Quantum kinetic theory

and its applications to chiral transports and spin polarizations

浦　实

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Recent review:

- Y. Hidaka, SP, Q.Wang, D.L. Yang, ***Foundations and Applications of Quantum Kinetic Theory***, Progress in Particle and Nuclear Physics, 127, 103989 (2022).

Outline

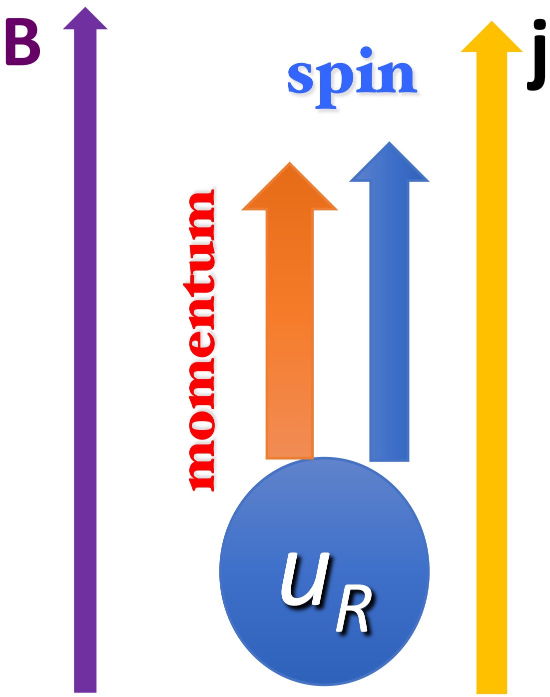
- **Part 1:**
  Chiral magnetic effect, Berry phase and kinetic theory
- **Part 2:**
  Wigner functions and the master equations
- **Part 3:**
  Quantum kinetic theory in massless limit and collisions
- **Part 4:**
  Applications to heavy ion physics

Part 2

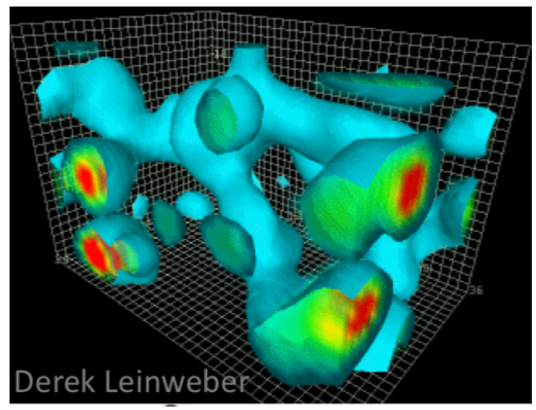
Wigner functions and the master equations

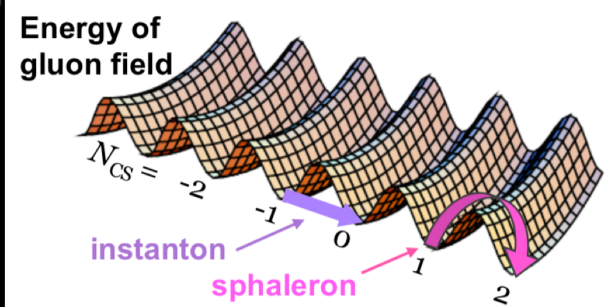
1. Definition of gauge invariant covariant Wigner function
   (1a) Gauge invariant covariant Wigner function
   (1b) Vector, chiral currents
   (1c) The choice of gauge link
2. Master equation for covariant Wigner function
   (2a) Closed-Time-Path formalism
   (2b) From Dirac equation to master equations for chiral fermions
   (2c) Master equations in general case (massive case)
3. Equal-time formulism for Wigner function

Chiral Magnetic Effect



- **Magnetic fields**
- **Nonzero axial chemical potential**
- **Number of Left handed fermions ≠ Number of Right handed fermions**





- **Charge current:** **charge separation**

$$\mathbf{j} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B},$$

Kharzeev, Fukushima, Warrigna, (08,09), etc. ...

Chiral vorticial effect Vs CME

$$j^\mu = \xi_B B^\mu + \xi\omega^\mu,$$
$$j_5^\mu = \xi_{5B} B^\mu + \xi_5\omega^\mu,$$

$$\xi = \frac{1}{\pi^2}\mu\mu_5,\ \xi_B = \frac{e}{2\pi^2}\mu_5,$$
$$\xi_5 = \frac{T^2}{6} + \frac{1}{2\pi^2}(\mu^2+\mu_5^2),\ \xi_{B5} = \frac{e}{2\pi^2}\mu.$$

Son, Surowka, PRL 2009;...

QFT: Vilenkin, PRD 22 (1980) 3080

- If we replace the Lorentz force by the Coriolis force,

  Stephanov, Yin, PRL 2012

$$\mathbf{B} \to 2m\omega \sim 2|\mathbf{p}|\omega \to 2\mu\omega$$

  CVE can be " derived" from CME expect $T^2/6$.

- Why have we gotten $T^2/6$?
  - Holographic models: it is the gravitational anomaly
  - Curved space QED: No...
- Will the $T^2/6$ Non-renormalizable?

  No. D.F. Hou, H. Liu, H.C. Ren, PRD 2012; S. Golkar, D.T. Son, JHEP 2015

Chiral kinetic equation

$$\sqrt{G}\partial_t f + \sqrt{G}\dot{\mathbf{x}} \cdot \nabla_x f + \sqrt{G}\dot{\mathbf{p}} \cdot \nabla_p f = C[f].$$

- **Particle's effective velocity:**

$$\sqrt{G}\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}} + \hbar \left(\frac{\partial \varepsilon}{\partial \mathbf{p}} \cdot \mathbf{\Omega} \right) \mathbf{B} + \hbar \mathbf{E} \times \mathbf{\Omega},$$

- **Effective force:**

$$\sqrt{G}\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar (\mathbf{E} \cdot \mathbf{B}) \mathbf{\Omega},$$

- **Berry curvature**

$$\sqrt{G} = 1 + \hbar \mathbf{B} \cdot \mathbf{\Omega}, \qquad \mathbf{\Omega} = \frac{\mathbf{p}}{2|\mathbf{p}|^3},$$

Classical kinetic theory for Weyl fermions

- In a non-interacting system of fermions without collisions

$$\frac{df(t,x,p)}{dt} = 0$$

$$\left[\sqrt{G}\partial_t + \sqrt{G}\dot{\mathbf{x}}\cdot\nabla_{\mathbf{x}} + \sqrt{G}\dot{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\right] f(t,\mathbf{x},\mathbf{p}) = 0.$$

$$\left\{ \boldsymbol{\nabla}_{\mathbf{p}}\cdot\boldsymbol{\Omega}_{\mathbf{p}} = 2\pi\delta^3(\mathbf{p}), \right\}$$

- Integral over momentum, we get

Chiral anomaly

$$\partial_t\rho + \boldsymbol{\nabla}\cdot\mathbf{J} = \frac{1}{2\pi}(\mathbf{E}\cdot\mathbf{B})f(\mathbf{p}=0) = \frac{1}{4\pi^2}(\mathbf{E}\cdot\mathbf{B}),$$

$$\rho = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{G}f(t,\mathbf{x},\mathbf{p}),$$

Number density for right handed fermions

$$\mathbf{J} = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\sqrt{G}\dot{\mathbf{x}}f(t,\mathbf{x},\mathbf{p}),$$

Right handed fermions current

Non-trivial Lorentz symmetry

- Quantum field theory

$$j^\mu = \overline{\psi}\sigma^\mu\psi \to \Lambda^\mu_\nu j^\nu$$

- Lorentz transformation

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu, \; p^{\mu\prime} = \Lambda^\mu_\nu p^\nu,$$

$$\left\{ f'(x', p', t') = f(x', p', t') + \hbar N^\mu(\partial^x_\mu + F_{\nu\mu}\partial^\nu_p) f, \right\}$$

Infinitesimal Lorentz Transform

$$\left\{ \begin{array}{rcl} \delta\mathbf{x} & = & \hbar\dfrac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}, \\ \delta\mathbf{p} & = & \hbar\dfrac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}\times\mathbf{B} \end{array} \right\}$$

Chen, Son, Stephanov, PRL, (2015); Y. Hidaka, SP, D.L. Yang, PRD (2016)

1. Definition of gauge invariant covariant Wigner function

(1a) Gauge invariant covariant Wigner function

How to define “position” and “momentum”?

- In a ”classical” many body system, we use distribution function f(x,p).

- In Quantum mechanics, we know,

$$[x, p_x] = i\hbar.$$

- If we still want to use the distribution function f(t,x,p) for a quantum many body system, how can we measure the x and p at the same time t?

(For simplicity, we use the natural unit $\hbar = c = k_B = 1$.)

"position" and "momentum"

- Assuming we have two points, x_1^μ and x_2^μ with $\mu = 0, 1, 2, 3$. (For simplicity, we neglect the upper index μ here)
- We introduce another two variables,

$$x = \frac{x_1 + x_2}{2}, \text{ and } y = x_1 - x_2$$

$$\partial_x = \partial_{x_1} + \partial_{x_2}$$

$$\partial_y = \frac{1}{2}(\partial_{x_1} - \partial_{x_2}) \quad \rightarrow \quad p = -i\partial_y$$

$$\rightarrow \quad [x, p] = [\frac{x_1 + x_2}{2}, -i\frac{1}{2}(\partial_{x_1} - \partial_{x_2})] = 0$$

Wigner-Weyl transformation

$$O(x_1, x_2) = \psi^*(x_1)\psi(x_2) \rightarrow O(x, p) = \int d^3y e^{-ip\cdot y}\psi^*\left(x + \frac{y}{2}\right)\psi\left(x - \frac{y}{2}\right)$$

Definition of Wigner function in Quantum Mechanics

$$W(\mathbf{x},\mathbf{p}) = \int d^3\mathbf{y} \exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{y}\right) \left\langle \mathbf{x}-\frac{\mathbf{y}}{2}\right| \widehat{\rho} \left|\mathbf{x}+\frac{\mathbf{y}}{2}\right\rangle ,$$

where ρ is the density matrix in QM.

$$\int d^3\mathbf{p} W(\mathbf{x},\mathbf{p}) = |\psi(\mathbf{x})|^2$$

$$\int d^3\mathbf{x} W(\mathbf{x},\mathbf{p}) = |\psi(\mathbf{p})|^2$$

$$\int d^3\mathbf{x} \int d^3\mathbf{p} W(\mathbf{x},\mathbf{p}) = \mathrm{Tr}\ \hat{\rho} = 1$$

Wigner function plays a role as the "probability" in phase space (x,p).

