Studies of Light Hadrons at Belle

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On behalf of Belle Collaboration

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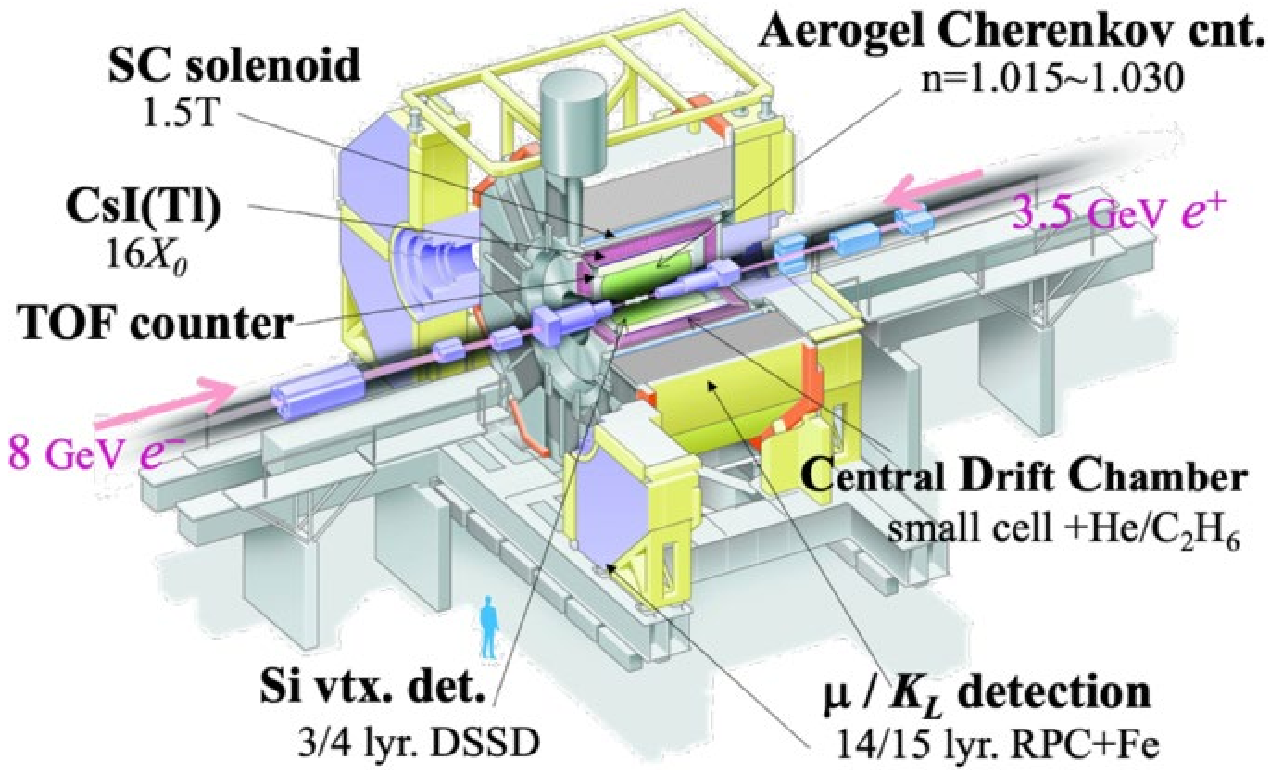
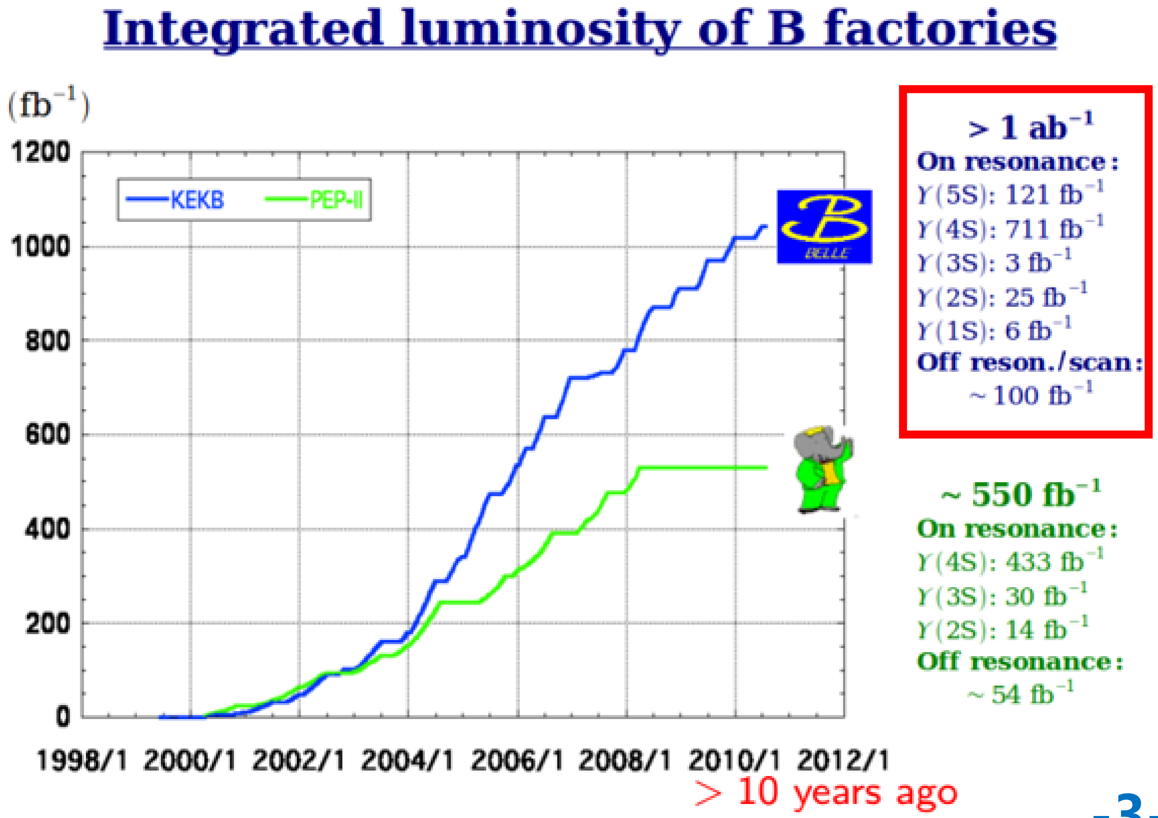
Outline

- ✓ Belle Detector and Data Samples
- ✓ Observation of $\Omega(2012)^- \rightarrow \Xi(1530)\bar{K}$ and Measurement of the Effective Couplings of $\Omega(2012)^- \rightarrow \Xi(1530)\bar{K}$ and $\Xi\bar{K}$.
  [arXiv: 2207.03090]
- ✓ Observation of a Threshold Cusp at the $\Lambda\eta$ Threshold in pK^- Mass Spectrum with $\Lambda_c^+ \rightarrow pK^-\pi^+$.
  [To be submitted]
- ✓ Study of $e^+e^- \rightarrow \eta\phi$ via Initial State Radiation at Belle.
  [To be submitted]
- ✓ Summary

Belle Detector and Data Samples

- KEKB is an asymmetric-energy e^+e^- collider operating near $\Upsilon(4S)$ mass peak ($\sim 10.58 GeV/c^2$, $> B\bar{B}$ threshold);
- Belle detector has good performances on momentum/vertex resolution; particle identification, etc;
- Accumulated data set of $\sim 1\ ab^{-1}$: not only taken a large $B\bar{B}$ sample as a B-factory, but also collected largest $\Upsilon(1S)$ and $\Upsilon(2S)$ samples.



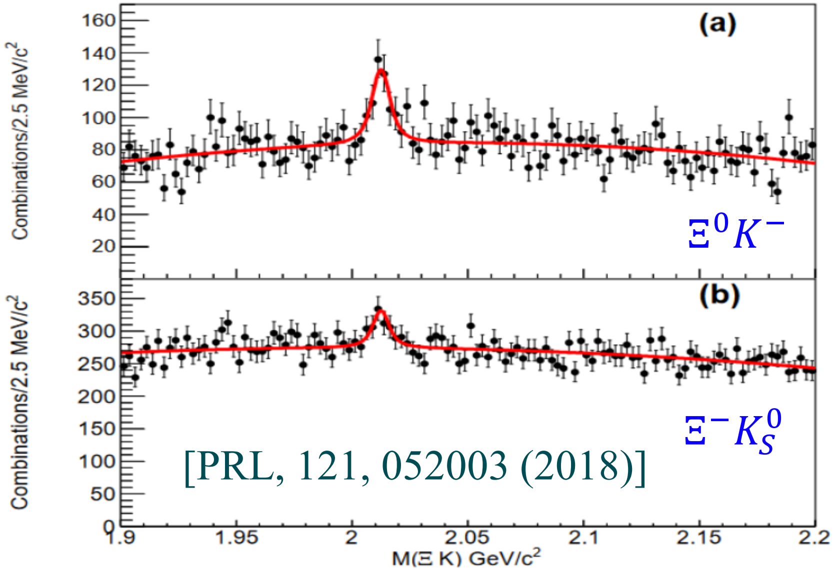


Observation of $\Omega(2012)^- \to \Xi(1530)\bar{K}$

Motivation:

