Quantum kinetic theory

and its applications to chiral transports and spin polarizations

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Recent review:

- Y. Hidaka, SP, Q.Wang, D.L. Yang, ***Foundations and Applications of Quantum Kinetic Theory****,* Progress in Particle and Nuclear Physics, 127, 103989 (2022).

Outline

- **Part 1:**

  Chiral magnetic effect, Berry phase and kinetic theory

- **Part 2:**

  Wigner functions and the master equations

- **Part 3:**

  Quantum kinetic theory in massless limit and collisions

- **Part 4:**

  Applications to heavy ion physics

Part 3

Quantum kinetic theory in massless limit and collisions

1. Solve quantum kinetic theory in gradient expansion
   - (1a) Gradient expansion
   - (1b) Leading order results and constrains from QKT
   - (1c) $\hbar$ order results
   - (1d) $\hbar^2$ order results
2. Discussions on the solution of Wigner function
   - (2a) CME, CVE, energy-momentum tensor and chiral anomaly
   - (2b) Chiral kinetic theory
   - (2c) Lorentz transformation and side jump
3. Collision effects
   - (3a) Kadanoff-Baym equation
   - (3b) General solution of Wigner function with collisions
   - (3c) Collision term for QED in HTL approximation

1. Solve quantum kinetic theory in gradient expansion

(1a) Gradient expansion

Summary for master equations of chiral fermions

- **The master equations read,**

$$\left\{\begin{array}{rcl} \Pi^{\mu}\mathscr{J}^{s}_{\mu}(x,p) & = & 0, \\ \nabla^{\mu}\mathscr{J}^{s}_{\mu}(x,p) & = & 0, \\ 2s\left(\Pi^{\mu}\mathscr{J}^{\nu}_{s}-\Pi^{\nu}\mathscr{J}^{\mu}_{s}\right) & = & -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}^{s}_{\sigma}, \end{array}\right\}$$

$$\mathscr{J}^{s}_{\mu}=\mathcal{V}_{\mu}+s\mathcal{A}_{\mu},\ \ s=\pm$$

+: right handed

-: left handed

- **We assume the EM fields are constant**

$$\nabla^{(0)}_{\mu}=\partial^{x}_{\mu}-F_{\mu\nu}\partial^{\nu}_{p},\ \ \Pi^{(0)}_{\mu}=p_{\mu}.$$

Gradient expansion

- A common way to handle the equations in many body system is gradient expansion.

- Gradient expansion: We assume that a field A(x) changes very slowly, i.e.

$$\frac{|\partial A|}{AL^{-1}} \ll 1$$

where L is one characteristic length (or time) of the system.

- A rough way to get the power of a term is to count the number of ∂.

Gradient expansion in relativistic hydrodynamics

- In relativistic hydrodynamics, L is the mean free path and $K \sim L_{mfp}\partial$ is the Knudsen number.
- The energy-momentum tensor in the gradient expansion is usually written as,

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$$

- Leading order is

$$T^{\mu\nu}_{(0)} = (\epsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu}$$

- Next leading order is

$$T^{\mu\nu}_{(1)} = -\Pi(g^{\mu\nu} - u^{\mu}u^{\nu}) + \pi^{\mu\nu}$$

$$\Pi = \zeta(\partial \cdot u),$$

$$\pi^{\mu\nu} = 2\eta\partial^{<\mu}u^{\nu>}$$

Bulk pressure

Shear viscous tensor

Gradient expansion VS $\hbar$ expansion

- If we go back to the equations for S(x,p),

$$\sigma\cdot\left(\frac{1}{2}i\hbar\nabla+\Pi\right)S(x,p)=0. \qquad \left(-\frac{1}{2}i\hbar\nabla+\Pi\right)S(x,p)\cdot\sigma=0.$$

$$\nabla_{\mu}^{(0)}=\partial_{\mu}^{x}-F_{\mu\nu}\partial_{p}^{\nu},\ \ \Pi_{\mu}^{(0)}=p_{\mu}.$$

we find the $\hbar$ is always with the space-time derivatives.

- For Wigner functions approaches, the gradient expansion is equivalent to the $\hbar$ expansion.
- EM fields appears at the order of $\hbar$.
- In general case (other systems), gradient expansion can be independent to the $\hbar$ expansion.

Strategy

- **We expand the Wigner function in the power series of $\hbar$**

$$\mathscr{J}^{\mu}(x,p)=\mathscr{J}^{\mu}_{(0)}(x,p)+\hbar\,\mathscr{J}^{\mu}_{(1)}(x,p)+\hbar^2\,\mathscr{J}^{\mu}_{(2)}(x,p)+\cdots\,,$$

and rewrite the master equations

$$\left\{\begin{aligned} p^{\mu}\,\mathscr{J}^{s}_{\mu}(x,p)&=0,\\ \nabla^{\mu}\,\mathscr{J}^{s}_{\mu}(x,p)&=0,\\ 2s\left(p^{\mu}\,\mathscr{J}^{\nu}_{s}-p^{\nu}\,\mathscr{J}^{\mu}_{s}\right)&=-\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}^{s}_{\sigma},\end{aligned}\right\}$$

Contracting p_λ →

$$p^2\,\mathscr{J}^{s}_{\mu}=\frac{s}{2}\epsilon_{\mu\nu\rho\sigma}p^{\nu}\nabla^{\rho}\,\mathscr{J}^{\sigma}_{s}$$

Master equations in the $\hbar$ expansion

$$p^\mu \mathscr{J}^{(n)}_{s,\mu}(x,p) = 0,$$

$$\nabla^\mu \mathscr{J}^{(n)}_{s,\mu}(x,p) = 0,$$

Constrain eq.

$$p^2 \mathscr{J}^{(n)}_{s,\mu} = \frac{s}{2}\epsilon_{\mu\nu\rho\sigma}p^\nu\nabla^\rho \mathscr{J}^{\sigma}_{s,(n-1)}$$

We can solve the third eq order by order.

$$\mathscr{J}^\mu_{(n)} = p^\mu f_{(n)}\delta(p^2) + X^\mu_{(n)}\delta(p^2) + \frac{s}{2p^2}\epsilon^{\mu\nu\rho\sigma}p_\nu\nabla_\rho\mathscr{J}^{(n-1)}_\sigma,$$

$$p^\mu X^{(n)}_\mu = 0.$$

Some useful expressions (I)

- With the fluid velocity vector u, for any vector A, we can have,

$$A^\mu = (u \cdot A) u^\mu + (g^{\mu\nu} - u^\mu u^\nu) A_\nu,$$

- For a rank-2 anti-symmetric tensor, we can decompose it as

$$\begin{aligned} A^{\mu\nu} &= a^\mu u^\nu - a^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha b_\beta \\ a^\mu &= A^{\mu\nu} u_\nu, \\ b^\mu &= \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu A_{\alpha\beta} \end{aligned}$$

Some useful expression (II)

- **We set the Levi-Civita tensor**

$$\epsilon^{0123} = -\epsilon_{0123} = 1$$

Contracting this 4-dim tensor is different with the one in 3-dim

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\rho} = (-3!)\delta^{\beta}_{\rho}$$

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma} = (-2!)\delta^{\alpha\beta}_{[\rho\sigma]} = (-2!)(\delta^{\alpha}_{\rho}\delta^{\beta}_{\sigma} - \delta^{\alpha}_{\sigma}\delta^{\beta}_{\rho})$$

$$\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\gamma\rho\sigma} = (-1!)\delta^{\nu\alpha\beta}_{[\gamma\rho\sigma]} = (-1!)(\delta^{\nu}_{\gamma}\delta^{\alpha\beta}_{[\rho\sigma]} - \delta^{\nu}_{\rho}\delta^{\alpha\beta}_{[\gamma\sigma]} + \delta^{\nu}_{\sigma}\delta^{\alpha\beta}_{[\gamma\rho]})$$

Decomposition of gradient u

$$\partial_\mu u_\nu = \partial_{(\mu} u_{\nu)} + \partial_{[\mu} u_{\nu]},$$

$$\partial_{(\mu} u_{\nu)} = \partial_{<\mu} u_{\nu>} - \frac{1}{2}[u_\mu (u\cdot\partial) u_\nu + u_\nu (u\cdot\partial) u_\mu] + \frac{1}{3}\Delta_{\mu\nu}(\partial\cdot u),$$

$$\partial_{[\mu} u_{\nu]} = \Delta_{\mu\alpha}\Delta_{\nu\beta}\partial^{[\alpha} u^{\beta]} - \frac{1}{2}[u_\mu (u\cdot\partial) u_\nu - u_\nu (u\cdot\partial) u_\mu],$$

$$A^{(\mu\nu)} = \frac{1}{2}(A^{\mu\nu} + A^{\nu\mu}),$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$A^{[\mu\nu]} = \frac{1}{2}(A^{\mu\nu} - A^{\nu\mu}),$$

$$A^{<\mu\nu>} = \frac{1}{2}[\Delta^{\mu\alpha}\Delta^{\nu\beta} + \Delta^{\nu\alpha}\Delta^{\mu\beta}]A_{\alpha\beta} - \frac{1}{3}\Delta^{\mu\nu}(\Delta^{\alpha\beta}A_{\alpha\beta}).$$

<>: Traceless

1. Solve quantum kinetic theory in gradient expansion

(1b) Leading order results and constrains from QKT

Zeroth order of $\hbar$

- **In the zeroth order of $\hbar$, we get**

$$\left\{ \begin{array}{r} \boxed{p^{\mu}\,\mathscr{J}^{(0)}_{s,\mu}(x,p)=0,} \\ \nabla^{\mu}\,\mathscr{J}^{(0)}_{s,\mu}(x,p)=0, \\ \boxed{p^{2}\,\mathscr{J}^{(0)}_{s,\mu}=0} \end{array} \right\}$$

$$\longrightarrow \quad \mathscr{J}^{\rho}_{(0)}(x,p)=p^{\rho}f_{(0)}(x,p)\delta(p^{2}),$$

$$f_{(0)}(x,p) \quad = \quad 2\pi\left\{\Theta(p_0)f_{\mathrm{FD}}(p_0-\mu_s)\ +\Theta(-p_0)\left[f_{\mathrm{FD}}(-p_0+\mu_s)-1\right]\right\}.$$

phase space distributions of massless fermions at the zeroth order

$$p_0\equiv u\cdot p,\ f_{\mathrm{FD}}(y)\equiv 1/[\exp(\beta y)+1]$$

$$\mu_s=\mu+s\mu_5$$

chemical potential with s = ±1 denoting the chirality

u: fluid velocity

Global equilibrium conditions

$$\begin{aligned}\nabla_\rho \mathscr{J}^{\rho}_{(0)} &= \delta(p^2)p^\rho\nabla_\rho f_{(0)}\\ &= \delta(p^2)f'_{(0)}\left[\frac{1}{2}p^\rho p^\sigma\left(\partial_\rho\beta_\sigma+\partial_\sigma\beta_\rho\right)-p^\rho\partial_\rho\left(\bar{\mu}+s\bar{\mu}_5\right)-F_{\rho\sigma}p^\rho\beta^\sigma\right]=0,\end{aligned}$$

$$f'_{(0)}\equiv\partial f_{(0)}/\partial(\beta\cdot p),\ \beta^\rho\equiv\beta u^\rho,\ \bar{\mu}_s\equiv\beta\mu_s,\ \bar{\mu}\equiv\beta\mu,\ \text{and}\ \bar{\mu}_5\equiv\beta\mu_5.$$

Global equilibrium conditions for a system under static and uniform vorticity and electromagnetic field:

$$\longrightarrow\left\{\begin{array}{lcl}\boxed{\partial_\rho\beta_\sigma+\partial_\sigma\beta_\rho = 0,} \\ \boxed{\partial_\rho\bar{\mu}+F_{\rho\sigma}\beta^\sigma = 0,\quad \partial_\lambda\partial_\rho\bar{\mu}=\partial_\rho\partial_\lambda\bar{\mu}\longrightarrow F^{\mu}{}_{\lambda}\Omega^{\nu\lambda}=F^{\nu}{}_{\lambda}\Omega^{\mu\lambda},} \\ \partial_\rho\bar{\mu}_5 = 0.\end{array}\right.$$

Killing condition for ideal fluid

integrability condition

$$\longrightarrow\left\{\begin{array}{lcl}\beta_\mu &=& -\Omega_{\mu\nu}x^\nu+c_\mu,\\ \bar{\mu} &=& -\frac{1}{2}F^{\mu\lambda}x_\lambda\Omega_{\mu\nu}x^\nu+c,\\ \bar{\mu}_5 &=& c_5,\end{array}\right.$$

$$\Omega_{\mu\nu}=\frac{1}{2}\left(\partial_\mu\beta_\nu-\partial_\nu\beta_\mu\right).$$

Thermal vorticity

1. Solve quantum kinetic theory in gradient expansion

(1c) $\hbar$ order results

Next-to-Leading order solutions

$$\begin{aligned}\mathscr{J}^{\mu}_{(1)} &= p^{\mu} f_{(1)}\delta(p^2) + X^{\mu}_{(1)}\delta(p^2) + \frac{s}{2p^2}\epsilon^{\mu\nu\rho\sigma} p_{\nu}\nabla_{\rho}\mathscr{J}^{(0)}_{\sigma} \\ &= X^{\mu}_{(1)}\delta(p^2) + s\widetilde{F}^{\mu\nu} p_{\nu} f_{(0)}\delta'(p^2),\end{aligned}$$

$$\widetilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \text{ and } \delta'(x) = -(1/x)\delta(x).$$

