Amplitude Bootstrap of The Energy-Energy Correlation in QCD

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Presentation Overview

Precision Test in QCD

- One of the interest in QCD is to see how the strong coupling constant α_s evolves:

$$\frac{\alpha_s(\mu)}{2\pi} = \frac{1}{\beta_0 \log \frac{\mu}{\Lambda_{QCD}}} \tag{1}$$

  Here $\beta_0 = 11 - \frac{2}{3}N_f$, where N_f is the number of quark flavours.
- The goal is to do **precision test**.
- To see how $\alpha_s(\mu)$ evolves we need to study the event shapes. In particular we are interested in studying the **energy-energy correlation**

Energy-Energy Correlation
Introduction

- There are numerous event shapes available out there.
- Thrust, C-Parameter, Energy-Energy Correlation (EEC), etc.
- Among these event shapes, the EEC is analytically the simplest.

What makes EEC good?

1. It is an IR-safe observable, being based on the e^+e^- annihilation.
2. Has been known analytically up to the NLO in QCD [*Dixon et al.; 1801.03219*], and NNLO in $\mathcal{N} = 4$ super Yang-Mills [*Henn et al.; 1903.05314*].

Energy-Energy Correlation

Properties

1. The EEC measures the angle χ between two jets

$$\frac{d\Sigma}{d\cos\chi} = \sum_{i,j}\int \frac{E_i E_j}{Q^2}\delta(\vec{n}_i \cdot \vec{n}_j - \cos\chi)d\sigma, \tag{2}$$

2. The EEC also obeys the sum rule

$$\int d\cos\chi \frac{d\Sigma}{d\cos\chi} = \sigma \tag{3}$$

3. At the LO, it is known as

$$\Sigma = \sigma_0 \frac{\alpha_s C_F}{2\pi}\frac{3-2z}{4(1-z)z^5} \times \Big(3z(2-3z) + 2(2z^2-6z+3)\log(1-z)\Big) + \mathcal{O}(\alpha_s^2). \tag{4}$$

where $z = \frac{1-\cos\chi}{2}$

Energy-Energy Correlation

A Brief History

- It was first calculated by Basham et al (*PhysRevLett.41.1585*). around the early 80s.
- NLO results were calculated numerically at late 80s [*R.K. Ellis et al.; Nucl.Phys.B 178 (1981) 421-456*], NNLO quite recently [*Del Duca et al;, 1606.03453*]
- Analytically, it has been studied in $\mathcal{N} = 4$ SYM by Belitsky et al. (1311.6800), in QCD by Dixon et al., up to the NLO.
- in QCD, the analytical method uses reverse unitarity and IBP for the Feynman integrals, with the colour structure comes from colour decomposition.

Energy-Energy Correlation

Review of Known Results

- The reverse unitarity transforms phase space integrals into loop integrals, i.e [*Dulat et al.; 1704.08220*].

$$\delta(p^2 - m^2) \to \frac{i}{p^2 - m^2} \tag{5}$$

- This way, the EEC measurement function can be written as

$$\begin{aligned} &\delta((1-\cos\chi)p_1 \cdot Q p_2 \cdot Q - Q^2 p_1 \cdot p_2) \\ &\to \frac{1}{2\pi i} \frac{1}{(1-\cos\chi)p_1 \cdot Q p_2 \cdot Q - Q^2 p_1 \cdot p_2} \end{aligned} \tag{6}$$

- The cut propagator is then used to reduce the master integrals by using the Integration-by-Parts (IBP) identity:

$$\int d^d k \frac{\partial}{\partial k^\mu} \frac{q^\mu}{k^2 \ldots} = 0 \to I_1 + I_2 + \cdots = 0 \tag{7}$$

- Then solved using differential equations, $(d + \vec{A})\vec{I} = 0$

Energy-Energy Correlation

Review of Known Results (cont'd)

- The result is then structured as

$$\frac{d\Sigma}{d\cos\chi} = \sigma_0\left(\frac{\alpha_s(\mu)}{2\pi}A(z)+\left(\frac{\alpha_s(\mu)}{2\pi}\right)^2\left(\beta_0\log\frac{\mu}{Q}+B(z)\right)+\mathcal{O}(\alpha_s^3)\right) \tag{8}$$

- with $B(z)$

$$B(z) = C_F^2 B_{lc}(z) + C_F(C_A - 2C_F)B_{nlc}(z) + C_F N_f T_f B_{N_f}(z) \tag{9}$$

Energy-Energy Correlation

What is the proposal?

- The analytical calculation of EEC in QCD is possible to do using reverse unitarity and IBP
- We would like to present an alternative method based on Symbol calculus, first studied by Goncharov et al (*1006.5703*). and extensively studied to calculate hexagon functions in $\mathcal{N} = 4$ SYM [*Dixon et al.; 1407.4724*].
- We call this the **bootstrap method**.

Boostrap

Introduction

- The bootstrap is an analytical method of calculating a QFT observable based on its general properties
- It is beneficial to use to bypass certain complications
- Extensively used to calculate planar loop diagrams in $\mathcal{N} = 4$ SYM
- In QCD, it has been used to calculate the soft anomalous dimension up to 3 loops [*Almelid et al.; 1706.10162*].