Note that, Wigner function itself could be negative sometimes.

Covariant Wigner function in quantum field theory

- (Covariant) Wigner operator

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}_\beta(x_1) U(x_1, x_2) \psi_\alpha(x_2)$$

with gauge link

$$U(x_1, x_2) = \exp\left[-iQ \int_{x_2}^{x_1} dz \cdot A(z)\right]$$

- Wigner function:

$$W(x,p) = \left\langle : \hat{W}(x,p) : \right\rangle$$

W operator in thermal ensemble average and normal ordering of the operators.

Vasak, Gyulassy, Elze, Ann. Phys. (N.Y.) 173, 462 (1987);

Elze, Heinz,Phys.Rep.183,81(1989).

1. Definition of gauge invariant covariant Wigner function

(1b) Vector, chiral currents

Decomposition of Wigner function

- **For massive spinors, Wigner function is a 4x4 matrix.**

$$W = \mathcal{F} + i\mathcal{P}\gamma^5 + \mathcal{V}^\mu\gamma_\mu + \mathcal{A}^\mu\gamma^5\gamma_\mu + \frac{1}{2}\mathcal{S}^{\mu\nu}\sigma_{\mu\nu},$$

$$\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$$

$$\mathcal{F} = \mathrm{Tr}\ W(x,p) \sim \langle\overline{\psi}\psi\rangle$$

$$\mathcal{P} = \mathrm{Tr}\ \gamma^5 W(x,p) \sim \langle\overline{\psi}\gamma^5\psi\rangle$$

$$\mathcal{V}^\mu = \mathrm{Tr}\ \gamma^\mu W(x,p) \sim \langle\overline{\psi}\gamma^\mu\psi\rangle$$

Vector current

$$\mathcal{A}^\mu = \mathrm{Tr}\ \gamma^\mu\gamma^5 W(x,p) \sim \langle\overline{\psi}\gamma^5\gamma^\mu\psi\rangle$$

Chiral current (Axial vector current)

$$\mathcal{S}^{\mu\nu} = \mathrm{Tr}\ \sigma^{\mu\nu} W(x,p) \sim \langle\overline{\psi}\sigma^{\mu\nu}\psi\rangle$$

$$W^\dagger = \gamma^0 W \gamma^0$$

All the coefficients, F, P, V, A, S are real!

$$\mathcal{V}^{\mu\dagger} = \mathrm{Tr}\ W^\dagger(\gamma^\mu)^\dagger = \mathrm{Tr}\ [(\gamma^0 W\gamma^0)(\gamma^0\gamma^\mu\gamma^0)] = \mathcal{V}^\mu$$

Vector and chiral currents

$$\mathcal{V}^{\mu}=\mathrm{Tr}\,\gamma^{\mu}W(x,p)=\int\frac{d^{4}y}{(2\pi)^{4}}e^{-ip\cdot y}\overline{\psi}\left(x+\frac{y}{2}\right)\gamma^{\mu}\exp\left[-i\int_{x-y/2}^{x+y/2}dz\cdot A(z)\right]\psi\left(x-\frac{y}{2}\right)$$

$$\mathcal{A}^{\mu}=\mathrm{Tr}\,\gamma^{\mu}\gamma^{5}W(x,p)=\int\frac{d^{4}y}{(2\pi)^{4}}e^{-ip\cdot y}\overline{\psi}\left(x+\frac{y}{2}\right)\gamma^{\mu}\gamma^{5}U\exp\left[-i\int_{x-y/2}^{x+y/2}dz\cdot A(z)\right]\psi\left(x-\frac{y}{2}\right)$$

- **Our definition means the vector and chiral currents are regularized. If we integrate over momentum,**

$$j_{5}^{\mu}=\int d^{4}p\mathcal{A}^{\mu}(x,p)\sim\lim_{y\to0}\overline{\psi}\left(x+\frac{y}{2}\right)\gamma^{\mu}\gamma^{5}U\exp\left[-i\int_{x-y/2}^{x+y/2}dz\cdot A(z)\right]\psi\left(x-\frac{y}{2}\right)$$

which is the chiral current in position space in Peskin's textbook QFT Sec. 19,

$$j^{\mu5}=\mathrm{symm}\lim_{\epsilon\to0}\left\{\overline{\psi}(x+\tfrac{\epsilon}{2})\gamma^{\mu}\gamma^{5}\exp\Big[-ie\int_{x-\epsilon/2}^{x+\epsilon/2}dz\cdot A(z)\Big]\psi(x-\tfrac{\epsilon}{2})\right\}.\quad(19.22)$$

That regularization and gauge link are the key to get chiral anomaly!

An example: 1+1 dim chiral anomaly in operator level

$$\partial_\mu j^{\mu 5} = \text{symm} \lim_{\epsilon \to 0} \left\{ \bar{\psi}(x+\tfrac{\epsilon}{2}) \left[-ie\gamma^\mu \epsilon^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \right] \gamma^5 \psi(x-\tfrac{\epsilon}{2}) \right\}.$$

Coming from gauge link

- In 1+1 dim, we can have

$$\bar{\psi}(x+\tfrac{\epsilon}{2})\Gamma\psi(x-\tfrac{\epsilon}{2}) = \frac{-i}{2\pi}\text{tr}\left[\frac{\gamma^\alpha \epsilon_\alpha}{\epsilon^2}\Gamma\right]$$

$$\psi(y)\bar{\psi}(z) = \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot(y-z)} \frac{i\not{k}}{k^2} = -\not{\partial}\left(\frac{i}{4\pi}\log(y-z)^2\right) = \frac{-i}{2\pi}\frac{\gamma^\alpha (y-z)_\alpha}{(y-z)^2}.$$

Regularization also plays a crucial role.

$$\partial_\mu j^{\mu 5} = \text{symm} \lim_{\epsilon \to 0} \left\{ \frac{-i}{2\pi} \text{tr}\left[\frac{\gamma^\alpha \epsilon_\alpha}{\epsilon^2}\gamma^\mu \gamma^5\right] \cdot (-ie\epsilon^\nu F_{\mu\nu}) \right\}.$$

$$= \frac{e}{2\pi} \text{symm} \lim_{\epsilon \to 0} \left\{ 2\frac{\epsilon^\mu \epsilon^\nu}{\epsilon^2} \right\} \epsilon^{\mu\alpha} F_{\nu\alpha}$$

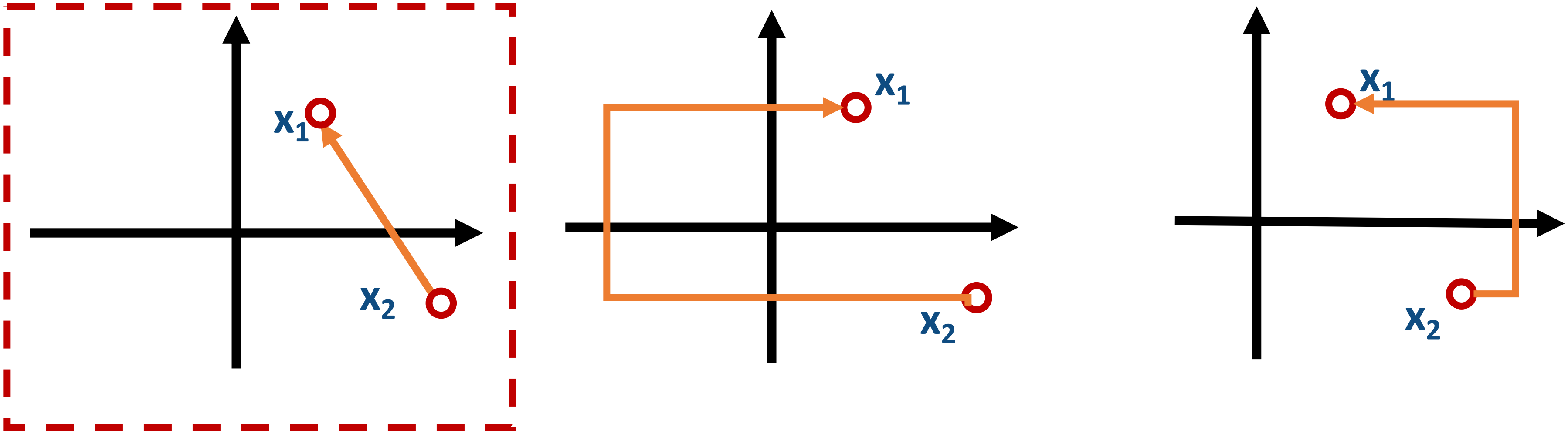
$$= \frac{e}{2\pi} \epsilon^{\mu\nu} F_{\mu\nu}$$

chiral anomaly in 1+1 dim

1. Definition of gauge invariant covariant Wigner function

(1c) The choice of gauge link

The choice of gauge link



A compact form for Wigner function (I)