- **We have set $f_{(1)} = 0$, which corresponds the normalization of distribution function.**
- **Using eq.**

$$2s\left(p^{\mu}\mathscr{J}^{(1)\nu}_{s} - p^{\nu}\mathscr{J}^{(1)\mu}_{s}\right) = -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}^{(0)s}_{\sigma},$$

to get $X_{(1)}$

Get $X_{(1)}$

Global equilibrium conditions

$$\begin{aligned}\partial_\rho \beta_\sigma + \partial_\sigma \beta_\rho &= 0,\\ \partial_\rho \bar{\mu} + F_{\rho\sigma}\beta^\sigma &= 0,\\ \partial_\rho \bar{\mu}_5 &= 0.\end{aligned}$$

$$\begin{aligned}(p^\mu X^\nu_{(1)} - p^\nu X^\mu_{(1)})\delta(p^2) &= -\frac{s}{2}\epsilon^{\mu\nu\rho\lambda} p_\lambda \delta(p^2)\nabla_\rho f_{(0)}\\ &= -\frac{s}{2}\epsilon^{\mu\nu\rho\lambda} f'_{(0)}\delta(p^2) p_\lambda p^\sigma \Omega_{\rho\sigma}\\ &= -\frac{s}{2}\left[p^\mu \widetilde{\Omega}^{\nu\lambda} p_\lambda - p^\nu \widetilde{\Omega}^{\mu\lambda} p_\lambda\right] f'_{(0)}\delta(p^2),\end{aligned}$$

Schouten identity

$$\epsilon^{\mu\nu\rho\lambda}p^\sigma + \epsilon^{\nu\rho\lambda\sigma}p^\mu + \epsilon^{\rho\lambda\sigma\mu}p^\nu + \epsilon^{\lambda\sigma\mu\nu}p^\rho + \epsilon^{\sigma\mu\nu\rho}p^\lambda = 0.$$

$$\Longrightarrow \quad X^\mu_{(1)} = -\frac{s}{2}\widetilde{\Omega}^{\mu\lambda} p_\lambda f'_{(0)}, \qquad \widetilde{\Omega}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma}\Omega_{\rho\sigma}$$

Next-to-Leading order solutions

$$\mathscr{J}^{\mu}_{(1)} = -\frac{s}{2}\widetilde{\Omega}^{\mu\lambda} p_{\lambda} f'_{(0)} \delta(p^2) + s\widetilde{F}^{\mu\nu} p_{\nu} f_{(0)} \delta'(p^2).$$

$$\widetilde{F}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$s = \pm$$

+: right handed

-: left handed

$$\widetilde{\Omega}^{\mu\nu} = (1/2)\epsilon^{\mu\nu\rho\sigma} \Omega_{\rho\sigma}$$

$$\delta'(x) = -(1/x)\delta(x).$$

1. Solve quantum kinetic theory in gradient expansion

(1d) $\hbar^2$ order results

$\hbar^2$ order solutions

$$\mathscr{J}_\mu^{(2)} = X_\mu^{(2)}\delta(p^2) + \frac{s}{2p^2}\epsilon_{\mu\nu\rho\sigma}p^\nu\nabla^\rho\mathscr{J}_{(1)}^\sigma$$

- By the same reason, we set $f_{(2)}$=0.
- We solve

$$2s\left(p^\mu\mathscr{J}_s^{(2)\nu} - p^\nu\mathscr{J}_s^{(2)\mu}\right) = -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_\rho\mathscr{J}_\sigma^{(1)s},$$

to get $X_{(2)}$.

$$\longrightarrow \qquad \left(p_\mu X_\nu^{(2)} - p_\nu X_\mu^{(2)}\right)\delta(p^2) = 0,$$

$\hbar^2$ order solutions

$$
\begin{aligned}
\mathscr{J}_{\mu}^{(2)} &= X_{\mu}^{(2)}\delta(p^2)+\frac{s}{2p^2}\epsilon_{\mu\nu\rho\sigma}p^{\nu}\nabla^{\rho}\mathscr{J}_{(1)}^{\sigma} \\
&= \frac{1}{4p^2}\left(p_{\mu}\Omega_{\gamma\beta}p^{\beta}-p^2\Omega_{\gamma\mu}\right)\Omega^{\gamma\lambda}p_{\lambda}f_{(0)}^{\prime\prime}\delta(p^2) \\
&\quad +\frac{1}{(p^2)^2}\left(p_{\mu}F_{\gamma\beta}p^{\beta}-p^2F_{\gamma\mu}\right)\Omega^{\gamma\lambda}p_{\lambda}f_{(0)}^{\prime}\delta(p^2) \\
&\quad +\frac{2}{(p^2)^3}\left(p_{\mu}F_{\gamma\beta}p^{\beta}-p^2F_{\gamma\mu}\right)F^{\gamma\lambda}p_{\lambda}f_{(0)}\delta(p^2),
\end{aligned}
$$

S.-Z. Yang, J.-H. Gao, Z.-T. Liang, Q. Wang, Phys. Rev. D 102 (11) (2020) 116024

2. Discussions on the solution of Wigner function

(2a) CME, CVE, energy-momentum tensor and chiral anomaly

Vector and chiral currents from Wigner function

- The right and left handed currents are given by integrating $\mathscr{J}_s^{\mu}(x,p)$ over momentum.

$$J_s^{\mu}(x) = 2\int \frac{d^4p}{(2\pi)^4}\mathscr{J}_s^{\mu}(x,p), \quad s=\pm$$

- We can also get the following currents,

$$J^{\mu} \quad = \quad J_+^{\mu} + J_-^{\mu} = J_{(0)}^{\mu} + \hbar J_{(1)}^{\mu} + \hbar^2 J_{(2)}^{\mu} + \cdots \,,$$

Vector current

$$J_5^{\mu} \quad = \quad J_+^{\mu} - J_-^{\mu} = J_{5,(0)}^{\mu} + \hbar J_{5,(1)}^{\mu} + \hbar^2 J_{5,(2)}^{\mu} + \cdots \,,$$

Chiral current

Leading order currents

- **Vector current**

$$J^{\mu}_{(0)} = \rho u^{\mu},$$

$$\rho = \frac{\mu}{3\pi^2}\left(\pi^2 T^2 + \mu^2 + 3\mu_5^2\right),$$

Charge number density

- **Chiral (axial vector) current**

$$J^{\mu}_{5,(0)} = \rho_5 u^{\mu},$$

$$\rho_5 = \frac{\mu_5}{3\pi^2}\left(\pi^2 T^2 + 3\mu^2 + \mu_5^2\right),$$

Axial charge number density

- **The leading order currents are consistent with those for ideal gas or ideal fluid.**

CME and CVE

- In (magneto-)hydrodynamics, we usually decompose $F_{\mu\nu}$ into the electric and magnetic vectors,

$$F_{\mu\nu} = E_\mu u_\nu - E_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma ,$$

with

$$E_\sigma = F_{\sigma\rho} u^\rho, \; B_\sigma = \frac{1}{2} \epsilon_{\sigma\mu\nu\rho} u^\mu F^{\nu\rho} ,$$

- Vector current

$$J^\mu_{(1)} = \underbrace{\xi \omega^\mu}_{\text{CVE}} + \underbrace{\xi_B B^\mu}_{\text{CME}} , \qquad \xi = \frac{\mu\mu_5}{\pi^2}, \qquad \xi_B = \frac{\mu_5}{2\pi^2}$$

- Chiral current

$$J^\mu_{5,(1)} = \xi_5 \omega^\mu + \xi_{B5} B^\mu ,$$

$$\xi_5 = \frac{1}{6\pi^2} \left[\pi^2 \boxed{T^2} + 3(\mu^2 + \mu_5^2) \right]$$

$$\xi_{B5} = \frac{\mu}{2\pi^2}$$

- We derive the CME and CVE from Wigner functions

Gao, Liang, SP, Wang, Wang, PRL 2012

Currents in the order of $\hbar^2$

- We can decompose the vorticity tensor into the electric and magnetic parts

$$T\Omega_{\mu\nu} = \varepsilon_\mu u_\nu - \varepsilon_\nu u_\mu + \epsilon_{\mu\nu\rho\sigma} u^\rho \omega^\sigma ,$$

- Vector current

$$\begin{aligned} J^\mu_{(2)} &= -\frac{\mu}{2\pi^2}(\varepsilon^2+\omega^2)u^\mu - \frac{1}{4\pi^2}(\varepsilon\cdot E+\omega\cdot B)u^\mu \\ &\quad -\frac{C}{12\pi^2}(E^2+B^2)u^\mu - \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho \omega_\sigma \\ &\quad -\frac{C}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma , \end{aligned}$$

- Chiral current

$$J^\mu_{5,(2)} = -\frac{\mu_5}{2\pi^2}(\varepsilon^2+\omega^2)u^\mu - \frac{C_5}{12\pi^2}(E^2+B^2)u^\mu - \frac{C_5}{6\pi^2}\epsilon^{\mu\nu\rho\sigma}u_\nu E_\rho B_\sigma ,$$

Coefficients can be found in

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Energy-momentum tensor up to $O(\hbar^1)$

$$T^{\mu\nu}=2\int\frac{d^4p}{(2\pi)^4}\mathcal{V}^\mu p^\nu=2\int\frac{d^4p}{(2\pi)^4}(\mathscr{J}_+^\mu+\mathscr{J}_-^\mu)p^\nu,$$

$$\begin{aligned}
T_{(0)}^{\mu\nu} &= \epsilon u^\mu u^\nu-\frac{1}{3}\epsilon\Theta^{\mu\nu},\\
T_{(1)}^{\mu\nu} &= \rho_5\left(u^\mu\omega^\nu+u^\nu\omega^\mu\right)+\frac{\xi}{2}\left(u^\mu B^\nu+u^\nu B^\mu+\epsilon^{\mu\nu\alpha\beta}u_\alpha E_\beta\right)\\
&\quad -\frac{1}{2}\rho_5\left(u^\mu\omega^\nu-u^\nu\omega^\mu-\epsilon^{\mu\nu\alpha\beta}u_\alpha\varepsilon_\beta\right),
\end{aligned}$$

Gao, Liang, SP, Wang, Wang, PRL 2012

$$\epsilon=\frac{T^4}{4\pi^2}\left[\frac{7}{15}\pi^4+2\pi^2\frac{\mu^2+\mu_5^2}{T^2}+\frac{\mu^4}{T^4}+6\frac{\mu^2\mu_5^2}{T^4}+\frac{\mu_5^4}{T^4}\right]$$

Energy density

Energy-momentum tensor up to $O(\hbar^2)$

$$T^{\mu\nu}_{(2)} = T^{\mu\nu}_{(2),\mathrm{vv}} + T^{\mu\nu}_{(2),\mathrm{ve}} + T^{\mu\nu}_{(2),\mathrm{ee}},$$

$$\begin{aligned}
T^{\mu\nu}_{(2),\mathrm{vv}} &= -\frac{1}{2}\xi_5\left[3u^\mu u^\nu(\omega^2+\varepsilon^2) - \Theta^{\mu\nu}(\omega^2+\varepsilon^2)\right. \\
&\left. -2(u^\mu\epsilon^{\nu\alpha\beta\gamma}+u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\varepsilon_\beta\omega_\gamma - 2(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha\varepsilon_\beta\omega_\gamma\right], \\
T^{\mu\nu}_{(2),\mathrm{ve}} &= -\frac{1}{2}\xi_{B5}\left[u^\mu u^\nu(\omega\cdot B+\varepsilon\cdot E) - (\omega^\mu B^\nu + E^\mu\varepsilon^\nu)\right. \\
&\left. -(u^\mu\epsilon^{\nu\alpha\beta\gamma}+u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha E_\beta\omega_\gamma - 2(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma})u_\alpha E_\beta\omega_\gamma\right],
\end{aligned}$$

$$\begin{aligned}
T^{\mu\nu}_{(2),\mathrm{ee,R}} &= \frac{1}{6\pi^2}\left(\hat{\kappa}_+ + \hat{\kappa}_- + \ln\frac{\Lambda}{T}\right)\left(\frac{1}{4}\eta^{\mu\nu}F_{\gamma\beta}F^{\gamma\beta} - F^{\gamma\mu}F_{\gamma}{}^{\nu}\right) \\
&+\frac{1}{24\pi^2}\left[u^\mu u^\nu E^2 - \Theta^{\mu\nu}\left(E^2+2B^2\right)\right. \\
&+4\left(E^\mu E^\nu + B^\mu B^\nu\right) + 3\left(u^\mu\epsilon^{\nu\alpha\beta\gamma}+u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma \\
&\left.+3\left(u^\mu\epsilon^{\nu\alpha\beta\gamma}-u^\nu\epsilon^{\mu\alpha\beta\gamma}\right)u_\alpha E_\beta B_\gamma\right]
\end{aligned}$$

$$\eta_{\mu\nu}T^{\mu\nu}_{(2)} = \eta_{\mu\nu}T^{\mu\nu}_{(2),\mathrm{ee,R}} = \frac{1}{24\pi^2}F_{\mu\nu}F^{\mu\nu}$$

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After the renormalization, it can verify the trace anomaly

Chiral anomaly and conservation laws

$$
\begin{aligned}
\partial^\mu J_\mu &= 0, \\
\boxed{\partial^\mu J^5_\mu = -\frac{1}{2\pi^2} E\cdot B,} \\
\partial^\mu T_{\mu\nu} &= F_{\nu\mu} J^\mu .
\end{aligned}
$$

2. Discussions on the solution of Wigner function

(2b) Chiral kinetic theory

Comments on f_0

$$\mathscr{J}^{\rho}_{(0)}(x,p) = p^{\rho} f_{(0)}(x,p)\delta(p^2),$$

- **In above discussion, we consider the f_0 as the distribution function at equilibrium.**

- **In general, the f_0 can be an arbitrary distribution. But, general form for the solutions of Wigner function holds.**

See. the study from an arbitrary distribution f_0

Y. Hidaka, SP, D.L. Yang, PRD(RC) (2017)

and Disentanglement of chiral Wigner functions (DWF) theorem:

J.-H. Gao, Z.-T. Liang, Q. Wang, X.-N. Wang, PRD 98 (3) (2018) 036019

Zeroth order - classical kinetic equation

$$
\begin{aligned}
\nabla_{\mu}^{(0)} \mathscr{J}_{(0)}^{\mu} &= \left(\partial_{\mu}^{x} - F_{\mu\nu}\partial_{p}^{\nu}\right) \mathscr{J}_{(0)}^{\mu} \\
&= \left(\partial_{\mu}^{x} - F_{\mu\nu}\partial_{p}^{\nu}\right) \left[p^{\mu} f_{(0)} \delta(p^2)\right] \\
&= \left(\partial_t + \mathbf{E} \cdot \nabla_p\right) \left[p_0 f_{(0)} \delta(p^2)\right] \\
&\quad + \left(\nabla_x + \mathbf{E}\partial_{p_0} + \mathbf{B} \times \nabla_p\right) \cdot \left[\mathbf{p} f_{(0)} \delta(p^2)\right] \\
&= 0,
\end{aligned}
$$

- By an integration of above result over p_0 in the range [0, ∞), we can get the classical Boltzmann equation

$$
\left(\partial_t + \mathbf{v} \cdot \nabla_x\right) f_{(0)} + \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \nabla_p f_{(0)} = 0,
$$

- $v = p/E_p$ is the velocity of the fermion.

