Muon $g - 2$ from lattice QCD

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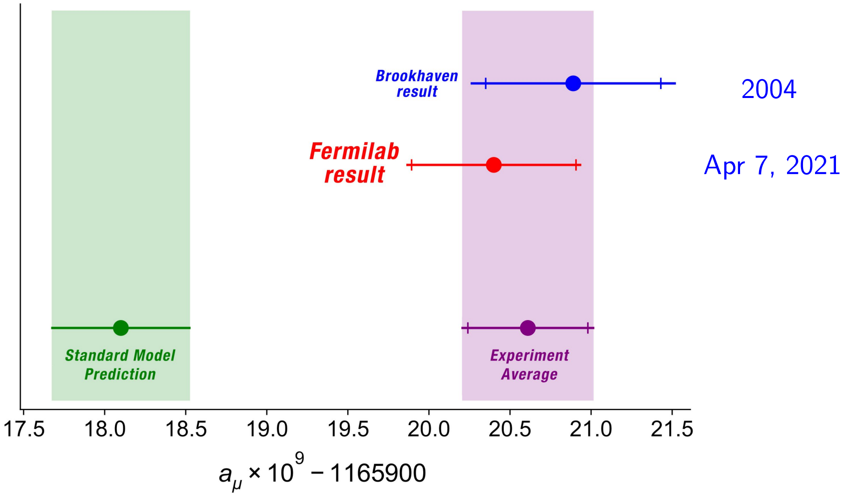
Aug 17, 2022

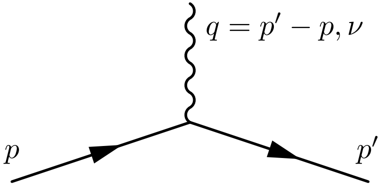
The 13th International Workshop on e+e- collisions from Phi to Psi

Outline

Muon anomalous magnetic moment $(g-2)$



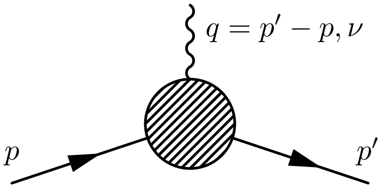
- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g-2$ experiment.



Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$



$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

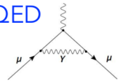
- The quantity a is called the anomalous magnetic moments.
- Its value comes from quantum correction.

Muon $g-2$ Theory Initiative White paper posted 10 June 2020.

132 authors from worldwide theory + experiment community. [Phys. Rept. 887 (2020) 1-166]

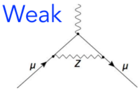
$$a_\mu = a_\mu(\text{QED}) + a_\mu(\text{Weak}) + a_\mu(\text{Hadronic})$$

QED



$+\ldots$ $116\,584\,718.9\,(1) \times 10^{-11}$ 0.001 ppm

Weak

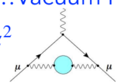


$+\ldots$ $153.6\,(1.0) \times 10^{-11}$ 0.01 ppm

Hadronic…

…Vacuum Polarization (HVP)

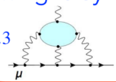
α^2



$+\ldots$ $6845\,(40) \times 10^{-11}$ [0.6%] 0.34 ppm

…Light-by-Light (HLbL)

