Heavy-quark-pair production at lepton colliders at NNNLO in QCD

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CEPC WorkShop, 27 October 2022

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Based on: [arXiv: 2209.14259]

Outline

1. **Introduction: Motivation and Background**

2. Computational Techniques

3. Numerical Results

4. Summary and Outlook

Motivation and Background

- Top quark plays a crucial role
  - Electroweak(EW) physics
  - physics beyond the Standard Model
- Unprecedented experiment precision
  - e^+e^- colliders offer a clean environment
  - Precision measurement: top-quark mass, cross section, A_{FB}^t
  - A few ‰ to % precision at the on-going colliders are possible.

The International Linear Collider. *(2013)*
The CEPC Study Group. *(2018)*
FCC-ee, A. Abada *et al.* *(2019)*
ILD Concept Group. *(2020)*

Theoretical efforts in literature

- NNLO

B. H. Smith, M. B. Voloshin. *Phys. Lett. B(1994)*
K. G. Chetyrkin, A. Kwiatkowski. *Z. Phys. C(1994)*
K. G. Chetyrkin, J. H. Kuhn, M. Steinhauser. *Nucl. Phys. B(1996)*

  - Near-threshold expansion and high energy expansion of cross section are known for more than two decades
  - Cross section and fully differential distributions in the continuum

Jun Gao, Hua Xing Zhu. *Phys. Rev. D(2014)*
Jun Gao, Hua Xing Zhu. *Phys. Rev. Lett. (2014)*
Long Chen, Oliver Dekkers, *et al. JHEP(2016)*

$\sqrt{s}$ [GeV]	360	381.3	400	500
Δ_1	0.627	0.352	0.266	0.127
Δ_2	0.281	0.110	0.070	0.020

$$\sigma_{NNLO} = \sigma_{LO}(1 + \Delta_1 + \Delta_2)$$

From Long Chen, Oliver Dekkers, *et al. JHEP(2016)*

  - The theoretical uncertainty is about 1% From Jun Gao, Hua Xing Zhu. *Phys. Rev. D(2014)*
  - Full computation of NNNLO QCD correction is necessary!

- NNNLO(partial results)
  - Near-threshold expansion M. Beneke, *et al. Phys. Rev. Lett. (2015)*
  - Massive form factors M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *Phys. Rev. Lett. (2022)*
    M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *Phys. Rev. D (2022)*

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The computational work-flow

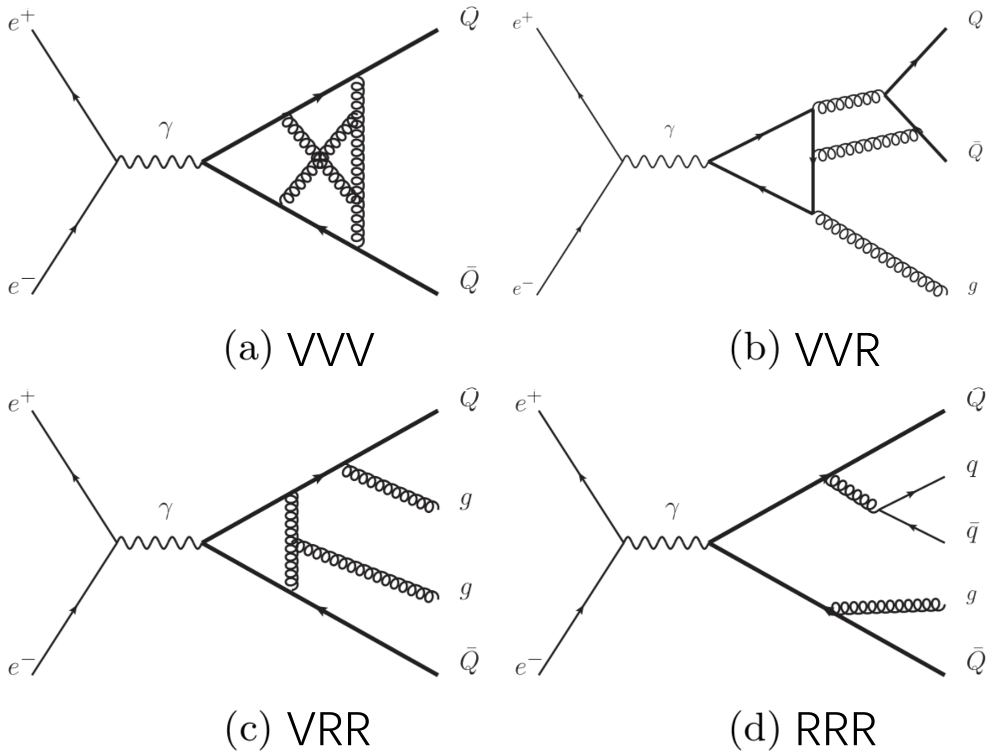
- Generating Feynman diagrams ($\mathcal{O}$(minutes))
- Dealing with Lorentz and color algebras ($\mathcal{O}$(few minutes to hours))
- IBP(integration-by-parts) reduction of loop integrals ($\mathcal{O}$(hours to days))
- Evaluating master integrals ($\mathcal{O}$(days) typically in frontier)