Crafting the Ansatzes

Introduction

We present a bootstrap approach to calculate the EEC in QCD. Inspired by the bootstrap approach for the two-loop form factor in $\mathcal{N} = 4$ SYM [*Guo et al.; 2106.01374*],the general ansatz of the master integral can be written in two components

1. Colour Structure Ansatz
2. Feynman Integral Ansatz

Colour Structure Ansatz

Inspired by the CoLorFulNNLO method [*Del Duca et al;, 1606.03453*], we present the colour structure ansatz as

$$B_{ans}(x,y) = \sum_{\{i\}} \left\{ \alpha_i J_i(x,y) \mathbf{T}_i^2 + \beta K(x,y) n_f T_f \right\} \mathbf{T}_i^2 + \sum_{\substack{i,j,l \\ j\neq l}} \left\{ \gamma_i L_i(x,y) \mathbf{T}_i^2 + \kappa M(x,y) n_f T_f \right\} \mathbf{T}_j \cdot \mathbf{T}_l \qquad (10)$$

The Reason

Colour Structure Ansatz

- The ansatz is grouped by the topology of the Feynman integrals: $\mathbf{T}_i^2$ represents the planar contribution and $\mathbf{T}_j \cdot \mathbf{T}_l$ is for the non-planar one, and $x = \sqrt{z}$, $y = \frac{i\sqrt{z}}{1-\sqrt{z}}$.
- It is simplified in a sense that we didn't craft the ansatz in full details, i.e. including the structure constant f^{abc} and directly using the trace of the group generators.
- We also use $\mathbf{T}_i^2 = C_i$, and $\mathbf{T}_j \cdot \mathbf{T}_l = \frac{C_A}{2} - C_F$. For practicality we only consider the case of $i, j, l \leq 2$, where index 1 represents the fundamental representation and 2 is for the adjoint one.
- One main reason to craft this kind of ansatz is not only to calculate the EEC bootstrap only at NLO, but hopefully for bootstraps at an even higher corrections in the future.

Constraining the Ansatz

Colour Structure Ansatz

- It is known that at the LO, the colour contribution for the scattering process comes only from the vertex correction of $e^+e^- \to q\bar{q}$, which means only the quadratic Casimir C_F shows up, in agreement to the leading order result of the EEC.
- However, at the NLO correction, we need to take account of three different processes:

$$e^+e^- \to \gamma* \to q\bar{q}gg,$$

$$e^+e^- \to \gamma* \to q\bar{q}q\bar{q},$$

$$e^+e^- \to \gamma* \to q\bar{q}q'\bar{q}'.$$

Where the last process involves the vacuum polarisation of gluons into pair of quarks q and q' [*R.K. Ellis et al.*].

Constraining the Ansatz
Colour Structure Ansatz

Two steps of constraining: Planar and Non-planar contribution

- Planar QED graph with C_F^2 term. For this, by running the indices in the first term of eq 10, we get something like

$$\alpha_1 J_1(x,y) C_F + \alpha_2 J_2(x,y) C_A.$$

  This is the process involving $e^+e^- \to \gamma* \to q\bar{q}gg$.
- We can set $\alpha_2 = 0$ because the C_A corresponds to the subleading order diagram. We can then identify $J_1(x,y)$ as the leading colour master integral for the EEC.
- In addition, the planar diagrams must take account of the light quark flavours, represented by $K(x,y)$, given by the process $e^+e^- \to \gamma* \to q\bar{q}q'\bar{q}'$. We can conveniently set $\beta = 1$ and $\alpha_1 = 1$.

Constraining the Ansatz

Colour Structure Ansatz

For the non-planar contribution

- Diagrams of this family are the $e^+e^- \to \gamma* \to q\bar{q}q\bar{q}$ process. Because there are no real emission of gluons in these diagrams, we can set $\kappa = 0$. Then, we can write $\mathbf{T}_j \cdot \mathbf{T}_l = \frac{C_A}{2} - C_F = \frac{1}{2}(C_A - 2C_F)$, and using the fact that there are two quark vertices in these diagrams as like the planar case, we set the term with C_A, $\gamma_2 = 0$. We have

$$\gamma_1 \frac{1}{2}(C_A - 2C_F) C_F L_1(x, y) \tag{11}$$

- Unlike the planar case, γ_1 needs to be set to 2.