α^3

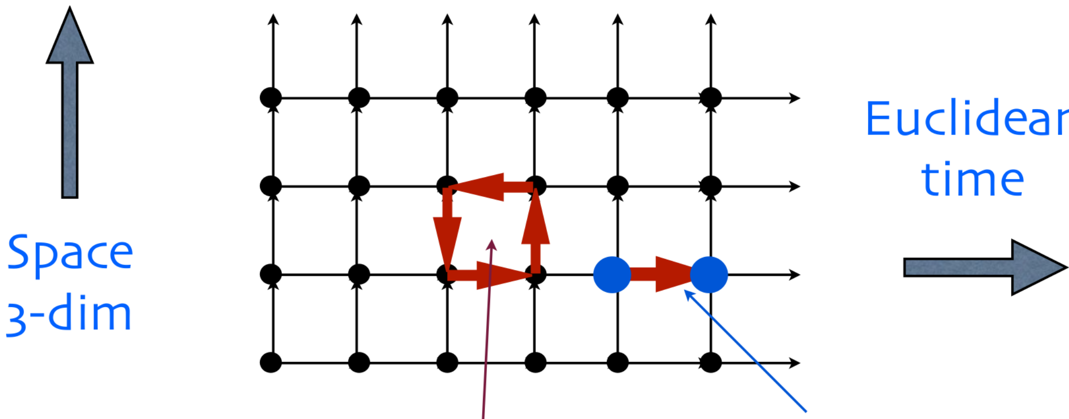


$+\ldots$ $92\,(18) \times 10^{-11}$ [20%] 0.15 ppm

- Two methods: dispersive + data $\leftrightarrow$ lattice QCD

From Aida El-Khadra's theory talk during the Fermilab $g-2$ result announcement.

Outline



$$S_E^{\mathrm{latt}} = -\sum_{\Box} \frac{6}{g^2}\,\mathrm{Re}\ \mathrm{tr}_N(U_{\Box,\mu\nu}) - \sum_q \overline{q}(D_\mu^{\mathrm{lat}}\gamma_\mu + a m_q)q$$

Wilson gauge action   Lattice fermion action

Figure credit: Stephen R. Sharpe.

Lattice QCD: Monte Carlo

$$\langle \mathcal{O}(U, q, \bar{q}) \rangle = \frac{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}} \mathcal{O}(U, q, \bar{q})}{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}}}$$

$$= \frac{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}} \gamma_\mu + am_q\right) \tilde{\mathcal{O}}(U)}{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}} \gamma_\mu + am_q\right)}$$

Monte Carlo:

- The integration is performed for all the link variables: U. Dimension is $L^3 \times T \times 4 \times 8$.
- Sample points the following distribution:

$$e^{-S_{\text{gauge}}^{\text{latt}}(U)} \prod_q \det\left(D_\mu^{\text{latt}}(U) \gamma_\mu + am_q\right)$$

- Therefore:

$$\langle \mathcal{O}(U, q, \bar{q}) \rangle = \frac{1}{N_{\text{conf}}} \sum_{k=1}^{N_{\text{conf}}} \tilde{\mathcal{O}}(U^{(k)})$$

- Parameters in lattice QCD calculations (e.g. isospin symmetric ($m_u = m_d = m_l$) and three flavor u, d, s theory):

$$g \qquad am_l \qquad am_s$$

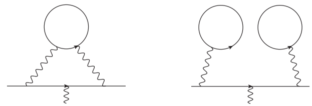
  Note that lattice spacing a is determined by g via the renormalization group equation.

- The experimental inputs needed to determine these parameters can be: m_π / m_Ω, m_K / m_Ω.

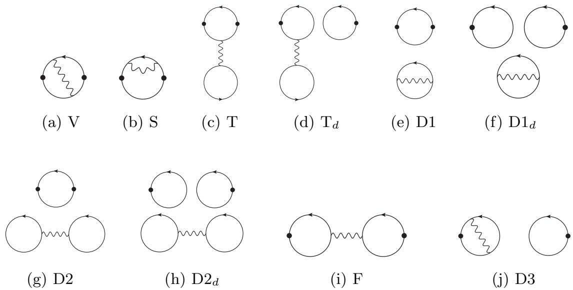
Outline

Muon $g-2$ HVP: all diagrams

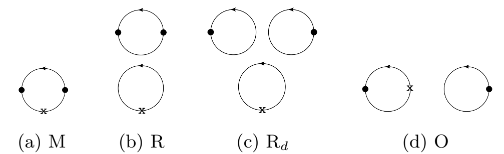
Isospin limit



QED corrections



Strong isospin breaking



- Need to calculate and cross check all the contributions.