Dealing with Feynman Amplitudes

- Generating Feynman diagrams
  - Tools: **QGRAF** P. Nogueira. *J. Comput. Phys.* (1993)
  - Sum all virtual and real correction at NNNLO
  - Focus: vector contribution, i.e. $e^+e^- \to \gamma^* \to t\bar{t}$
- Dealing with Feynman amplitudes
  - Lorentz and color algebra
  - Phase-space integrals $\Rightarrow$ reverse unitarity

$$\delta(x) \to \frac{1}{2\pi i}\left(\frac{1}{x-i0} - \frac{1}{x+i0}\right)$$

  - Express squared amplitudes as linear combinations of scalar integrals
  - Total 475 families (VVV, VVR, VRR, RRR)



"V" means "Virtual", "R" means "Real"

$$|M|^2 = \sum_{i=1}^{\mathcal{O}(10^4)} a_i \times I_i$$

Integrals reduction

- Strategy:
  - Laporta's algorithm S. Laporta. *Int. J. Mod. Phys. A (2000)*
  - Tools: **LiteRed** and **FiniteFlow** ⟹ IBP system (over finite field) R. N. Lee. *J. Phys. Conf. Ser. (2014)* T. Peraro. *JHEP (2019)*
  - Numerical samplings ⟹ reconstruct the reduction coefficients

    Small number (typically 50)

    Large number (typically 1000)

    Block-triangular relations X. Guan, X. Liu, Y.-Q. Ma. *Chin. Phys. C (2020)*
  - Block-triangular form is an improved linear system
    - ✓ Contains as many equations as target integrals
    - ✓ Several orders of magnitude less than plain IBP system
  - Applied to this work and our recent another work X. Chen, X. Guan, C.-Q. He, Z. Li, X. Liu, Y.-Q. Ma. *arXiv:2209.14953*

    [⟶ Talk by Xin Guan]

Integrals reduction computing resources

- Computing resources:
  - Total 475 families

Contribution	Number of families	Computing resources (the most expensive family)
VVV	62	10 hours on 16 cores
VVR	147	5 hours on 16 cores
VRR	167	4 hours on 16 cores
RRR	99	2 hours on 16 cores

  - The computational time of the most expensive family of VVV contribution is about one order of magnitude shorter than that reported in literature

M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *Phys. Rev. D (2022)*

  - Block-triangular form reduction is much more efficient!

Evaluating master integrals

R. N. Lee, V. A. Smirnov. *JHEP(2018)*
R. Bonciani. G. Degrassi, P. P. Giardino, R. Grober. *Comput. Phys. Commun (2019)*
H. Frellesvig, M. Hidding, L. Maestri, F. Moriello, G. Salvatori. *JHEP(2020)*

- Method: using **numerical differential equations** based on power series expansion

Long Chen, M. Czakon, M. Niggetiedt. *JHEP(2021)*
Martijin Hidding. *Comput. Phys. Commun. (2021)*
X. Liu, Y.-Q. Ma. *arXiv:2201.11669(2021)*
M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *Phys. Rev. Lett. (2022)*

- General procedure
  - Set up the differential equations with respect to $x = \frac{4m^2}{s}$ by IBP reducing derivatives

$$\frac{dI_i(\epsilon, x)}{dx} = \sum_j A_{ij} I_j(\epsilon, x)$$

  - Obtain boundary condition (**non-trivial**)