- For a function A(a+x) with a small x, we can expand it as

$$A(a+x) = A(a) + A'(a)x + \frac{1}{2}A''(a)x^2 + \ldots \frac{1}{n!}\frac{d^{(n)}A}{dx^n}x^n = e^{x\cdot\partial}A(a)$$

- Similarly, we have

$$\bar{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right) \to \bar{\psi}_\beta(x)e^{+\frac{1}{2}y\cdot\overleftarrow{\partial}} \times e^{-\frac{1}{2}y\cdot\partial}\psi_\alpha(x) = \bar{\psi}_\beta(x)\exp\left[-\frac{1}{2}y\cdot(\partial-\overleftarrow{\partial})\right]\psi_\alpha(x)$$

- Without gauge link, the Wigner function can be formally written as

$$\hat{W}_{\beta\alpha} = \int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\bar{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right) = \int\frac{d^4y}{(2\pi)^4}\bar{\psi}_\beta(x)\exp\left[iy\cdot(p-\hat{p})\right]\psi_\alpha(x)$$

$$\sim \bar{\psi}_\beta(x)\delta^{(4)}(p^\mu-\hat{p}_c^\mu)\psi_\alpha(x)$$

$$\hat{p}_c^\mu \equiv \frac{1}{2}i(\partial^\mu - \overleftarrow{\partial}^\mu)$$ Momentum

Elze, Gyulassy, Vasak, Nucl.Phys.B 276 (1980), 706

A compact form for Wigner function (II)

- **With the straight line type gauge link, we find**

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}_\beta\left(x+\frac{y}{2}\right) \exp\left[-i\int_{x_2}^{x_1} dz\cdot A(z)\right] \psi_\alpha\left(x-\frac{y}{2}\right)$$

$$\to \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}_\beta(x) \exp\left[-\frac{1}{2} y\cdot(D-\overleftarrow{D})\right] \psi_\alpha(x)$$

$$\sim \overline{\psi}_\beta(x)\delta^{(4)}(p^\mu - \hat{p}^\mu)\psi_\alpha(x) \qquad D_\mu = \partial_\mu + iA_\mu$$

$$\hat{p}^\mu = \frac{i}{2}(D^\mu - \overleftarrow{D}) = \hat{p}_c^\mu - A^\mu$$

Kinetic momentum

canonical momentum

The straight line type gauge link corresponds the momentum in W(x,p) is the kinetic one.

Elze, Gyulassy, Vasak, Nucl.Phys.B 276 (1980), 706

2. Master equation for covariant Wigner function

(2a) Closed-Time-Path formalism

References for Closed-Time-Path formalism

Review:

K.-c. Chou, Z.-b. Su, B.-l. Hao, L. Yu, Phys. Rept. 118 (1985)

周光召、苏肇冰、郝柏林、于渌， 统一描述平衡与非平衡体系的格林函数理论研究

- 1999，中国科学院自然科学奖一等奖
- 2000，国家自然科学奖二等奖

PHYSICS REPORTS (Review Section of Physics Letters) 118, nos. 1 & 2 (1985) 1–131. North-Holland, Amsterdam

EQUILIBRIUM AND NONEQUILIBRIUM FORMALISMS MADE UNIFIED

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Contents:

Schrodinger, Heisenberg's and interaction picture

- **Schrodinger picture:**

  States are time dependent, operators do not depend on t.

- **Heisenberg's picture:**

  States do not depend on t, operators are time dependent.

$$\hat{O}(t) = e^{iH(t-t_0)}\hat{O}(t_0)e^{-iH(t-t_0)}$$

- **Interaction picture:** $H = H_0 + H_{\text{int}}$

$$\hat{O}_I(t) = e^{iH_0(t-t_0)}\hat{O}(t_0)e^{-iH_0(t-t_0)}$$

$$\begin{aligned}\hat{O}(t) &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\hat{O}_I(t)e^{iH_0(t-t_0)}e^{-iH(t-t_0)}\\ &\equiv U^\dagger(t,t_0)\hat{O}_I(t)U(t,t_0)\end{aligned}$$

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$

Interaction picture for perturbative QFT

- **Let us consider the scalar field. Note that, scalar field is operator.**

$$\phi(t,\mathbf{x}) = U^\dagger(t,t_0)\phi_I(t,\mathbf{x})U(t,t_0)$$

- **By solving the eq.**

$$i\partial_t U(t,t_0) = H_I(t)U(t,t_0)$$

One can get

$$U(t,t_0) = T\left\{\exp\left[-i\int_{t_0}^{t} dt' H_I[\phi_I(t')]\right]\right\}$$

T: time ordering

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3);$$

$$U(t_1,t_3)\left[U(t_2,t_3)\right]^\dagger = U(t_1,t_2).$$

Green function in zero temperature

- The two point function in zero temperature is defined as

$$G(x_1, x_2) = \langle \Omega | T\phi(x_1)\phi(x_2) | \Omega \rangle$$

T is time ordering and |Ω> is the ground state of H.

- Usually, we express the scalar fields and ground state by them in free cases.

$$\phi(t, \mathbf{x}) = U^\dagger(t, t_0)\phi_I(t, \mathbf{x})U(t, t_0)$$

$$|\Omega\rangle \propto \lim_{\mathcal{T}\to\infty(1-i\epsilon)} e^{-iH\mathcal{T}}|0\rangle \propto \lim_{\mathcal{T}\to\infty(1-i\epsilon)} U(t_0, -\mathcal{T})|0\rangle$$

$$G(x_1, x_2) = \lim_{\mathcal{T}\to\infty(1-i\epsilon)} \langle 0| \boxed{U^\dagger(-\mathcal{T}, \mathcal{T})} \left[T\phi_I(x_1)\phi_I(x_2)U(-\mathcal{T}, \mathcal{T})\right]|0\rangle$$

In zero temperature case, $U^+(t, t_0) = U^{-1}(t, t_0)$ is the quantum fluctuations. Note that $U\,|0>$ only gives a trivial phase factor. As a consequence, we do not need to consider the disconnected Feynman diagrams.

Green correlation function at finite temperature (I)

- In non-equilibrium quantum field theory, the two-point correlation function is defined through the density operator at an equilibrium state (at time t),

$$G(x_1, x_2) = \langle T\phi(x_1)\phi(x_2)\rangle = \frac{1}{\mathrm{Tr}\hat{\rho}(t)}\mathrm{Tr}[\hat{\rho}(t)T\phi(x_1)\phi(x_2)]$$

ϕ (x): field operator in Heisenberg's picture
ρ(t): density operator at the initial time t
T : (standard) time ordering

$$\hat{\rho}(t) = \exp[\beta(\hat{H} - \mu\hat{N})]\Big|_{\beta=-it}$$

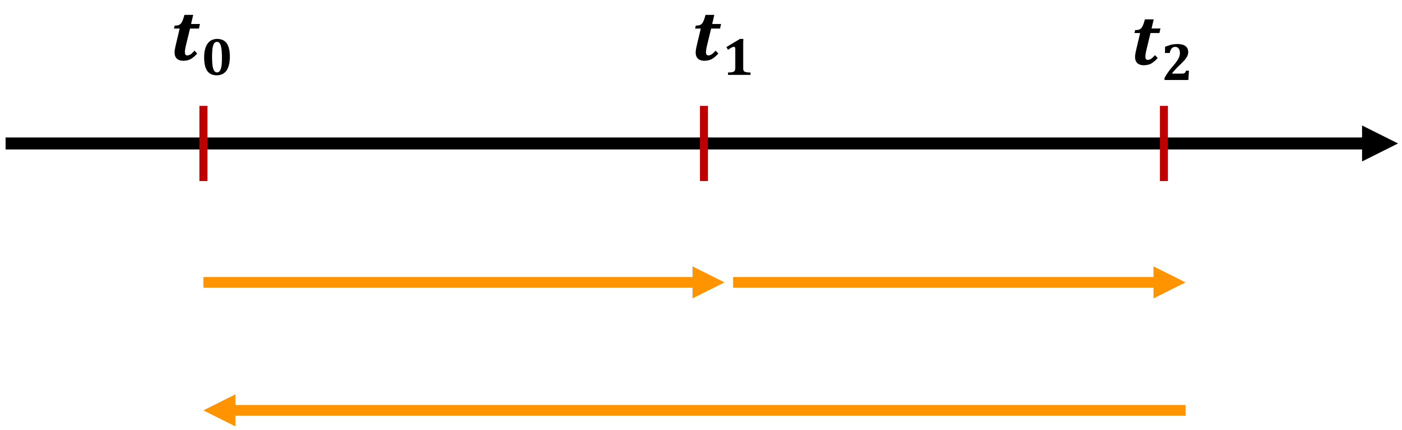
- We can use the similar interaction picture for G(x1,x2). But, note that, we also need to express the density matrix ρ(t) in interaction picture.

