The 2022 international workshop on the high energy Circular Electron-Positron Collider

Principle of maximum conformality and its application to the determination of QCD coupling

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Based on arXiv: 2209.03546; 2112.06212; 1701.08245; in collaboration with

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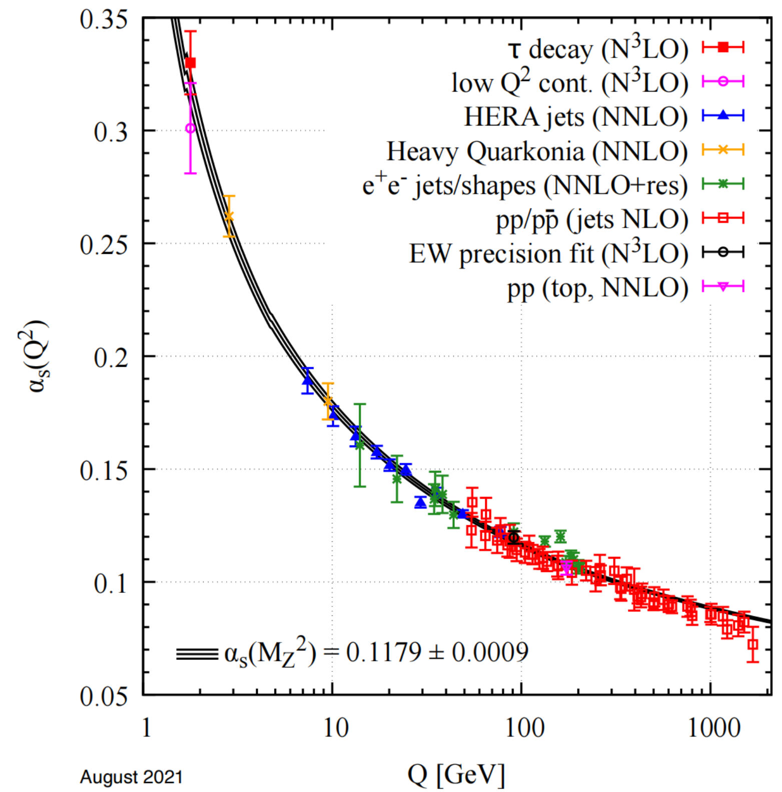
Outline

- Introduction
- Principle of maximum conformality (PMC)
- Bayesian Analysis
- Determination of α_s at e^+e^- colliders
- Conclusion

Introduction

$$\frac{d\alpha_s(\mu_r)}{d\ln\mu_r^2} = -\alpha_s^2(\mu_r)\sum_{i=0}^{\infty}\beta_i\alpha_s^i(\mu_r)$$

Asymptotic freedom:
the QCD coupling between quarks and gluons becomes weak at short distances, allowing perturbative calculations of physical observables involving large momentum transfer.



PDG 2022

Introduction

Renormalization in pQCD calculations

Regularization

Redefining integrals in a way to control the divergences,
i.e., $\int \mathrm{d}^4 p \to \mu^{2\epsilon} \int \mathrm{d}^{4-2\epsilon} p$, divergences parameterized as, $1/\epsilon$

Renormalization

Redefining parameters to remove the well-defined divergences,
i.e., replacing the bare gauge coupling as, $\alpha_0 = \mu^{2\epsilon} Z_{\alpha_S} \alpha_S, \cdots$

In addition to the evaluation of high-order loops, the precision and predictive power of pQCD predictions depends on two important issues:

- ✓ how to achieve a reliable, convergent fixed-order series
- ✓ how to reliably estimate the contributions of unknown higher-order terms

Introduction

Our Calculation technology, arXiv: 2209.03546, 1701.08245

- Using Principle of Maximal Conformality (PMC) to calculating the fixed-order pQCD series ; (central value)
- Using Bayesian analysis to estimating the uncalculated higher-order contribution. (theoretical error)

Regularization

Redefining integrals in a way to control the divergences,
i.e., $\int d^4p \to \mu^{2\epsilon} \int d^{4-2\epsilon}p$, divergences parameterized as, $1/\epsilon$

Renormalization

Redefining parameters to remove the well-defined divergences,
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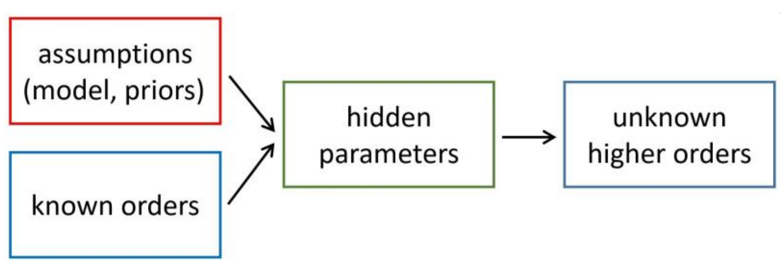
Renormalization scale setting

Redefining α_S by absorbing the nonconformal parts of the pQCD series,
i.e., PMC-s suggests, $\alpha_S(\mu) \to \alpha_S(Q_\star)$, $\quad r_i(\mu) \to r_{i,0}$.

