Modern Methods for Multi-Scale Scattering Amplitudes

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Precise Perturbative Predictions

- Perturbation theory is our major tool for making predictions.

$$\sigma \sim \sigma_{\mathsf{LO}} + \alpha\delta\sigma_{\mathsf{NLO}} + \alpha^2\delta\sigma_{\mathsf{NNLO}} + \mathcal{O}(\alpha^3).$$

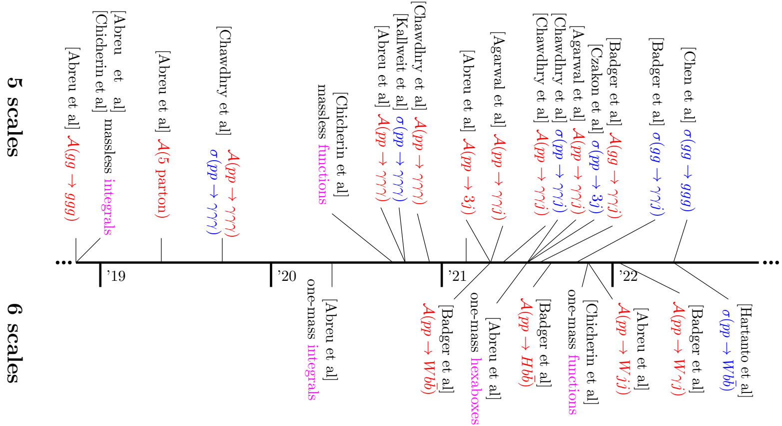
- At fixed order σ is built from phase space integrals over amplitudes:

$$\sigma \sim \int \mathrm{d}\Phi |\mathcal{A}|^2.$$

For-high precision, we need multi-loop scattering amplitudes.

Large Progress in Multi-Scale NNLO Calculations

In hadron-collider context, we have witnessed large advances:



Can we also apply these advances to lepton colliders?

Calculating Multi-Scale Loop Amplitudes

The Modern Perspective and Challenges

Aim: efficient, stable evaluation of amplitude over phase space.

(e.g. at worst 4 digits precision).

- Too many integrals $\Rightarrow$ write $A^{(2)}$ in terms of master integrals:

$$A^{(2)}(p_1, \ldots, p_5) = \sum_k \underbrace{\mathcal{C}_k(p_1, \ldots, p_5)}_{\text{rational functions}} \underbrace{\mathcal{I}_k(s_{12}, s_{23}, s_{34}, s_{45}, s_{51})}_{\text{master integrals}}.$$

- Coefficients $\mathcal{C}_k$ are process specific, integrals $\mathcal{I}_k$ are universal.

Challenges

- $\mathcal{I}_k$: analytic complexity $\Rightarrow$ efficient, stable evaluation difficult.
- $\mathcal{C}_k$: intermediate expression swell, poorly understood structure.

Master Integrals

Master Integrals – the Problems

Aim: As $\mathcal{I}_k$ must be combined with $\mathcal{C}_k$, need strong control of precision.

[Monte Carlo hard as low control over precision.]

Roadblocks to a "good representation":

- Integrals are special functions with complicated branch cuts.

  $\Rightarrow$ Analytic continuation to physical region is complex.

- With many scales, one finds a myriad of square root branch cuts.

  $\Rightarrow$ Limited use of known special functions.

- With massive fields, special functions are poorly understood.

The Workhorse – Differential Equations

Framework:

- Feynman integrals can be computed from their differential equations

$$\mathrm{d}\mathcal{I}_k = \mathbf{M}_{kl}(\epsilon, \vec{s})\mathcal{I}_l.$$

[Gehrmann, Remiddi '01]

- For (pure) polylogarithmic integrals, **M** in "canonical form" [Henn '13].

$$\mathbf{M}_{kl}(\epsilon, \vec{s}) = \underbrace{\epsilon}_{\text{regulator}} \sum_{\alpha,l} \underbrace{M^{\alpha}_{kl}}_{\text{rational numbers}} \mathrm{d}\log\Big(\underbrace{W_\alpha[\vec{s}]}_{\text{algebraic functions}} \Big).$$

The steps:

1. Find pure $\mathcal{I}_k$. (Many approaches, though unsolved problem).
2. Construct alphabet from cut differential equations.
   [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]
3. Compute M^{α}_{kl} via Ansatz, boundary condition from physics.
   [Abreu, BP, Zeng '18] [Abreu, Ita, Moriello, BP, Tschernow, Zeng '20] [Liu, Ma '21]

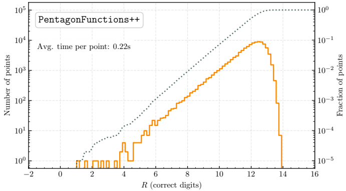
Open question: How to canonicalize beyond polylogarithms?