J. Schwinger, J.Math. Phys.2(1961) 407
Keldysh, J. Exptl. Theoret. Phys. 47, (1964) 1515-1527

Green correlation function at finite temperature (II)

- We assume that the interaction H_I start from t_0.

$$G(x_1,x_2)=\frac{1}{\mathrm{Tr}\hat{\rho}(t_0)}\mathrm{Tr}\left\{\hat{\rho}(t_0)U(t_0,t)T[U(t,t_1)\phi_I(x_1)U(t_1,t_2)\phi_I(x_2)U(t_2,t)]U(t,t_0)\right\}$$

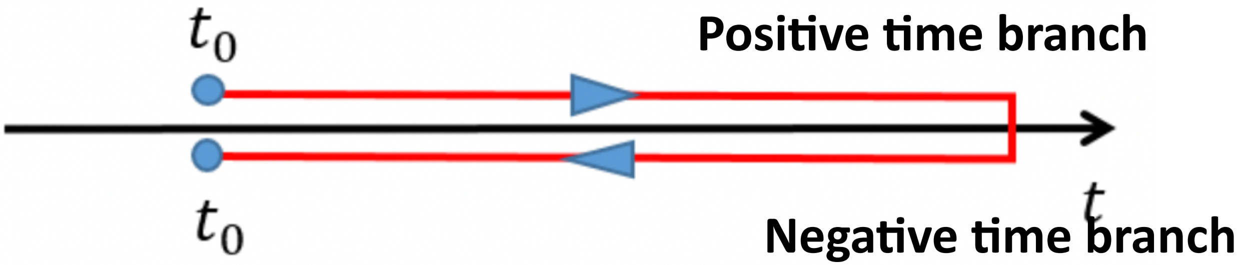


- Note that the $U(t_0,t)$ and $U(t,t_0)$ are not no longer trivial phase factors. It is difficult to handle them.

Schwinger-Keldysh contour

$$G(x_1, x_2) = \frac{1}{\mathrm{Tr}\hat{\rho}(t_0)} \mathrm{Tr}\left\{\hat{\rho}(t_0) T[\phi_I(x_1)\phi_I(x_2)U_{CTP}(t_0)]\right\}$$

- **Schwinger-Keldysh contour**



$$U_{CTP}(t_0) \equiv T_P \left[\exp\left(-i\int_{CTP} dt H_I(t)\right)\right]$$
$$= T_P \left[\exp\left(-i\int_{t_0}^{\infty} dt_+ H_I(t_+) + i\int_{t_0}^{\infty} dt_- H_I(t_-)\right)\right]$$

J. Schwinger, J.Math. Phys.2(1961) 407
Keldysh, J. Exptl. Theoret. Phys. 47, (1964) 1515-1527
Calzetta, Hu, PRD, 37 (1988), 10
Jordan, PRD, 33 (1986), 15

Two point Green functions

$$G(x_1,x_2) = \begin{pmatrix} G^{++}(x_1,x_2) & G^{+-}(x_1,x_2) \\ G^{-+}(x_1,x_2) & G^{--}(x_1,x_2) \end{pmatrix} = \begin{pmatrix} G^F(x_1,x_2) & \pm G^<(x_1,x_2) \\ G^>(x_1,x_2) & G^{\overline{F}}(x_1,x_2) \end{pmatrix}$$

$G^{ij}(x_1,x_2)$:

i=+,- : first time argument x_1^0 lives on the time branch i

j=+,- : second time argument x_2^0 lives on the time branch j

+: positive time branch;

-: negative time branch

- For spin-1/2 fermions, the definitions of the various two-point correlation functions are

$$\begin{aligned} G^F_{\alpha\beta}(x_1,x_2) &= \langle T\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle\,, \\ G^{\overline{F}}_{\alpha\beta}(x_1,x_2) &= \langle T_A\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle\,, \\ G^<_{\alpha\beta}(x_1,x_2) &= \langle \overline{\psi}_\beta(x_2)\psi_\alpha(x_1)\rangle\,, \\ G^>_{\alpha\beta}(x_1,x_2) &= \langle \psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle\,, \end{aligned}$$

$$G^F + G^{\overline{F}} = G^> - G^<,$$

$$\begin{aligned} -iG^R &= G^F + G^< = -G^{\overline{F}} + G^>, \\ -iG^A &= G^F - G^> = -G^{\overline{F}} - G^<, \end{aligned}$$

R: Retarded

A: Advanced

2. Master equation for covariant Wigner function

(2b) From Dirac equation to master equations for chiral fermions

Dirac equation for chiral fermions

- **Let us start from the Lagrangian for massless fermions in a background electromagnetic field,**

$$\mathcal{L} = \overline{\psi} i\gamma \cdot D\psi = \chi_R^\dagger i\sigma \cdot D\chi_R + \chi_L^\dagger i\overline{\sigma} \cdot D\chi_L,$$

$$\psi = (\chi_L, \chi_R)^T$$

L, R: left or right handed Pauli spinors

$$\sigma^\mu = (1, \boldsymbol{\sigma}), \ \overline{\sigma}^\mu = (1, -\boldsymbol{\sigma}), \ \text{and} \ D_\mu = \hbar\partial_\mu + iA_\mu$$

- **We concentrate on right handed fermions and suppress the subscript R**

$$i\sigma \cdot D\chi_R = 0, \qquad -\chi_R^\dagger i\sigma \cdot \overleftarrow{D}^\dagger = 0,$$

Blaizot, Iancu, Phys. Rept. 359 (2002) 355–528

Equation for $S^<$

- **We define $S^>$ and $S^<$ for right-handed fermions**

$$
\begin{aligned}
S^>_{ab}(x_1,x_2) &= \left\langle \chi_a(x_1)\chi_b^\dagger(x_2) \right\rangle, \\
S^<_{ab}(x_1,x_2) &= \left\langle \chi_b^\dagger(x_2)\chi_a(x_1) \right\rangle,
\end{aligned}
$$

$$S = S^<.$$

$$
\begin{aligned}
i\sigma\cdot D\chi_R &= 0, \\
-\chi_R^\dagger i\sigma\cdot \overleftarrow{D}^\dagger &= 0,
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
i\sigma\cdot D_{x_1} S(x_1,x_2) &= 0, \\
-S(x_1,x_2) i\sigma\cdot \overleftarrow{D}^\dagger_{x_2} &= 0,
\end{aligned}
$$

Gauge invariant S

- We can define the gauge invariant two-point function by including gauge links,

$$\widetilde{S}(x,y) = U\left(x, x_1\right) S(x_1, x_2) U\left(x_2, x\right), \qquad x = \frac{1}{2}(x_1 + x_2),$$

$$U(x_1, x_2) = \mathcal{P}\exp\left[-i\frac{1}{\hbar}\int_{x_2}^{x_1} dz \cdot A(z)\right], \qquad y = x_1 - x_2,$$

- If electromagnetic fields are classical background ones, $U(x_1, x_2)$ is just a phase factor

$$\widetilde{S}(x_1, x_2) = S(x_1, x_2) U(x_2, x_1) = U(x_2, x_1) S(x_1, x_2).$$

$$\Rightarrow \quad S(x_1, x_2) = U(x_2, x_1)^\dagger \tilde{S}(x_1, x_2) = U(x_1, x_2) \tilde{S}(x_1, x_2)$$

Equation for gauge invariant S (I)

$$
\begin{aligned}
i\sigma\cdot D_{x_1}S(x_1,x_2) &= i\sigma\cdot[\hbar\partial_{x_1}+ieA(x_1)]S(x_1,x_2)\\
&= i\sigma\cdot D_{x_1}U\left(x_1,x_2\right)\widetilde{S}(x_1,x_2)\\
&= i\sigma\cdot\left[D_{x_1}U\left(x_1,x_2\right)\right]\widetilde{S}(x_1,x_2)+U\left(x_1,x_2\right)i\hbar\sigma\cdot\partial_{x_1}\widetilde{S}(x_1,x_2),
\end{aligned}
$$

$$
\begin{aligned}
D_{x_1,\mu}U\left(x_1,x_2\right) &= -U\left(x_1,x_2\right)i(x_1^\nu-x_2^\nu)\int_0^1 ds\,sF_{\mu\nu}[z(s)]\\
&= -U\left(x_1,x_2\right)iy^\nu\int_0^1 ds\,sF_{\mu\nu}[z(s)],
\end{aligned}
$$

See next page for Details.

$$z(s)=x+(s-1/2)y \text{ with } z(0)=x_2 \text{ and } z(1)=x_1.$$

- **Also see Eq. (3.16) of Elze, Gyulassy, Vasak, Nucl. Phys. B276 (1986) 706**

Get rid of $\partial_{x_1} U(x_1, x_2)$

$$\int_{x_2}^{x_1} dz \cdot A(z) = y^\nu \int_0^1 ds A_\nu \left(x + (s - \frac{1}{2})y\right)$$

$x = \frac{x_1+x_2}{2}$,
$y = x_1 - x_2$

$$\partial_\mu^{x_1} \int_{x_2}^{x_1} dz \cdot A(z) = \left(\frac{1}{2}\partial_\mu^x + \partial_y\right) \left[y^\nu \int_0^1 ds A_\nu \left(x + (s - \frac{1}{2})y\right)\right]$$
$$= \int_0^1 ds A_\mu(z(s)) + \frac{1}{2} y^\nu \int_0^1 ds \partial_\mu^x A_\nu(z(s)) + y^\nu \int_0^1 ds \partial_\mu^y A_\nu(z(s))$$
$$= \int_0^1 ds A_\mu(z(s)) + y^\nu \int_0^1 dss[\partial_\mu A_\nu(z(s))]$$
$$= \int_0^1 ds A_\mu(z(s)) + y^\nu \int_0^1 dss[F_{\mu\nu}(z(s)) + \partial_\nu A_\mu(z(s))]$$
$$= \int_0^1 ds A_\mu(z(s)) + y^\nu \int_0^1 dss[F_{\mu\nu}(z(s))] + \int_0^1 dss \frac{d}{ds} A_\mu(z(s))$$
$$= y^\nu \int_0^1 dss[F_{\mu\nu}(z(s))] + A_\mu(x_1)$$

$$y^\nu \partial_\nu^z A_\mu(z(s)) = (s - \frac{1}{2}) y^\nu \partial_\nu^y A_\mu(z) = \frac{d}{ds} A_\mu(z(s))$$
$$\frac{d}{ds} A_\mu(z(s)) = \frac{\partial z^\alpha}{\partial s} \partial_\alpha^z A_\mu = y^\alpha \partial_\alpha^z A_\mu$$