- The Particle Data Group (PDG) lists only four excited Ω^- baryons: $\Omega(2012)^-$, $\Omega(2250)^-$, $\Omega(2380)^-$, and $\Omega(2470)^-$.
- In 2018, the $\Omega(2012)^-$ decaying into $\Xi^0 K^-$ and $\Xi^- K_S^0$ was first observed by Belle in $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ data samples.

$$M[\Omega(2012)^-] = (2012.4 \pm 0.9)\ \text{MeV}$$
$$\Gamma[(2012)^-] = (6.4^{+3.0}_{-2.7})\ \text{MeV}$$



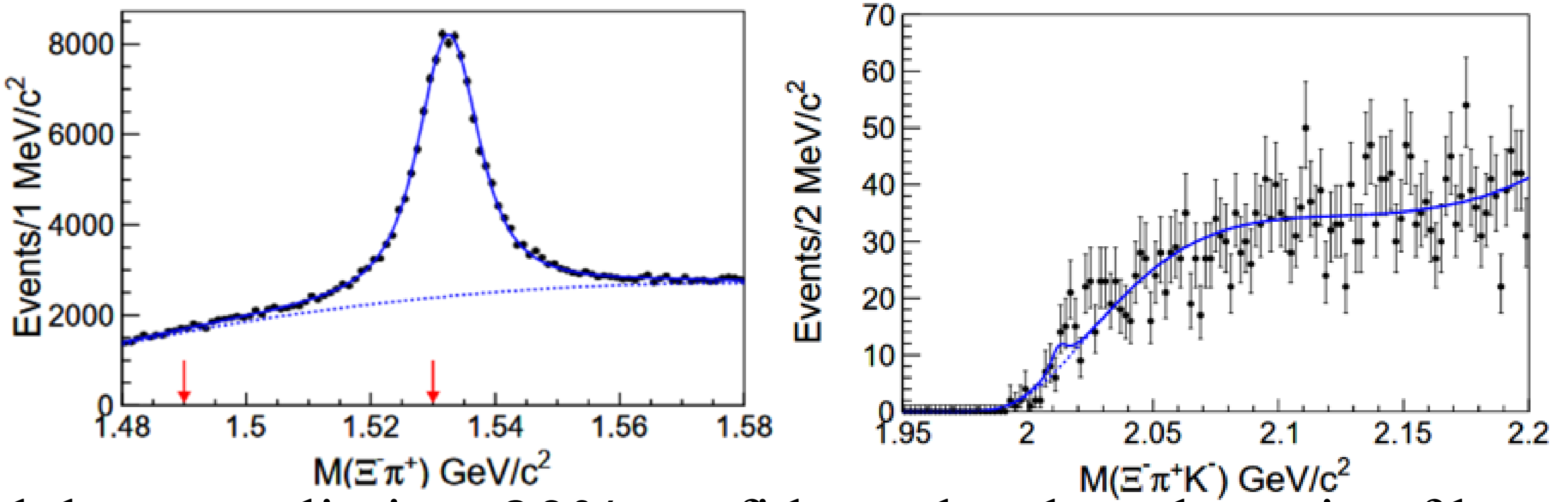
- The $\Omega(2012)^-$ has been interpreted as a standard baryon or a $\Xi(1530)\bar{K}$ molecule.

Interpretations	References	Comments
Standard baryon	PRD 98, 034004 (2018), EPJC 78, 894 (2018), PRD 98, 114023 (2018), PRD 101, 016002 (2020), PRD 105, 094006 (2022), PRC 103, 025202 (2021), PRD 98, 014031 (2018), PLB 792, 315 (2019), arXiv: 2203.04458 (2022), arXiv: 2201.10427 (2022)	The $\Omega(2012)^-$decays dominantly to $\Xi\bar{K}$
$\Xi(1530)\bar{K}$ molecule	PRD 98, 054009 (2018), EPJC 78, 857 (2018), PRD 98, 076012 (2018), JPG 48, 025001 (2021), PRD 98, 056013 (2018), PRD 101, 094016 (2020), EPJC 80, 361 (2020), PRD 102, 074025 (2020), arXiv: 2204.13396 (2022)	The $\Omega(2012)^-$ decays equally to $\Xi\bar{K}$ and $\Xi(1530)\bar{K}$. Or the $\Xi(1530)\bar{K}$ decay mode is dominant.

- Measurement of $\Omega(2012)^- \to \Xi(1530)\bar{K} \to \Xi\pi\bar{K}$ can give us information about its internal structure.

Search for $\Omega(2012)^- \to \Xi(1530)\bar{K}$

- The decay $\Omega(2012)^- \to \Xi(1530)\bar{K} \to \pi\Xi\bar{K}$ was searched for by Belle in $\Upsilon(1S)$, $\Upsilon(2S)$, and $\Upsilon(3S)$ data samples, and no significant signals were found. [PRD 100, 032006 (2019)]



- Belle determined the upper limit at 90% confidence level on the ratio of branching fraction:

$$\mathcal{R}^{\Xi\pi K}_{\Xi K} = \frac{\mathcal{B}(\Omega(2012) \to \Xi(1530)(\to \Xi\pi)K)}{\mathcal{B}(\Omega(2012) \to \Xi K)} < 11.9\%$$

- However, we realized that

① the requirement of $M(\Xi\pi)$ includes large non-$\Xi(1530)$ decay backgrounds;

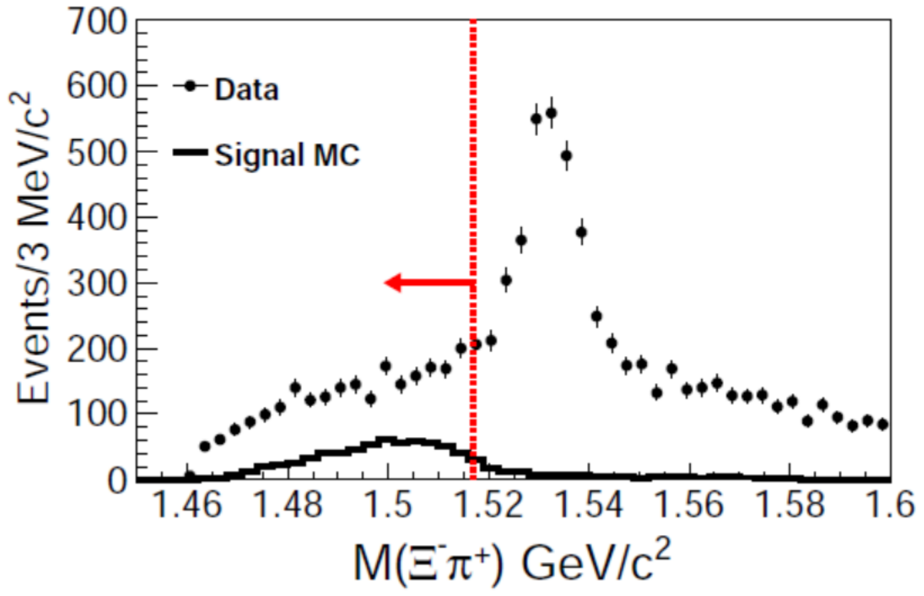
② do not consider a three-body phase space in $M(\Xi\pi\bar{K})$, which increases sharply due to the unstable $\Xi(1530)$ constituent.

Revisit $\Omega(2012)^- \to \Xi(1530)\bar{K}$

- In the contrast to the previous study [PRD 100, 032006 (2019)]:

① We require $M(\Xi\pi) < 1.517$ GeV;

② We parameterize the $\Omega(2012)^-$ signal shape with a Flatté-like function with a three-body phase space included.

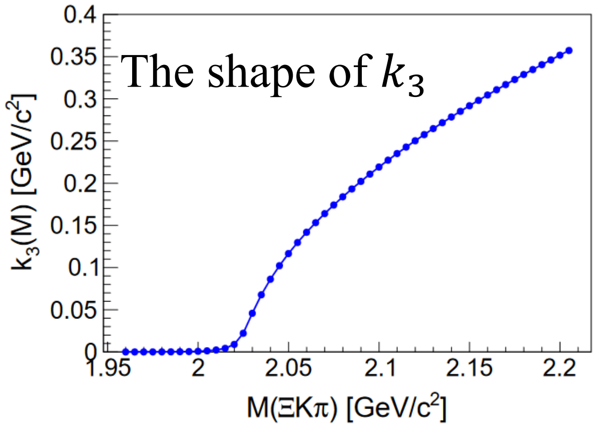


- The black solid histogram shows the expected lineshape of the $\Xi(1530)$ from $\Omega(2012)^-$ decay.
- The number of signal MC events is scaled to three times the yield of $\Omega(2012)^- \to \Xi^-\pi^+K^-$ in data to make it more visible.

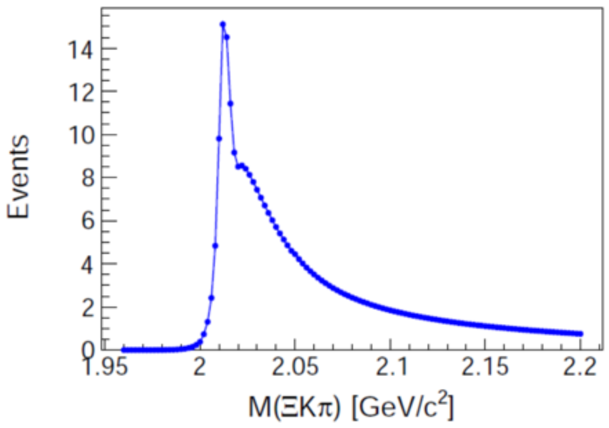
- The Flatté-like function [PRD 81, 094028 (2010)]:

$$T_n(M) = \frac{g_n k_n(M)}{|M - m_{\Omega(2012)} + \frac{1}{2}\sum_{j=2,3} g_j[\kappa_j(M) + ik_j(M)]|^2}$$

- g_n is the effective coupling of $\Omega(2012)^-$ to the n-body final state
- κ_n and k_n parameterize the real and imaginary parts of the $\Omega(2012)^-$ self-energy.