Final result

Colour Structure Ansatz

The final result is heavily constrained, as also implied by Almelid et al

$$B_{ans}(x,y) = (J_1(x,y)C_F + K(x,y)n_f T_f)C_F + (C_A - 2C_F)C_F L_1(x,y) \quad (12)$$

In agreement with [1801.03219]

Feynman Integral Ansatz

The EEC deals with the process of virtual photon decay. Within it, integrals that appear in the higher order correction often manifest in four-point two-loop order functions.
At the two loop, the function rely on four independent momenta: q, p_1, p_2, p_3, and p_4., corresponding to the process

$$e^+ + e^- \to \gamma^*(q) \to k(p_1) + l(p_2) + m(p_3) + n(p_4), \quad p_i^2 = 0 \tag{13}$$

where *klmn* can either be *qqqq*, *qqq′q′*, or *qqgg*. We also define for the angle between two jets, we write $z = \frac{1-\cos\chi}{2} = \frac{p_1 \cdot p_3}{2 q\cdot p_1 q\cdot p_3}$ [*O. Gituliar, S. Moch; 1711.05549*]

The Reason

Feynman Integral Ansatz

In order to bootstrap the EEC, we craft the master integral ansatz based on the argument $x = \sqrt{z}$ and $y = \frac{i\sqrt{z}}{1-\sqrt{z}}$. namely

$$\mathcal{F}^{ans}(x, y; \epsilon) = \sum_i \mathcal{R}_i(x, y; \epsilon)\mathcal{H}_i(x, y; \epsilon) \tag{14}$$

Given that the Symbol alphabet of the EEC in QCD is given by

$$\{1 - x, y, 1 - y, 1 + y\}, \tag{15}$$

The Symbol for $\mathcal{H}$ is then given as the sum of the alphabet's tensor product

$$\mathcal{S}(\mathcal{H}) = \sum_I \otimes_{i=1}^{k} w_{I_i}, \tag{16}$$

where w_{I_i} is the symbol alphabet.

Constraining The Ansatz

Feynman Integral Ansatz

In order to constrain the ansatz, we need to do these steps:

1. Generate the necessary topologies for each processes [*Garland et al.; hep-ph/0112081*]. $qqqq$, $qqq'q'$, or $qqgg$ would yield different diagrams.
2. The possible arguments are given by the symbol alphabet.
3. Using FIRE6 [*A. V. Smirnov, F.S. Chukarev; 1901.07808*] and/or FUCHSIA [*O. Gituliar; 1701.04269*], we reduce each of the two-loop topologies into the sum of its master integrals.
4. We'll turn to the Symbol formalism of the master integrals, and apply our constraints there.

Constraining The Ansatz

Feynman Integral Ansatz

List of possible constraints:

1. Symmetry. We check for the terms that survive $\sqrt{z} \to -\sqrt{z}$
2. End-point Kinematics. For this, we perform the series expansion of Symbols, given by [*Dixon et al.; 1402.3300*]

$$[\mathcal{S}(R)]_1 = \int_0^T dT_0 k(T_0) \left[\frac{R_0'(T_0)}{R_0(T_0)} A_0(T_0) \right]_0 + \int_0^T \frac{dT_0}{T_0} \int_0^{T_0} dT_1 k(T_1) \left[\frac{R_1'(T_1)}{R_1(T_1)} A_1(T_1) \right]_0 + \cdots \quad (17)$$

For any symbol given as

$$\mathcal{S}(R) = k(T_0)[A_0(T_0) \otimes R_0(T_0)] + k(T_1)[A_1(T_1) \otimes R_1(T_1)] \otimes T_1 + \cdots \quad (18)$$

Constraining The Ansatz

Feynman Integral Ansatz

As an example, for Li_2, that is $(1-z) \otimes z$ at $z \to 0$, we need to first identify the symbol expansion at the next-to-leading order. Identify the symbol alphabet as $R_1(z) = (1-z)$, $k(T_1) = 1$, $T_1 = z$, and A_1 is empty, thus we can set it as 1, then using the second term integral of the series expansion, we can find the expansion of the symbol in collinear limit:

$$\begin{aligned}[\mathcal{S}(\text{Li}_2(z))]_{1(coll.)} &= \int_0^z \frac{dz_0}{z_0} \int_0^{z_0} dz_1 \left[\frac{R_1'(z_1)}{R_1(z_1)} A_1(z_1) \right]_0 \\ &= \int_0^z \frac{dz_0}{z_0} \int_0^{z_0} dz_1 (-1) \\ &= -z\end{aligned} \tag{19}$$

Constraining The Ansatz

Feynman Integral Ansatz

The expansion gives an extra symbol of $-z$ for $\mathrm{Li}_2(z)$, indicating that at $z \to 0$, $\mathrm{Li}_2(z)$ behaves as $-\log z$. To check the back-to-back limit at $z \to 1$, one would need to redefine T as $y = 1 - T$, in order to keep the series expansion at $T \to 0$ well defined. For example:

$$\begin{aligned}
[\mathcal{S}(\mathrm{Li}_2(z))]_{1(btb)} &= \int_0^y \frac{dy_0}{y_0} \int_0^{y_0} dy_1 \left[\frac{R_1'(y_1)}{R_1(y_1)} A_1(y_1) \right]_0 \\
&= \int_0^y \frac{dy_0}{y_0} \int_0^{y_0} dy_1 (-1) \\
&= -(1-z)
\end{aligned} \tag{20}$$

Discussion

What to expect?

1. The symmetry and end-point kinematics constraints the ansatz
2. The result would be (nearly) identical with the reverse unitarity method
3. The bootstrap method could be extended to the NNLO correction and beyond.

Future plans...

1. Extend the bootstrap strategy to elliptic integrals
2. Bootstrap of EEC at NNLO (contains Elliptic Integrals) in either (or perhaps both) QCD and $N = 4$ SYM