PMC

central value

Bayesian analysis



theoretical error

Principle of Maximal Conformality (PMC)

MS-like scheme (R_δ scheme)

PRL 110,192001(2013)
PRD 89,014027(2014)

$$\alpha_{s,B} \longmapsto \mu^{2\epsilon}\left(\frac{e^{\gamma_E+\delta}}{4\pi}\right)^{\epsilon} Z_{\alpha_s}\alpha_s$$

$$\mathcal{R}_0 = \overline{\mathrm{MS}},$$

$$\mathcal{R}_{\ln 4\pi-\gamma_E} = \mathrm{MS},$$

$$\mathcal{R}_{-2} = \mathrm{G},$$

$$\begin{aligned}\rho_\delta(Q) =& r_1\alpha_s(\mu)^p + [r_2 + p\beta_0 r_1\delta]\,\alpha_s(\mu)^{p+1} + \left[r_3 + p\beta_1 r_1\delta + (p+1)\beta_0 r_2\delta + \frac{p(p+1)}{2}\beta_0^2 r_1\delta^2\right]\alpha_s(\mu)^{p+2}\\ &+\left[r_4 + p\beta_2 r_1\delta + (p+1)\beta_1 r_2\delta + (p+2)\beta_0 r_3\delta + \frac{p(3+2p)}{2}\beta_1\beta_0 r_1\delta^2 + \frac{(p+1)(p+2)}{2}\beta_0^2 r_2\delta^2\right.\\ &\left.+\frac{p(p+1)(p+2)}{3!}\beta_0^3 r_1\delta^3\right]\alpha_s(\mu)^{p+3} + \cdots,\end{aligned}$$

$$\frac{\partial\rho_\delta}{\partial\delta} = -\beta(\alpha_s)\frac{\partial\rho_\delta}{\partial\alpha_s} \quad\Rightarrow\quad If\ \ \beta = 0, \qquad then\ \ \frac{\partial\rho_\delta}{\partial\delta} = 0$$

$$\begin{aligned}\rho(Q) =& r_{1,0}\alpha_s(\mu)^p + [r_{2,0} + p\beta_0 r_{2,1}]\,\alpha_s(\mu)^{p+1} + \left[r_{3,0} + p\beta_1 r_{2,1} + (p+1)\beta_0 r_{3,1} + \frac{p(p+1)}{2}\beta_0^2 r_{3,2}\right]\alpha_s(\mu)^{p+2}\\ &+\left[r_{4,0} + p\beta_2 r_{2,1} + (p+1)\beta_1 r_{3,1} + \frac{p(3+2p)}{2}\beta_1\beta_0 r_{3,2} + (p+2)\beta_0 r_{4,1} + \frac{(p+1)(p+2)}{2}\beta_0^2 r_{4,2}\right.\\ &\left.+\frac{p(p+1)(p+2)}{3!}\beta_0^3 r_{4,3}\right]\alpha_s(\mu)^{p+3} + \cdots,\end{aligned}$$

$$r_{i,j} = \sum_{k=0}^{j} C_j^k \hat{r}_{i-k,j-k}\ln^k\frac{\mu^2}{Q^2},\quad \hat{r}_{i,j} = r_{i,j}\big|_{\mu=Q}$$

PMC single-scale-setting

$$\rho(Q) = r_{1,0}\alpha_s(\mu)^p + [r_{2,0} + p\beta_0 r_{2,1}]\,\alpha_s(\mu)^{p+1} + \left[r_{3,0} + p\beta_1 r_{2,1} + (p+1)\beta_0 r_{3,1} + \frac{p(p+1)}{2}\beta_0^2 r_{3,2}\right]\alpha_s(\mu)^{p+2}$$
$$+\left[r_{4,0} + p\beta_2 r_{2,1} + (p+1)\beta_1 r_{3,1} + \frac{p(3+2p)}{2}\beta_1\beta_0 r_{3,2} + (p+2)\beta_0 r_{4,1} + \frac{(p+1)(p+2)}{2}\beta_0^2 r_{4,2}\right.$$
$$\left.+\frac{p(p+1)(p+2)}{3!}\beta_0^3 r_{4,3}\right]\alpha_s(\mu)^{p+3} + \cdots,$$

All non-conformal β-terms are absorbed into the redefinition of the renormalization scale (or α_s)