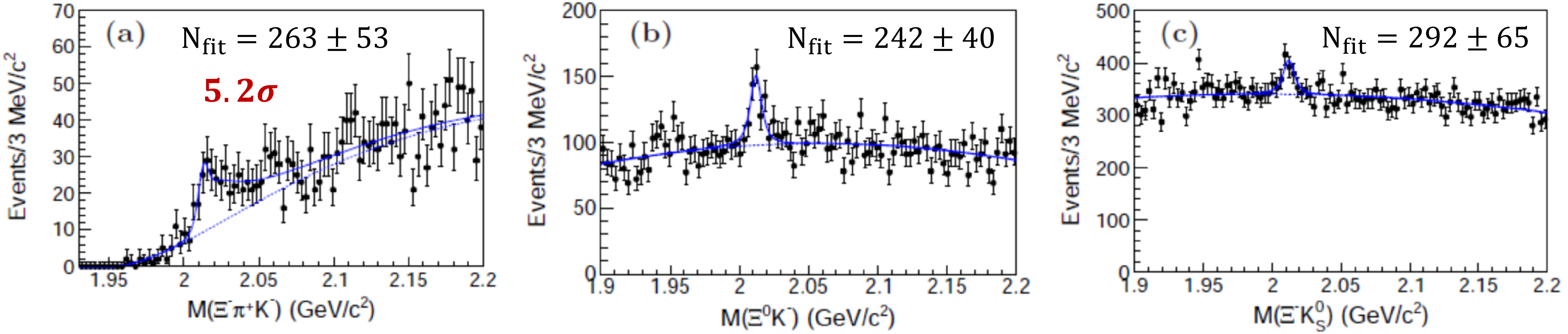
The phase space k_3 increases sharply above 2.02 GeV$/c^2$.



Lineshape for $\Omega(2012)^-$ with $\Xi K\pi$ final state

Revisit $\Omega(2012)^- \to \Xi(1530)\bar{K}$

- We fit simultaneously to the binned $\Xi^-\pi^+K^-$, Ξ^0K^-, and $\Xi^-K_S^0$ mass distributions from $\Upsilon(1S, 2S, 3S)$ data samples. [arXiv:2207.03090]



The mass and effective couplings:

$\Omega(2012)^-$ mass	$(2012.5 \pm 0.7 \pm .0.5)$ MeV
The coupling to $\Xi\bar{K}$	$(1.7 \pm 0.3 \pm 0.3) \times 10^{-2}$
The coupling to $\Xi(1530)\bar{K}$	$(41.1 \pm 35.8 \pm 6.0) \times 10^{-2}$

✓ Our result is consistent with the molecular model of $\Omega(2012)^-$, which predicts similar branching fractions for $\Omega(2012)^-$ decay to $\Xi(1530)\bar{K}$and $\Xi\bar{K}$.

The ratio of the branching fraction for the three-body to two-body decays:

$$\mathcal{R}^{\Xi\pi\bar{K}}_{\Xi\bar{K}} = \frac{\mathcal{B}(\Omega(2012)^- \to \Xi(1530)\bar{K} \to \Xi\pi\bar{K})}{\mathcal{B}(\Omega(2012)^- \to \Xi\bar{K})}$$

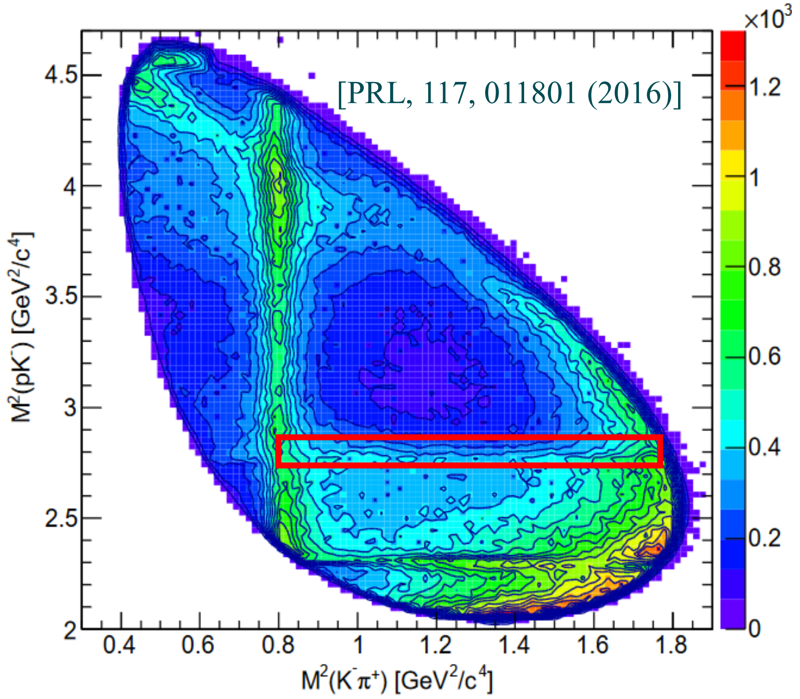
⇓

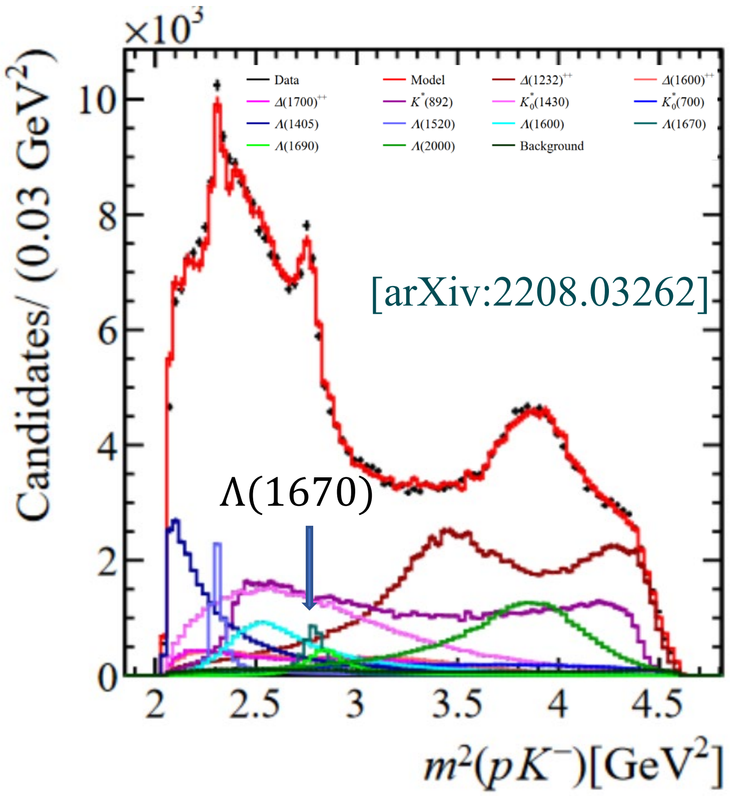
$$0.97 \pm 0.24(\text{stat.}) \pm 0.07(\text{syst.})$$

Observation of a Threshold Cusp at the $\Lambda\eta$ Threshold in $M(pK^-)$

Motivation:

- A trace of a peak structure is observed in the pK^- mass spectrum in the previous analysis of $\Lambda_c^+ \to pK^-\pi^+$ decay by the Belle. [PRL, 117, 011801 (2016)]
- Very recently, LHCb performed an amplitude analysis of $\Lambda_c^+ \to pK^-\pi^+$. A similar structure is also seen. LHCb explained the structure using a BW form with fixed mass and width. [arXiv:2208.03262]





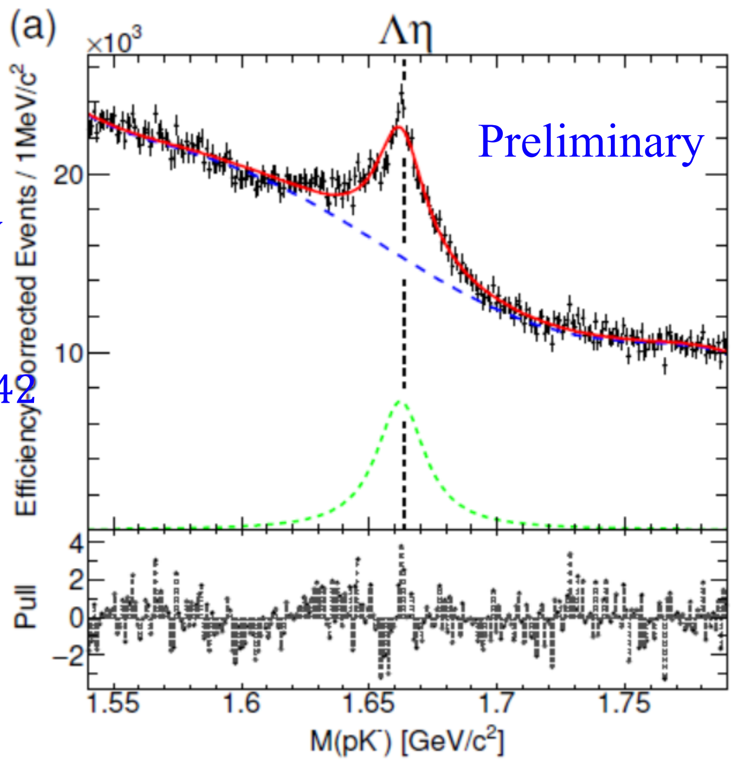
- In this analysis, we report a newly discovered peaking structure in the pK^- mass spectrum near the $\Lambda\eta$ mass threshold, and approach this peak considering two possible cases:
  ① BW-type peak;
  ② A visible $\Lambda\eta$ threshold cusp enhanced by the $\Lambda(1670)$ pole nearby.