Next to leading order – CKT (I)

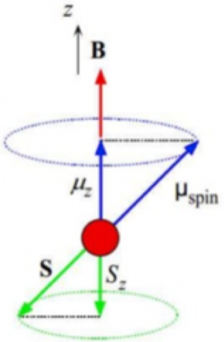
- **We can add the zeroth and first order equations together**

$$p_\mu \nabla^\mu_{(0)} \left[f\delta(\tilde{p}^2) \right] + \hbar \frac{s}{2} \boldsymbol{\nabla}_{(0)} \cdot \left\{ \frac{1}{p_0} \boldsymbol{\nabla}_{(0)} \times \left[\mathbf{p} f_{(0)} \delta(p^2) \right] \right\} = 0,$$

$$\tilde{p}^2 \equiv p^2 + s\hbar(\mathbf{p}\cdot\mathbf{B})/p_0$$

Zeeman effect

$$E_p^{(\pm)} = \pm E_p(1 \mp s\hbar\mathbf{B}\cdot\boldsymbol{\Omega}_p),$$



Spin magnetic moment:

$$\mu_S = -g_S \frac{e}{2m_e} \mathbf{S} \rightarrow -\frac{e}{|\mathbf{p}|} \mathbf{S} \rightarrow \mp \frac{e}{|\mathbf{p}|} \frac{\mathbf{p}}{2|\mathbf{p}|}$$

Massless

Chirality

Zeeman effect:

$$\Delta\varepsilon = -\hbar\mu_S \cdot \mathbf{B} = \mp\hbar \frac{|e|}{|\mathbf{p}|} \frac{\mathbf{p}\cdot\mathbf{B}}{2|\mathbf{p}|}$$

Next to leading order – CKT (II)

- **We can add the zeroth and first order equations together**

$$p_\mu \nabla^\mu_{(0)} \left[f\delta(\widetilde{p}^2)\right] + \hbar\frac{s}{2}\boldsymbol{\nabla}_{(0)}\cdot\left\{\frac{1}{p_0}\boldsymbol{\nabla}_{(0)}\times\left[\mathbf{p}f_{(0)}\delta(p^2)\right]\right\} = 0,$$

$$\widetilde{p}^2 \equiv p^2 + s\hbar(\mathbf{p}\cdot\mathbf{B})/p_0$$

- **By an integration of above result over p_0 in the range [0, ∞), we can get the chiral kinetic theory for on-shell particles**

$$\left\{\begin{array}{r} (1+\hbar s\boldsymbol{\Omega}_p\cdot\mathbf{B})\,\partial_t f(x,E_p,\mathbf{p}) \\ +\left[\mathbf{v}+\hbar s(\widetilde{\mathbf{E}}\times\boldsymbol{\Omega}_p)+\hbar s\dfrac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]\cdot\boldsymbol{\nabla}_x f(x,E_p,\mathbf{p}) \\ +\left[\widetilde{\mathbf{E}}+\mathbf{v}\times\mathbf{B}+\hbar s(\widetilde{\mathbf{E}}\cdot\mathbf{B})\boldsymbol{\Omega}_p\right]\cdot\boldsymbol{\nabla}_p f(x,E_p,\mathbf{p}) = 0, \end{array}\right\}$$

$\mathbf{v}\equiv\boldsymbol{\nabla}_p E_p^{(+)}$ **Effective velocity** $E_p^{(\pm)} = \pm E_p(1\mp s\hbar\mathbf{B}\cdot\boldsymbol{\Omega}_p),$

$\widetilde{\mathbf{E}}\equiv\mathbf{E}-\boldsymbol{\nabla}_x E_p^{(+)}$ **Effective electric field**

Next to leading order – CKT (III)

$$\left\{\begin{aligned} &(1+\hbar s \boldsymbol{\Omega}_p \cdot \mathbf{B})\,\partial_t f(x, E_p, \mathbf{p}) \\ &+ \left[\mathbf{v} + \hbar s(\widetilde{\mathbf{E}} \times \boldsymbol{\Omega}_p) + \hbar s \frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right] \cdot \boldsymbol{\nabla}_x f(x, E_p, \mathbf{p}) \\ &+ \left[\widetilde{\mathbf{E}} + \mathbf{v} \times \mathbf{B} + \hbar s(\widetilde{\mathbf{E}} \cdot \mathbf{B})\boldsymbol{\Omega}_p\right] \cdot \boldsymbol{\nabla}_p f(x, E_p, \mathbf{p}) = 0, \end{aligned}\right\}$$

$\mathbf{v} \equiv \boldsymbol{\nabla}_p E_p^{(+)}$ **Effective velocity**

$$E_p^{(\pm)} = \pm E_p(1 \mp s\hbar \mathbf{B} \cdot \boldsymbol{\Omega}_p),$$

$$\boxed{\widetilde{\mathbf{E}} \equiv \mathbf{E} - \boldsymbol{\nabla}_x E_p^{(+)} \qquad \textbf{Effective electric field}}$$

- **In the quantum field theory at finite temperature or condensed matter systems, the energy of a single particle may depend on space-time.**
    - **e.g. in QFT at finite temperature, with interactions, the mass becomes Debye mass, which is a integration of moment of distributions f(x,p) over momentum.**
    - **In condensed matter, if we set a space-time dependent potential V(t,x), then particle's energy is modified by the V(t,x) and becomes space-time dependent.**

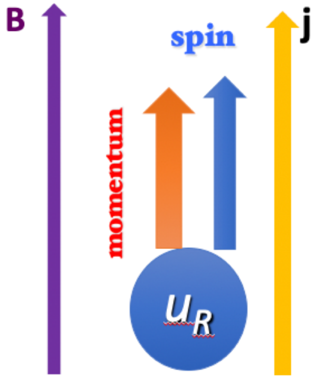
Next to leading order – CKT (IV)

$$\left\{\begin{aligned} &(1+\hbar s\boldsymbol{\Omega}_p\cdot\mathbf{B})\,\partial_t f(x,E_p,\mathbf{p}) \\ &+\boxed{\left[\mathbf{v}+\hbar s(\widetilde{\mathbf{E}}\times\boldsymbol{\Omega}_p)+\hbar s\frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]}\cdot\boldsymbol{\nabla}_x f(x,E_p,\mathbf{p}) \\ &+\left[\widetilde{\mathbf{E}}+\mathbf{v}\times\mathbf{B}+\hbar s(\widetilde{\mathbf{E}}\cdot\mathbf{B})\boldsymbol{\Omega}_p\right]\cdot\boldsymbol{\nabla}_p f(x,E_p,\mathbf{p})=0, \end{aligned}\right\}$$

$\mathbf{v}\equiv\boldsymbol{\nabla}_p E_p^{(+)}$ **Effective velocity** $E_p^{(\pm)}=\pm E_p(1\mp s\hbar\mathbf{B}\cdot\boldsymbol{\Omega}_p),$

$\widetilde{\mathbf{E}}\equiv\mathbf{E}-\boldsymbol{\nabla}_x E_p^{(+)}$ **Effective electric field**

- Correction to effective velocity/w.o. E fields



Particles move parallel or anti-parallel to B

$$\Delta\dot{\mathrm{x}}\propto\mathrm{B}$$

Dimension analysis

$$\Delta\dot{\mathrm{x}}\propto\frac{\mathrm{B}}{|\mathbf{p}|^2}$$

Final results:

$$\Delta\dot{\mathrm{x}}=\hbar\frac{\mathrm{B}}{2|\mathbf{p}|^2}$$

- Correction to effective velocity with E fields

$$\Delta\dot{\mathbf{x}}=\hbar\frac{\mathbf{B}}{2|\mathbf{p}|^2}$$

For moving particles, they feel like:

$$\mathbf{B}\rightarrow\mathbf{B}+\mathbf{E}\times\mathbf{v}$$

$$\Delta\dot{\mathbf{x}}=\hbar\frac{\mathbf{B}}{2|\mathbf{p}|^2}+\hbar\frac{1}{2|\mathbf{p}|^2}\mathbf{E}\times\mathbf{v}$$

Next to leading order – CKT (V)

$$\left\{ \begin{aligned} &(1+\hbar s\boldsymbol{\Omega}_p\cdot\mathbf{B})\,\partial_t f(x,E_p,\mathbf{p}) \\ &+\left[\mathbf{v}+\hbar s(\widetilde{\mathbf{E}}\times\boldsymbol{\Omega}_p)+\hbar s\frac{1}{2|\mathbf{p}|^2}\mathbf{B}\right]\cdot\boldsymbol{\nabla}_x f(x,E_p,\mathbf{p}) \\ &+\boxed{\left[\widetilde{\mathbf{E}}+\mathbf{v}\times\mathbf{B}+\hbar s(\widetilde{\mathbf{E}}\cdot\mathbf{B})\boldsymbol{\Omega}_p\right]}\cdot\boldsymbol{\nabla}_p f(x,E_p,\mathbf{p})=0, \end{aligned} \right\}$$

$\mathbf{v}\equiv\boldsymbol{\nabla}_p E_p^{(+)}$ **Effective velocity**

$$E_p^{(\pm)}=\pm E_p(1\mp s\hbar\mathbf{B}\cdot\boldsymbol{\Omega}_p),$$

$\widetilde{\mathbf{E}}\equiv\mathbf{E}-\boldsymbol{\nabla}_x E_p^{(+)}$ **Effective electric field**

- **Are there corrections to effective force?**

$$\dot{\mathbf{p}}=\mathbf{E}+\frac{\partial\varepsilon}{\partial\mathbf{p}}\times\mathbf{B}+\hbar\ldots$$

- **History: in condensate matter physics:**

Could be neglected!

D. Xiao, M.C. Chang, Q. Niu, Rev. Mod. Phys. 82, 1959 (2010)

- **QFT: Chiral anomaly!**

Son, Yamamoto, PRL, (2012); PRD (2013)

Stephanov, Yin, PRL (2012);

J.W. Chen, SP, Q. Wang, X.N. Wang, PRL (2013);

2. Discussions on the solution of Wigner function

(2c) Lorentz transformation and side jump

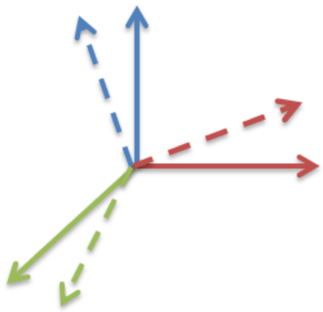
Non-trivial Lorentz transformation

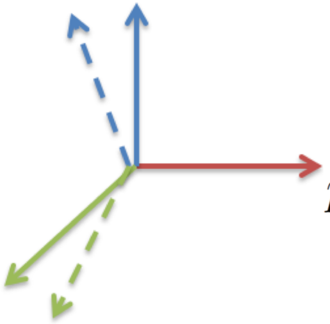
- **The subgroup for Lorentz group for massless fermions and massive fermions are different.**

Massive particles:
(Rest frame)

$$p^{\mu} = (m, 0, 0, 0)$$

Subgroup: SO(3)

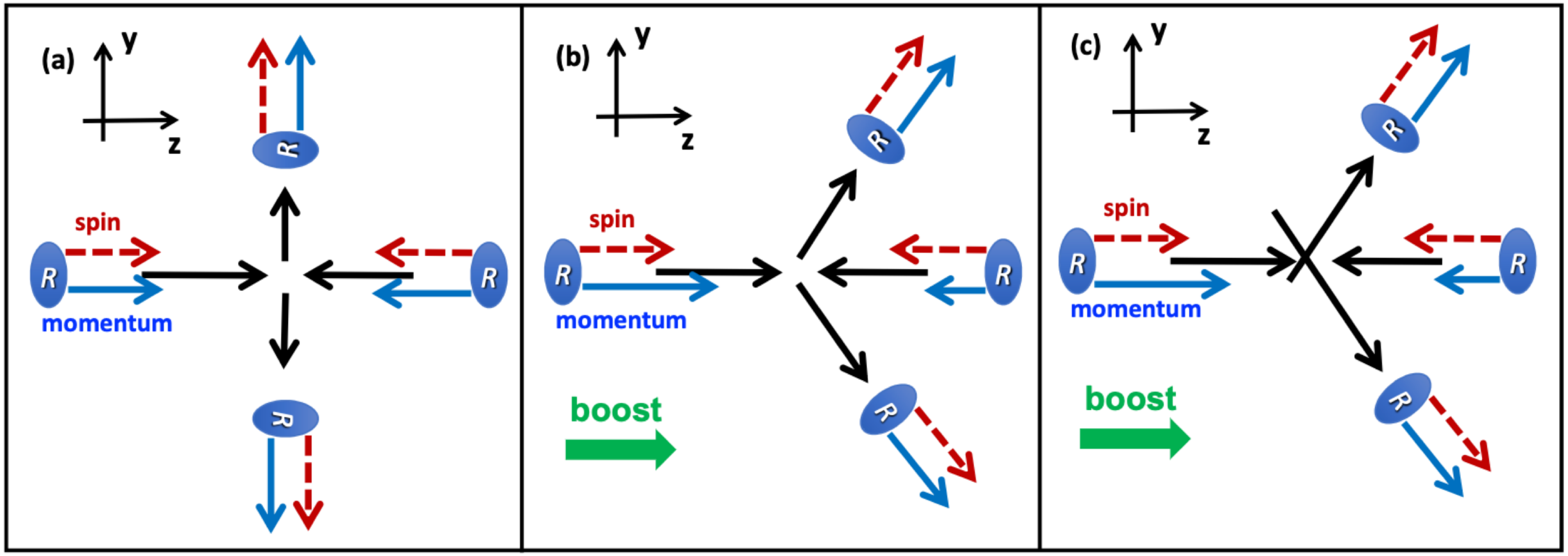




Massless particles:
(No rest frame)

$$p^{\mu} = (|p_z|, 0, 0, p_z)$$

Subgroup: ISO(2)