Equation for gauge invariant S (II)

$$\sigma^\mu \left\{\frac{1}{2}i\hbar\frac{\partial}{\partial x^\mu} + i\hbar\frac{\partial}{\partial y^\mu} + y^\nu \int_0^1 ds\, sF_{\mu\nu}[z(s)]\right\}\widetilde{S}(x,y) = 0.$$

- We can expand $F_{\mu\nu}$ in powers of y,

$$\begin{aligned}\int_0^1 ds\, sF_{\mu\nu}[z(s)] &= \sum_{n=0} \frac{1}{n!}\int_0^1 ds\, s\left(s-\frac{1}{2}\right)^n (y\cdot\partial_x)^n F_{\mu\nu}(x) \\ &= \sum_{n=0} \frac{[(-1)^n+3+2n]}{4(n+2)!}\left(\frac{1}{2}y\cdot\partial_x\right)^n F_{\mu\nu}(x).\end{aligned}$$

Wigner transformation to gauge invariant S

- **Wigner transformation and its inverse transformation**

$$\widetilde{S}(x,p) = \int d^4y \exp\left(\frac{i}{\hbar}p\cdot y\right)\widetilde{S}(x,y),$$

$$\widetilde{S}(x,y) = \int \frac{d^4p}{(2\pi\hbar)^4}\exp\left(-\frac{i}{\hbar}p\cdot y\right)\widetilde{S}(x,p).$$

For simplicity, we define $S(x,p) \equiv S^<(x,p) \equiv \widetilde{S}(x,p).$

- **Then, we get**

$$\begin{cases} \sigma\cdot\left(\frac{1}{2}i\hbar\nabla+\Pi\right)S(x,p)=0. \\ \left(-\frac{1}{2}i\hbar\nabla+\Pi\right)S(x,p)\cdot\sigma=0. \end{cases}$$

$$\nabla_\mu = \partial_\mu^x - j_0(\Delta)F_{\mu\nu}(x)\partial_p^\nu,$$

$$\Pi_\mu = p_\mu - \frac{1}{2}\hbar j_1(\Delta)F_{\mu\nu}(x)\partial_p^\nu,$$

$$\Delta = (1/2)\hbar\partial_x\cdot\partial_p$$

$$j_0(z)=\sin z/z \text{ and } j_1(z)=(\sin z - z\cos z)/z^2$$

Constant EM fields limits

$$
\nabla^\mu = \sum_{n=0}^{\infty} \hbar^{2n} \nabla^\mu_{(2n)}, \qquad \nabla^{(0)}_\mu = \partial^x_\mu - F_{\mu\nu}\partial^\nu_p, \quad \Pi^{(0)}_\mu = p_\mu .
$$

$$
\Pi^\mu = \sum_{n=0}^{\infty} \hbar^{2n} \Pi^\mu_{(2n)}.
$$

$$
\begin{aligned}
\nabla^\mu_{(2n)} &= (-1)^{n+1} \frac{1}{(2n+1)!} \left(\frac{1}{2}\hbar\partial_x \cdot \partial_p\right)^{2n} F_{\mu\nu}(x)\partial^\nu_p, \\
\Pi^\mu_{(2n)} &= (-1)^{n} \frac{n}{(2n+1)!} \left(\frac{1}{2}\hbar\partial_x \cdot \partial_p\right)^{2n-1} F_{\mu\nu}(x)\partial^\nu_p.
\end{aligned}
$$

- **In constant background fields, we find**

$$
\begin{aligned}
\sigma^\mu \left(\frac{1}{2} i\hbar \nabla^{(0)}_\mu + p_\mu\right) S(x,p) &= 0, \\
\left(-\frac{1}{2} i\hbar \nabla^{(0)}_\mu + p_\mu\right) S(x,p)\sigma^\mu &= 0.
\end{aligned}
$$

Decomposition of S

Since S(x,p) is a 2x2 matrix, we can use the 1 and Pauli matrix to decompose it,

$$\left\{\begin{array}{rcl} S_R(x,p) & = & \bar{\sigma}^\mu \mathscr{J}_\mu^+, \\ S_L(x,p) & = & \sigma^\mu \mathscr{J}_\mu^-, \end{array}\right\} \quad \mathrm{Tr}\,(\sigma^\mu \bar{\sigma}^\nu) = 2\eta^{\mu\nu} \quad \longrightarrow \quad \left\{\begin{array}{l} \mathscr{J}_\mu^+ = \frac{1}{2}\mathrm{Tr}\,(\sigma_\mu S_R), \\ \mathscr{J}_\mu^- = \frac{1}{2}\mathrm{Tr}\,(\bar{\sigma}_\mu S_L), \end{array}\right\}$$

$$\sigma^\mu = (1, \boldsymbol{\sigma}),\ \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}),$$

± represent right/left-handed components of Wigner functions

which are connected to the V and A in Wigner functions

$$\left\{ \quad \mathcal{V}_\mu = \mathscr{J}_\mu^+ + \mathscr{J}_\mu^-,\ \ \mathcal{A}_\mu = \mathscr{J}_\mu^+ - \mathscr{J}_\mu^-. \quad \right\}$$

Master eq. for massless Wigner function

By using identities

$$\begin{aligned}\sigma^{\mu}\overline{\sigma}^{\nu} &= \eta^{\mu\nu}-\frac{1}{2}i\epsilon^{\mu\nu\lambda\rho}\sigma_{\lambda}\overline{\sigma}_{\rho},\\ \overline{\sigma}^{\nu}\sigma^{\mu} &= \eta^{\mu\nu}+\frac{1}{2}i\epsilon^{\mu\nu\lambda\rho}\sigma_{\lambda}\overline{\sigma}_{\rho},\end{aligned}$$

and insert the decomposition of S into the eqs.

$$\begin{aligned}\sigma^{\mu}\left(\frac{1}{2}i\hbar\nabla_{\mu}^{(0)}+p_{\mu}\right)S(x,p) &= 0,\\ \left(-\frac{1}{2}i\hbar\nabla_{\mu}^{(0)}+p_{\mu}\right)S(x,p)\sigma^{\mu} &= 0.\end{aligned}$$

We can get the master equation for massless Wigner function

$$\begin{aligned}\Pi^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0,\\ \nabla^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0,\\ 2s\left(\Pi^{\mu}\mathscr{J}_{s}^{\nu}-\Pi^{\nu}\mathscr{J}_{s}^{\mu}\right) &= -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}_{\sigma}^{s},\end{aligned}$$

s = ± is the chirality index for right-handed (+) and left-handed (−) fermions

Summary for Master eq.

$$\begin{aligned}\Pi^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0,\\ \nabla^{\mu}\mathscr{J}_{\mu}^{s}(x,p) &= 0,\\ 2s\left(\Pi^{\mu}\mathscr{J}_{s}^{\nu}-\Pi^{\nu}\mathscr{J}_{s}^{\mu}\right) &= -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}_{\sigma}^{s},\end{aligned}$$

s = ± is the chirality index for right-handed (+) and left-handed (−) fermions

- **Assumption:**
  - **Constant homogenous EM fields** $\partial_\alpha F^{\mu\nu} = 0$
  - **No-interactions between particles**

2. Master equation for covariant Wigner function

(2c) Master equations in general case (massive case)

General cases

- If we start from a 4x4 Wigner function, by implementing the Dirac equations, we can get, in a classical EM fields,

$$(\gamma\cdot K-m)W=0$$

$$K^\mu=\Pi^\mu+\frac{1}{2}i\nabla^\mu$$

$$\begin{aligned}\nabla_\mu &= \partial_\mu^x-j_0(\Delta)F_{\mu\nu}(x)\partial_p^\nu,\\ \Pi_\mu &= p_\mu-\frac{1}{2}\hbar j_1(\Delta)F_{\mu\nu}(x)\partial_p^\nu,\end{aligned}$$

- Next, we need to insert the decomposition of W into the main equations

$$W=\mathcal{F}+i\mathcal{P}\gamma^5+\mathcal{V}^\mu\gamma_\mu+\mathcal{A}^\mu\gamma^5\gamma_\mu+\frac{1}{2}\mathcal{S}^{\mu\nu}\sigma_{\mu\nu},$$

Vasak, Gyulassy, Elze, Ann. Phys. (N.Y.) 173, 462 (1987);
Elze, Heinz,Phys.Rep.183,81(1989).