From the perspective of a new resonance

➢ We perform a binned least-χ^2 fit to the efficiency-corrected $M(pK^-)$ distribution

- Fit to $M(pK^-)$ distribution using non-relativistic BW function.

$$\frac{dN}{dm} \propto |BW(m)|^2 = \left|\frac{1}{(m-m_0)+i\frac{\Gamma_0}{2}}\right|^2$$

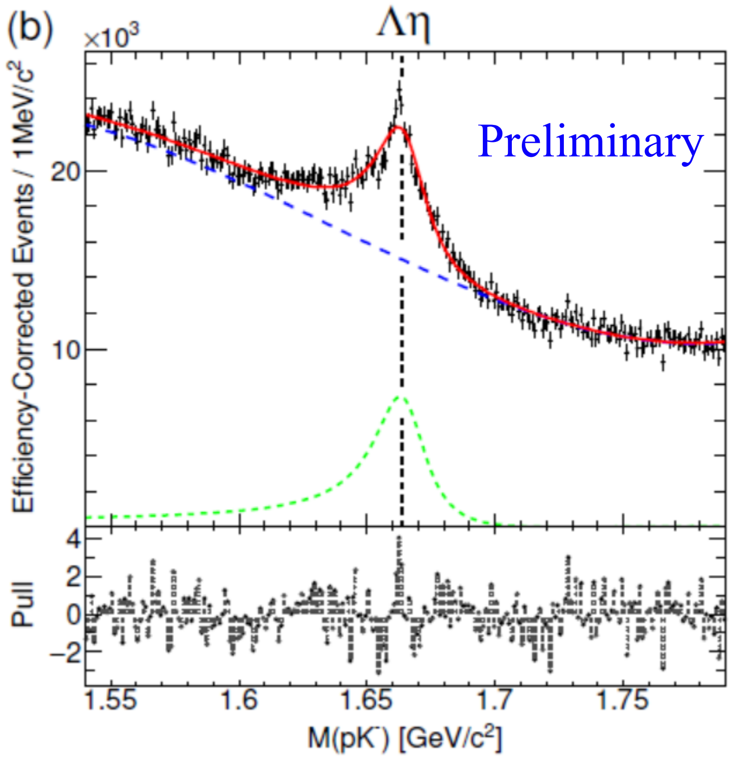


$m_0 = (1662.4 \pm 0.3)$ MeV
$\Gamma_0 = (22.6 \pm 1.5)$ MeV

Reduced $\chi^2/ndf = 328/242 = 1.35$

- Fit to $M(pK^-)$ using BW with complex constant added coherently, leading to constructive interference below the $\Lambda\eta$ threshold and destructive above that.

$$\frac{dN}{dm} \propto \left|BW(m) + re^{i\theta}\right|^2 = \left|\frac{1}{(m-m_0)+i\frac{\Gamma_0}{2}} + re^{i\theta}\right|^2$$



$m_0 = (1665.4 \pm 0.5)$ MeV
$\Gamma_0 = (23.8 \pm 1.2)$ MeV

Reduced $\chi^2/ndf = 308/243 = 1.27$

From the perspective of a cusp at the $\Lambda\eta$ threshold

- Another possibility is that the peak structure is a cusp at the $\Lambda\eta$ threshold enhanced by the $\Lambda(1670)$ pole nearby.
- We fit the efficiency-corrected $M(pK^-)$distribution using a non-relativistic Flatté function [PLB, 63, 224 (1976), EPJA, 23, 523 (2005)]:

$$\frac{dN}{dm} \propto |f(m)|^2 = \left|\frac{1}{m - m_f + \frac{i}{2}(\Gamma' + \bar{g}_{\Lambda\eta} k)}\right|^2$$

- m_f is a parameter corresponding to the nominal mass of $\Lambda(1670)$.
- Γ' is a parameter for the sum of the partial widths of the decay modes other than $\Lambda\eta$, and is approximated as a constant.
- k is the decay momentum in the $\Lambda\eta$ channel, and $\bar{g}_{\Lambda\eta} k$ represents the partial decay width of the $\Lambda\eta$ channel.

- We fix m_f when we perform a fit and repeat the fit with various m_f values.

- We take into account an interference with another S-wave amplitude such as a tail of $\Lambda(1405)$. We perform a binned least-χ^2 fit with the combined function, $\frac{dN}{dm} \propto \left|f(m) + re^{i\theta}\right|^2$.

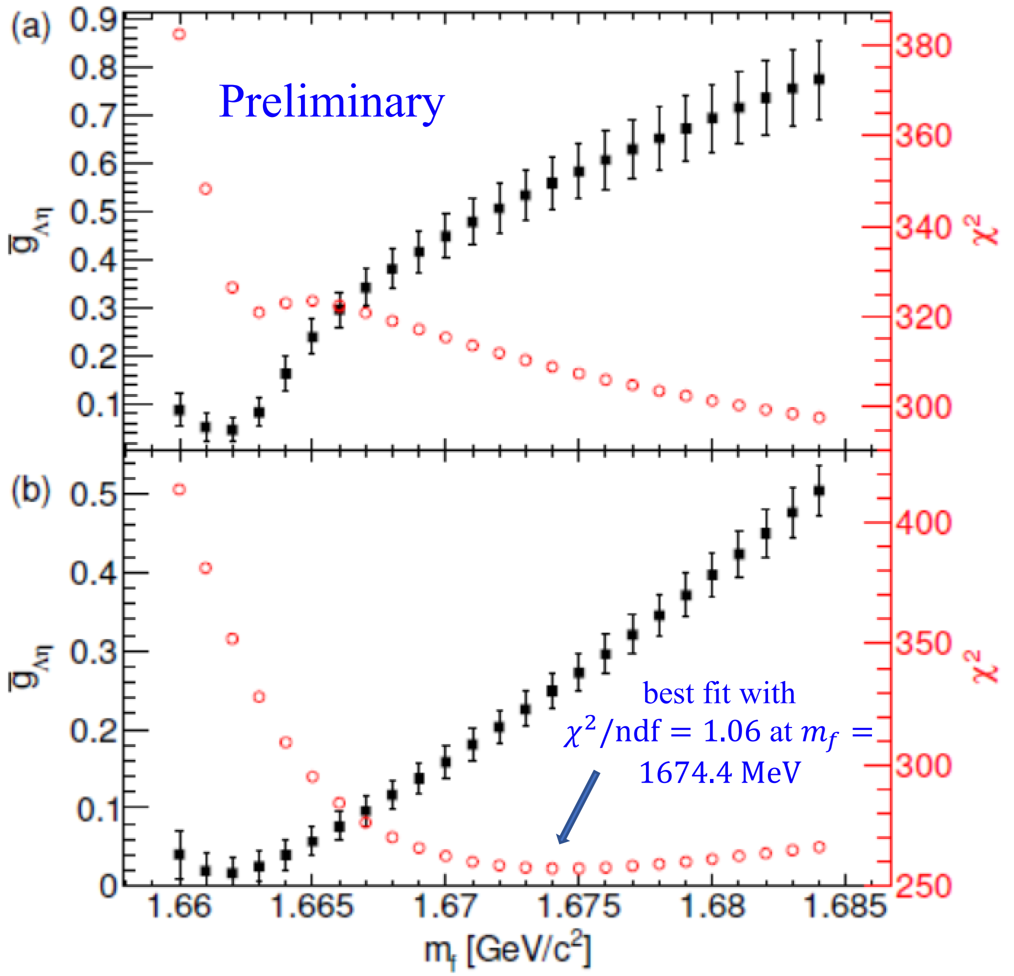


Figure :$\bar{g}_{\Lambda\eta}k$ and χ^2 from Flatté model (a) without and (b) with the interference as a function of fixed m_f .