PRD 95, 094006 (2017)

$$\rho(Q)|_{\rm PMCs} = r_{1,0}\alpha_s^p(Q_*) + r_{2,0}\alpha_s^{p+1}(Q_*) + r_{3,0}\alpha_s^{p+2}(Q_*) + r_{4,0}\alpha_s^{p+3}(Q_*) + \cdots.$$

$$\ln\frac{Q_*^2}{Q^2} = \sum_{i=0}^{n-2} S_i \alpha_s^2(Q_*)$$

$$S_0 = -\frac{\hat{r}_{2,1}}{\hat{r}_{1,0}},$$
$$S_1 = \frac{(p+1)(\hat{r}_{2,0}\hat{r}_{2,1} - \hat{r}_{1,0}\hat{r}_{3,1})}{p\hat{r}_{1,0}^2} + \frac{(p+1)(\hat{r}_{2,1}^2 - \hat{r}_{1,0}\hat{r}_{3,2})}{2\hat{r}_{1,0}^2}\beta_0,$$
$$S_2 = \frac{(p+1)^2\left(\hat{r}_{1,0}\hat{r}_{2,0}\hat{r}_{3,1} - \hat{r}_{2,0}^2\hat{r}_{2,1}\right) + p(p+2)\left(\hat{r}_{1,0}\hat{r}_{2,1}\hat{r}_{3,0} - \hat{r}_{1,0}^2\hat{r}_{4,1}\right)}{p^2\hat{r}_{1,0}^3}$$
$$+\frac{(p+1)^2(\hat{r}_{1,0}\hat{r}_{2,0}\hat{r}_{3,2} - \hat{r}_{2,0}\hat{r}_{2,1}^2) + (p+1)(p+2)\left(2\hat{r}_{1,0}\hat{r}_{2,1}\hat{r}_{3,1} - \hat{r}_{2,0}\hat{r}_{2,1}^2 - \hat{r}_{1,0}^2\hat{r}_{4,2}\right)}{2p\hat{r}_{1,0}^3}\beta_0$$
$$+\frac{(p+1)(p+2)\left(3\hat{r}_{1,0}\hat{r}_{2,1}\hat{r}_{3,2} - 2\hat{r}_{2,1}^3 - \hat{r}_{1,0}^2\hat{r}_{4,3}\right)}{6\hat{r}_{1,0}^3}\beta_0^2 + \frac{(p+2)\left(\hat{r}_{2,1}^2 - \hat{r}_{1,0}\hat{r}_{3,2}\right)}{2\hat{r}_{1,0}^2}\beta_1.$$

Bayesian analysis

$$\rho_k = \sum_{i=l}^{k} c_i \alpha_s^i \Longrightarrow \delta_{k+1} = c_{k+1}\alpha_s^{k+1}$$

To obtain a probability density function (p.d.f.) for unknown c_{k+1}

Basic assumptions (JHEP 09, 039 (2011)) :

all coefficients are finite and bounded by a common number,

$$|c_i| < \bar{c}, \qquad (\bar{c} > 0) \qquad \forall\, i$$

- The order of magnitude of $\bar{c}$ is equally probable for all values.

$$g_0(\bar{c}) = \frac{1}{2|\ln\epsilon|}\frac{1}{\bar{c}}\,\theta\left(\frac{1}{\epsilon} - \bar{c}\right)\theta(\bar{c} - \epsilon),$$

  taking the limit $\epsilon \to 0$ for the final result.

- The conditional p.d.f. $h_0(c_i|\bar{c})$ is assumed as a uniform distribution

$$h_0(c_i|\bar{c}) = \frac{1}{2\bar{c}}\theta(\bar{c} - |c_i|), \quad \forall\, i,$$

- All the coeffcients $c_i (i = l, l+1, \cdots)$ are mutually independent

$$h(c_j, c_k|\bar{c}) = h_0(c_j|\bar{c})h_0(c_k|\bar{c}), \ \ \forall\, j, k,\ \ j \neq k.$$

Bayesian analysis: posterior distribution

The conditional p.d.f. $f_c(c_n|c_l, c_{l+1}, \cdots c_k), (n > k)$

$$f_c(c_n|c_l,\cdots,c_k) = \int h_0(c_n|\bar{c}) f_{\bar{c}}(\bar{c}|c_l,\cdots,c_k)\mathrm{d}\bar{c},$$

arXiv: 2209.03546

$$f_{\bar{c}}(\bar{c}|c_l,\cdots,c_k) = \frac{h(c_l,\cdots,c_k|\bar{c})g_0(\bar{c})}{\int h(c_l,\cdots,c_k|\bar{c})g_0(\bar{c})\mathrm{d}\bar{c}},$$