Chen, Son, Stephanov, PRL, (2015); Y. Hidaka, SP, D.L. Yang, PRD (2016)

Distribution function in classical theory

- **Let us consider a current**

$$\underset{\substack{\uparrow\\ \text{four}\\ \text{vector}}}{j^\mu} = \int d^4p\,\delta(p^2 - m^2)p^\mu f(x,p) = \int \underset{\substack{\uparrow\\ \text{Lorentz}\\ \text{invariant}}}{\boxed{\frac{d^3\mathbf{p}}{(2\pi)^3 2E_p}}}\,\underset{\substack{\uparrow\\ \text{four}\\ \text{vector}}}{p^\mu}\,\underset{\substack{\uparrow\\ \text{scalar}}}{f(x,p)}$$

- **It is obviously that classical distribution function is a scalar.**

Wave function under Lorentz transformation (I)

- General discussion following **Weinberg's textbook**.
- We start from a spinor

$$\psi(x)=\int\frac{d^3p}{(2\pi\hbar)^3\sqrt{2|\mathbf{p}|}}e^{-ip\cdot x/\hbar}v_+(p)a_{\mathbf{p}},$$

- The wave function in free case is given by

$$v_+(p)=\begin{pmatrix}\sqrt{|\mathbf{p}|+p_3}\\ \frac{p_1+ip_2}{\sqrt{|\mathbf{p}|+p_3}}\end{pmatrix}$$

- Then, we consider the infinitesimal Lorentz transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + (\boldsymbol{\beta}\cdot\mathbf{x}, \boldsymbol{\beta}t+\boldsymbol{\theta}\times\mathbf{x}) \equiv x^\mu + \omega^\mu_\nu x^\nu,$$

$$p^\mu \rightarrow p'^\mu = p^\mu + \omega^\mu_\nu p^\nu,$$

Wave function under Lorentz transformation (II)

- Under the infinitesimal Lorentz transformation, we know the standard results

$$\psi_R \to U(\Lambda)\psi_R, \quad U(\Lambda) = 1 + \frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\beta} - \frac{i}{2}\boldsymbol{\sigma}\cdot\theta.$$

- However, we can also compute the wave function directly under the transformation,

$$v_+(\Lambda p) = e^{i\varphi(\Lambda,p)}U(\Lambda)v_+(p),$$

where there is an extra phase factor,

$$\varphi = \frac{\theta_3(|\mathbf{p}|+p_3) - \beta_1 p_2 + \beta_2 p_1 + \theta_1 p_1 + \theta_2 p_2}{2(|\mathbf{p}|+p_3)}$$

- The Lorentz transformation for massless fermions gives an extra phase.

General discussion for f

- For simplicity, we neglect the EM fields,

$$\mathscr{J}_{+}^{\mu} \sim \left\langle : \int d^4 q e^{iq\cdot x} v_{+,p+q}^{\dagger} \sigma^{\mu} v_{+,p-q} (a_{p+q}^{\dagger} a_{p-q}) : \right\rangle$$

- Under Lorentz transformation,

$$\mathscr{J}_{+}^{\mu} \rightarrow \Lambda_{\nu}^{\mu} \mathscr{J}_{+}^{\nu}$$

$$v_{+,p+q}^{\dagger} \sigma^{\mu} v_{+,p-q} \rightarrow \Lambda_{\nu}^{\mu} \exp[i\varphi_{p-q}(\Lambda) - i\varphi_{p+q}(\Lambda)] v_{+,p+q}^{\dagger} \sigma^{\nu} v_{+,p-q}$$

which means

$$(a_{p+q}^{\dagger} a_{p-q}) \rightarrow \exp[-i\varphi_{p-q}(\Lambda) + i\varphi_{p+q}(\Lambda)](a_{p+q}^{\dagger} a_{p-q}).$$

Recalling the definition for distribution function

$$f(x,p) = \int d^4 p \left\langle : (a_{p+q}^{\dagger} a_{p-q}) : \right\rangle e^{ip\cdot x}$$

For massless fermions, the distribution function up to $O(\hbar^1)$ is not a scalar.

Y. Hidaka, SP, D.L. Yang, PRD(RC) (2017)

Side-jump (I)

- Let us focus on the non-trivial terms in the solutions of Wigner function

$$S^{\mu}=\delta(p^2)\left(p^{\mu}f+\hbar\epsilon^{\mu\nu\alpha\beta}\frac{p_{\nu}u_{\alpha}}{2(u\cdot p)}\nabla_{\beta}f\right)+\frac{1}{2}\hbar\epsilon^{\mu\nu\alpha\beta}p_{\nu}F_{\alpha\beta}\frac{\partial\delta(p^2)}{\partial p^2}f.$$

where u is the frame vector in frame F.

- Next, let us consider another frame F’, whose frame vector is u’ at the original frame F. We assume that the distribution function in F’ becomes to f’

$$f'=f+\hbar N^{\mu}\partial^{x}_{\mu}f+\hbar N^{\mu}_{q}\partial^{q}_{\mu}f,$$

- Now, we consider the inverse transformation from F’ to F.

$$\begin{aligned}0&=\Lambda^{\mu}_{\nu}S'^{\nu}-S^{\mu}\\&=\hbar\delta(p^2)\left[p^{\mu}N\cdot\partial_x+p^{\mu}N^{\mu}_{q}\cdot\partial_p+\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}p_{\nu}\left(\frac{u'_{\alpha}}{u'\cdot p}-\frac{u_{\alpha}}{u\cdot p}\right)\nabla_{\beta}f\right],\end{aligned}$$

→ $$N^{\mu}=\frac{\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u'_{\beta}u_{\nu}}{2(u'\cdot p)(u\cdot p)},\quad N^{\mu}_{q}=N_{\nu}F^{\mu\nu}$$

Side-jump (II)

$$f'(x',p') = f(x,p) + \hbar N^\mu \nabla_\mu f(x,p). \; N^\mu = \frac{\epsilon^{\mu\nu\alpha\beta} p_\alpha u'_\beta u_\nu}{2(u'\cdot p)(u\cdot p)}, \; N_q^\mu = N_\nu F^{\mu\nu}$$

Y. Hidaka, SP, D.L. Yang, PRD(RC) (2017)
J.-H. Gao, Z.-T. Liang, Q. Wang, X.-N. Wang, PRD 98 (3) (2018) 036019
Also see:
Duval, Elbistan, Horváthy, Zhang, Phys. Lett. B 742 (2015) 322
Stone, Dwivedi, Zhou, Phys. Rev. Lett. 114 (21) (2015) 210402

- If we choose u=(1,0) and u'=(1,β), under the infinitesimal Lorentz transform

$$\begin{aligned} f'(x',p',t') &= f(x+\boldsymbol{\beta} t, p+\boldsymbol{\beta}\varepsilon, t+\boldsymbol{\beta}\cdot\mathbf{x}) + \hbar N^\nu_{uu'}\Delta_\nu f \\ &= f(x+\delta x, p+\delta p, t+\delta t), \end{aligned}$$

$$\begin{aligned} \delta\mathbf{x} &= \boldsymbol{\beta} t + \hbar\frac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}, \\ \delta\mathbf{p} &= \boldsymbol{\beta}\varepsilon + \hbar\frac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}\times\mathbf{B}, \end{aligned}$$

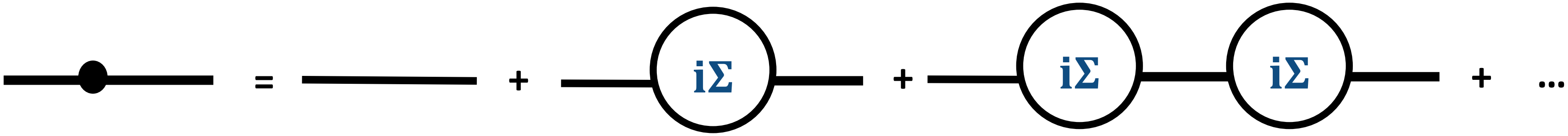
It is the side-jump in condensed matter physics.
Chen, Son, Stephanov, Yee, Y. Yin, PRL 2014

3. Collision effects

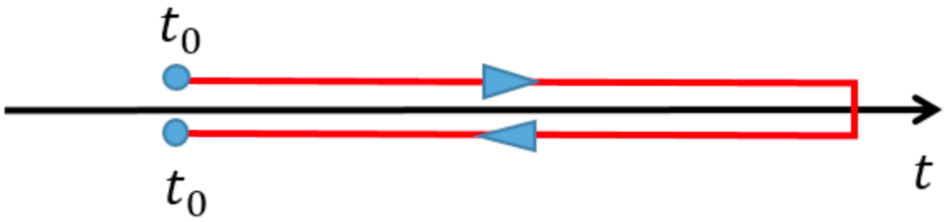
(3a) Kadanoff-Baym equation

Dyson-Schwinger equation in CTP

$$S = S_0 - iS_0\Sigma S = S_0 - iS\Sigma S_0,$$



- **S: full propagator**
- **S_0: free propagator**
- **Σ: self energy**



$$G(x_1,x_2) = \begin{pmatrix} G^{++}(x_1,x_2) & G^{+-}(x_1,x_2) \\ G^{-+}(x_1,x_2) & G^{--}(x_1,x_2) \end{pmatrix} = \begin{pmatrix} G^{F}(x_1,x_2) & \pm G^{<}(x_1,x_2) \\ G^{>}(x_1,x_2) & G^{\bar{F}}(x_1,x_2) \end{pmatrix}$$

$$\begin{aligned} G^{F}_{\alpha\beta}(x_1,x_2) &= \langle T\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle\,, \\ G^{\bar{F}}_{\alpha\beta}(x_1,x_2) &= \langle T_A\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle\,, \\ G^{<}_{\alpha\beta}(x_1,x_2) &= \langle\overline{\psi}_\beta(x_2)\psi_\alpha(x_1)\rangle\,, \\ G^{>}_{\alpha\beta}(x_1,x_2) &= \langle\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\rangle\,, \end{aligned}$$

$$\left\{ \begin{aligned} \Sigma^R &= \Sigma^{++} - \Sigma^{+-} = \Sigma^{-+} - \Sigma^{--}, \\ \Sigma^A &= \Sigma^{++} - \Sigma^{-+} = \Sigma^{+-} - \Sigma^{--}, \\ i\Sigma^{+-} &= -\Sigma^{<}, \\ i\Sigma^{-+} &= \Sigma^{>}, \end{aligned} \right\}$$

Equation of motion for S (I)

- We rewrite the S in CTP and get

$$\begin{aligned} S^< &= S_0^< - S_0^R \Sigma^R S^< + S_0^R \Sigma^< S^A - S_0^< \Sigma^A S^A \\ &= S_0^< - S^R \Sigma^R S_0^< + S^R \Sigma^< S_0^A - S^< \Sigma^A S_0^A . \end{aligned}$$

- Similar to the QFT at zero temperature, the free parts of retarded and advanced propagators satisfy

$$\begin{aligned} -i\sigma \cdot D_{x_1} S_0^R(x_1, x_2) &= \delta^{(4)}(x_1 - x_2), \\ S_0^A(x_1, x_2) i\sigma \cdot \overleftarrow{D}^\dagger_{x_2} &= \delta^{(4)}(x_1 - x_2). \end{aligned}$$

which leads to

$$\begin{aligned} i\sigma \cdot D_{x_1} S^< &= \Sigma^R S^< - \Sigma^< S^A, \\ -S^< i\sigma \cdot \overleftarrow{D}^\dagger_{x_2} &= -S^R \Sigma^< + S^< \Sigma^A, \end{aligned}$$

Equation of motion for S (II)

- The retarded and advanced functions are further decomposed into

$$S^R = \overline{S} + \frac{i}{2}(S^> + S^<), \quad S^A = \overline{S} - \frac{i}{2}(S^> + S^<),$$

$$\Sigma^R = \overline{\Sigma} - \frac{i}{2}(\Sigma^> + \Sigma^<), \quad \Sigma^A = \overline{\Sigma} + \frac{i}{2}(\Sigma^> + \Sigma^<),$$

$$\overline{S} \equiv (S^R + S^A)/2 \text{ and } \overline{\Sigma} \equiv (\Sigma^R + \Sigma^A)/2.$$

- we obtain the quantum kinetic equation with collisions in coordinate space,

$$\left\{\begin{array}{rcl} (i\sigma \cdot D_{x_1} - \overline{\Sigma})S^< + \Sigma^< \overline{S} & = & -\frac{i}{2}(\Sigma^> S^< - \Sigma^< S^>), \\ S^<(-i\sigma \cdot \overleftarrow{D}^{\dagger}_{x_2} - \overline{\Sigma}) + \overline{S}\Sigma^< & = & \frac{i}{2}(S^< \Sigma^> - S^> \Sigma^<). \end{array}\right\}$$

$$\left\{\begin{array}{rcl} (i\sigma \cdot D_{x_1} - \overline{\Sigma})S^> + \Sigma^> \overline{S} & = & -\frac{i}{2}(\Sigma^< S^> - \Sigma^> S^<), \\ S^>(-i\sigma \cdot \overleftarrow{D}^{\dagger}_{x_2} - \overline{\Sigma}) + \overline{S}\Sigma^> & = & \frac{i}{2}(S^> \Sigma^< - S^< \Sigma^>). \end{array}\right\}$$