From the perspective of a cusp at the $\Lambda\eta$ threshold

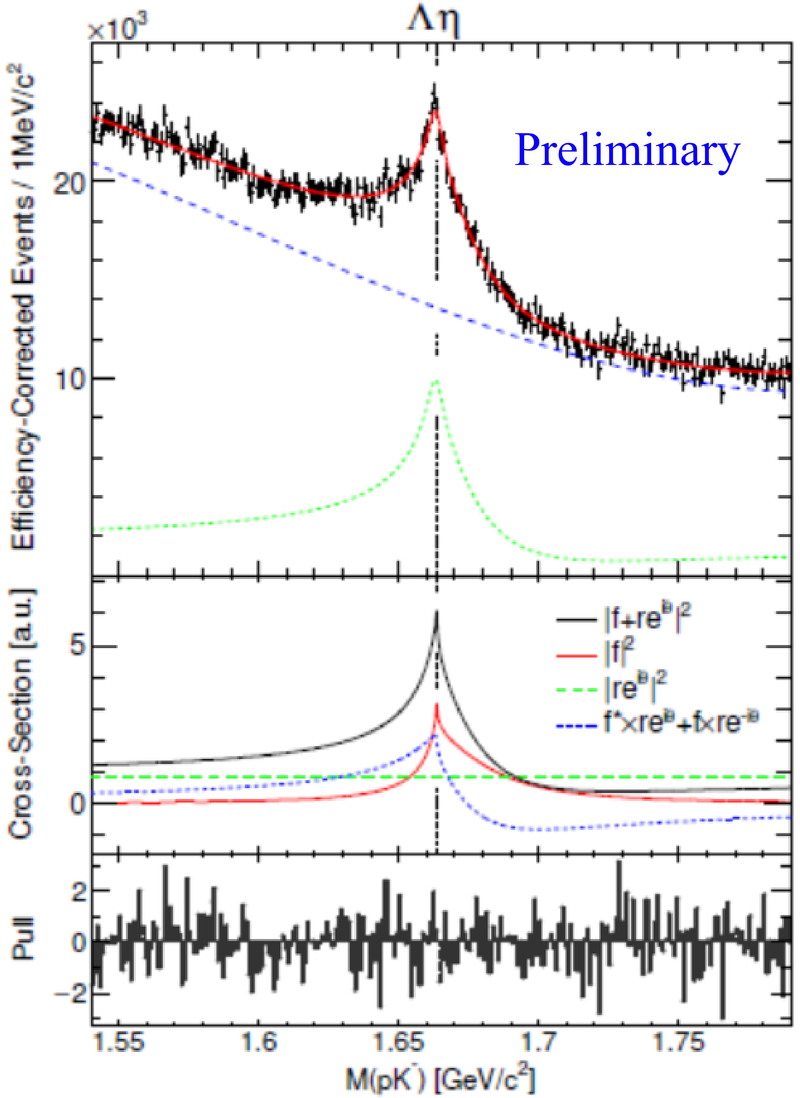


Figure: Best fit result with $\chi^2/\text{ndf} = 1.06$ at $m_f = 1674.4\ \text{MeV}/c^2$

- The best fit with χ^2/ndf=1.06 (257/243) is obtained at m_f=1674.4 MeV/c^2.
- The measured: $\Gamma' = (27.2 \pm 1.9^{+5.0}_{-3.9})$ MeV, $\bar{g}_{\Lambda\eta} = \left(258 \pm 23^{+61}_{-75}\right) \times 10^{-3}$

	Our measurement	$\Lambda(1670)$ [PRD 103, 052005 (2021)]
mass	Fix m_f =1674.4 MeV/c^2	$(1674.3 \pm 0.8 \pm 4.9)\text{MeV}/c^2$
Total width	$(50.3 \pm 2.9^{+4.2}_{-4.0})$ MeV	$(36.1 \pm 2.4 \pm 4.8)$MeV

- The fit result with the Flatté function to which the constant is coherently added shows the best reduced χ^2/ndf of 1.06 (257/243, p = 0.25), while 1.27 (308/243, p = 3.1×10^{-3}) from the best BW fit.
- The best fit explains the structure as a cusp at the Λη threshold.
- The obtained parameters are consistent with the known properties of $\Lambda(1670)$.

Study of $e^+e^- \to \eta\phi$ via Initial State Radiation at Belle

Motivation:

- The Y(2175) [now called 'ϕ(2170)'] was discovered in $e^+e^- \to \pi^+\pi^-\phi$ via ISR by BaBar, and later confirmed by Belle. [PRL 91, 262001 (2003), PRL 122, 222001 (2019)]

- Babar studied the $e^+e^- \to \eta\phi$ process vis ISR using a $232 fb^{-1}$ data sample and an excess was observed around 2.1 GeV/c^2, called the ϕ''. [PRD 76, 092005 (2007), PRD 77, 092002 (2008)]

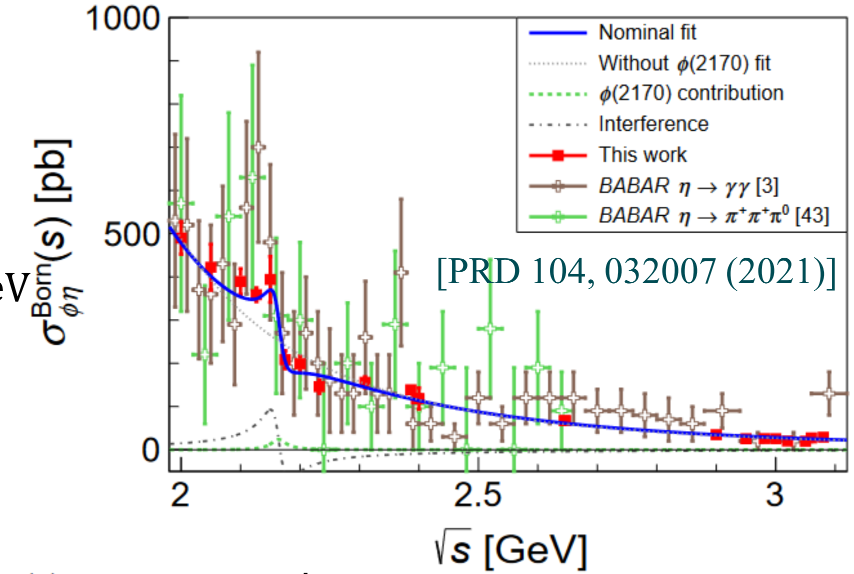
- BESIII also measured the Born cross section of $e^+e^- \to \eta\phi$ and determined the resonant parameters of ϕ(2170).

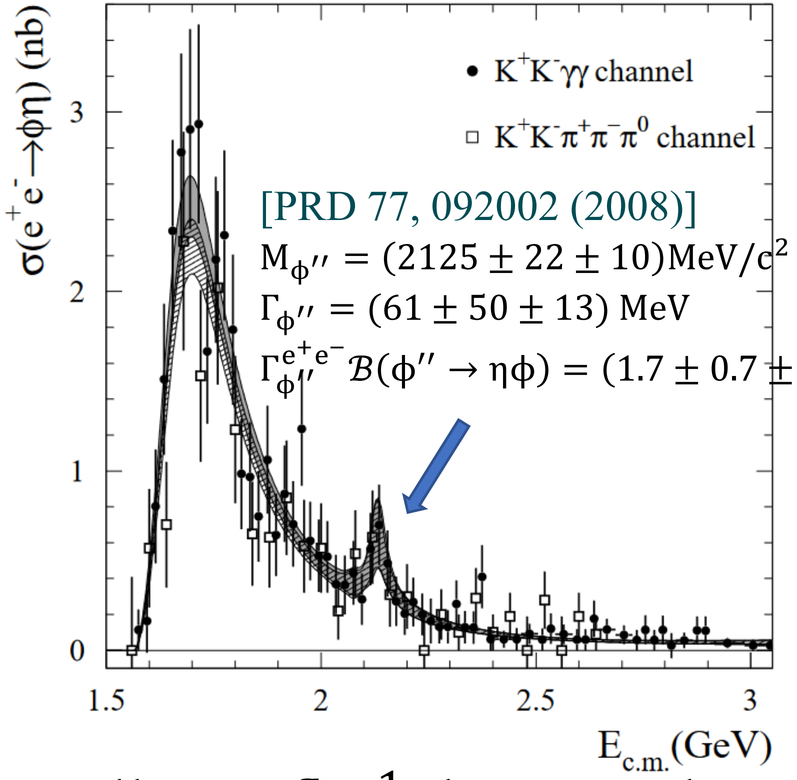
$M_{\phi(2170)} = (2163.5 \pm 6.2 \pm 3.0)$ MeV/c^2

$\Gamma_{\phi(2170)} = (31.1^{+21.1}_{11.6} \pm 1.1)$ MeV

$\Gamma^{e^+e^-}_{\phi(2170)} \mathcal{B}(\phi(2170) \to \phi\eta) = (0.24^{+0.12}_{-0.07})$ eV

or $(10.11^{+3.87}_{-3.13})$ eV





[PRD 77, 092002 (2008)]

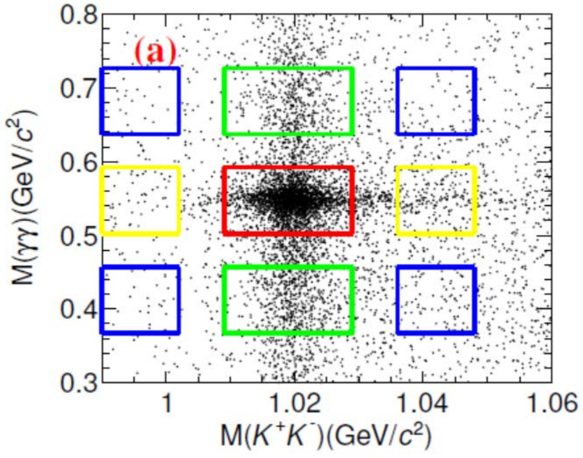
$M_{\phi''} = (2125 \pm 22 \pm 10)$MeV/$c^2$

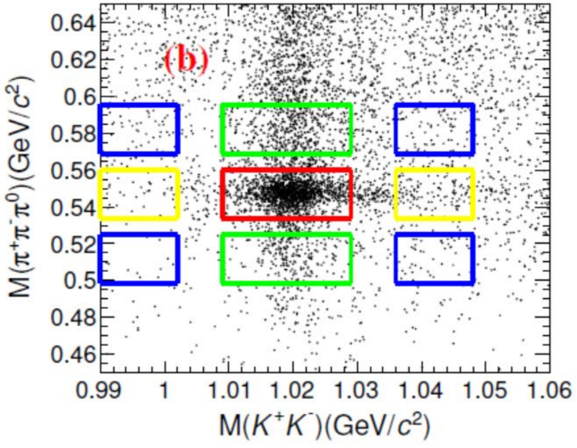
$\Gamma_{\phi''} = (61 \pm 50 \pm 13)$ MeV

$\Gamma^{e^+e^-}_{\phi''} \mathcal{B}(\phi'' \to \eta\phi) = (1.7 \pm 0.7 \pm 1.3)$eV

- In this analysis, we study the process $e^+e^- \to \eta\phi$ via ISR using all 980 fb^{-1} data sample.