Master equations in massive case (I)

- **Inserting the decomposition of W into the main equations, yields**

$$K^\mu = \Pi^\mu + \frac{1}{2} i \nabla^\mu$$

$$\begin{aligned}
K \cdot \mathcal{V} - m\mathcal{F} &= 0, \\
iK \cdot \mathcal{A} + m\mathcal{P} &= 0, \\
K_\mu \mathcal{F} - iK^\nu \mathcal{S}_{\mu\nu} - m\mathcal{V}_\mu &= 0, \\
iK_\mu \mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} K^\nu \mathcal{S}^{\rho\sigma} - m\mathcal{A}_\mu &= 0, \\
i(K_\mu \mathcal{V}_\nu - K_\nu \mathcal{V}_\mu) - \epsilon_{\mu\nu\rho\sigma} K^\rho \mathcal{A}^\sigma + m\mathcal{S}_{\mu\nu} &= 0.
\end{aligned}$$

Master equations in massive case (II)

- **Since the coefficients V,A,F,P,S are real, we can get**

Real part of main eq. gives

$$\begin{aligned} \Pi\cdot\mathcal{V} &= m\mathcal{F}, \\ \nabla\cdot\mathcal{A} &= 2m\mathcal{P}, \\ \Pi_\mu\mathcal{F} - iK^\nu\mathcal{S}_{\mu\nu} &= m\mathcal{V}_\mu, \\ -\nabla_\mu\mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathcal{S}^{\rho\sigma} &= 2m\mathcal{A}_\mu, \\ \frac{1}{2}(\nabla_\mu\mathcal{V}_\nu - \nabla_\nu\mathcal{V}_\mu) + \epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathcal{A}^\sigma &= m\mathcal{S}_{\mu\nu}. \end{aligned}$$

Imaginary part of main eq. gives

$$\begin{aligned} \nabla\cdot\mathcal{V} &= 0, \\ \Pi\cdot\mathcal{A} &= 0, \\ \frac{1}{2}\nabla_\mu\mathcal{F} - \Pi^\nu\mathcal{S}_{\mu\nu} &= 0, \\ \Pi_\mu\mathcal{P} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu\mathcal{S}^{\rho\sigma} &= 0, \\ (\Pi_\mu\mathcal{V}_\nu - \Pi_\nu\mathcal{V}_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathcal{A}^\sigma &= 0. \end{aligned}$$

Master eqs. in massless limit (I)

- **In massless limit, V, A are decoupled with others.**

$$\nabla \cdot \mathcal{V} = 0,$$
$$\Pi \cdot \mathcal{A} = 0,$$
$$\Pi \cdot \mathcal{V} = 0,$$
$$\nabla \cdot \mathcal{A} = 0,$$
$$(\Pi_\mu \mathcal{V}_\nu - \Pi_\nu \mathcal{V}_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathcal{A}^\sigma = 0.$$

$\epsilon^{\mu\nu\alpha\beta}\times$ →

$$\frac{1}{2}(\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu) + \epsilon_{\mu\nu\rho\sigma}\Pi^\rho \mathcal{A}^\sigma = 0.$$

$$\frac{1}{2}\nabla_\mu \mathcal{F} - \Pi^\nu \mathcal{S}_{\mu\nu} = 0,$$
$$\Pi_\mu \mathcal{P} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu \mathcal{S}^{\rho\sigma} = 0,$$
$$\Pi_\mu \mathcal{F} - iK^\nu \mathcal{S}_{\mu\nu} = 0,$$
$$-\nabla_\mu \mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Pi^\nu \mathcal{S}^{\rho\sigma} = 0.$$

Master eqs. in massless limit (II)

- Eqs. for V and A are similar

$$
\left\{
\begin{aligned}
\nabla \cdot \mathcal{V} &= 0, \\
\Pi \cdot \mathcal{A} &= 0, \\
\Pi \cdot \mathcal{V} &= 0, \\
\nabla \cdot \mathcal{A} &= 0, \\
(\Pi_\mu \mathcal{V}_\nu - \Pi_\nu \mathcal{V}_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathcal{A}^\sigma &= 0. \\
(\Pi_\mu \mathcal{A}_\nu - \Pi_\nu \mathcal{A}_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho \mathcal{V}^\sigma &= 0.
\end{aligned}
\right\}
$$

Master equations for chiral fermions

- **Naturally, we introduce the Wigner function for right and left handed fermions,**

$$\mathscr{J}^{s}_{\mu} = \mathcal{V}_{\mu} + s\mathcal{A}_{\mu}, \;\; s = \pm$$

+: right handed
-: left handed

$$\left\{ \begin{array}{rcl} \Pi^{\mu}\mathscr{J}^{s}_{\mu}(x,p) & = & 0, \\ \nabla^{\mu}\mathscr{J}^{s}_{\mu}(x,p) & = & 0, \\ 2s\left(\Pi^{\mu}\mathscr{J}^{\nu}_{s} - \Pi^{\nu}\mathscr{J}^{\mu}_{s}\right) & = & -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}^{s}_{\sigma}, \end{array} \right\}$$

3. Equal-time formulism for Wigner function

Equal-time Wigner function

- **Equal-time Wigner function**

I. Bialynicki-Birula, P. Gornicki and J. Rafelski, Phys. Rev. D44, 1825 (1991).

C. Best, P. Gornicki and W. Greiner, Ann. Phys. (N.Y.) 225, 169(1993)

Zhuang, Heinz, Phys. Rev. D 57 (1998) 6525

Ochs, Heinz, Annals Phys. 266 (1998) 351

- **Application to Schwinger pair production:**

Hebenstreit, Alkofer, Gies, Phys.Rev.D82:105026,2010

Sheng, Fang, Wang, Rischke, Phys.Rev.D 99 (2019) 5, 056004

- **Application to QKT:**

Chen, Wang, Zhuang, arXiv:2101.07596.

Definition of equal time Wigner function

- **Equal time Wigner function is integration of covariant Wigner function over p^0,**

$$W_{\alpha\beta}(t,\mathbf{x},\mathbf{p}) = \int dp^0 W_{\alpha\beta}(x,p)$$
$$= \int \frac{d^3y}{(2\pi)^3} e^{i\mathbf{p}\cdot\boldsymbol{y}} \left\langle \overline{\psi}_\beta\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right) \exp\left[iQ\int_{\mathbf{x}-\mathbf{y}/2}^{\mathbf{x}+\mathbf{y}/2} d\mathbf{s}\cdot\mathbf{A}(t,\mathbf{s})\right] \psi_\alpha\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right\rangle$$

- **Then, the two spinors are set to be at equal time.**

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}_\beta(x_1) U(x_1,x_2) \psi_\alpha(x_2)$$

Equal time formalism Vs covariant formalism

- Covariant Wigner function:
  - Lorentz covariance
  - Might be hard to be simulated directly.

- Equal time Wigner function:
  - Easy to set up as an initial problem
  - No Lorentz covariance
  - e.g. Schwinger pair production

Zhuang, Heinz, Phys. Rev. D 57 (1998) 6525

Dirac equations and gauge fixing

- Next, we will derive the master eq. for equal time Wigner function.
- We start from the Dirac equation,

$$(i\gamma\cdot D - m + \mu\gamma^0)\psi = 0,$$
$$\overline{\psi}[-i\gamma\cdot(\overleftarrow{\partial} - ieA) - m + \mu\gamma^0] = 0.$$

$$i(\partial_t + ieA_t)\psi = [-i\gamma^0\boldsymbol{\gamma}\cdot(\boldsymbol{\partial} - ie\mathbf{A}) + m\gamma^0 - \mu]\psi,$$
$$\Rightarrow \quad -i(\partial_t - ieA_t)\overline{\psi} = \overline{\psi}[i\boldsymbol{\gamma}\cdot(\overleftarrow{\boldsymbol{\partial}} + ie\mathbf{A})\gamma^0 + m\gamma^0 - \mu].$$

- For simplicity, we choose the gauge fixing condition

$$A^t = 0.$$

$$\mathbf{E} = F^{i0} = \partial^i A^0 - \partial^0 A^i = -\partial_t \mathbf{A},$$
$$\mathbf{B} = -\frac{1}{2}\epsilon^{ijk}F^{jk} = -\epsilon^{ijk}\partial^j A^k = \nabla\times\mathbf{A}.$$

Equations for equal time Wigner function (I)

- For simplicity, we set EM fields are homogenous in space.

$$
\begin{aligned}
D_t W_{\alpha\beta} &= [\partial_t + e\mathbf{E}(t)\cdot\nabla_{\mathbf{p}}]W_{\alpha\beta} \\
&= \int \frac{d^3y}{(2\pi)^3}\exp\left[i\boldsymbol{y}\cdot\mathbf{p} + ie\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{s}\cdot\mathbf{A}(t,\mathbf{s})\right]\left\langle \partial_t\left[\overline{\psi}_\beta\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi_\alpha\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]\right\rangle \\
&= -im\left[W,\gamma^0\right]_{\alpha\beta} \\
&\quad -\int \frac{d^3y}{(2\pi)^3}\exp\left[i\boldsymbol{y}\cdot\mathbf{p} + ie\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{s}\cdot\mathbf{A}(t,\mathbf{s})\right] \\
&\quad \times\left\langle [\overline{\psi}(\boldsymbol{\gamma}\cdot(\overleftarrow{\boldsymbol{\partial}} + ie\mathbf{A})\gamma^0]_\beta\psi_\alpha + \overline{\psi}_\beta[\gamma^0\boldsymbol{\gamma}\cdot(\boldsymbol{\partial}_X - ie\mathbf{A})\psi]_\alpha\right\rangle
\end{aligned}
$$

- Integration by part and after a long calculation, we get,

$$
[\partial_t + e\mathbf{E}(t)\cdot\nabla_{\mathbf{p}}]W = \frac{1}{2}\left(\nabla_x + e\mathbf{B}\times\nabla_{\mathbf{p}}\right)\left[W,\gamma^0\boldsymbol{\gamma}\right] - i\mathbf{p}\cdot\left\{W,\gamma^0\boldsymbol{\gamma}\right\} + im\left[W,\gamma^0\right].
$$

Equation for equal time Wigner function (II)

- We summary the master eqs. here,

$$D_t W = \frac{1}{2}\mathbf{D}_x \cdot [W, \gamma^0 \boldsymbol{\gamma}] - i\boldsymbol{\Pi} \cdot \{W, \gamma^0 \boldsymbol{\gamma}\} + im\left[W, \gamma^0\right],$$

$$D_t = \partial_t + e\mathbf{E}(t) \cdot \nabla_{\mathbf{p}},$$
$$\mathbf{D}_x = \nabla_x + e\mathbf{B} \times \nabla_{\mathbf{p}},$$
$$\boldsymbol{\Pi} = \mathbf{p}.$$

- Similarly, we decompose the Wigner function as

$$W = \mathcal{F} + i\mathcal{P}\gamma^5 + \mathcal{V}^\mu \gamma_\mu + \mathcal{A}^\mu \gamma^5 \gamma_\mu + \frac{1}{2}\mathcal{S}^{\mu\nu}\sigma_{\mu\nu},$$
$$\mathbf{T} = (\mathcal{S}^{10}, \mathcal{S}^{20}, \mathcal{S}^{30}),$$
$$\mathbf{S} = (\mathcal{S}^{23}, \mathcal{S}^{31}, \mathcal{S}^{12}).$$