Numerically Solving the Differential Equation

Aim: numerically evaluate $\mathcal{I}_j$ in ϵ expansion, $\mathcal{I}_j(\vec{p},\epsilon) = \sum_{k=-4}^{0} d_{j,k}\epsilon^k h_k + \mathcal{O}(\epsilon)$.

Pentagon functions: [Gehrmann, Henn, lo Presti '18]

- Dedicated iterated integral code:
$$h_k \sim \int_0^1 \mathrm{d}\log(W_n[t_n]) \cdots \int_0^{t_2} \mathrm{d}\log(W_1[t_1]).$$
- Relations between h_k known.
- Very efficient, only dynamical scales.

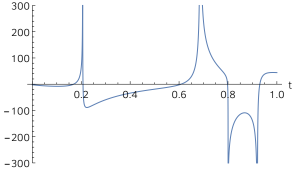


[Chicherin, Sotnikov '20]

Series expansions:

- Patch together h_k from series solutions:
$$h_k \sim \sum_{j_1,j_2} (t-t_0)^{j_1/2} \log(t-t_0)^{j_2}.$$
- Implementations: DiffExp/AMFlow.

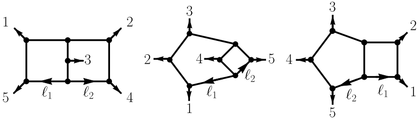
  [Hidding '20] [Liu, Ma '22]
- Proof-of-concept, promising for masses.



[Abreu, Ita, Moriello, BP, Tschernow, Zeng '20]

Master Integrals – State of the Art (Two-Loop Five-Point)

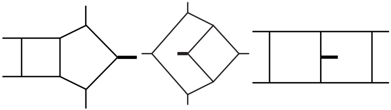
- Five-point Massless



[Papadopoulos, Tommasini, Wever '15]
[Gehrmann, Henn, lo Presti '18]
[Abreu, Page, Zeng '18]
[Chicherin, Gehrmann, Henn, lo Presti, Mitev, Wasser '18]

[Abreu, Dixon, Herrmann, Page, Zeng '18]
[Chicherin, Gehrmann, Henn, Wasser, Zhang, Zoia '18]
[Gehrmann, Henn, lo Presti '18]
[Chicherin, Sotnikov '20]

- Five-point One-Mass



[Abreu, Ita, Moriello, Page Tschernow, Zeng '20]
[Abreu, Ita, Page, Tschernow '21]
[Papadopoulos, Tommasini, Wever 15]

[Canko Papadopoulos, Syrrakos 20]
[Kardos, Papadopoulos, Smirnov, Syrrakos, Wever 19]
[Chicherin, Sotnikov, Zoia '21]

[Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia. to appear]

Great progress for many dynamical scales, no fixed scales.

Integral Coefficients

The Analytic Reconstruction Approach

- Given known h_k, must now determine $\mathcal{C}_{j,k}$.

$$\mathcal{A} = \sum_{j=-4}^{0} \epsilon^j \sum_{k \in B} \mathcal{C}_{j,k} h_k + O(\epsilon).$$

- Strategy: reconstruct analytic $\mathcal{C}_{j,k}$ from numerical samples (mod p).

[Schabinger, von Manteuffel '14; Peraro '16]

$$\mathcal{C}_{j,k}(p_1, \ldots, p_n) = \sum_{l=1}^{N} c_{j,k,l} \mathfrak{a}_{j,k,l}(p_1, \ldots, p_n), \qquad c_{j,k,l} \in \mathbb{Q}.$$

Two steps:

1. Construct linear Ansatz.
2. Gather numerical data and fix unknowns from numerical data.

Runtime cost $\sim N \times T_{\mathrm{eval}}$.

Ansätze for Rational Functions

- Coefficients $\mathcal{C}_{j,k}(p_1, \ldots, p_n)$ are rational functions of kinematics.

$$C_{j,k}(p_1, \ldots, p_n) = \frac{\mathcal{N}_{j,k}(p_1, \ldots, p_n)}{D_{j,k}(p_1, \ldots, p_n)}$$

- Denominators simple when we know the integrals: product of letters.

$$D_{j,k} = \prod_l W_l^{q_{jkl}}$$

[Abreu, Dormans, Ita, Febres Cordero, Page '18]

- Simpler problem – reconstruct polynomial $\mathcal{N}_{j,k}$. Multiple techniques:
  - Newton reconstruction. [Peraro '16]
  - Vandermonde reconstruction. [Abreu et al '21]
  - Univariate partial fractions. [Badger, Hartanto, Zoia '21]
  - p-adic numbers. [de Laurentis, BP '22]

Major challenge: Constructing compact Ansätze.