$$\begin{aligned} f_c(c_n|c_l,\ldots,c_k) &= \lim_{\epsilon\to 0} \frac{\int h_0(c_n|\bar{c})\prod_{i=l}^{k} h_0(c_i|\bar{c})g_0(\bar{c})\mathrm{d}\bar{c}}{\int \prod_{i=l}^{k} h_0(c_i|\bar{c})g_0(\bar{c})\mathrm{d}\bar{c}} \\ &= \frac{1}{2}\frac{n_c}{n_c+1}\frac{\bar{c}_{(k)}^{n_c}}{(\max\{|c_n|,\bar{c}_{(k)}\})^{n_c+1}} \\ &= \begin{cases} \frac{n_c}{2(n_c+1)\bar{c}_{(k)}}, & |c_n| \le \bar{c}_{(k)} \\ \frac{n_c \bar{c}_{(k)}^{n_c}}{2(n_c+1)|c_n|^{n_c+1}}, & |c_n| > \bar{c}_{(k)} \end{cases}. \end{aligned}$$

$$f_\delta(\delta_{k+1}|c_l,\ldots,c_k) = \left(\frac{n_c}{n_c+1}\right)\frac{1}{2\alpha_s^{k+1}\bar{c}_{(k)}} \begin{cases} 1, & |\delta_{k+1}| \le \alpha_s^{k+1}\bar{c}_{(k)} \\ \left(\frac{\alpha_s^{k+1}\bar{c}_{(k)}}{|\delta_{k+1}|}\right)^{n_c+1}, & |\delta_{k+1}| > \alpha_s^{k+1}\bar{c}_{(k)} \end{cases},$$

Determination of α_s at e^+e^- colliders

Example : $R(s) = \frac{\sigma(e^+e^- \to \text{hadrons}, s)}{\sigma(e^+e^- \to \mu^+\mu^-, s)} = R_{\text{EW}}(s)\,(1 + \delta_{\text{QCD}}(s))$

PRL 101, 012002 (2008); PRL 104, 132004 (2010);
PLB 714, 62 (2012); JHEP07,017(2012)

The pQCD predictions of $R(s)$

$$D(Q^2) = -12\pi^2 Q^2 \frac{d}{dQ^2}\Pi(Q^2) = \int_0^\infty \frac{Q^2 R(s) ds}{(s+Q^2)^2},$$

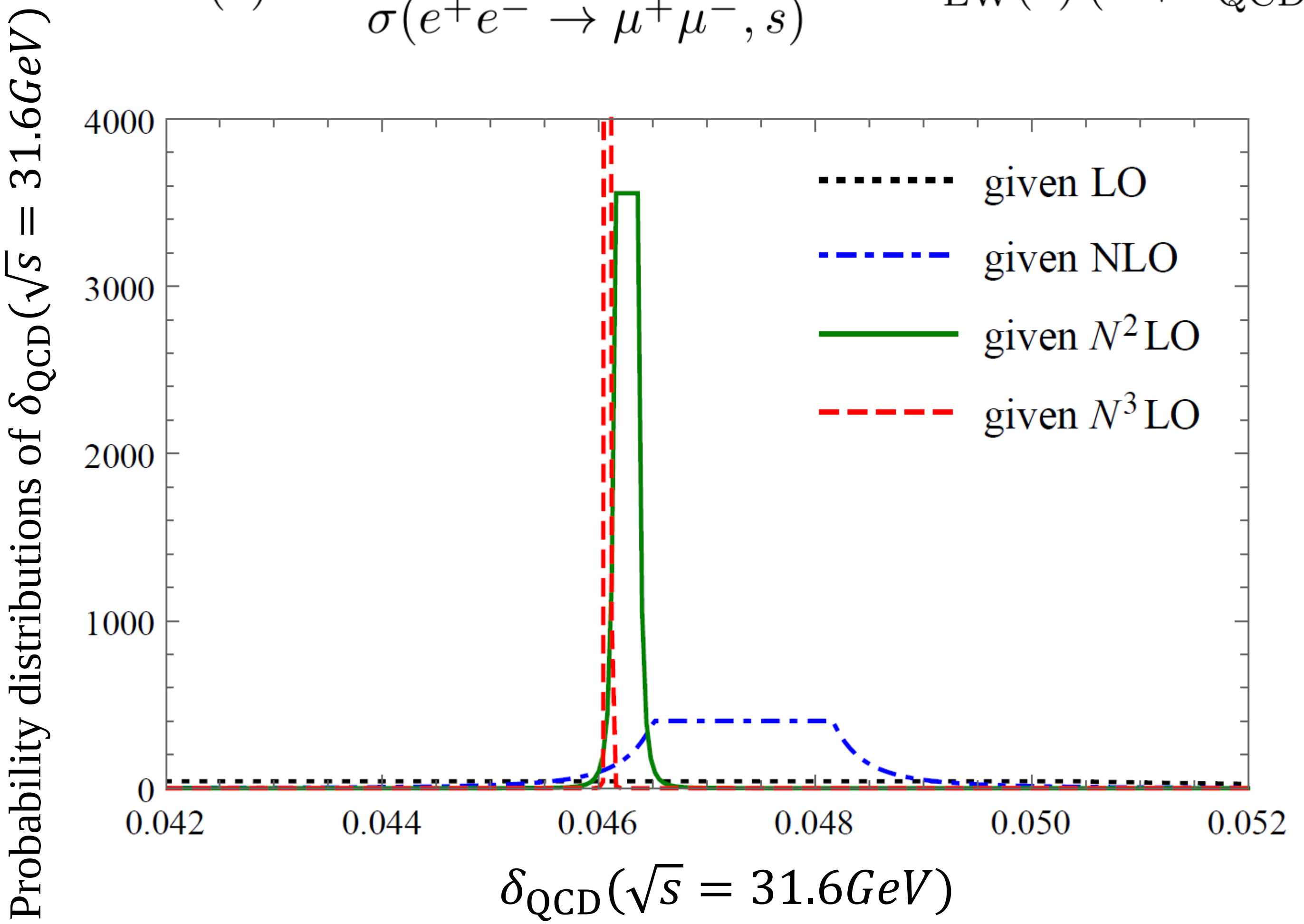
$$R(s) = D(s) - \pi^2\beta_0^2\left\{\frac{d_1}{3}a_s^3 + \left(d_2 + \frac{5}{6\beta_0}d_1\beta_1\right)a_s^4\right\} + \ldots.$$