X. Liu, Y.-Q. Ma. *arXiv:2201.11669(2021)*

We use **AMFlow** to calculate master integrals at $x = \frac{4}{23}$ (arbitrary regular point) [⟶ Talk by Xiao Liu]

  - Generalized series expansions around singularities

$$I(\epsilon, x) = \sum_{\mu,k,n} c_{\mu,k,n}(\epsilon) x^{\mu(\epsilon)} \log^k x \, x^n$$

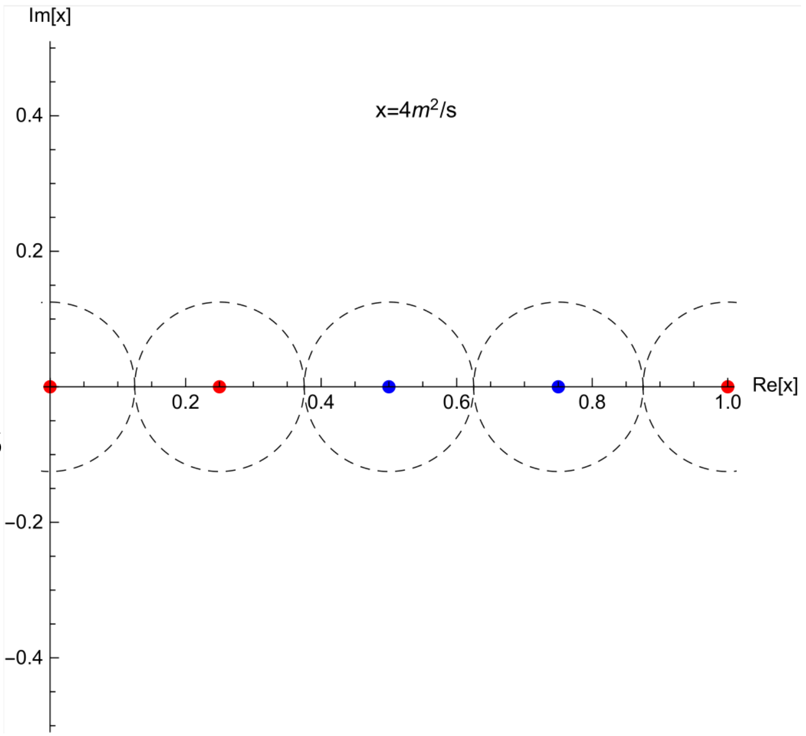
  - Taylor expansions around regular points

(the two expansions above) $\Longrightarrow$ Cover physical region

  - Match between two neighboring expansions

Results for master integrals

- Constructing a piecewise function of $x(=\frac{4m^2}{s})$ for each master integral
  - 5 deeply expanded power series at the following points
$$x \to \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$$
  - $0, \frac{1}{4}, 1$ are physical singularities, $\frac{1}{2}$ and $\frac{3}{4}$ are regular points
  - The expansion at each x_0 is **valid** in $x \in \left(x_0 - \frac{1}{8}, x_0 + \frac{1}{8}\right)$
  - The whole physical region $x \in (0,1)$ is covered



Renormalization

- α_s in the $\overline{\boldsymbol{MS}}$ **scheme**: multiplicative
- Wave-function renormalization in the **on-shell** scheme: multiplicative
- Quark-mass renormalization in the **on-shell** scheme:
  - Replace bare mass by renormalized mass via $m^b = m^{OS} + \delta m$
  - Keep on-shell condition of external top quark momentum $p^2 = (m^{OS})^2$ untouched
  - In spirit of perturbation theory, δm is a small quantity and can be expanded

$$\sigma(m^b) = \sigma(m^{OS}) + \left.\frac{\partial\sigma(m)}{\partial m}\right|_{m=m^{OS}} \delta m + \cdots, \qquad \delta m = (Z_m^{OS} - 1)m^{OS}$$