T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\mathrm{QCD}}$$

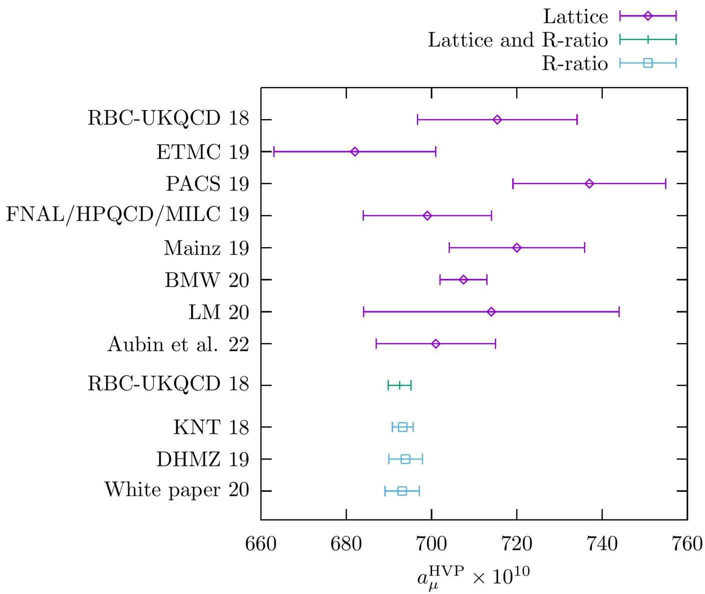
$$a_\mu^{\mathrm{HVP\ LO}} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2) = \sum_{t=0}^{+\infty} w(t)C(t)$$

QED and strong isospin breaking

$a_\mu^{\mathrm{HVP,\,LO}}(ud)$	$a_\mu^{\mathrm{HVP,\,LO}}(s)$	$a_\mu^{\mathrm{HVP,\,LO}}(c)$	$a_{\mu,\mathrm{disc}}^{\mathrm{HVP,\,LO}}$	$\delta a_\mu^{\mathrm{HVP}}$
650.2(11.6)	53.2(0.3)	14.6(0.1)	–13.7(2.9)	7.2(3.4)

- From muon $g-2$ theory initiative white paper (2020). Value in unit of 10^{-10}
- Light quark connected diagram has the largest contribution and largest uncertainty.

- Dispersive method via R-ratio (red points) is mature and reproducible.
- Lattice (blue points) errors are limited by statistics. Except for BMW, which beats down the statistical error, result is limited by systematic error: BMW 20: $707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}$
- Lattice-QCD calculations of comparable precision needed.
- Consistency is needed to claim new physics.



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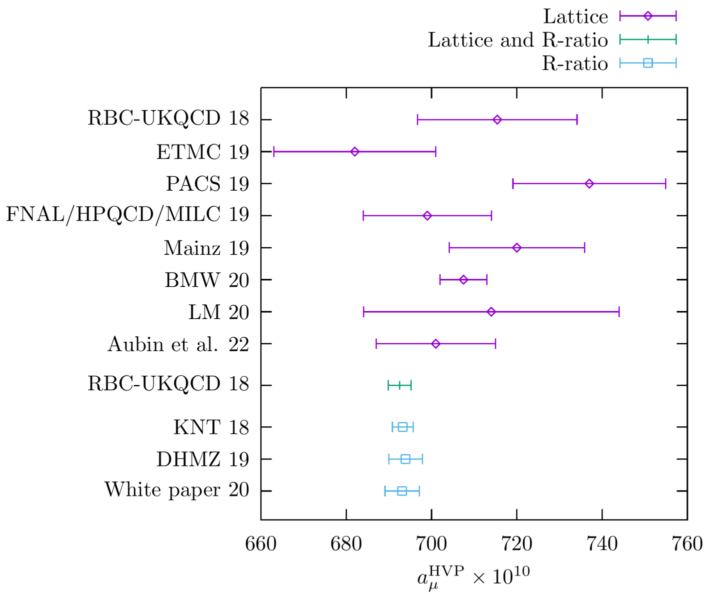

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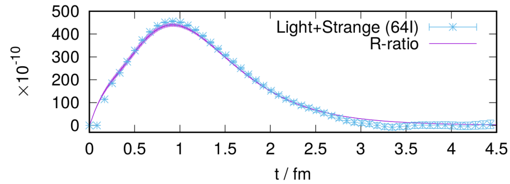
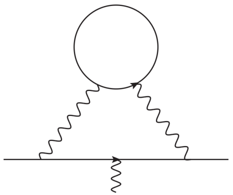
$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$