We define the perturbative expansions

$$D(Q^2) = \sum_{i=0}^{\infty} d_i a_s^i(Q^2), \qquad R(s) = \sum_{i=0}^{\infty} r_i a_s^i(s),$$

Determination of α_s at e^+e^- colliders

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons}, s)}{\sigma(e^+e^- \to \mu^+\mu^-, s)} = R_{\text{EW}}(s)\,(1 + \delta_{\text{QCD}}(s))$$

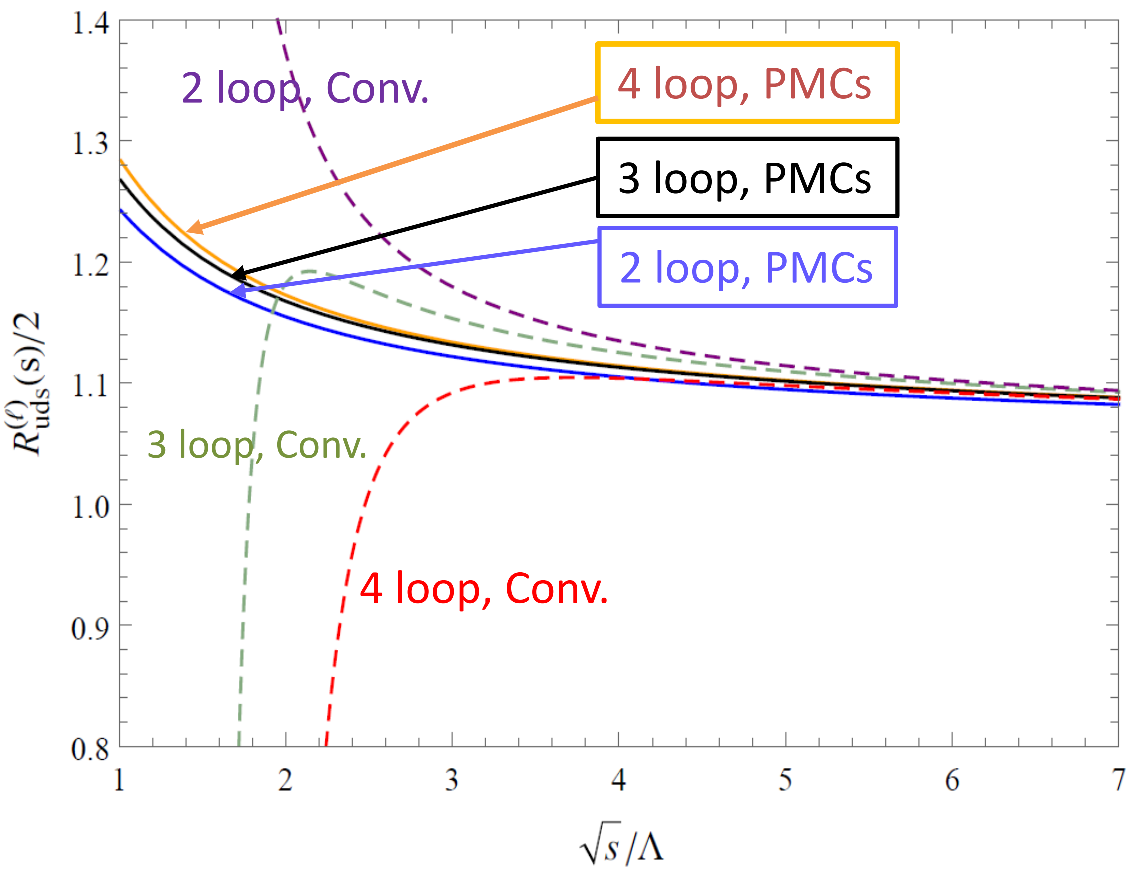


Determination of α_s at e^+e^- colliders

The pQCD predictions of $R_{uds}(s)$

$$\ln\frac{Q_*^2}{Q^2} = 0.2249 + 1.5427\alpha_s(Q_*^2) + 2.4933\alpha_s^2(Q_*^2)$$

$$\frac{1}{2}R_{\rm uds}(s)|_{\rm PMCs} = 1 + \frac{\alpha_s(Q_*^2)}{\pi} + 0.2174\alpha_s^2(Q_*^2) + 0.1108\alpha_s^3(Q_*^2) + 0.0698\alpha_s^4(Q_*^2),$$



the effects due to continuation of the spacelike perturbative results into the timelike domain are only partially accounted for in conventional scale setting (Conv.)

After using PMCs approach :

Scale-fixed prediction with

improved convergence

Determination of α_s at e^+e^- colliders

Determination of α_s ① solving equation ② least squares (LS)

solving the equation $R_{uds}^{(\text{data})} = R_{uds}^{(\text{theo.})}(\Lambda)$

TABLE I. The values of the QCD coupling and the QCD scale parameter $\Lambda^{(n_f=3)}$ at various loop levels ($\ell = 2, 3, 4$) extracted from a single measurement $R_{\text{uds}}(\sqrt{s_0} = 2.444) = 2.175$ [34] using the PMCs and conventional (conv.) scale setting, respectively.

$\ell = 1$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$
$\Lambda_{(\ell)}^{(n_f=3)}\|_{\text{PMCs}}$ (MeV)	193	406	345	342