Master equation for equal time Wigner function

$$D_t\mathcal{F} = 2\mathbf{\Pi}\cdot\mathbf{T},$$

$$D_t\mathcal{P} = -2\mathbf{\Pi}\cdot\mathbf{S} + 2m\mathcal{A}^0,$$

$$D_t\mathcal{V}^0 = -\mathbf{D}_x\cdot V,$$

$$D_t\mathbf{V} = -\mathbf{D}_x V^0 + 2\mathbf{\Pi}\times\mathbf{A} - 2m\mathbf{T},$$

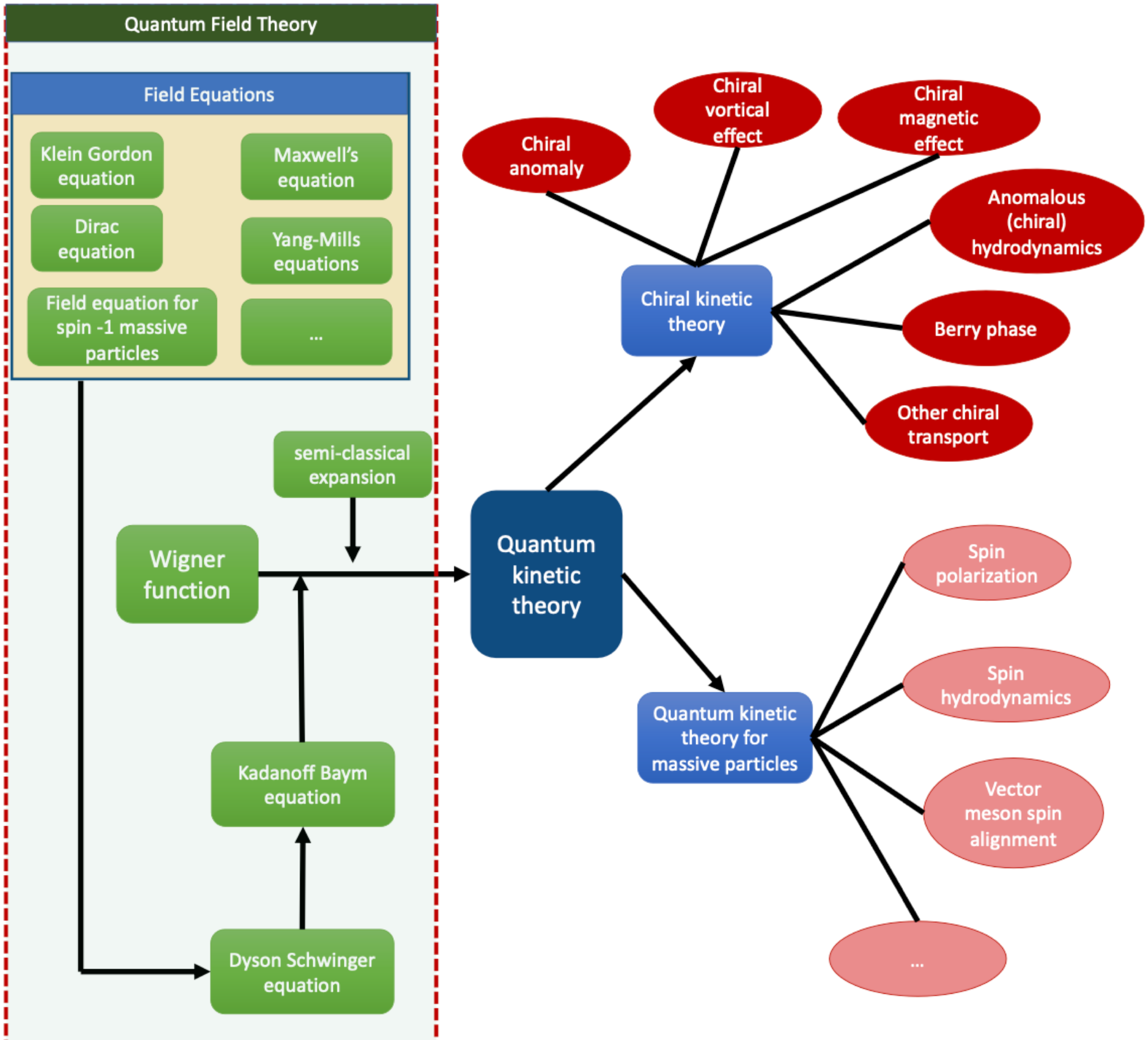
$$D_t\mathcal{A}^0 = -\mathbf{D}_x\cdot\mathbf{A} - 2m\mathcal{P},$$

$$D_t\mathbf{A} = -\mathbf{D}_x\mathcal{A}^0 + 2\mathbf{\Pi}\times\mathbf{V},$$

$$\mathbf{D}_t\mathbf{T} = \mathbf{D}_x\times\mathbf{S} - 2\mathbf{\Pi}\mathcal{F} + 2m\mathbf{V},$$

$$\mathbf{D}_t\mathbf{S} = \mathbf{D}_x\times\mathbf{T} + 2\mathbf{\Pi}\mathcal{P}.$$

Summary



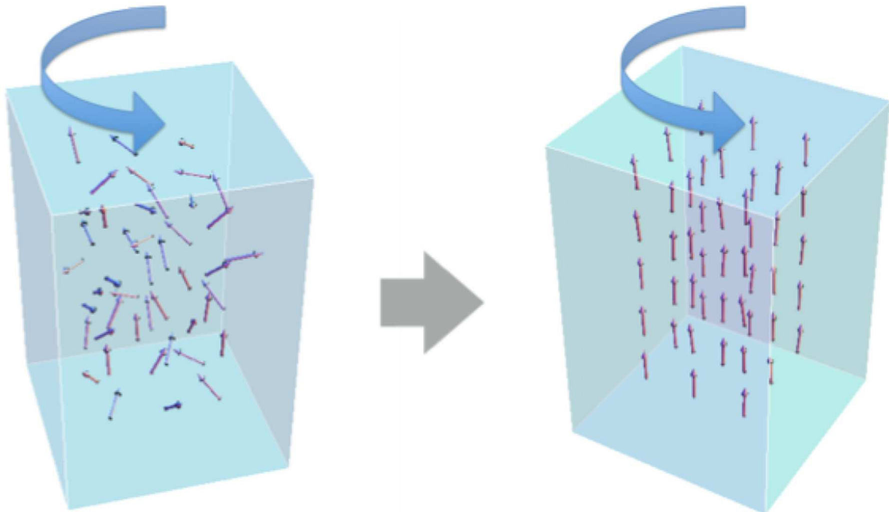
Quantum Field Theory
Field Equations
Klein Gordon equation
Maxwell's equation
Dirac equation
Yang-Mills equations
Field equation for spin -1 massive particles
...
semi-classical expansion
Wigner function
Kadanoff Baym equation
Dyson Schwinger equation
Quantum kinetic theory
Chiral kinetic theory
Chiral anomaly
Chiral vortical effect
Chiral magnetic effect
Anomalous (chiral) hydrodynamics
Berry phase
Other chiral transport
Quantum kinetic theory for massive particles
Spin polarization
Spin hydrodynamics
Vector meson spin alignment
...

Thank you for your time!

欢迎批评指正！

Backup

Barnet effects and Einstein-de Hass effects



Barnett effect:

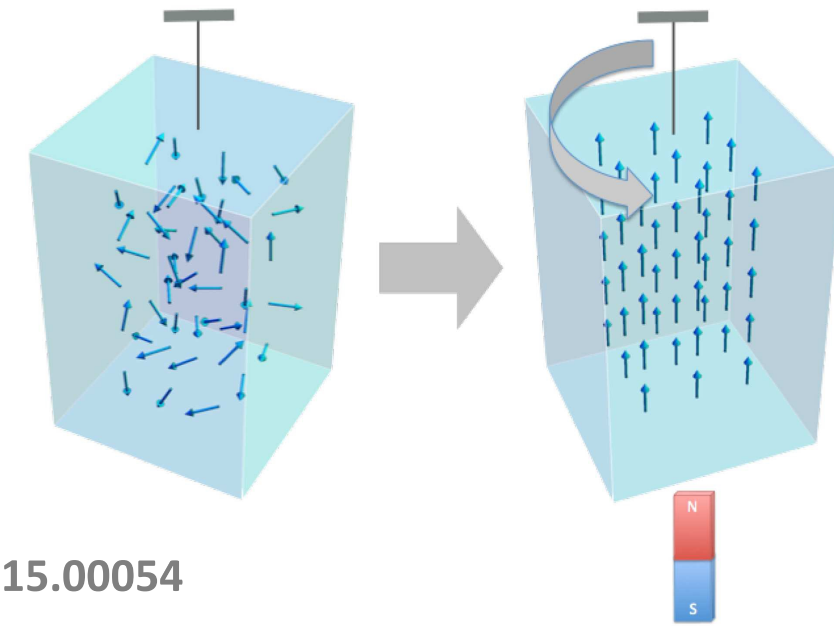
Rotation $\Longrightarrow$ Magnetization

Barnett, Magnetization by rotation, Phys Rev. (1915) 6:239–70.