  - Substitute renormalization constant

Verification

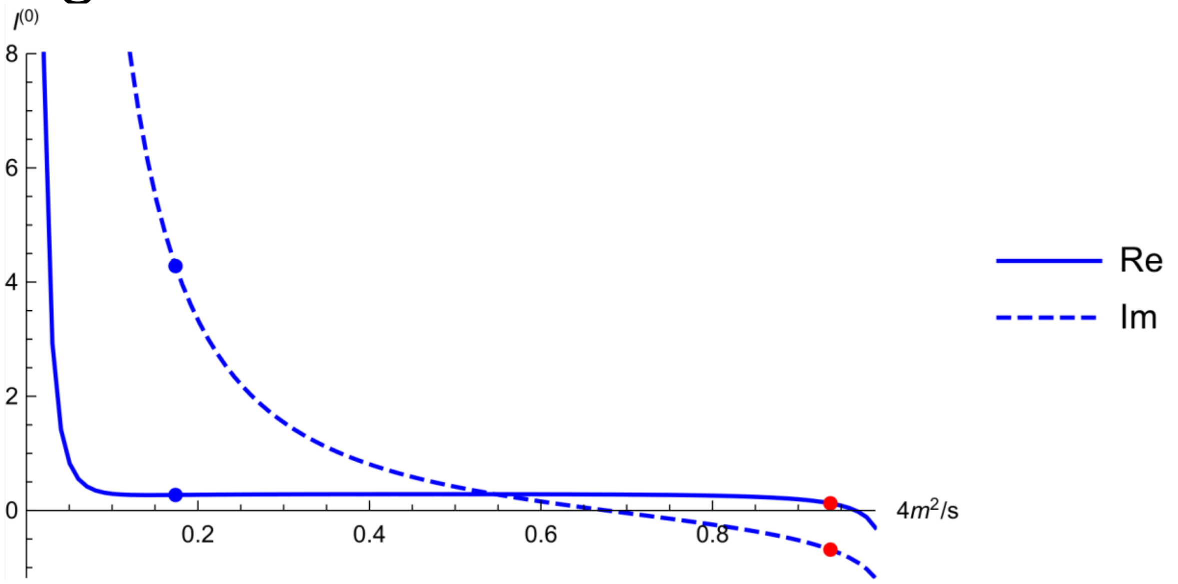
- Free of ultraviolet and infrared divergences

- Check master integrals

All MIs obtained by solving DEs with boundary conditions $I\left(x=\frac{4}{23},\epsilon\right)$

‖ checked

Evaluating master integrals by **AMFlow** at $x=\frac{15}{16}$ (another arbitrary point)



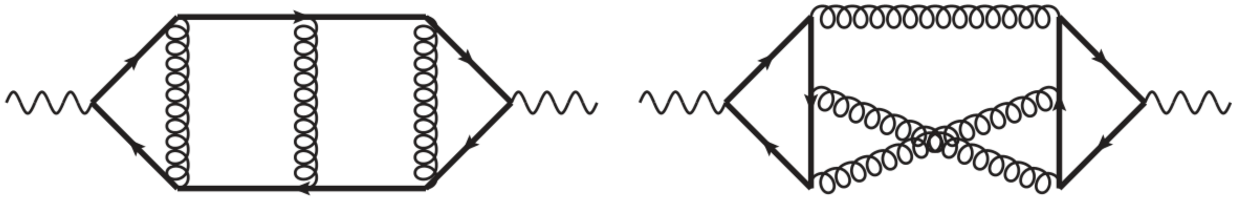
The blue lines correspond to the results of differential equations, the blue points correspond to boundary condition, the red points correspond to check point.

- Three-loop contributions agree with massive form factors

M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *Phys. Rev. Lett. (2022)*
M. Fael, F. Lange, K. Schonwald, M. Steinhauser. *Phys. Rev. D (2022)*

Verification

- We calculate the cross section in another totally different way
  - Optical theorem: imaginary part of forward scattering amplitude $\Longrightarrow$ cross section
  - Same computational strategy
  - Sample Feynman diagrams:



  - Using block-triangular relations is expected to be 2 orders of magnitude faster than plain IBP system
  - Contributions from **massless final states** without any top quark must be subtracted
  - Contributions from **four-heavy-quark production** should be subtracted when we consider the region above its threshold, i.e. $x < 1/4$