Gauge link in Wigner transformation (I)

$$\Sigma S = \int d^4 x \Sigma(x_1, z) S(z, x_2)$$

- **To consider the gauge link, we consider the following gauge-invariant product,**

$$\int d^4 z U(x_2, x_1) A(x_1, z) B(z, x_2) = \int d^4 z U_C(x_1, x_2, z) \widetilde{A}(x_1, z) \widetilde{B}(z, x_2),$$

$$\widetilde{A}(x_1, z) = A(x_1, z) U(z, x_1), \ \widetilde{B}(z, x_2) = B(z, x_2) U(x_2, z);$$

$$U_C(x_2, x_1, z) = \exp\left(-\frac{i}{\hbar}\int_L dx^\mu A_\mu\right) = U(x_2, x_1) U(x_1, z) U(z, x_2).$$

L represents the closed path: $z \to x_1 \to x_2 \to z$

- **Let us consider the Wigner transformation of the above product**

$$\begin{aligned} \widetilde{A}(x,p) \star \widetilde{B}(x,p) \quad \equiv \quad & \int d^4 y \exp\left(\frac{i}{\hbar} p \cdot y\right) \int d^4 z U_C\left(x - \frac{y}{2}, x + \frac{y}{2}, z + x\right) \\ & \times \widetilde{A}\left(x + \frac{y}{2}, z + x\right) \widetilde{B}\left(z + x, x - \frac{y}{2}\right), \end{aligned}$$

Gauge link in Wigner transformation (II)

- After a long calculation, we can get

$$\begin{aligned}\widetilde{A}(x,p)\star\widetilde{B}(x,p) &= \exp\left[-\frac{i\hbar}{2}H(-i\hbar\partial_{p''}\cdot\partial_x, i\hbar\partial_{p'}\cdot\partial_x)F_{\mu\nu}(x)\partial^\mu_{p'}\partial^\nu_{p''}\right] \\ &\quad\times\exp\left[\frac{i\hbar}{2}(\partial_{x''}\cdot\partial_{p'}-\partial_{x'}\cdot\partial_{p''})\right]\times\widetilde{A}(x',p')\widetilde{B}(x'',p'')\Big|_{x'=x''=x,p'=p''=x},\end{aligned}$$

$$H(a,b)=\frac{2}{ab(a-b)}\left[b\exp\left(\frac{b-a}{2}\right)-a\exp\left(\frac{a-b}{2}\right)\right]+\frac{2}{ab}\exp\left(\frac{a+b}{2}\right)$$

- Up to $O(\hbar)$, the Moyal product is expanded as

$$\left\{\begin{aligned}\widetilde{A}(x,p)\star\widetilde{B}(x,p) &= \widetilde{A}(x,p)\widetilde{B}(x,p)+\frac{i\hbar}{2}\left\{\widetilde{A}(x,p),\widetilde{B}(x,p)\right\}_{\text{P.B.}} \\ &\quad-\frac{i\hbar}{2}F_{\mu\nu}\partial^\mu_p\widetilde{A}(x,p)\partial^\nu_p\widetilde{B}(x,p)+O(\hbar^2),\end{aligned}\right\}$$

$$\left\{\widetilde{A},\widetilde{B}\right\}_{\text{P.B.}}\equiv(\partial^\mu_p\widetilde{A})(\partial^x_\mu\widetilde{B})-(\partial^x_\mu\widetilde{A})(\partial^\mu_p\widetilde{B}).$$

Kadanoff-Baym equation

- After performing the Wigner transformation of the equations of motion for S and adding the gauge linke, we can get

$$\left\{\begin{array}{rcl}\sigma^\mu\left(\Pi_\mu+\frac{1}{2}i\hbar\nabla_\mu\right)S^<-\hbar\overline{\Sigma}\star S^<+\hbar\Sigma^<\star\overline{S} & = & \frac{i\hbar}{2}(\Sigma^<\star S^>-\Sigma^>\star S^<),\\ \left(\Pi_\mu-\frac{1}{2}i\hbar\nabla_\mu\right)S^<\sigma^\mu-\hbar S^<\star\overline{\Sigma}+\hbar\overline{S}\star\Sigma^< & = & -\frac{i\hbar}{2}(S^>\star\Sigma^<-S^<\star\Sigma^>).\end{array}\right\}$$

Note that, in the above equations, S is gauge invariant full propagator.

CKT with collisions

$$\left\{ \begin{array}{rcl} \nabla_\mu S^{<,\mu} + i\left[\overline{\Sigma}_\mu, S^{<,\mu}\right]_\star - i\left[\Sigma^<_\mu, \overline{S}^\mu\right]_\star & = & C^\mu_\mu, \\ \Pi_\mu S^{<,\mu} - \frac{\hbar}{2}\left\{\overline{\Sigma}_\mu, S^{<,\mu}\right\}_\star + \frac{\hbar}{2}\left\{\Sigma^<_\mu, \overline{S}^\mu\right\}_\star & = & -\frac{\hbar^2}{4}D^\mu_\mu, \\ \Pi^{[\nu}S^{<,\mu]} - \frac{\hbar}{2}\left\{\overline{\Sigma}^{[\nu}, S^{<,\mu]}\right\}_\star + \frac{\hbar}{2}\left\{\Sigma^{<,[\nu}, \overline{S}^{\mu]}\right\}_\star & = & \frac{\hbar}{2}\epsilon^{\mu\nu\rho\sigma}\left(\nabla_\rho S^<_\sigma - C_{\rho\sigma}\right) + \frac{\hbar^2}{4}D^{[\mu\nu]}, \end{array} \right\}$$

$$C_{\mu\nu} = \frac{1}{2}\left(\left\{\Sigma^<_\mu, S^>_\nu\right\}_\star - \left\{\Sigma^>_\mu, S^<_\nu\right\}_\star\right),$$

$$D_{\mu\nu} = \frac{1}{i\hbar}\left(\left[\Sigma^<_\mu, S^>_\nu\right]_\star - \left[\Sigma^>_\mu, S^<_\nu\right]_\star\right),$$

$$\{A, B\}_\star = A \star B + B \star A,\ [A, B]_\star = A \star B - B \star A,$$

$$A^{[\mu}B^{\nu]} \equiv A^\mu B^\nu - A^\nu B^\mu.$$

3. Collision effects

(3b) General solution of Wigner function with collisions

CKT with collisions

$$\left\{ \begin{array}{rcl} \nabla_\mu S^{<,\mu} + i\left[\cancel{\overline{\Sigma}_\mu}, S^{<,\mu}\right]_\star - i\left[\Sigma^<_\mu, \cancel{\overline{S}^\mu}\right]_\star & = & C^\mu_\mu, \\ \Pi_\mu S^{<,\mu} - \frac{\hbar}{2}\left\{\cancel{\overline{\Sigma}_\mu}, S^{<,\mu}\right\}_\star + \frac{\hbar}{2}\left\{\Sigma^<_\mu, \cancel{\overline{S}^\mu}\right\}_\star & = & -\frac{\hbar^2}{4}D^\mu_\mu, \\ \Pi^{[\nu}S^{<,\mu]} - \frac{\hbar}{2}\left\{\cancel{\overline{\Sigma}^{[\nu}}, S^{<,\mu]}\right\}_\star + \frac{\hbar}{2}\left\{\Sigma^{<,[\nu}, \cancel{\overline{S}^{\mu]}}\right\}_\star & = & \frac{\hbar}{2}\epsilon^{\mu\nu\rho\sigma}\left(\nabla_\rho S^<_\sigma - C_{\rho\sigma}\right) + \frac{\hbar^2}{4}D^{[\mu\nu]}, \end{array} \right\}$$

$$C_{\mu\nu} = \frac{1}{2}\left(\left\{\Sigma^<_\mu, S^>_\nu\right\}_\star - \left\{\Sigma^>_\mu, S^<_\nu\right\}_\star\right),$$

$$D_{\mu\nu} = \frac{1}{i\hbar}\left(\left[\Sigma^<_\mu, S^>_\nu\right]_\star - \left[\Sigma^>_\mu, S^<_\nu\right]_\star\right),$$

$$\{A, B\}_\star = A \star B + B \star A,\ [A, B]_\star = A \star B - B \star A,$$

$$A^{[\mu}B^{\nu]} \equiv A^\mu B^\nu - A^\nu B^\mu.$$

CKT with collisions up to $O(\hbar)$

- For S

$$S^{<,\mu} = 2\pi p^\mu f\delta(p^2) + \pi\hbar\epsilon^{\mu\nu\rho\sigma}p_\nu F_{\rho\sigma} f\delta'(p^2) + 2\pi\hbar S^{\mu\nu}_{(n)}(\nabla_\nu f - \boxed{C_\nu})\delta(p^2) + O(\hbar^2),$$

$$C_\mu = \Sigma^<_{(0)\mu}(1 - f_{(0)}) - \Sigma^>_{(0)\mu} f_{(0)}.$$

- For $S^>$, one simply needs to replace f by (1 – f).
- The CKT becomes

$$0 = \delta\left(p^2 - \hbar\frac{B_{(n)}\cdot p}{p\cdot n}\right)\left[\left(p\cdot\hat{\mathcal{D}} + \frac{\hbar S^{\mu\nu}_{(n)}E_{(n)\mu}}{p\cdot n}\hat{\mathcal{D}}_\nu + \hbar S^{\mu\nu}_{(n)}(\partial_\mu F_{\rho\nu})\partial^\rho_p + \hbar(\partial_\mu S^{\mu\nu}_{(n)})\hat{\mathcal{D}}_\nu\right)f - \hbar S^{\mu\nu}_{(n)}\left((1 - f)\nabla_\mu\Sigma^<_\nu - f\nabla_\mu\Sigma^>_\nu\right)\right],$$

$$\hat{\mathcal{D}}_\mu f = \nabla_\mu f - C_\mu$$

Y. Hidaka, S. Pu, D.-L. Yang, PRD(RC), 2017

3. Collision effects

(3c) Collision term for QED in HTL approximation

Quantum kinetic theory (massive fermions)

- Collision term with quantum corrections

  Weickgenannt, Sheng, Speranza, Wang, Rischke, PRD (2019); PRL (2021)

  Hattori, Hidaka, Yang, PRD100, 096011 (2019); Yang, Hattori, Hidaka, arXiv: 2002.02612.

  Liu, Mameda, Huang, arXiv:2002.03753.

  Wang, Guo, Shi, Zhuang, PRD100, 014015 (2019); Wang, Guo, Zhuang, EPJC (2021);Wang, Zhuang, arXiv:2105.00915

  Li ,Yee, PRD100, 056022 (2019)

  Hou, Lin, arXiv: 2008.03862; Lin, arXiv: 2109.00184

  Fang, SP, Yang, PRD (2022)

  Z.Y. Wang, arXiv:2205.09334; Lin, Wang, arXiv:2206.12573

Recent reviews:

  Gao, Ma, SP, Wang, NST 31 (2020) 9, 90

  Gao, Liang, Wang, IJMPA 36 (2021), 2130001

  Hidaka, SP, Yang, Wang, arXiv:2201.07644

Collisional kernel

- **An example for collision kernel of NJL type interactions:**

Eq. for Particle distribution function

$$\frac{1}{E_p} p\cdot\partial_x \mathrm{tr}\left[f^{(1)}(x,p)\right] = -\frac{1}{\pi\hbar}\int_0^\infty dp_0\, \mathrm{Im}\,\mathrm{Tr}\left(I^{(2)}_{\mathrm{coll}}\right) - \frac{1}{2\pi\hbar m}\mathrm{Re}\,\mathrm{Tr}\left(\gamma\cdot\partial_x I^{(1)}_{\mathrm{coll}}\right)$$

$$\equiv \mathscr{C}_{\mathrm{scalar}}\left(\Delta I^{(1)}_{\mathrm{coll,\,qc}}\right) + \mathscr{C}_{\mathrm{scalar}}\left(\Delta I^{(1)}_{\mathrm{coll},\,\nabla}\right) + \mathscr{C}_{\mathrm{scalar}}\left(I^{(0)}_{\mathrm{coll,PB}}\right) + \mathscr{C}_{\mathrm{scalar}}\left(\partial_x I^{(1)}_{\mathrm{coll}}\right) ,$$

Eq. for Spin distribution function

$$\frac{1}{E_p} p\cdot\partial_x \mathrm{tr}\left[n_j^{(+)\mu}\tau_j^T f^{(1)}(x,p)\right] = \frac{1}{2\pi\hbar m}\int_0^\infty dp_0 \left[\epsilon^{\mu\nu\alpha\beta}p_\nu \mathrm{Im}\,\mathrm{Tr}\left(\sigma_{\alpha\beta} I^{(2)}_{\mathrm{coll}}\right) + \mathrm{Re}\,\mathrm{Tr}\left(\gamma^5 \partial_x^\mu I^{(1)}_{\mathrm{coll}}\right)\right]$$

$$\equiv \mathscr{C}^\mu_{\mathrm{pol}}\left(\Delta I^{(1)}_{\mathrm{coll,\,qc}}\right) + \mathscr{C}^\mu_{\mathrm{pol}}\left(\Delta I^{(1)}_{\mathrm{coll},\,\nabla}\right) + \mathscr{C}^\mu_{\mathrm{pol}}\left(I^{(0)}_{\mathrm{coll,PB}}\right) + \mathscr{C}^\mu_{\mathrm{pol}}\left(\partial_x I^{(1)}_{\mathrm{coll}}\right) .$$

$\mathscr{C}_{\mathrm{scalar}}\left(\Delta I^{(1)}_{\mathrm{coll,\,qc}}\right)$ $\mathscr{C}_{\mathrm{scalar}}\left(\Delta I^{(1)}_{\mathrm{coll},\,\nabla}\right)$ $\mathscr{C}_{\mathrm{scalar}}\left(\partial_x I^{(1)}_{\mathrm{coll}}\right)$ $\mathscr{C}_{\mathrm{scalar}}\left(I^{(0)}_{\mathrm{coll,PB}}\right)$

$\mathscr{C}^\mu_{\mathrm{pol}}\left(\Delta I^{(1)}_{\mathrm{coll,\,qc}}\right)$ $\mathscr{C}^\mu_{\mathrm{pol}}\left(\Delta I^{(1)}_{\mathrm{coll},\,\nabla}\right)$ $\mathscr{C}^\mu_{\mathrm{pol}}\left(\partial_x I^{(1)}_{\mathrm{coll}}\right)$ $\mathscr{C}^\mu_{\mathrm{pol}}\left(I^{(0)}_{\mathrm{coll,PB}}\right)$

Perturbative Correction to Ordinary terms

Non-local terms related to the space derivatives may be the key to describe the spin-orbital transformation.