$\alpha_s^{(\ell)}(\sqrt{s_0})\|_{\text{PMCs}}$	0.2749	0.2794	0.2717	0.2718
$\Lambda_{(\ell)}^{(n_f=3)}\|_{\text{conv.}}$ (MeV)	193	303	308	357
$\alpha_s^{(\ell)}(\sqrt{s_0})\|_{\text{conv.}}$	0.2749	0.2438	0.2580	0.2774

Determination of α_s at e^+e^- colliders

$$\chi^2(\Lambda) = (\mathbf{e}-\mathbf{t})^T V^{-1} (\mathbf{e}-\mathbf{t})$$

Error: $\chi^2(\Lambda) = \chi^2_{\rm min} + 1$

LS fitting

$$\mathbf{e} = (R^{\rm exp.}_{\rm uds}(Q_1), R^{\rm exp.}_{\rm uds}(Q_2), \cdots, R^{\rm exp.}_{\rm uds}(Q_N))$$

$$\mathbf{t} = (R^{\rm the.}_{\rm uds}(Q_1), R^{\rm the.}_{\rm uds}(Q_2), \cdots, R^{\rm the.}_{\rm uds}(Q_N))$$

TABLE III. The fitted Λ (in unit of MeV) from $R_{\rm uds}$ data below the $D\bar{D}$ threshold measured by KEDR collaboration [34].

$R^{\rm the.}_{\rm uds}$	$\chi^2_{\rm min}/n_{\rm d.o.f.}$	$\Lambda^{(n_f=3)}$	$\alpha_s(M_Z^2)$
$R^{(2)}_{\rm uds}\vert_{\rm PMCs}$	$10.5935/21$	$478^{+244+28}_{-218-24}$	$0.1252^{+0.0118+0.0015}_{-0.0137-0.0013}$
$R^{(3)}_{\rm uds}\vert_{\rm PMCs}$	$10.5079/21$	416^{+217+6}_{-192-6}	$0.1235^{+0.0121+0.0004}_{-0.0136-0.0010}$
$R^{(4)}_{\rm uds}\vert_{\rm PMCs}$	$10.5706/21$	406^{+207+2}_{-186-2}	$0.1227^{+0.0117+0.0002}_{-0.0132-0.0002}$

the 1st and 2nd errors are the experimental and theoretical uncertainties

$$\alpha_s(M_Z^2)\Big|_{\rm KEDR} = 0.1227^{+0.0117}_{-0.0132}(expe.) \pm 0.0002(theo.)$$

Event shape observables at CEPC

We also calculated the classical event shapes at the CEPC at 91.2, 160 and 240 GeV.