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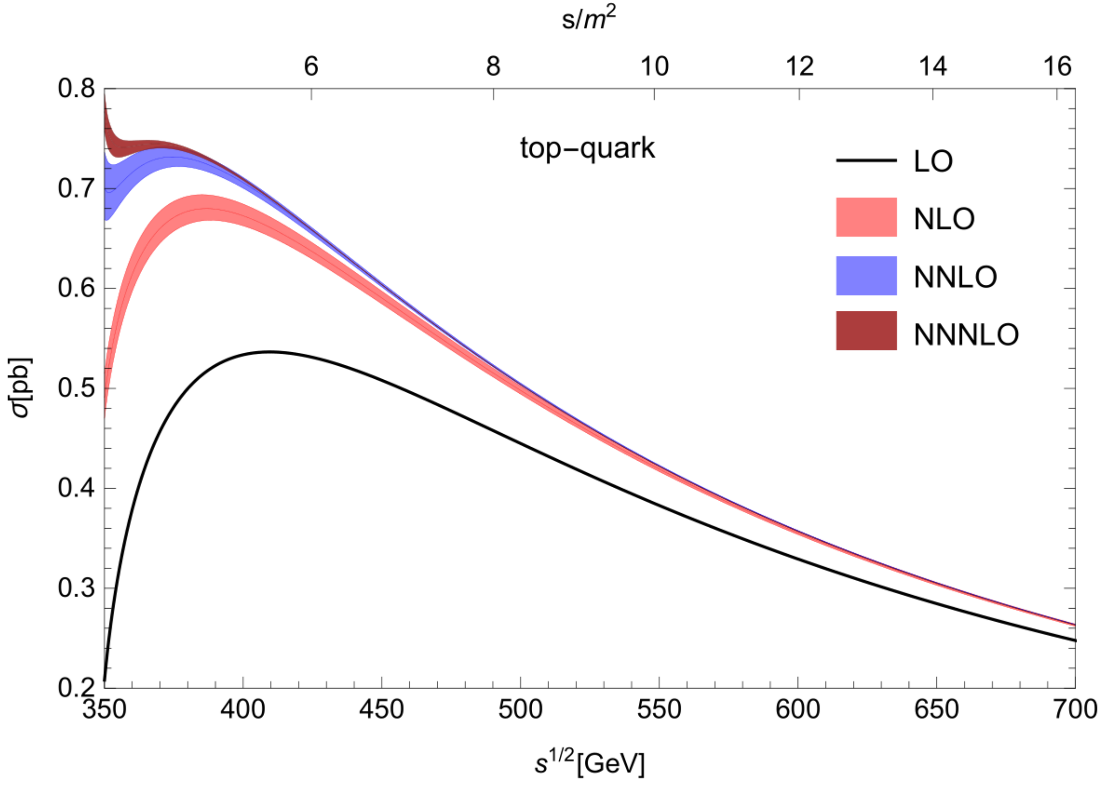
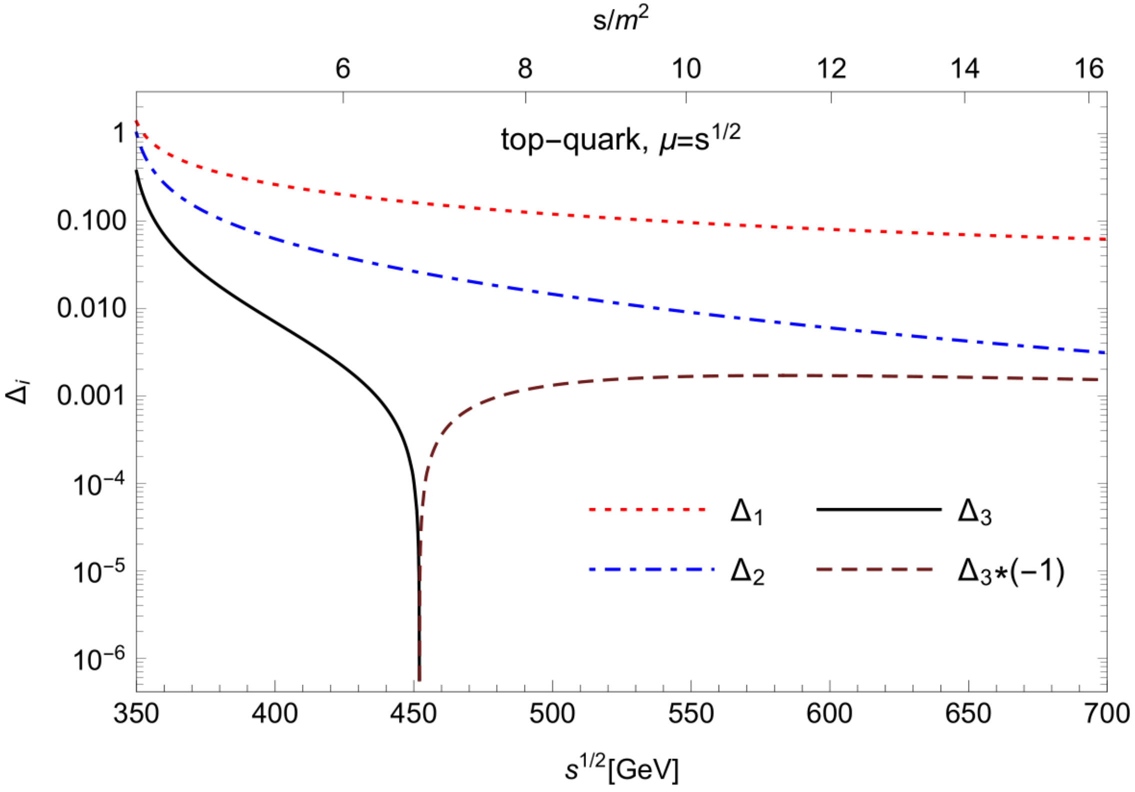
Numerical Results for NNNLO cross section