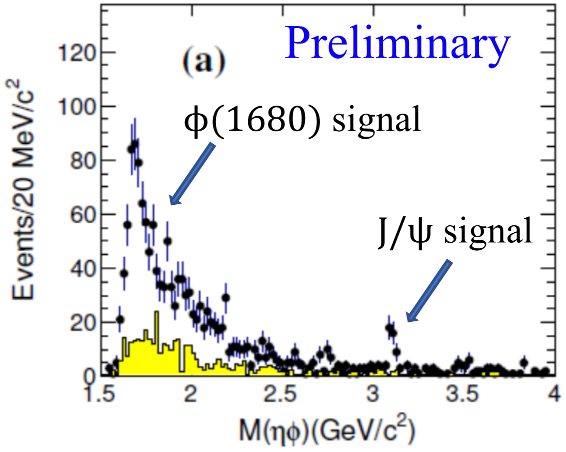
$M(\eta\phi)$ distributions from ISR production

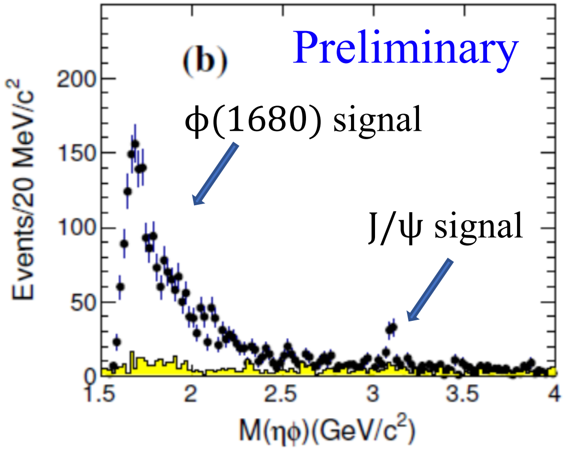




$\phi(\to K^+K^-)\eta(\to \gamma\gamma)$ mode

$\phi(\to K^+K^-)\eta(\to \pi^+\pi^-\pi^0)$ mode



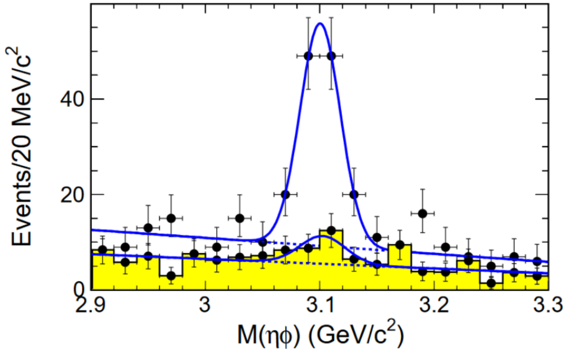


- The number of the $\eta\phi$ signal events we obtained is about seven times of that in the previous work. [PRD 77, 092002 (2008)]
- No significant $\phi(2170)$ signal is seen.
- There are clear J/ψ signals in both the $\pi^+\pi^-\pi^0$ mode and the $\gamma\gamma$ mode, and the branching fraction are determined to be

$$\mathcal{B}(J/\psi \to \eta\phi) = \frac{N_{sig}^{fit}}{\sigma_{ISR}^{prod} \times L \times \epsilon \times \mathcal{B}(\phi \to K^+K^-) \times \mathcal{B}(\eta \to \gamma\gamma/\pi^+\pi^-\pi^0)}$$

$$= (0.71 \pm 0.10 \pm 0.05\,) \times 10^{-3},$$

which agrees well with the world average value $(0.74 \pm 0.08) \times 10^{-3}$.



Fit to J/ψ signal

Extraction of Resonant Parameters

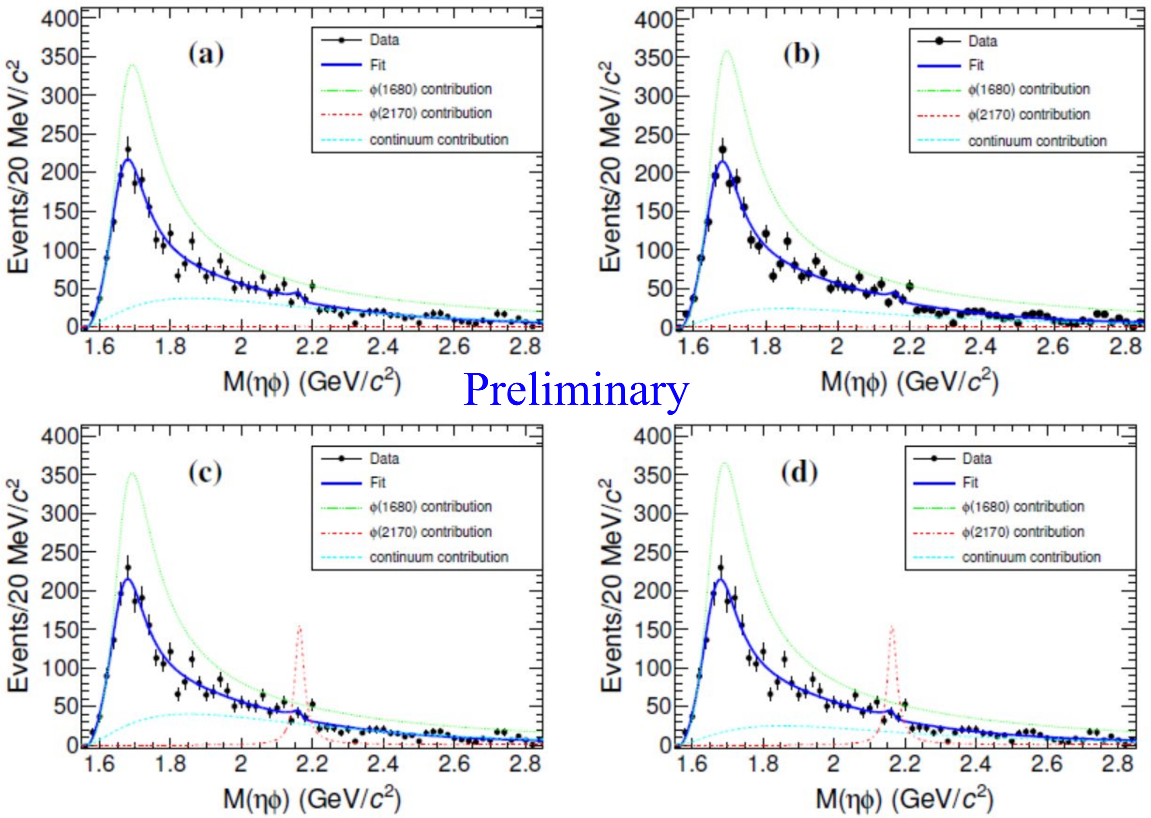
- The parametrization for the cross section of $e^+e^- \to \eta\phi$ at $\sqrt{s}$ takes the form [PRD 77, 092002 (2008)]

phase space of final state — $\phi(1680)$ amplitude

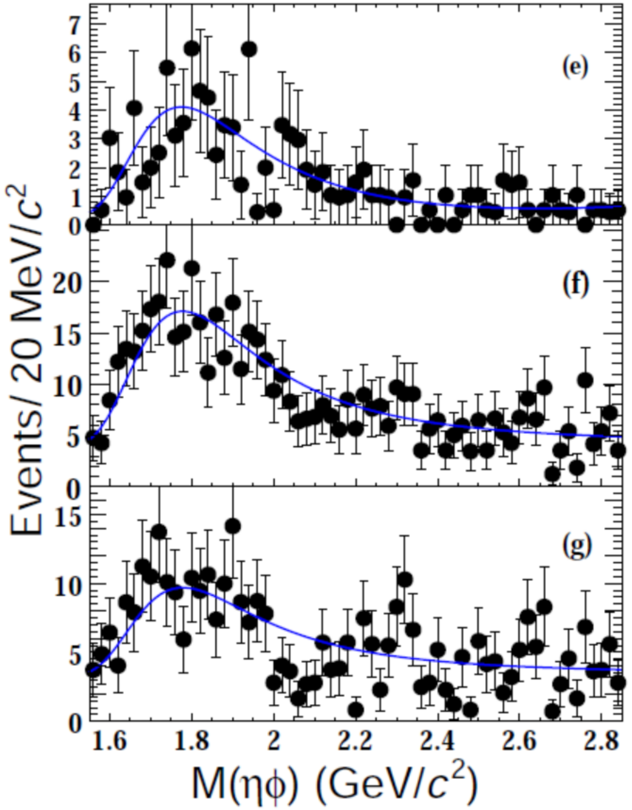
$$\sigma_{\eta\phi}(\sqrt{s}) = 12\pi P_{\eta\phi}(\sqrt{s})|A_{\eta\phi}^{n.r.}(\sqrt{s}) + A_{\eta\phi}^{\phi(1680)}(\sqrt{s}) + A_{\eta\phi}^{\phi(2170)}(\sqrt{s})|^2$$

Non-resonant contribution — $\phi(2170)$ amplitude

- We assume that the $\phi(2170)$ exists, and fix its mass and width based on the values measured by BESIII. There are four solutions of equivalent quality, having the same $M_{\phi(1680)}$ and $\Gamma_{\phi(1680)}$.
- The statistical significance of $\phi(2170)$ is only 1.7σ.