Sheng, Weickgenannt, Speranza, Rischke, Wang PRD (2021)

Challenge the collisional kernel

- **Theory:**
    - Collison kernel needs to be further simplified.

- **Simulations: it is changeling to simulate the QKT:**
    - Collison kernel is high dimensional integrals.
    - One needs to consider the non-local terms.

- **Usually, to solve kinetic theory, one can use the cross section + MC sampling instead of integrating the collision kernel. But, it would fail in quantum kinetic theory with collisions.**
- **We may need to face the high dimensional integrations in collision kernel.**

Theory: Collisions for gauge fields and spin polarization (I)

- We have derived collision kernel for QED in HTL approximation.

Eq. for Particle distribution function

$$(p\cdot\partial)f_V^{<}(x,p)=\mathcal{C}_V^{\mathrm{HTL}}[f_V]+\mathcal{O}(\hbar^2),$$

Eq. for Spin distribution function

$$(p\cdot\partial)f_A^{<}(x,p)+\hbar\partial_\mu S_{(u)}^{\mu\nu}\partial_\nu f_V^{<}(x,p)=\mathcal{C}_A^{\mathrm{HTL}}[f_V,f_A]+\mathcal{O}(\hbar^2),$$

- For the first time, the real QED type collision kernel for axial part:

$$\begin{aligned}\mathcal{C}_A[f_V,f_A]=&-\frac{e^4\delta(p^2)}{8\pi^2|\boldsymbol{p}|}\ln\frac{T}{m_D}\left\{\frac{2\pi^2}{3\beta^2}|\boldsymbol{p}|F(p)f_A^{<}(p)+\frac{\pi^2}{3\beta^2}|\boldsymbol{p}|^2F(p)[(\hat{p}_\perp\cdot\partial_{p_\perp})-\frac{1}{\beta}(\partial_{p_\perp}\cdot\partial_{p_\perp})]f_A^{<}(p)\right.\\&-\frac{2\pi^2}{3\beta^2}|\boldsymbol{p}|^2f_A^{<}(p)(\hat{p}_\perp\cdot\partial_{p_\perp})f_V^{<}(p)+\hbar F(p)|\boldsymbol{p}|H_{3,\alpha}\partial_{p_\perp}^{\alpha}f_V^{<}(p)\\&-\hbar\frac{\pi^2}{12\beta^2}F(p)|\boldsymbol{p}|\epsilon^{\rho\alpha\nu\beta}\hat{p}_{\perp,\nu}u_\beta\partial_{p_\perp,\rho}\partial_\alpha f_V^{<}(p)+\hbar\frac{\pi^2}{6\beta^3}\epsilon^{\rho\alpha\nu\beta}\hat{p}_{\perp,\rho}u_\beta\partial_{p_\perp,\nu}\partial_\alpha f_V^{<}(p)\\&+\hbar\frac{\pi^2}{6\beta^2}\epsilon^{\mu\xi\lambda\kappa}p_\lambda u_\kappa\partial_\xi f_V^{<}(p)\partial_{p_\perp,\mu}f_V^{<}(p)\\&\left.-\hbar\frac{\pi^2}{12\beta^3}|\boldsymbol{p}|\epsilon^{\rho\alpha\nu\beta}\hat{p}_{\perp,\nu}u_\beta\hat{p}_{\perp,(\gamma}g_{\lambda)\rho}\hat{p}_{\perp,\lambda}\partial_{p_\perp}^{\lambda}\partial_{p_\perp}^{\gamma}\partial_\alpha f_V^{<}(p)\right\}+\mathcal{O}(\hbar^2).\end{aligned}\qquad(65)$$

S. Fang, SP, D.L. Yang, PRD (2022)

Theory: Collisions for gauge fields and spin polarization (II)

- **We have proved that dynamical spin polarization for a probe is much slower than its thermalization.**

$$\frac{\text{Spin polarization time}}{\text{Thermalization time}} \approx \frac{\Gamma_A(p)}{\Gamma_V(p)} \approx \frac{\hbar H_{3,\alpha}}{T^2|\boldsymbol{p}|} \sim \mathcal{O}\left(\frac{\partial}{|\boldsymbol{p}|}\right),$$

Also see Wang, Guo, Zhuang, EPJC (2021);Wang, Zhuang, arXiv:2105.00915

- **We also derive the Boltzmann equation for spin evolution:**

$$(p\cdot\partial)f_A^{<}(x,p) + \hbar\partial_\mu S_{(u)}^{\mu\nu}\partial_\nu f_{V,leq}^{<}(x,p) = \mathcal{C}_A^{\mathrm{HTL}}[f_{V,leq}, f_A] + \mathcal{O}(\hbar^2),$$

$$\begin{aligned}
C_A^{\mathrm{HTL}}[f_{V,leq}, f_A] = &-\frac{e^4}{16\pi^3}\frac{\pi^2}{3\beta^2}\ln\frac{T}{m_D}\Big\{2\left(f_{V,leq}^{>}(p) - f_{V,leq}^{<}(p)\right) + 2|\boldsymbol{p}|\beta f_{V,leq}^{<}(p) f_{V,leq}^{>}(p)\\
&+|\boldsymbol{p}|\left[\left(f_{V,leq}^{>}(p) - f_{V,leq}^{<}(p)\right)\hat{p}_\perp\cdot\partial_{p_\perp} - \frac{1}{\beta}(\partial_{p_\perp}\cdot\partial_{p_\perp})\right]\Big\} f_A^{<}(p)\\
&+\hbar\frac{e^4}{16\pi^3|\boldsymbol{p}|}\frac{\pi^2}{3\beta^3}\ln\frac{T}{m_D} S_{(u)}^{\alpha\nu}\Omega_{\alpha\nu} f_{V,leq}^{<}(p) f_{V,leq}^{>}(p) + \mathcal{O}(\hbar^2),
\end{aligned}$$

S. Fang, SP, D.L. Yang, PRD (2022)

Outline

- **Part 1:**

  Chiral magnetic effect, Berry phase and kinetic theory

- **Part 2:**

  Wigner functions and the master equations

- **Part 3:**

  Quantum kinetic theory in massless limit and collisions

- **Part 4:**

  Applications to heavy ion physics

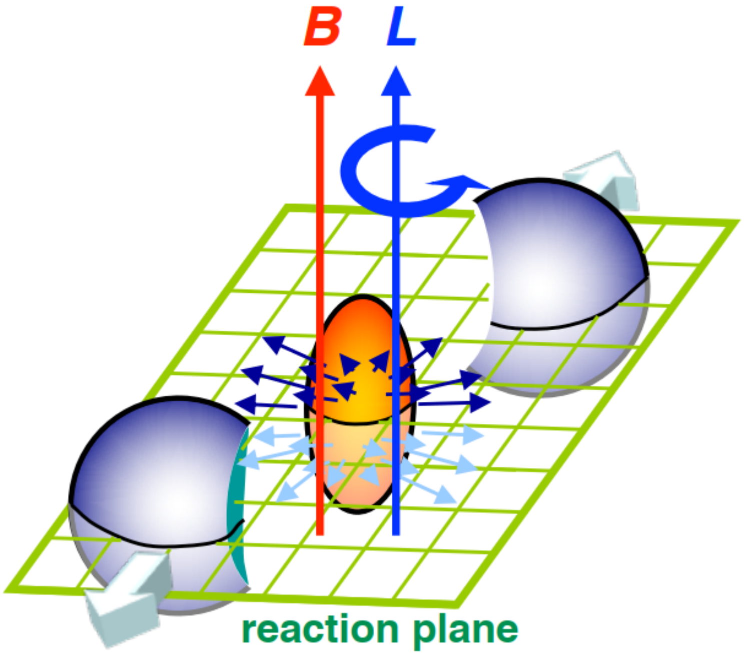
Part 4

Applications to heavy ion physics

1. Spin polarization in relativistic heavy ion collisions
2. Recent development on QKT
3. Applications to spin polarization

1. Spin polarization in relativistic heavy ion collisions

Huge angular momentum

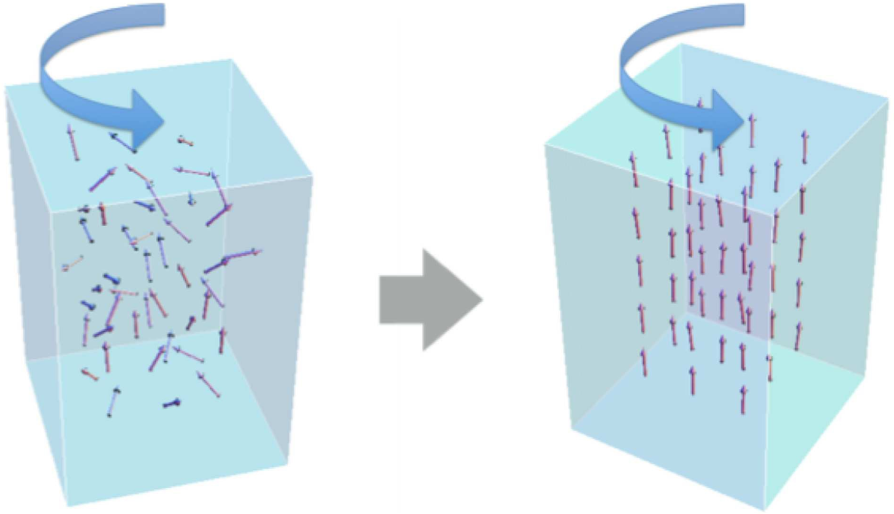


- Huge global orbital angular momenta are produced

$$L \sim 10^5 \hbar$$

- How do orbital angular momenta be transferred to the matter created?

Barnet effects and Einstein-de Hass effects



Barnett effect:

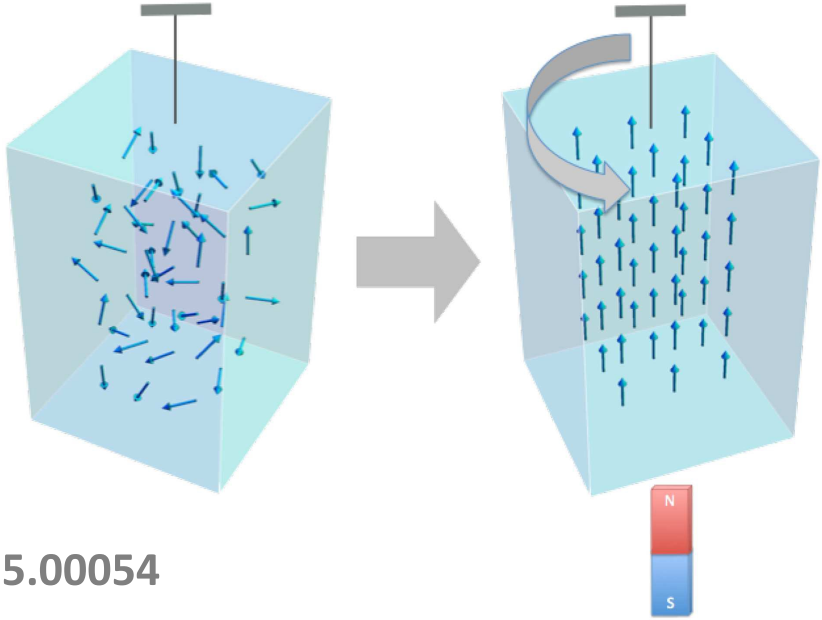
Rotation $\Longrightarrow$ Magnetization

Barnett, Magnetization by rotation, Phys Rev. (1915) 6:239–70.

Einstein-de Haas effect:

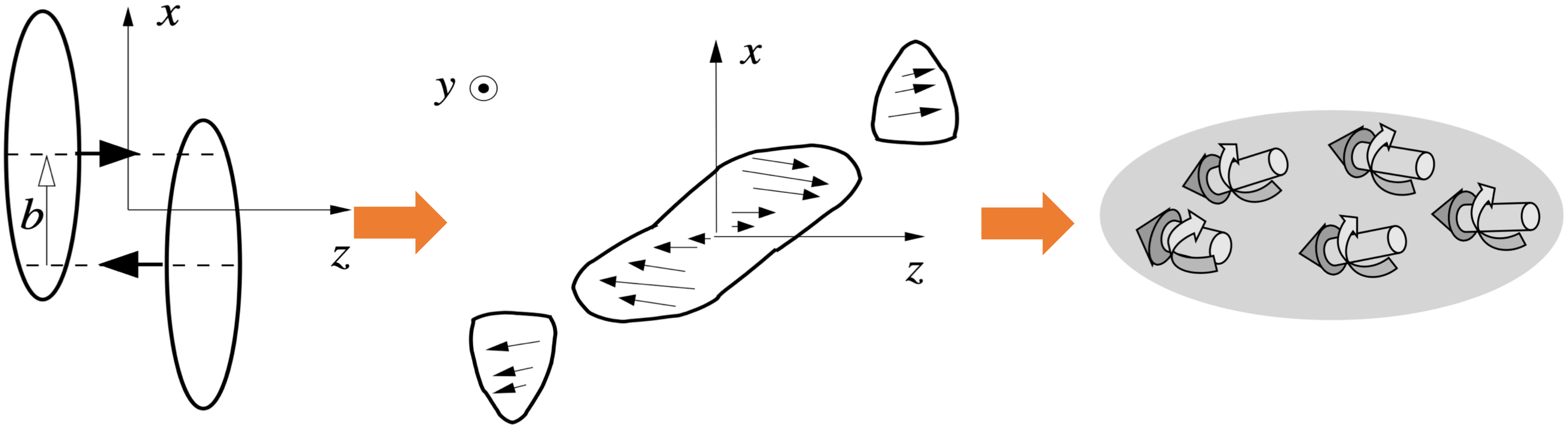
Magnetization $\Longrightarrow$ Rotation

Einstein, de Haas, Experimental proof of the existence of Ampere's molecular currents. Verh Dtsch Phys Ges. (1915) 17:152.