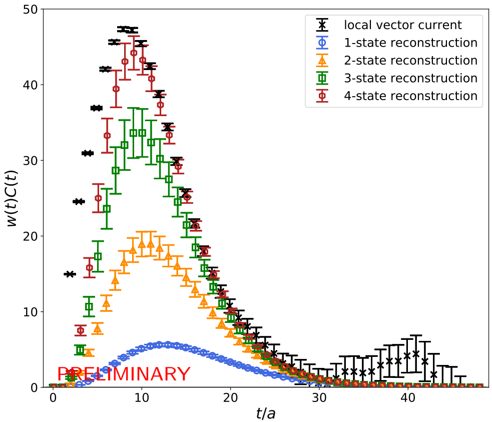
- Statistical error is mostly from:
  Light quark connected diagram at $t \gtrsim 1.5$ fm
  - More configurations (BMW 20 used $\sim 20,000$).
  - Use low modes averaging to gain full volume average. ✓
  - Bounding method on the long distance tail. ✓
  - Study the $\pi\pi$ system spectrum to calculate $C(t)$ large t.
    - Not used in any published work yet!
    - On-going efforts with promising initial results.
- Systematic error is mostly from the **continuum extrapolation**.

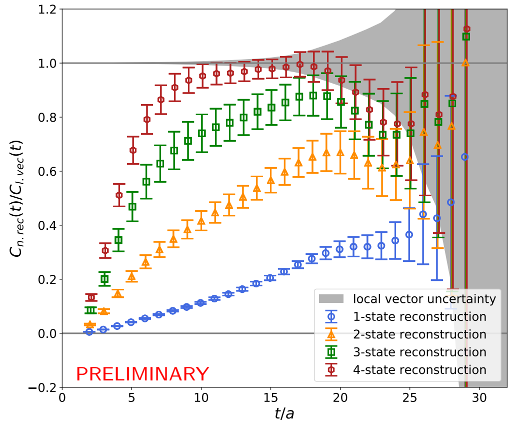


- Main idea is that: one does not have to calculate the long distance part of the correlation function directly.

$$\begin{aligned} C(t) &= \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle \\ &= \sum_n \frac{V}{3}\sum_{j=0,1,2}\langle 0|J_j(0)|n\rangle\langle n|J_j(0)|0\rangle e^{-E_n t} \end{aligned}$$

- The summation over n is limited to zero momentum states and states are normalized to "1".
- At large t, only lowest few states contribute. We only need the matrix elements $\langle n|J_j(0)|0\rangle$ and the corresponding energy E_n.
- Need to study the spectrum of the $\pi\pi$ system!
- Can reduce the statistical error beyond the gauge noise limit!





GEVP results to reconstruct long-distance behavior of local vector correlation function needed to compute connected HVP

Explicit reconstruction good estimate of correlation function at long-distance, missing excited states at short-distance

More states $\Longrightarrow$ better reconstruction, can replace $C(t)$ at shorter distances

RBC-UKQCD by Aaron Meyer and Christoph Lehner

Preliminary

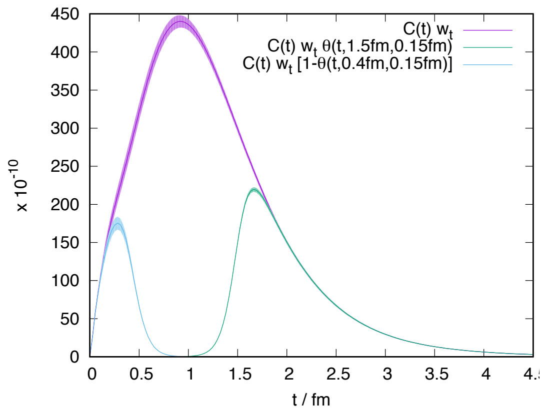
RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t) \qquad w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
  - short distance $\Leftarrow$ discretization effects
  - long distance $\Leftarrow$ noisy $\pi\pi$ tail
  - intermediate (Window): sweet spot
- Can form windows from $R(e^+e^-)$ dispersive analysis too.