➢Top quark

- Input values: $m_t = 173.34GeV$, $\alpha = 1/132.2$, $\alpha_s^{n_f=5}(m_Z = 91.1876GeV) = 0.1181$
- $\sigma_{NNNLO} = \sigma_{LO}(1 + \Delta_1 + \Delta_2 + \Delta_3)$





The middle lines correspond to $\mu = \sqrt{s}$, the upper and lower lines correspond to $\mu = 2\sqrt{s}$ and $\mu = \sqrt{s}/2$

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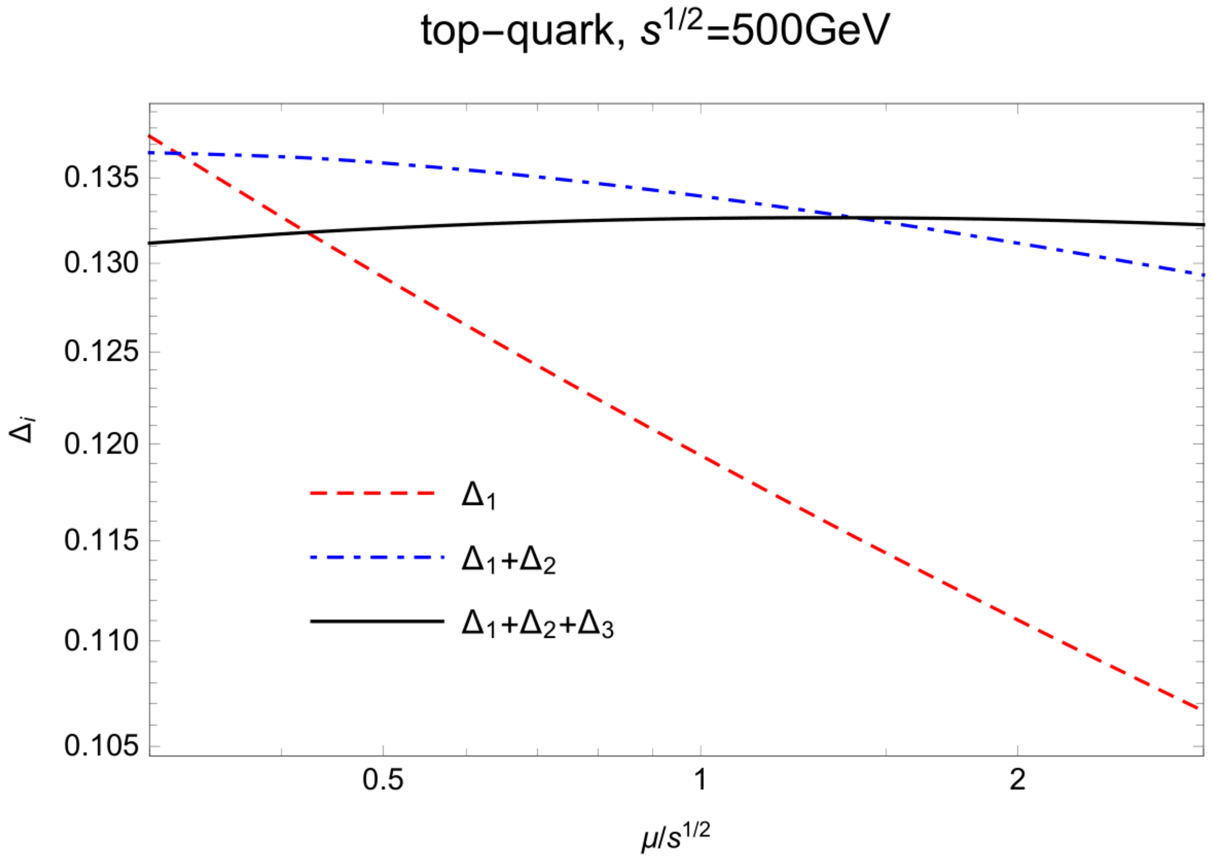