Figures: copy from paper doi: 10.3389/fphy.2015.00054

Global orbital angular momentum in HIC



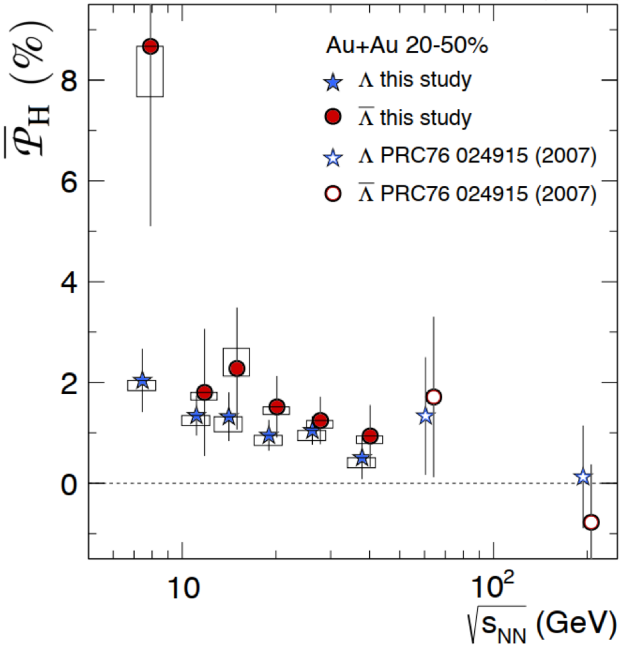
- Global orbital angular momentum leads to the polarizations of Λ hyperons and vector mesons through spin-orbital coupling.

Liang and Wang, PRL 94,102301(2005); PLB 629, 20(2005)

Gao, Chen, Deng, Liang, Wang, Wang, PRC (2008)

Global Polarization of Λ and $\overline{\Lambda}$

STAR, Nature 548, 62 (2017)

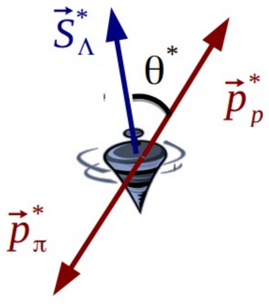


parity-violating decay of hyperons

In case of Λ's decay, daughter proton preferentially decays in the direction of Λ's spin (opposite for anti-Λ)

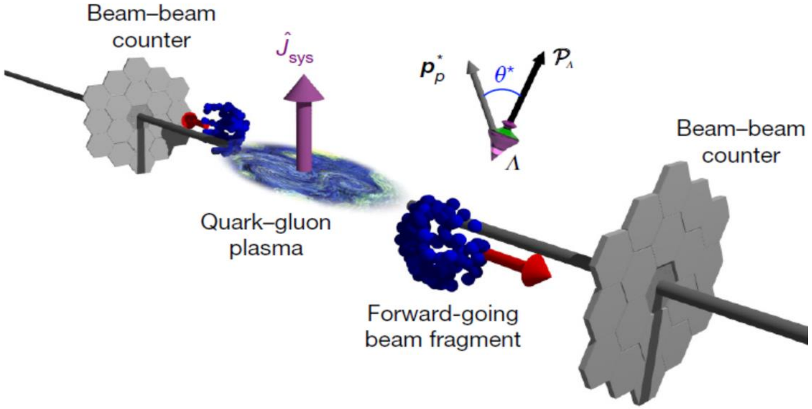
$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha \mathbf{P_\Lambda} \cdot \mathbf{p_p^*})$$

α: Λ decay parameter (=0.642±0.013)
P_Λ: Λ polarization
p_p^*: proton momentum in Λ rest frame



$$\Lambda \to p + \pi^+$$

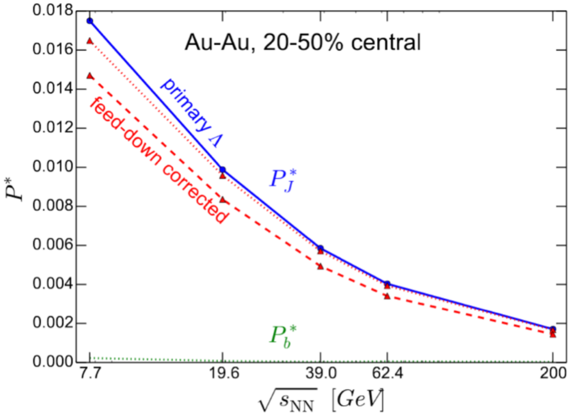
(BR: 63.9%, cτ~7.9 cm)



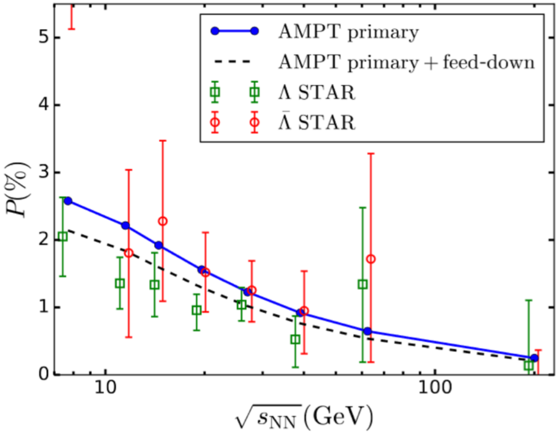
- **The lower energy, the stronger polarization effects.**
- **ω = (9 ± 1)x10^{21}/s, greater than previously observed in any system.**

Liang, Wang, PRL (2005)
Betz, Gyulassy, Torrieri, PRC (2007)
Becattini, Piccinini, Rizzo, PRC (2008)
Becattini, Karpenko, Lisa, Upsal, Voloshin, PRC (2017)
Fang, Pang, Q. Wang, X. Wang, PRC (2016)

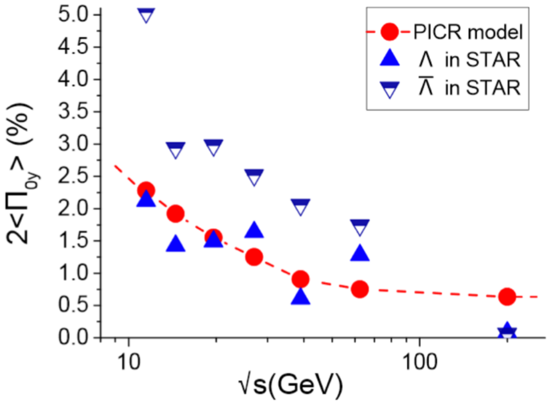
Global Polarization from different models



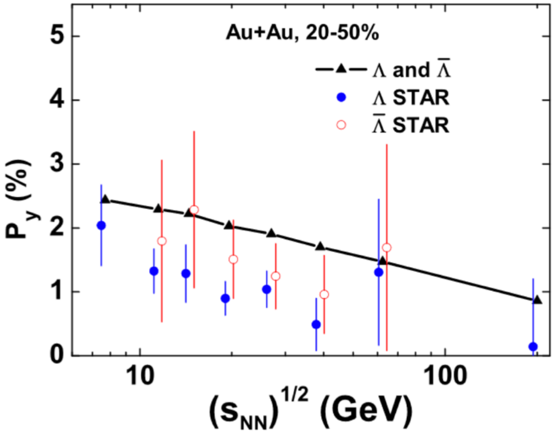
Karpenko, Becattini, EPJC(2017)



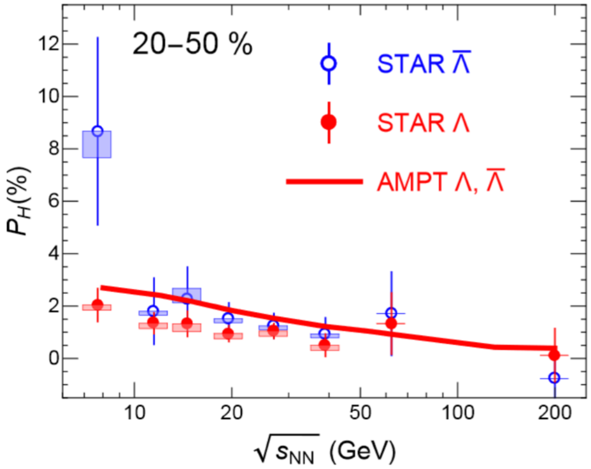
Li, Pang, Wang, Xia PRC(2017)



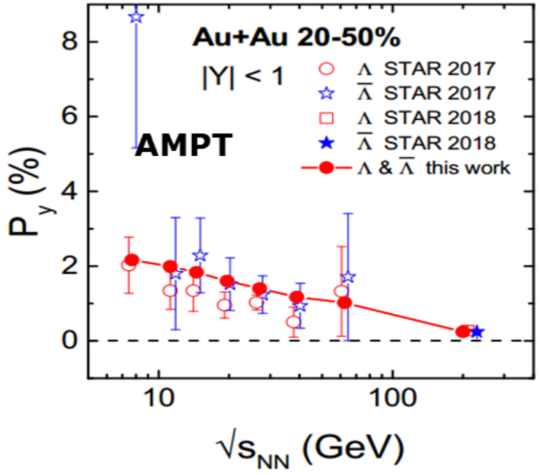
Xie, Wang, Csernai, PRC(2017)



Sun, Ko, PRC(2017)

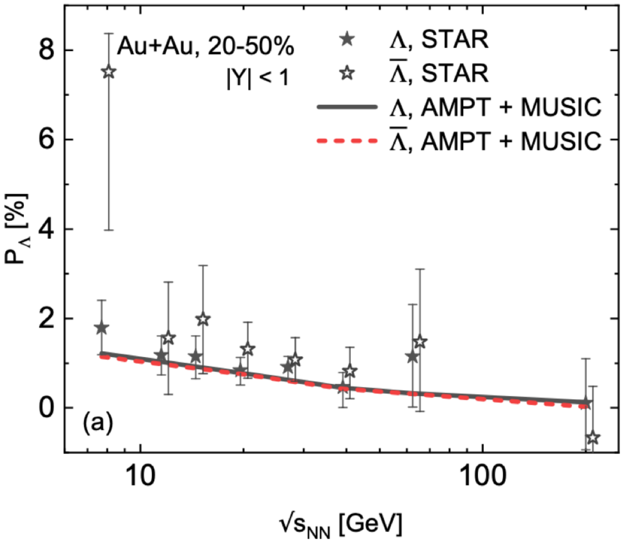


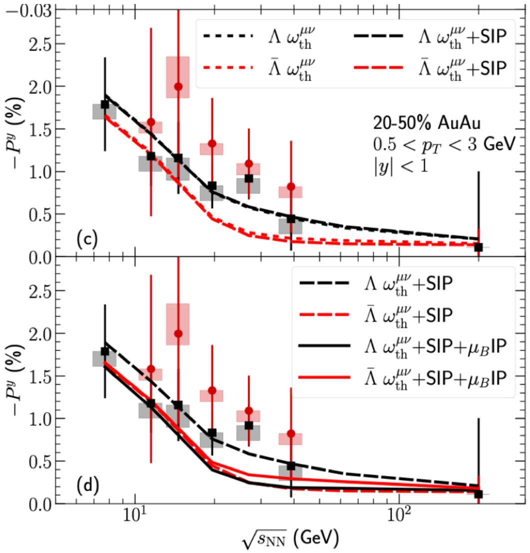
Shi, Li, Liao, PLB(2018)

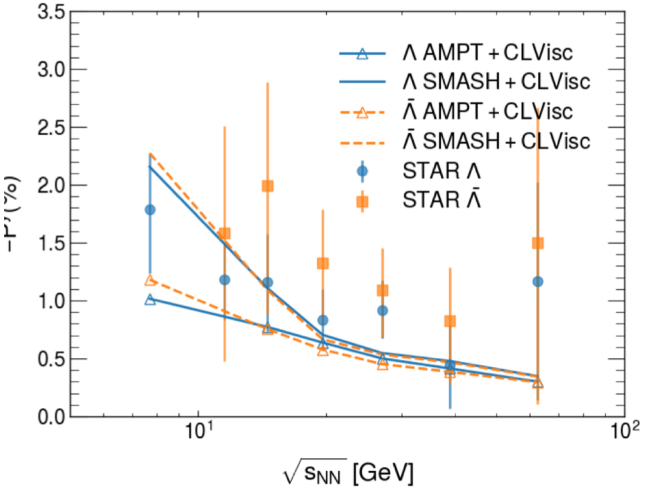


Wei, Deng, Huang, PRC(2019)