  Combine “window” from lattice with dispersive analysis.
- Compare “window” among lattice-QCD calculations



Muon $g-2$ HVP: controlling systematics

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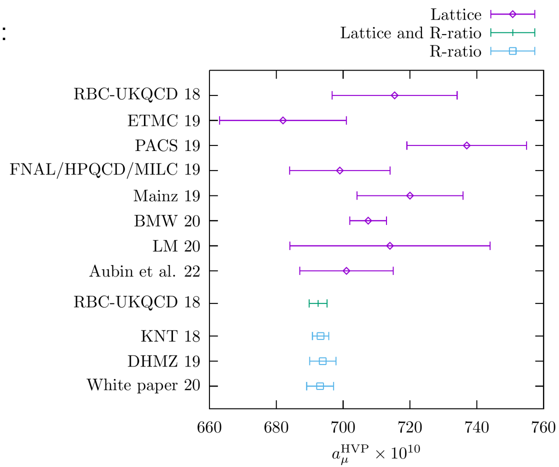

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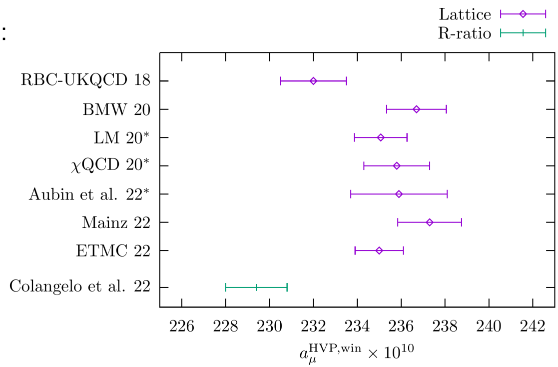
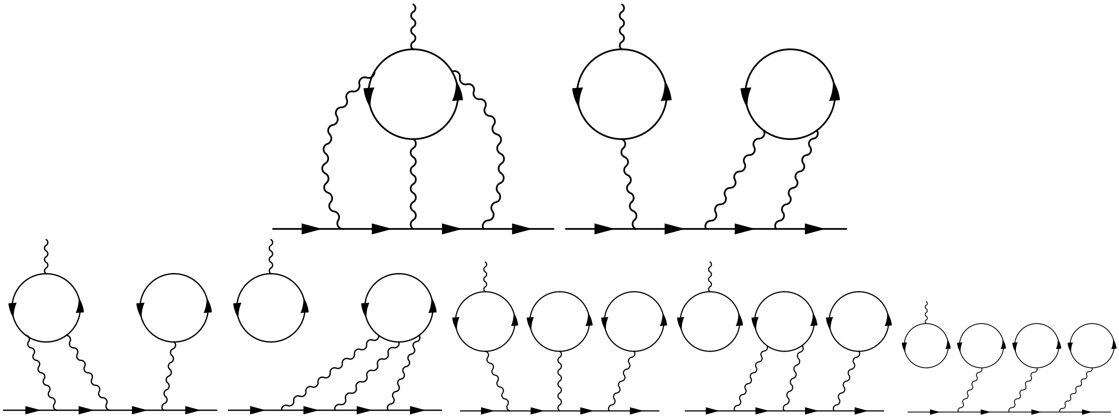


- The white paper result for the HVP is a community-vetted method average for the data-driven approach. It accounts for spreads in sub-contributions between individual results (KNT/DHMZ) that may not be visible in the agreement of looking at the final results for the HVP. It should be noted that its error estimate also accounts for the tension between BaBar and KLOE experimental inputs.
- We are now in the fortunate situation that we have a first lattice result with sub-percent precision (BMW). It is clear that to safely assess systematic uncertainties, most notably the one related to the choice of the lattice regulator, calculations by other lattice groups with a similar precision will be essential. The importance of having more than one lattice calculations of the same quantity and obtained with different lattice discretizations is well understood inside the lattice community.
- On the way to a method average for lattice QCD, it is prudent to also look at individual sub-contributions and their agreement, similar to what was done for the data driven approach. Here the tension in the window results need to be addressed. Also, individual QED corrections should be cross checked (currently there are some tensions). This is possible since these sub-contributions are already available by different collaborations at adequate precision for such a comparison.
- Finally, it should also not be ignored that for the standard window result, there are larger ($> 3\sigma$) tensions between different determinations.