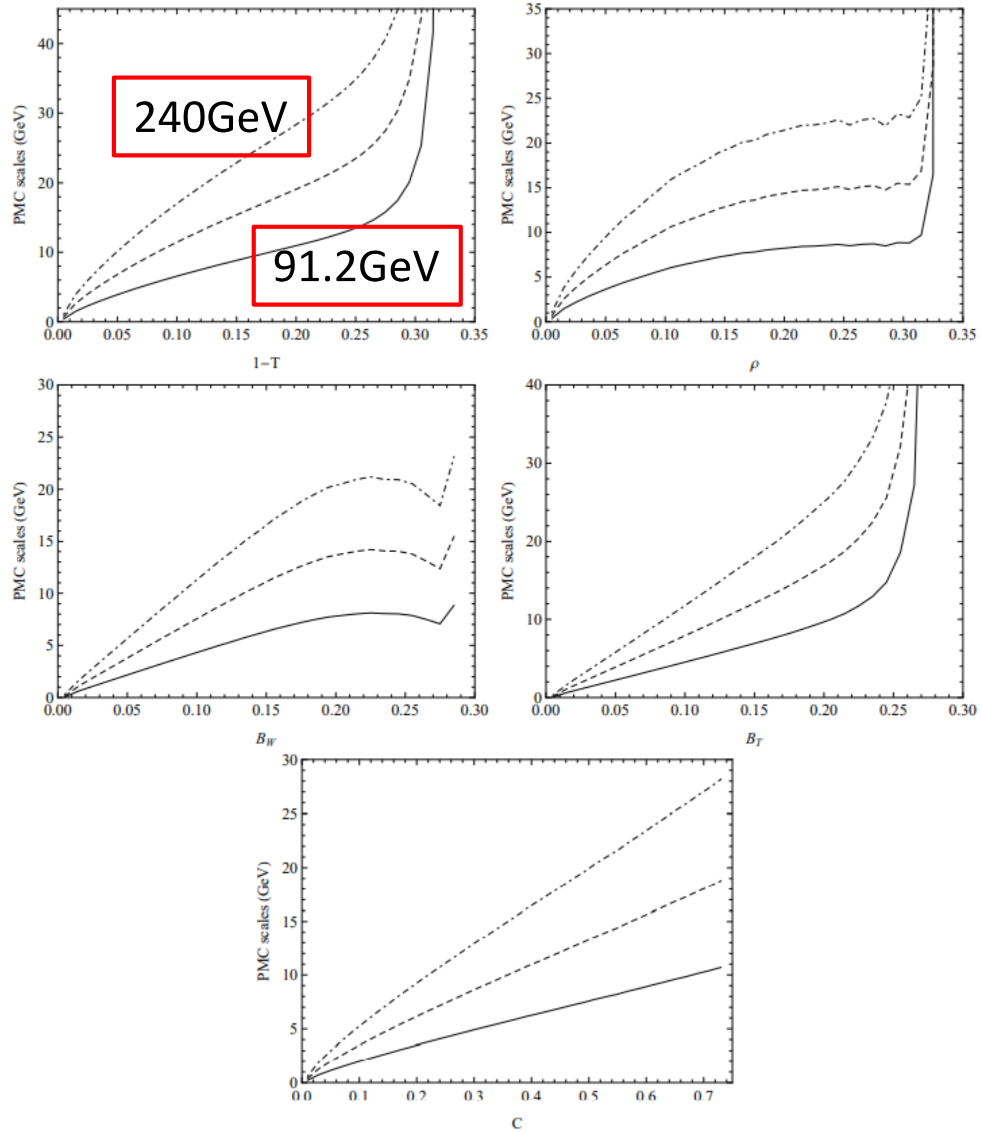
arXiv: 2112.06212

$$\frac{1}{\sigma_h}\frac{d\sigma}{d\tau} = \bar{A}(\tau)\, a_s(Q) + \bar{B}(\tau)\, a_s^2(Q) + \mathcal{O}(a_s^3).$$

$$\frac{1}{\sigma_h}\frac{d\sigma}{d\tau} = \bar{A}(\tau) a_s(\mu_r^{\rm pmc}) + \bar{B}(\tau,\mu_r)_{\rm con} a_s^2(\mu_r^{\rm pmc}) + \mathcal{O}(a_s^3)$$