Preliminary



Extraction of Resonant Parameters

- Fit results with both ϕ(1680) and ϕ(2170) included, or without ϕ(2170). The mass and width of ϕ(2170) are fixed from the prior BESIII measurement.

Parameters	with $\phi(2170)$				without $\phi(2170)$	
	Solution I	Solution II	Solution III	Solution IV	Solution I	Solution II
χ^2/ndf	77/56				85/60	
a_0	-4.1 ± 0.5	5.0 ± 0.7	-5.0 ± 0.5	-4.8 ± 0.2	-3.2 ± 0.7	5.0 ± 0.1
a_1	2.7 ± 0.1	2.6 ± 0.1	2.7 ± 0.1	2.6 ± 0.1	2.9 ± 0.1	2.6 ± 0.1
$\mathcal{B}_{\eta\phi}^{\phi(1680)}\Gamma_{e^+e^-}^{\phi(1680)}$(eV)	122 ± 6	219 ± 15	163 ± 11	203 ± 12	75 ± 10	207 ± 16
$M_{\phi(1680)}$(MeV/c^2)	1683 ± 7				1696 ± 8	
$\Gamma_{\phi(1680)}$(MeV)	149 ± 12				175 ± 13	
$\mathcal{B}_{\eta\phi}^{\phi(1680)}$	0.18 ± 0.02	0.19 ± 0.04	0.21 ± 0.02	0.17 ± 0.04	0.25 ± 0.12	0.23 ± 0.10
$\mathcal{B}_{\eta\phi}^{\phi(2170)}\Gamma_{e^+e^-}^{\phi(2170)}$(eV)	0.09 ± 0.05	0.06 ± 0.02	16.7 ± 1.2	17.0 ± 1.2	—	
$M_{\phi(2170)}$(MeV/c^2)	2163.5(*fixed*)				—	
$\Gamma_{\phi(2170)}$(MeV)	31.1(*fixed*)				—	
$\theta_{\phi(1680)}$(°)	-89 ± 2	96 ± 6	-92 ± 1	-86 ± 7	-87 ± 15	108 ± 22
$\theta_{\phi(2170)}$(°)	37 ± 14	-102 ± 11	-167 ± 6	-155 ± 5	—	

- The upper limits at 90% C.L. on the $\Gamma^{e^+e^-}_{\phi(2170)}\mathcal{B}^{\eta\phi}_{\phi(2170)}$ are determined to be < 0.17eV or < 18.6eV, both are consistent the BESIII measurement. [PRD 104, 032007 (2021)]

- Since the ϕ(2170) is not significant, we also fit M(ϕη) distribution without ϕ(2170) component.

Cross section of $e^+e^- \to \eta\phi$

➢ The cross section of $e^+e^- \to \eta\phi$ for each $M_{\eta\phi}$ bin is calculated according to

$$\sigma_i = \frac{n_i^{obs} - n_i^{bkg}}{L_i \times \Sigma_j \epsilon_{ij} \mathcal{B}_j}$$

➢ The cross sections for $e^+e^- \to \eta\phi$ are around 2.6 nb and 0.4 nb at the $\phi(1680)$ and $\phi(2170)$ peaks, respectively.

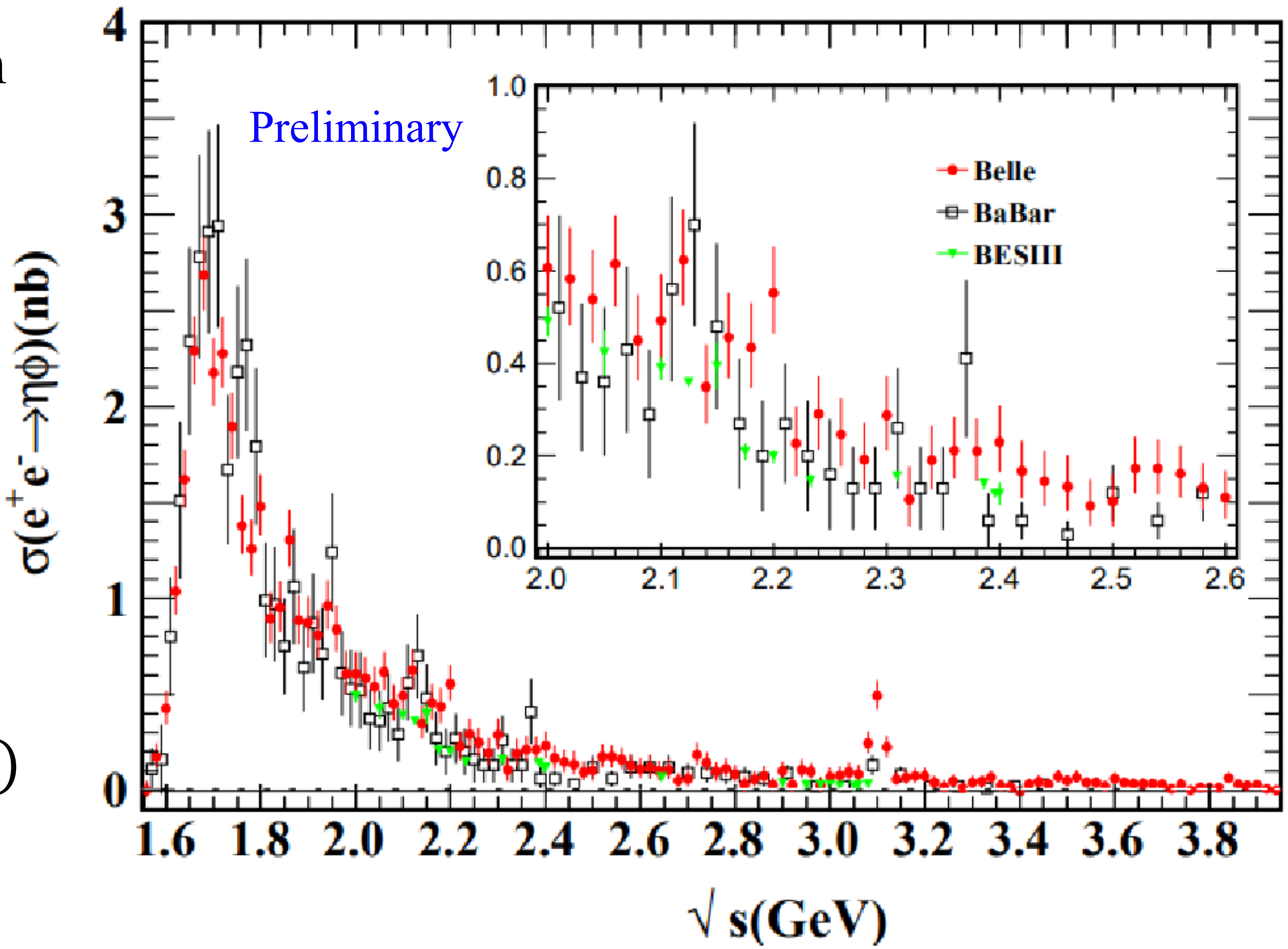


Figure: Cross section of $e^+e^- \to \eta\phi$ from threshold to 3.95 GeV.

Summary

- Although Belle has stopped data taking for ~12 years ago, we are still producing exciting results;

- Using $\Upsilon(1S), \Upsilon(2S)$ and $\Upsilon(3S)$ data samples, we discover a new resonant three-body decay $\Omega(2012)^- \to \Xi(1530)^0 K^- \to \Xi^- \pi^+ K^-$ with a significance of 5.2σ. The measured ratio of the branching fraction for the resonant three-body decay to that for the two-body decay is consistent with the molecular model of $\Omega(2012)$.

- We observe a narrow peaking structure in the pK^- invariant-mass spectrum with the $\Lambda_c^+ \to pK^-\pi^+$ decay near the $\Lambda\eta$ threshold. The best fit explains the structure as a cusp at the $\Lambda\eta$ threshold and the obtained parameters are consistent with the known properties of $\Lambda(1670)$.

- We measure the cross section of $e^+e^- \to \eta\phi$ via ISR from threshold to 3.95 GeV. The resonant parameters of $\phi(1680)$ are determined, and no $\phi(2170)$ signal is observed.

Thanks for your attentions!