Einstein-de Haas effect:

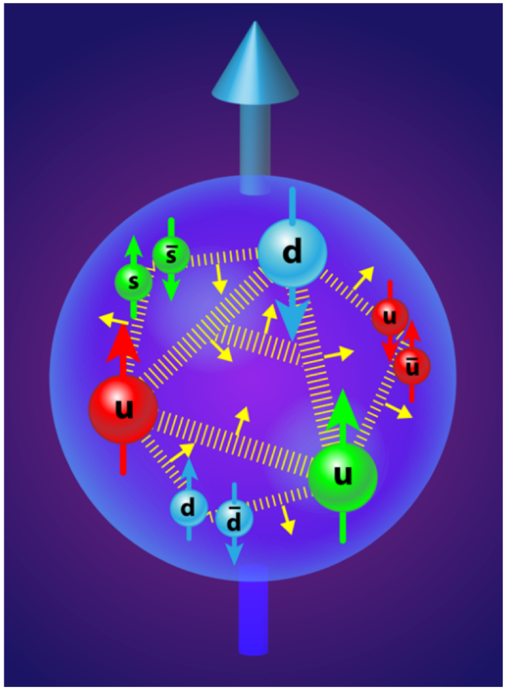
Magnetization $\Longrightarrow$ Rotation

Einstein, de Haas, Experimental proof of the existence of Ampere's molecular currents. Verh Dtsch Phys Ges. (1915) 17:152.



Figures: copy from paper doi: 10.3389/fphy.2015.00054

Connection to "spin physics" (QCD)



- **Proton spin problem:**

  (slides from Hatta's talk)

$$\frac{1}{2} = \frac{1}{2}\Delta\Sigma + \Delta G + L^q + L^g$$

Quarks' helicity

Gluons' helicity

Orbital angular Momentum (OAM)

Total angular momentum conservation

- **Nöther's theorem：**

$$x^\mu \;\rightarrow\; x'^\mu = \Lambda^\mu_{\ \nu} x^\nu = (\delta^\mu_\nu + \epsilon^\mu_{\ \nu}) x^\nu\,, \qquad A^\mu(x) \rightarrow A'^\mu(x) = \Lambda^\mu_{\ \nu} A^\nu(\Lambda^{-1}x),$$

$$\psi(x) \rightarrow \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x),$$

$$\partial_\lambda (J_A^{\lambda\mu\nu} + J_\psi^{\lambda\mu\nu}) = 0$$

- **Nöther current**

Gauge part

$$J_A^{\lambda\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda A^\alpha)} \Delta A^{\mu\nu\alpha} = -F^\lambda_{\ \alpha}(x^\mu\partial^\nu - x^\nu\partial^\mu)A^\alpha - F^{\lambda\mu}A^\nu + F^{\lambda\nu}A^\mu\,.$$

Fermionic part

$$J_\psi^{\lambda\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\lambda \psi)} \Delta\psi^{\mu\nu} = \bar{\psi} i\gamma^\lambda \left(x^\mu\partial^\nu - x^\nu\partial^\mu - i\Sigma^{\mu\nu}\right)\psi$$

- **How to define the orbital and spin parts?**

Table of two different forms

- **Canonical (Jaffe-Manohar) decomposition**

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} + \underbrace{\boldsymbol{E}\times\boldsymbol{A}}_{\Delta G} \underbrace{-i\psi^\dagger(\boldsymbol{x}\times\nabla)\psi}_{L^q_{\mathrm{can}}} + \underbrace{\boldsymbol{E}(\boldsymbol{x}\times\nabla)\boldsymbol{A}}_{L^g_{\mathrm{can}}}$$

$$T^{\mu\nu}_{\psi,\mathrm{can}} = \bar{\psi}i\gamma^\mu\partial^\nu\psi - g^{\mu\nu}\bar{\psi}(i\gamma^\alpha D_\alpha - m)\psi$$

non-symmetric
Not gauge invariant

- **Belinfante (Ji) decomposition**

$$\boldsymbol{J} = \underbrace{-\frac{1}{2}\bar{\psi}\gamma_5\boldsymbol{\gamma}\psi}_{\frac{1}{2}\Delta\Sigma} \underbrace{-i\psi^\dagger(\boldsymbol{x}\times\boldsymbol{D})\psi}_{L^q_{\mathrm{Ji}}} + \underbrace{\boldsymbol{x}\times(\boldsymbol{E}\times\boldsymbol{B})}_{J^g_{\mathrm{Ji}}}$$

$$T^{\mu\nu}_{\psi,\mathrm{Bel}} \equiv \bar{\psi}i\gamma^\mu\overleftrightarrow{D}^\nu\psi + \frac{1}{4}\varepsilon^{\mu\nu\lambda\rho}\partial_\lambda(\bar{\psi}\gamma_5\gamma_\rho\psi).$$

Connected by pseudo gauge transformation

Symmetric
Gauge invariant

GLW decomposition

- **Another Pseudo gauge transformation**

$$T^{\mu\nu}_{\rm can} = T^{\mu\nu}_{\rm GLW} + \frac{1}{2}\partial_\lambda\left(\Phi^{\lambda,\mu\nu}_{\rm can} + \Phi^{\mu,\nu\lambda}_{\rm can} + \Phi^{\nu,\mu\lambda}_{\rm can}\right)$$

$$S^{\lambda,\mu\nu}_{\rm can} = S^{\lambda,\mu\nu}_{\rm GLW} - \Phi^{\lambda,\mu\nu}_{\rm can} \qquad \Phi^{\lambda,\mu\nu}_{\rm can} \equiv S^{\mu,\lambda\nu}_{\rm GLW} - S^{\nu,\lambda\mu}_{\rm GLW}$$

$$\partial_\lambda S^{\lambda,\mu\nu}_{\rm can}(x) = T^{\nu\mu}_{\rm can} - T^{\mu\nu}_{\rm can} = -\partial_\lambda S^{\mu,\lambda\nu}_{\rm GLW}(x) + \partial_\lambda S^{\nu,\lambda\mu}_{\rm GLW}(x).$$

$$T^{\mu\nu}_{\rm Bel} = T^{\mu\nu}_{\rm GLW} - \frac{1}{2}\partial_\lambda\left(S^{\nu,\lambda\mu}_{\rm GLW} + S^{\mu,\lambda\nu}_{\rm GLW}\right)$$

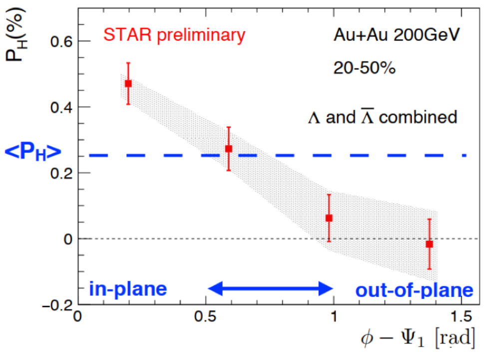
Textbook written by de Groot, van Leeuven, and van Weert

Review: *W. Florkowski, R. Ryblewski and Avdhesh Kumar, Prog. Part. Nucl. Phys. 108 (2019) 103709*

- **Microscopic kinetic theory: GLW is the classical one.**

$$T^{\mu\nu}_{\rm GLW}(x) = \frac{1}{m}{\rm tr}_4\int d^4k\, k^\mu\, k^\nu \mathcal{W}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu \mathcal{F}(x,k).$$

Polarization induced by thermal vorticity



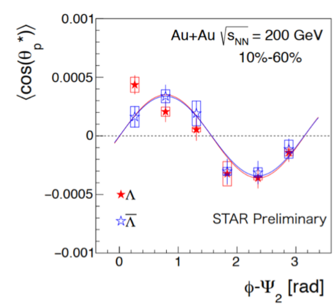
Thermal vorticity

$$\omega^{\text{th}}_{\rho\sigma} = \frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right) - \partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Distribution function: f_0

$$\mathcal{S}^\mu(p) = \frac{1}{8m_\Lambda}\epsilon^{\mu\nu\rho\sigma}p_\nu\frac{\int d\Sigma_\lambda p^\lambda f_0(1-f_0)\omega^{\text{th}}_{\rho\sigma}}{\int d\Sigma_\lambda p^\lambda f_0}$$

Freezeout surface



Karpenko, F. Becattini, Eur. Phys. J. C 77 (2017) 213
R.-H. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, Phys. Rev. C94, 024904 (2016)

Polarization and axial current

- **Recalling the original equations**

$$\mathcal{S}^\mu(\mathbf{p}) = \frac{\int d\Sigma \cdot p \mathcal{J}_5^\mu(p, X)}{2m_\Lambda \int d\Sigma \cdot \mathcal{N}(p, X)},$$

- **For massless fermions, the left and right handed currents read**

$$\mathcal{J}^\mu_\lambda(p,X) \;=\; 2\pi \mathrm{sign}(u\cdot p)\left\{p^\mu + \lambda\frac{\hbar}{2}\delta(p^2)\left[u^\mu(p\cdot\omega) - \omega^\mu(u\cdot p)\right.\right.$$
$$\left.\left. -2S^{\mu\nu}_{(u)}\tilde{E}_\nu\right]\partial_{u\cdot p} + \lambda\frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\partial^p_\nu\delta(p^2)\right\} f^{(0)}_\lambda,$$

$\lambda = \pm$

+: right

-: left

$$S^{\mu\nu}_{(u)} \;=\; \epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta/(2u\cdot p),$$

$$\tilde{E}_\nu \;=\; E_\nu + T\partial_\nu\frac{\mu_\lambda}{T} + \frac{(u\cdot p)}{T}\partial_\nu T - p^\sigma\left[\partial_{<\sigma}u_{\nu>} + \frac{1}{3}\Delta_{\sigma\nu}(\partial\cdot u) + u_\nu D u_\sigma\right].$$

$$f^{(0)}_\lambda = 1/(e^{(u\cdot p-\mu_\lambda)/T} + 1),$$

Y. Hidaka, SP, and D.L. Yang, Phys. Rev. D97, 016004 (2018)