above provide first order corrections to $\mathcal{F}^{(0)}$ and $\mathcal{A}^{(0)\beta}$ in (83) through MVSD $f^{(\pm,1)}$.

Gradient and collision part of two-point functions at $O(\hbar)$

On the other hand, $\mathcal{F}_{\nabla}^{(1)}$ and $\mathcal{A}_{\nabla}^{(1)\beta}$ contain spacetime derivatives of the zeroth-order Wigner functions and the collision term $I_{\mathrm{coll}}^{(1)}$

$$
\begin{aligned}
\mathcal{P}_{\nabla}^{(1)} &= -\frac{1}{2m}\partial_x^{\mu}\mathcal{A}_{\mu}^{(0)} + \frac{1}{m}\mathrm{Re\,Tr}\left(i\gamma^5 I_{\mathrm{coll}}^{(1)}\right) \\
\mathcal{V}_{\nabla,\mu}^{(1)} &= -\frac{1}{2m}\partial_x^{\nu}\mathcal{S}_{\nu\mu}^{(0)} - \frac{1}{m}\mathrm{Re\,Tr}\left(\gamma_{\mu} I_{\mathrm{coll}}^{(1)}\right) \\
\mathcal{S}_{\nabla,\mu\nu}^{(1)} &= \frac{1}{2m}\partial_{x[\mu}\mathcal{V}_{\nu]}^{(0)} - \frac{1}{m}\mathrm{Re\,Tr}\left(\sigma_{\mu\nu} I_{\mathrm{coll}}^{(1)}\right)
\end{aligned}
\qquad (109)
$$

Note that we have set $\mathcal{F}_{\nabla}^{(1)} = 0$ and $\mathcal{A}_{\nabla}^{(1)\mu} = 0$. If $\mathcal{F}_{\nabla}^{(1)}$ and $\mathcal{A}_{\nabla}^{(1)\mu}$ are non-vanishing, it can be absorbed into quasi-classical contributions of WFs by a redefinition of the first-order MVSD.

So $G_\nabla^{\lessgtr(1)}$ can be put into the form

$$\begin{aligned} G_\nabla^{<(1)} = G_\nabla^{>(1)} &\approx -\frac{1}{8m} i\gamma^5(\partial_x \cdot \mathcal{A}^{(0)}) \\ &\quad -\frac{1}{8m}\gamma^\nu \partial_x^\mu \mathcal{S}_{\mu\nu}^{(0)} + \frac{1}{8m}\sigma^{\mu\nu}\partial_{x,\mu}\mathcal{V}_\nu^{(0)} \\ &\approx \frac{i}{4m}\left[\gamma^\mu \partial_{x,\mu}, G^{<(0)}\right] \end{aligned} \tag{110}$$

Boltzmann equations at the next-to-leading order

$$\frac{1}{E_p} p \cdot \partial_x \mathrm{tr}\left[f^{(1)}(x,p)\right] = \mathscr{C}_{\mathrm{scalar}}\left(\Delta I^{(1)}_{\mathrm{coll,\,kin}}\right) + \mathscr{C}_{\mathrm{scalar}}\left(\Delta I^{(1)}_{\mathrm{coll,\,nonkin}}\right) + \mathscr{C}_{\mathrm{scalar}}\left(I^{(0)}_{\mathrm{coll,PB}}\right) + \mathscr{C}_{\mathrm{scalar}}\left(\partial_x I^{(1)}_{\mathrm{coll}}\right) \quad (111)$$

$$\frac{1}{E_p} p \cdot \partial_x \mathrm{tr}\left[n_j^{(+)\mu} \tau_j^T f^{(1)}(x,p)\right] = \mathscr{C}^{\mu}_{\mathrm{pol}}\left(\Delta I^{(1)}_{\mathrm{coll,\,kin}}\right) + \mathscr{C}^{\mu}_{\mathrm{pol}}\left(\Delta I^{(1)}_{\mathrm{coll,\,nonkin}}\right) + \mathscr{C}^{\mu}_{\mathrm{pol}}\left(I^{(0)}_{\mathrm{coll,PB}}\right) + \mathscr{C}^{\mu}_{\mathrm{pol}}\left(\partial_x I^{(1)}_{\mathrm{coll}}\right) \quad (112)$$

Solve MVSD order by order in $\hbar$

MVSD as a series of $\hbar$

$$
\begin{aligned}
f(x,p) &= f^{(0)}(x,p) + \hbar f^{(1)}(x,p) + \cdots \\
&= \frac{1}{2}\mathrm{Tr}\,(f) + \frac{1}{2}\boldsymbol{\tau}\cdot\mathrm{Tr}\,(\boldsymbol{\tau} f)
\end{aligned} \tag{113}
$$

The space-time evolution of the polarization part can given by $\mathrm{Tr}\,(\boldsymbol{\tau} f)$. We see in Eq. (112) that the polarization part depends on space-time derivative of $f^{(0)}(x,p)$ which can provide shear and vorticity contribution.

Summary and conclusion

- SBE in terms of MVSD can be derived from KBE.
- The MVSD can be solved order by order in $\hbar$.
- A rigorous approach to spin dynamics from first principle of quantum field theory.