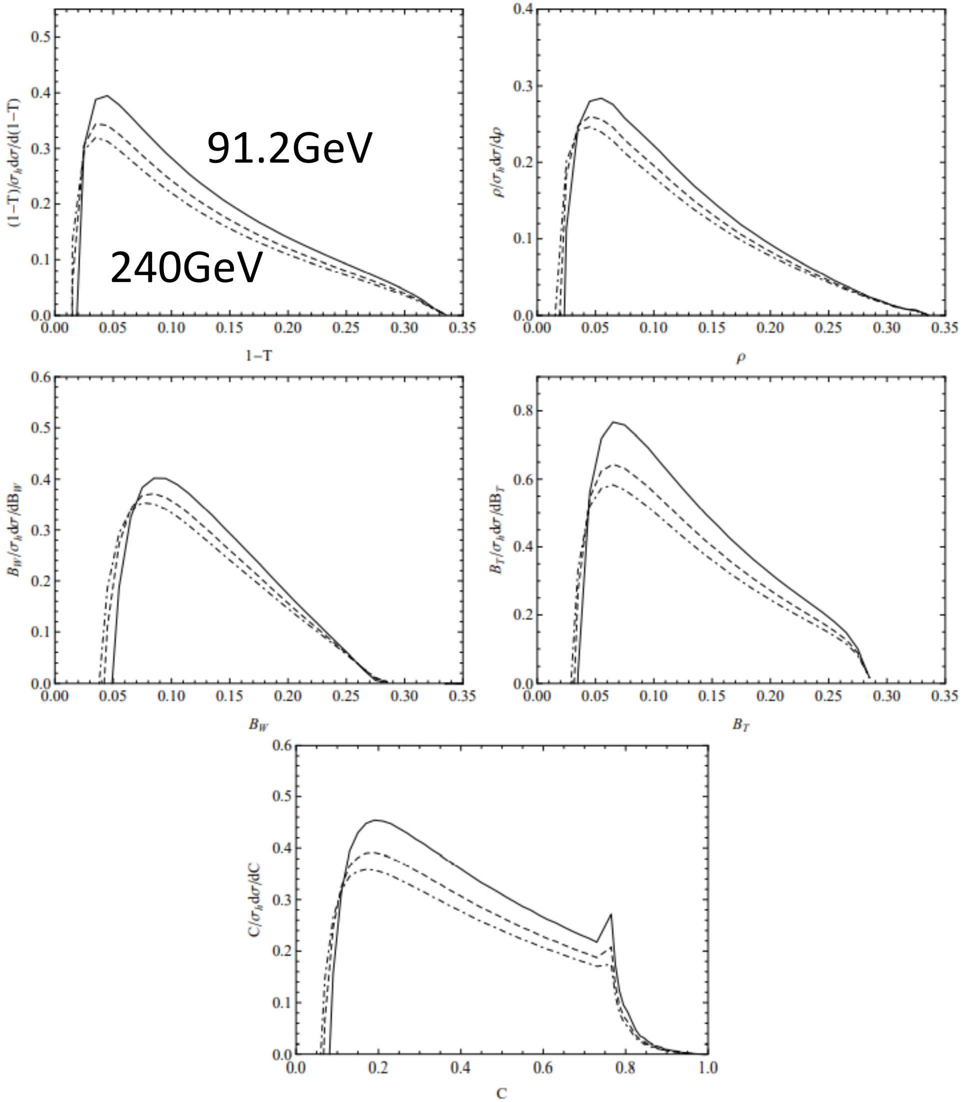
$$\bar{B}(\tau,\mu_r)_{\rm con} = \frac{11 C_A}{4 T_R}\bar{B}(\tau,\mu_r)_{n_f} + \bar{B}(\tau,\mu_r)_{\rm in},$$

$$\mu_r^{\rm pmc} = \mu_r \exp\left[\frac{3\bar{B}(\tau,\mu_r)_{n_f}}{4 T_R \bar{A}(\tau)} + \mathcal{O}(a_s)\right].$$



PMC scales for event shape observables at CEPC

Event shape observables at CEPC



Our precise and scale-independent predictions for event shape observables, provides a novel way to verify the running of αs(Q^2) at CEPC.

More detail See Wang's Talk on 10.27

Conclusion

- The resulting PMC series is a renormalon-free and scale-invariant conformal series; it thus achieves precise fixed-order pQCD predictions and provides a reliable basis for predicting unknown higher-order (UHO) contributions.
- The Bayesian analysis provides a compelling approach for estimating the UHOs from the known fixed-order series by adopting a probabilistic interpretation.
- Using the PMC, in combination with the Bayesian analysis, one can consistently achieve high degree of reliability predictions for fixed-order pQCD calculation.
- The combination of PMC and Bayesian analysis provides a reliable theoretical basis for the precise determination of the QCD running coupling.
- Future precise R_{uds} measurements at Tau-Charm Facility will provide a reliable and independent determination of α_s.
- Our precise and scale-independent predictions for event shape observables call for the precise measurements at CEPC.

Thank you for your attention !