Outline

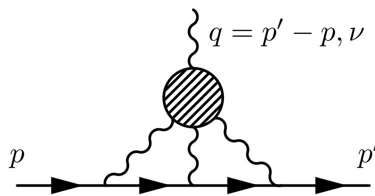


- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional different permutations of photons not shown.
- The second row diagrams are suppressed by flavor SU(3) symmetry (and small charge factors, $1/N_c$, etc). The contributions are numerically very small.

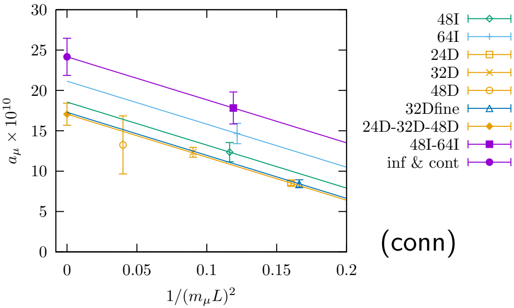
Contribution	PdRV(09) [471]	N/JN(09) [472, 573]	J(17) [27]	Our estimate
π^0, η, η'-poles	114(13)	99(16)	95.45(12.40)	93.8(4.0)
π, K-loops/boxes	−19(19)	−19(13)	−20(5)	−16.4(2)
S-wave $\pi\pi$ rescattering	−7(7)	−7(2)	−5.98(1.20)	−8(1)
subtotal	88(24)	73(21)	69.5(13.4)	69.4(4.1)
scalars	–	–	–	} − 1(3)
tensors	–	–	1.1(1)	
axial vectors	15(10)	22(5)	7.55(2.71)	6(6)
u, d, s-loops / short-distance	–	21(3)	20(4)	15(10)
c-loop	2.3	–	2.3(2)	3(1)
total	105(26)	116(39)	100.4(28.2)	92(19)

Table 15: Comparison of two frequently used compilations for HLbL in units of 10^{-11} from 2009 and a recent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus"); N/JN = Nyffeler / Jegerlehner, Nyffeler; J = Jegerlehner.

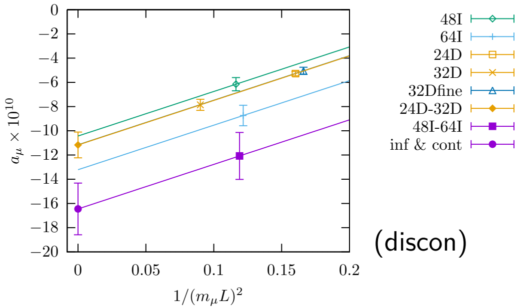
- Values in the table is in unit of 10^{-11}.
- Uncertainty of the analytically approach mostly come from the short distance part.



$$a_\mu(L, a^{\text{I}}, a^{\text{D}}) \quad = \quad a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\text{I}}(a^{\text{I}}\ \text{GeV})^2 - c_1^{\text{D}}(a^{\text{D}}\ \text{GeV})^2 + c_2^{\text{D}}(a^{\text{D}}\ \text{GeV})^4\right)$$



(conn)