Backup

k_3 and κ_3 are

PRD 81, 094028 (2010)

$$k_3(M) = \frac{g_l}{2\pi\mu_p}\int_0^{\sqrt{2\mu_p q(M)}} p^2 dp \times \frac{(q(M) - \frac{p^2}{2\mu_p})^{(2l+1)/2}}{(M_R - q(M) + \frac{p^2}{2\mu_p})^2 + \frac{g_l^2}{4}(q(M) - \frac{p^2}{2\mu_p})^{2l+1}},$$

$$\kappa_3(M) = \kappa(q(M)) + \kappa'(q(M)) - \kappa(q(m)) - \kappa'(q(m)),$$

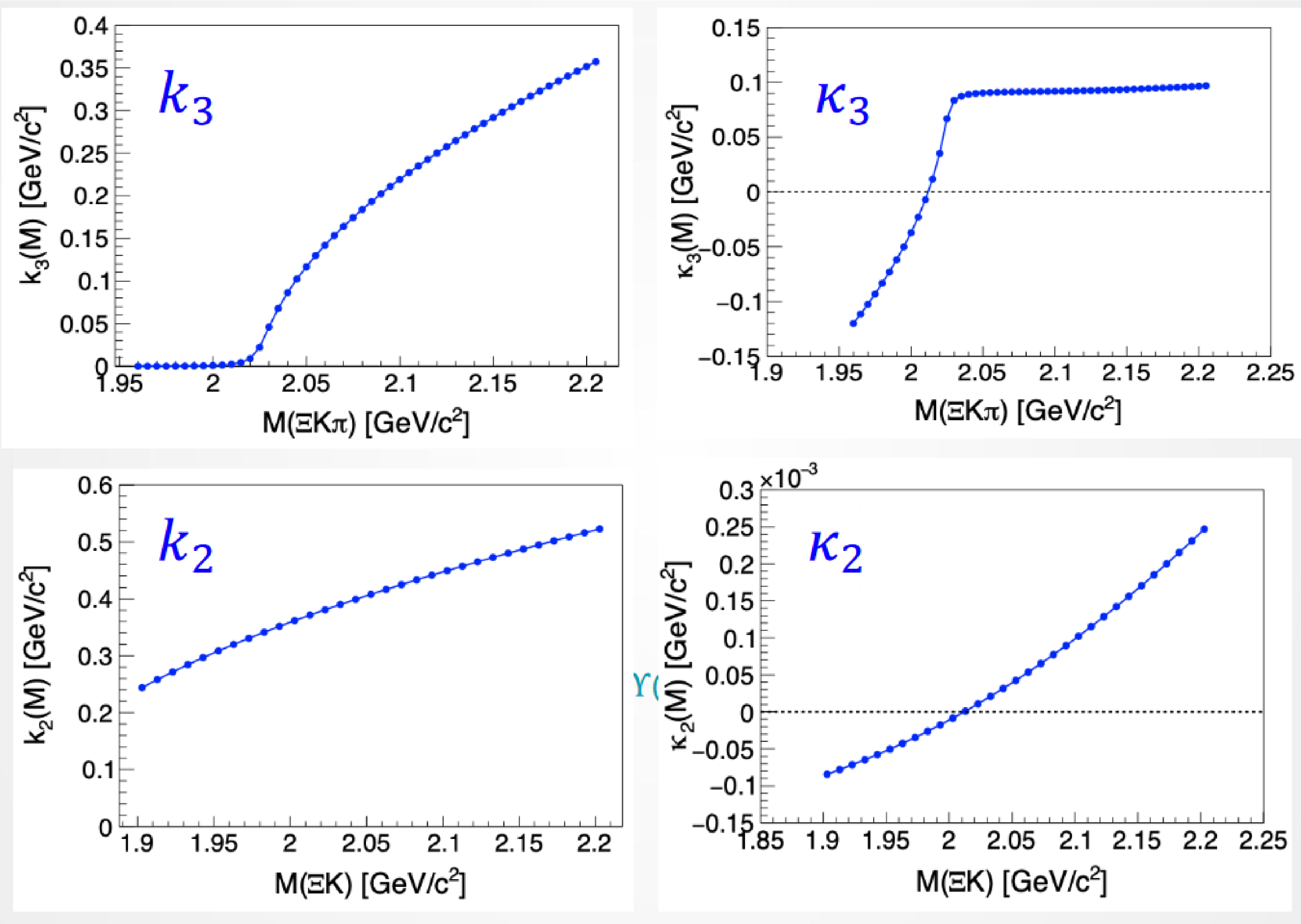
$$\kappa(M) = \frac{1}{\pi\mu_p}\int_0^{\infty} p^2 dp \times \frac{M_R - q(M) + \frac{p^2}{2\mu_p}}{(M_R - q(M) + \frac{p^2}{2\mu_p})^2 + \frac{g_l^2}{4}(q(M) - \frac{p^2}{2\mu_p})^{2l+1}},$$

$$\kappa'(M) = -\frac{g_l}{2\pi\mu_p}\int_{\sqrt{2\mu_p q(M)}}^{\infty} p^2 dp \times \frac{(\frac{p^2}{2\mu_p} - q(M))^{(2l+1)/2}}{(M_R - q(M) + \frac{p^2}{2\mu_p})^2 + \frac{g_l^2}{4}(q(M) - \frac{p^2}{2\mu_p})^{2l+1}}.$$

Here, $q(M) = M(\Xi\pi\overline{K}) - m_\Xi - m_\pi - m_K$, $q(m) = m_{\Omega(2012)^-} - m_\Xi - m_\pi - m_K$, $\mu_p = \frac{m_K(m_\pi + m_\Xi)}{m_\Xi + m_\pi + m_K}$ is the reduced mass of the $\Xi\overline{K}$ system, $M_R = m_{\Xi(1530)} - m_\Xi - m_\pi$ is the mass of the unstable constituent, the coupling g_l is $\Gamma_R/E_R^{l+1/2}$ (Γ_R is the width of Ξ(1530)), the orbital angular momentum of $\overline{K}$ in the Ξ(1530)$\overline{K}$ system is l = 1, and p is the $\overline{K}$ momentum in the Ξ(1530)$\overline{K}$ center-of-mass system.

The functions k_2 and κ_2 are identical to k_3 and κ_3 with Ξ(1530) replaced with Ξ, followed by $\Xi \to \Lambda\pi$.

Backup



k_3
k_3(M) [GeV/c^2]
M(ΞKπ) [GeV/c^2]
κ_3
κ_3(M) [GeV/c^2]
M(ΞKπ) [GeV/c^2]
k_2
k_2(M) [GeV/c^2]
M(ΞK) [GeV/c^2]
κ_2
$\times 10^{-3}$
κ_2(M) [GeV/c^2]
M(ΞK) [GeV/c^2]

Fitting the M($\eta\phi$)

- The parametrization for the cross section of $e^+e^- \to \eta\phi$

$$\sigma_{\eta\phi}(\sqrt{s}) = 12\pi\mathcal{P}_{\eta\phi}(\sqrt{s})\left|A_{\eta\phi}{}^{n.r.}(\sqrt{s}) + A_{\eta\phi}{}^{\phi(1680)}(\sqrt{s}) + A_{\eta\phi}{}^{\phi(2170)}(\sqrt{s})\right|^2$$

$A_{\eta\phi}{}^{n.r.}(\sqrt{s}) = a_0/s^{a_1}$ is used to describe the non-resonant contribution

$$A_{\eta\phi}^{\phi(1680)}(\sqrt{s}) = \sqrt{\mathcal{B}^{\eta\phi}_{\phi(1680)}\Gamma^{e^+e^-}_{\phi(1680)}}\frac{\sqrt{\Gamma_{\phi(1680)}/\mathcal{P}_{\eta\phi}(M^2_{\phi(1680)})}e^{i\theta_{\phi(1680)}}}{M^2_{\phi(1680)} - s - i\sqrt{s}\Gamma_{\phi(1680)}(\sqrt{s})}$$

$$\Gamma_{\phi(1680)}(\sqrt{s}) = \Gamma_{\phi(1680)}[\frac{\mathcal{P}_{KK^*(892)}(\sqrt{s})}{\mathcal{P}_{KK^*(892)}(M_{\phi(1680)})}\mathcal{B}^{KK^*(892)}_{\phi(1680)} + \frac{\mathcal{P}_{\eta\phi}(\sqrt{s})}{\mathcal{P}_{\eta\phi}(M_{\phi(1680)})}\mathcal{B}^{\eta\phi}_{\phi(1680)} + (1 - \mathcal{B}^{\eta\phi}_{\phi(1680)} - \mathcal{B}^{KK^*(892)}_{\phi(1680)})].$$

$\mathcal{B}_{\phi(1680)}{}^{KK^*(892)} \approx 2 \times \mathcal{B}_{\phi(1680)}{}^{\eta\phi}$ from Ref. [1] directly

$$A_{\eta\phi}^{\phi(2170)}(s) = \sqrt{\mathcal{B}^{\eta\phi}_{\phi(2170)}\Gamma^{e^+e^-}_{\phi(2170)}}\frac{\sqrt{\Gamma_{\phi(2170)}/\mathcal{P}_{\eta\phi}(M^2_{\phi(2170)})}e^{i\theta_{\phi(2170)}}}{M^2_{\phi(2170)} - s - i\sqrt{s}\Gamma_{\phi(2170)}} \cdot \frac{B(p)}{B(p')}$$