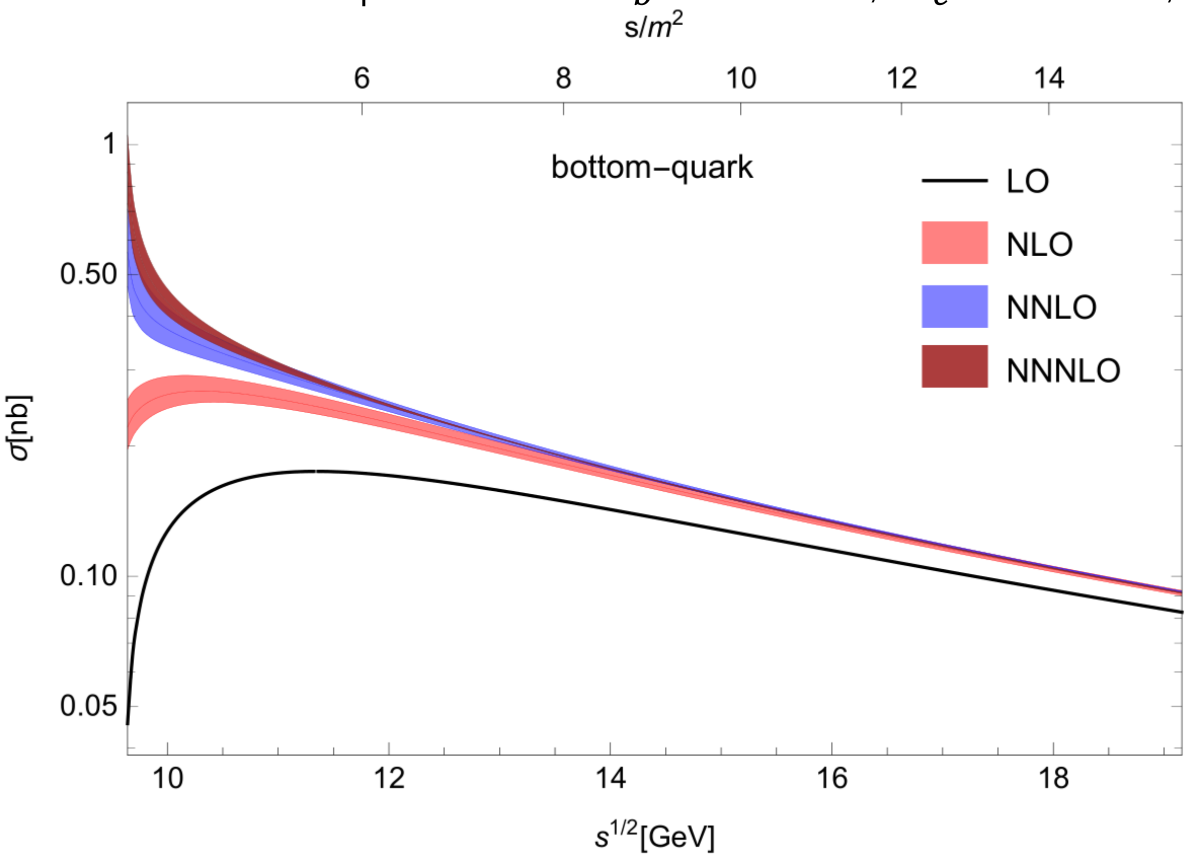
The scale dependence has been reduced from 0.72% at **NNLO** to 0.15% at **NNNLO** for a collider energy of $\mathbf{500GeV}$, which meets the precision (1‰ to 1%) requested by future lepton colliders!