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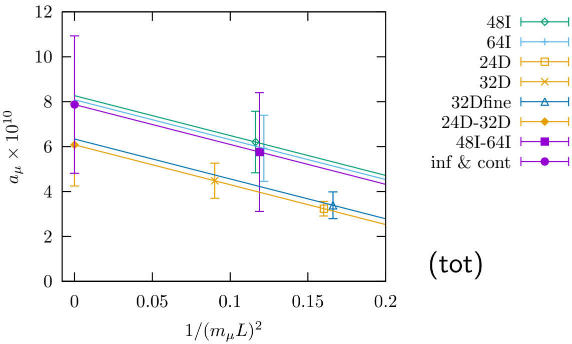
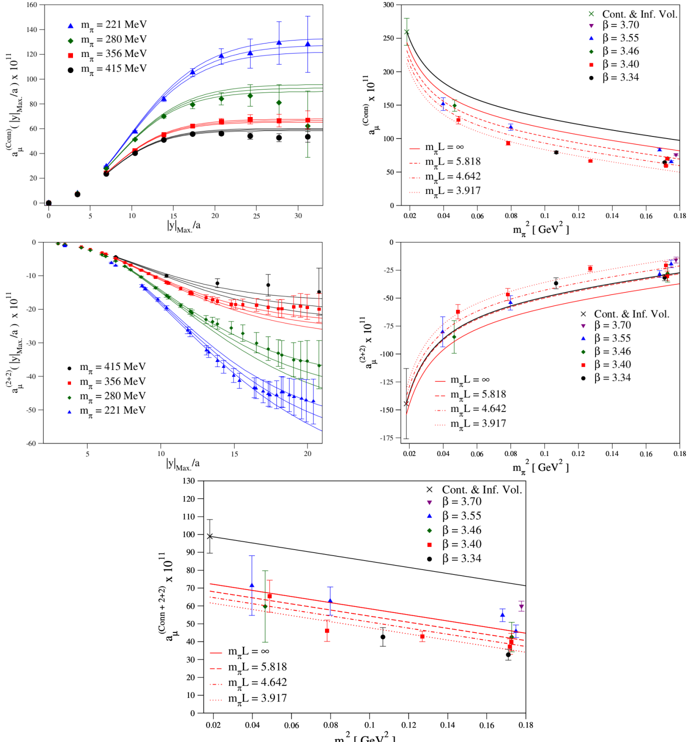


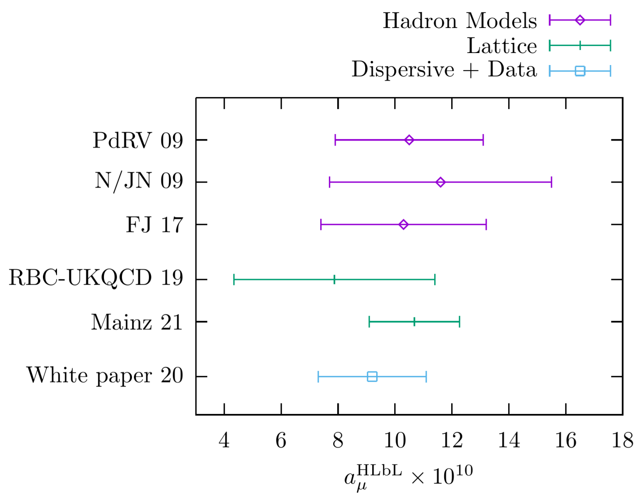
(tot)

T. Blum et al 2020. (PRL 124, 13, 132002)



E.H. Chao et al 2021. (EPJC 81, 7, 651)

- Mainz 21 is the most recent lattice result. It uses heavier pion mass with infinite volume QED kernel and extrapolate to the physical pion mass.
- RBC-UKQCD 19 is the first lattice result. It uses physical pion mass in the finite volume QED_L scheme and extrapolate to the infinite volume.



- White paper 2020 result uses dispersive relations and data. It is the sum of the contributions from different cuts and poles.
- These three results have different systematics and agree well with each other. Uncorrelated average gives: $a_\mu^{\text{HLbL}} = 9.77(1.16) \times 10^{-10}$.
- Hadronic light-by-light contribution cannot be the source of the muon $g-2$ puzzle.

Outline

Summary

- The errors of lattice QCD calculations comes from:

  1. finite statistics → statistical error
  2. non-zero lattice spacing → discretization error
     - smaller lattice spacing $a \lesssim 0.06$ fm
     - compare different lattice actions: Staggered, Wilson, Domain Wall, etc
  3. finite lattice size → finite volume error
  4. non-physical pion mass → Chiral extrapolation
     Many lattice calculations are now performed with physical pion mass, eliminating this source of the systematic errors.

- Lattice QCD calculation is playing important role in determining the hadronic contribution to muon $g-2$ and many other physical observables.

- More accurate lattice results are expected when Fermilab releases their final result.

- Need more investigation on the tensions between the different HVP determinations and the window quantities.

Thank You!