Gathering Numerical Data

Two major paths:

"The Feynman way"

- Feynman diagrams via QGRAF. Integrands via FORM.
- Projectors + form factors. [Chen '19; Peraro, Tancredi '19 '20]
- Finite field IBP reduction. FiniteFlow [Peraro '19]

"Numerical Unitarity"

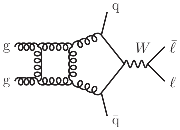
- Parameterize integrand with master/surface integrands. [Ita '15]
- Generate integrand from unitarity cuts.

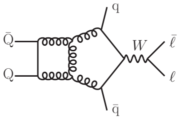
Use to compute coefficients numerically on a phase-space point $\vec{p}_0$:

$$\mathcal{A}(\vec{p}_0) = \sum_i \mathcal{C}_j(\vec{p}_0, \epsilon)\mathcal{I}_j(\vec{p}, \epsilon) = \sum_{j=-4}^{0} \epsilon^j \sum_{k \in B} \underbrace{\mathcal{C}_{j,k}(\vec{p}_0)}_{\text{numerical data}} h_k(\vec{p}) + \mathcal{O}(\epsilon).$$

Principal bottleneck: IBP reduction.

$pp \to Vjj$ Case Study

$\mathcal{A}_g =$ 

$+ \cdots,$ $\quad \mathcal{A}_Q =$ 

$+ \cdots.$

- Leading-colour helicity-amplitude finite remainders.
- Planar pentagon functions from [Chicherin, Sotnikov, Zoia '21].
- Ansatz: Univariate partial fractions, Vandermonde sampling.
- $\mathcal{O}(10^6)$ samples for most complex $\mathcal{A}_n$. Sample time: $\mathcal{O}(3)$ minutes.

Analytic results at http://www.hep.fsu.edu/~ffebres/W4partons

Towards e^+e^- Colliders

From $pp \to Vjj$ to $e^+e^- \to 4j$ (QCD corrections)

$pp \to Vjj$ amplitudes computed in terms of QCD current $\mathcal{A}^\mu$:

$$\mathcal{A} = \mathcal{A}^\mu J_\mu, \qquad J_\mu \sim \frac{\overline{u}(p_6)\gamma_\mu v(p_5)}{s_{56} - M_v^2 + i\Gamma_v M_v}.$$

Steps to achieve crossing:

- Attach appropriate leptonic current.
- Analytically continue from $2 \to 3$ to $1 \to 4$ kinematics.

Nonplanar contributions:

- Terms subleading in N_c not included, likely suppressed.
- Contribution proportional to $\sum_f Q_f$ potentially required.
  (Non-planar integrals: [Abreu, Chicherin, Ita, Tschernow, BP, Sotnikov, Zoia. to appear])

Outlook for Curated Processes

No progress required:

- $e^+e^- \to 4j$ at $\mathcal{O}(\alpha_s^2)$. [just crossing!]

Incremental progress required:

- $e^+e^- \to 3j$ at $\mathcal{O}(\alpha_s^3)$. [+ 1 dynamical scales]
- $e^+e^- \to QQj$ at $\mathcal{O}(\alpha_s^2)$ [elliptic, low # dynamical scales].
- $e^+e^- \to 5j$ at $\mathcal{O}(\alpha_s^2)$ [+ 1 leg, + 3 dynamical scales].

Breakthrough required? [Elliptic integrals, many dynamical scales.]

- $e^+e^- \to \nu\nu H$ at $\mathcal{O}(\alpha^2)$.
- $e^+e^- \to l^+l^-H$ at $\mathcal{O}(\alpha^2)$.

Summary/Discussion

Status

- Loop amplitudes remain a major bottleneck to precise predictions.
- Large progress in multi-scale amplitudes will benefit e^+e^- colliders.

Lessons learned

- Large final-state phase space $\Rightarrow$ need efficient/stable amplitudes.
- Calculations based on structural understanding of Feynman integrals.

The Next Frontier?

- Large improvements desired for cases with massive particles (elliptics).
- Time to go fully numerical?