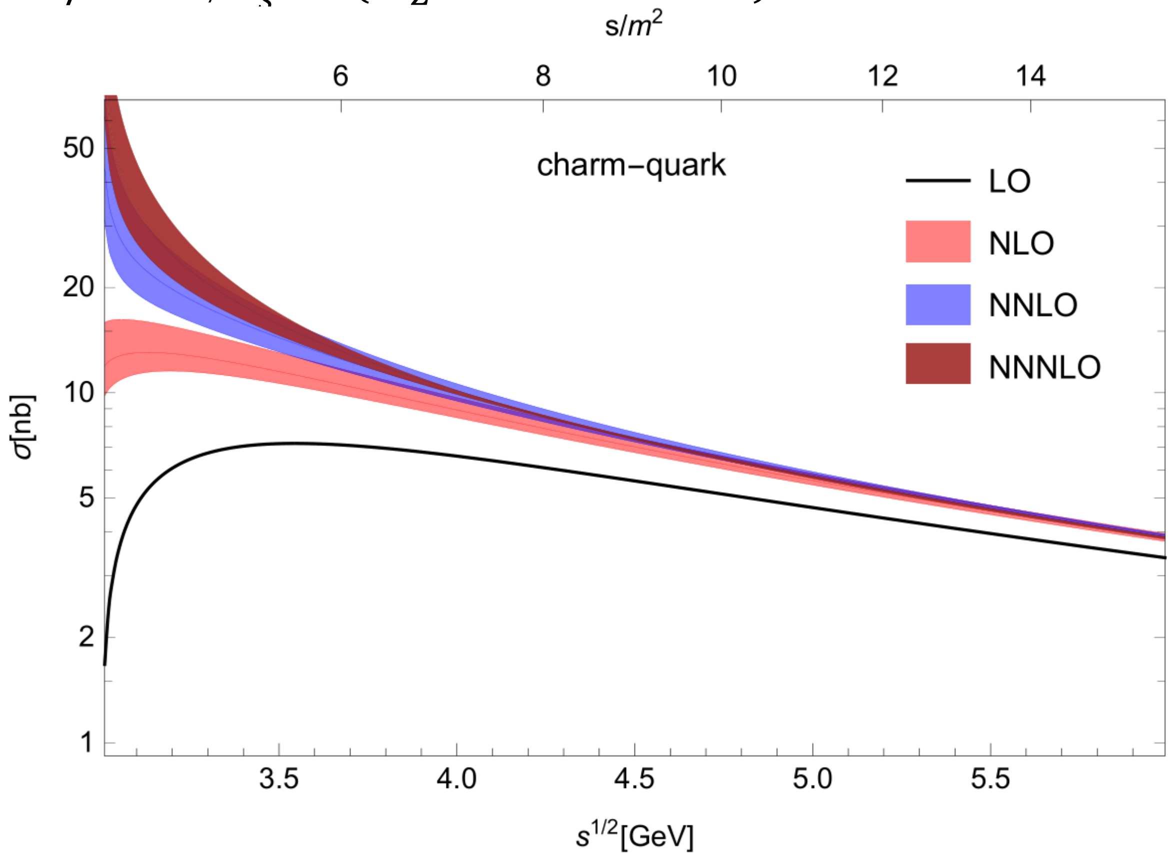
ILD Concept Group. *(2020)*

Numerical Results for NNNLO cross section

➢Bottom quark and Charm quark

- Input values: $m_b = 4.8GeV$, $m_c = 1.5GeV$, $\alpha = 1/132.2$, $\alpha_s^{n_f=5}(m_Z = 91.1876GeV) = 0.1181$





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Summary and Outlook

➢Summary

- We provide the **first NNNLO perturbative correction** to $t\bar{t}$ production cross section at lepton colliders mediated by a **virtual photon**.
- We combine several powerful techniques:
  - ✓ **Block-triangular relations**
  - ✓ **Auxiliary mass flow**
  - ✓ Numerical sampling of ϵ
  - ✓ A novel way to perform renormalization
- The calculation are done **in two totally different way** for cross check.
- Our result significantly **reduce theoretical uncertainty**, which **meets the precision** requested by future lepton colliders.

➢Outlook

- Contribution of top-quark-pair production mediated by a virtual **Z boson** will be calculated in the near future.
- Our strategy is **applicable** for many other process.