Global Polarization from different models





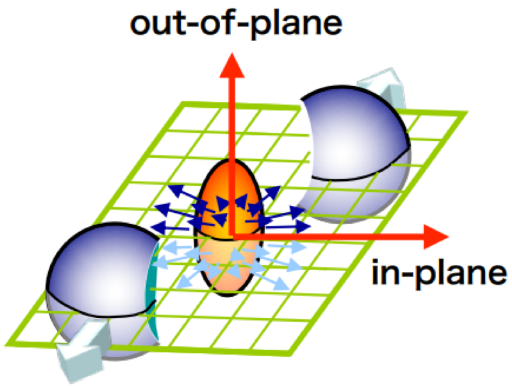


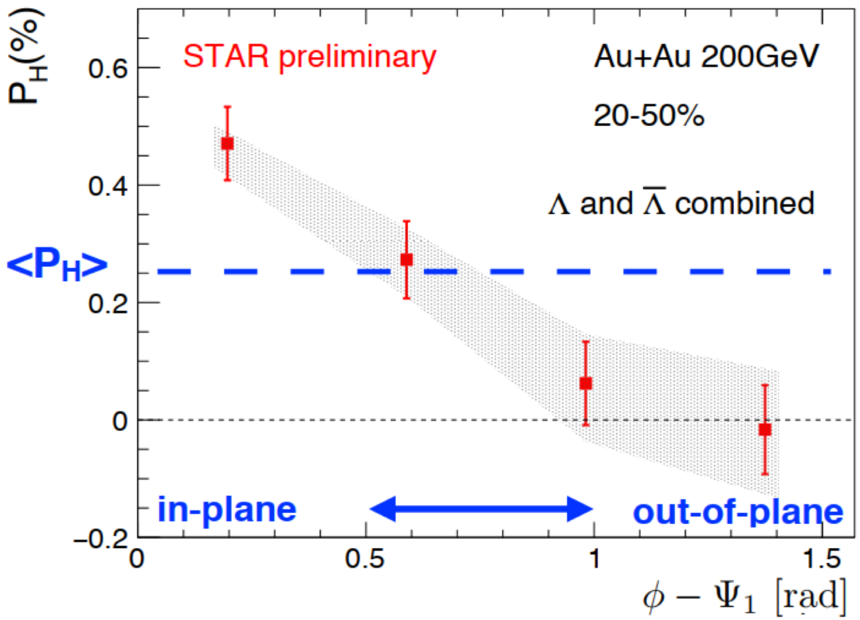
B.C. Fu, K. Xu, X.G. Huang, H.C. Song, Phys. Rev. C 103, 024903 (2021)

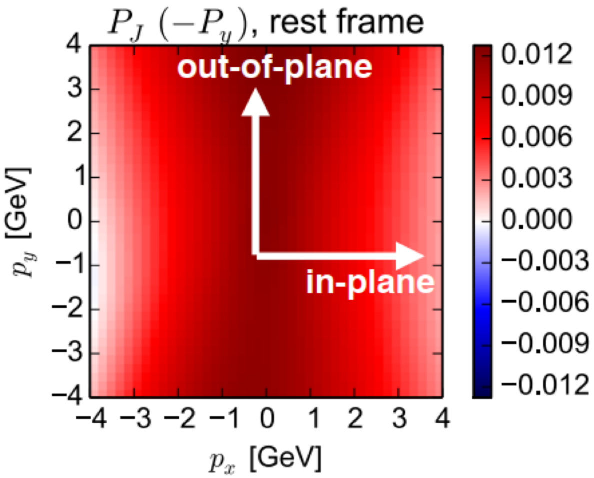
S. Ryu, V. Jupic, C. Shen, arXiv:2106.08125

Y.X. Wu, C. Yi, G.Y. Qin, SP arXiv:2204.02218

Local Polarization







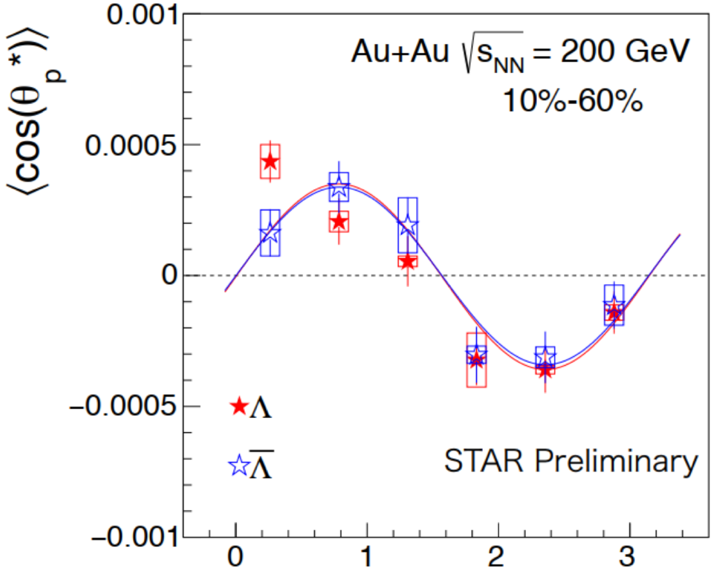
- **Exp data:**

  P_H in-plane > P_H out-of-plane

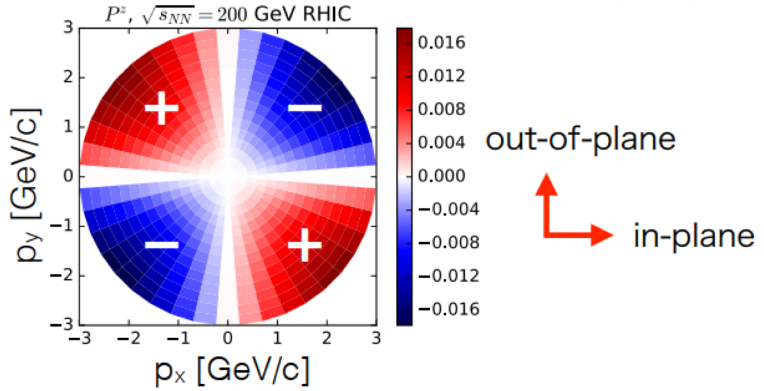
- **Simulations:**

  P_H out-of-plane > P_H in-plane

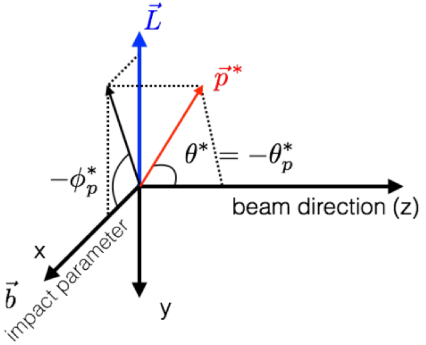
Local Polarization alone beam direction



Sign problem in polarization.



UrQMD : ***Becattini, Karpenko, PRL (2018)***



$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_H \mathbf{P_H} \cdot \mathbf{p}_p^*)$$

$$\langle \cos\theta_p^* \rangle = \int \frac{dN}{d\Omega^*} \cos\theta_p^* d\Omega^*$$

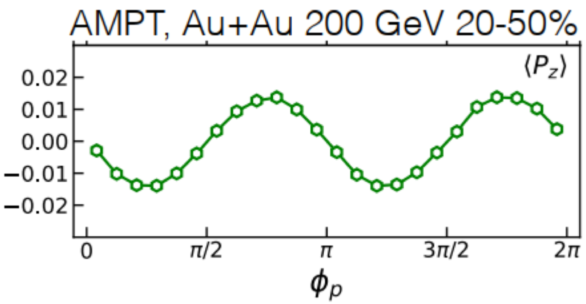
$$= \alpha_H P_z \langle (\cos\theta_p^*)^2 \rangle$$

$$\therefore \; P_z = \frac{\langle \cos\theta_p^* \rangle}{\alpha_H \langle (\cos\theta_p^*)^2 \rangle}$$

$$= \frac{3\langle \cos\theta_p^* \rangle}{\alpha_H} \quad \text{(if perfect detector)}$$

α_H: hyperon decay parameter

θ_p^*: θ of daughter proton in Λ rest frame

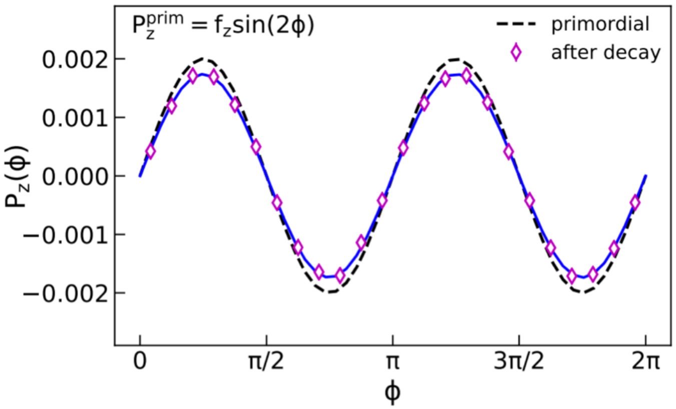


AMPT: ***Xia, Li, Tang, Wang, PRC (2018)***

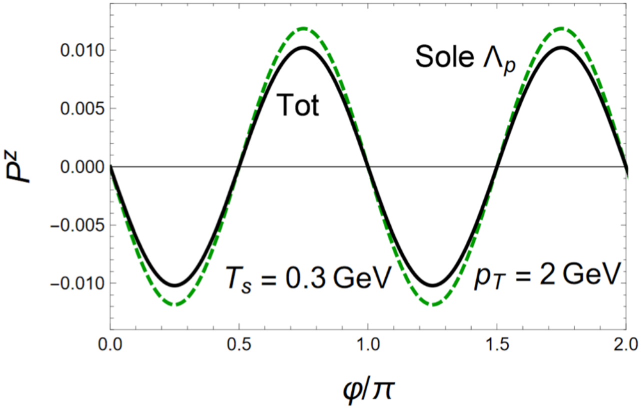
Feed-down effects: NO!

- **Feed-down effects**

Lambda may come from decays of heavier particles



Xia, Li, Huang, Huang PRC(2019)



Becattini, Cao, Speranza, EPJC(2019)

Different approaches

- Spin hydrodynamics

  Florkowski, Friman, Jaiswal, Ryblewski, Speranza (2017-2018);

  Montenegro, Tinti, Torrieri (2017-2019);

  Hattori, Hongo, Huang, Matsuo, Taya PLB(2019) ; arXiv: 2201.12390; arXiv: 2205.08051

  Fukushima, SP, Lecture Note (2020); PLB(2021); Wang, Fang, SP, PRD(2021); Wang, Xie, Fang, SP, PRD (2022)

  S.Y. Li, M.A Stephanov, H.U Yee, arXiv:2011.12318

  D. She, A. Huang, D.F. Hou, J.F Liao, arXiv: 2105.04060

- Quantum kinetic theory for massive fermions and collisions

  Weickgenannt, Sheng, Speranza, Wang, Rischke, PRD 100, 056018 (2019)

  Hattori, Hidaka, Yang, PRD100, 096011 (2019); Yang, Hattori, Hidaka, arXiv: 2002.02612.

  Liu, Mameda, Huang, arXiv:2002.03753.

  Gao, Liang, PRD 2019

  Wang, Guo, Shi, Zhuang, PRD100, 014015 (2019) ；Z.Y. Wang, arXiv:2205.09334;

  Li ,Yee, PRD100, 056022 (2019)

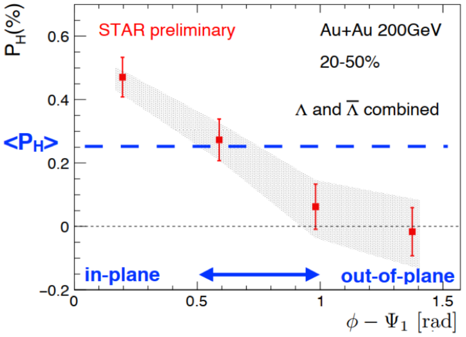
  Hou, Lin, arXiv: 2008.03862; Lin, arXiv: 2109.00184; Lin, Wang, arXiv:2206.12573

  Fang, SP, Yang, PRD (2022)

- Other approaches:
  - Side-jump effect **Liu, Sun, Ko PRL(2020)**
  - Mesonic mean-field **Csernai, Kapusta, Welle, PRC(2019)**
  - Using different vorticity **Wu, Pang, Huang, Wang, PRR (2019)**

3. Applications to spin polarization

Modified Cooper-Frye formula



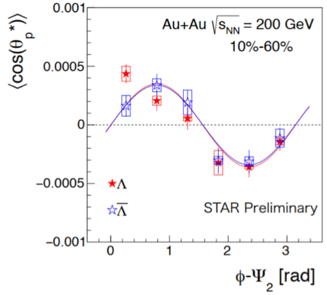
Thermal vorticity

$$\omega^{\text{th}}_{\rho\sigma} = \frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right) - \partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Distribution function: f_0

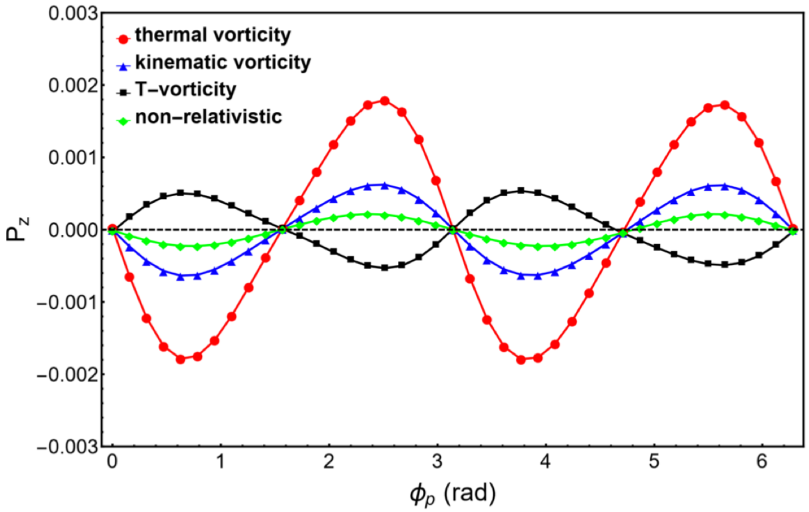
$$\mathcal{S}^\mu(p) = \frac{1}{8m_\Lambda}\epsilon^{\mu\nu\rho\sigma}p_\nu\frac{\int d\Sigma_\lambda p^\lambda f_0(1-f_0)\omega^{\text{th}}_{\rho\sigma}}{\int d\Sigma_\lambda p^\lambda f_0}$$

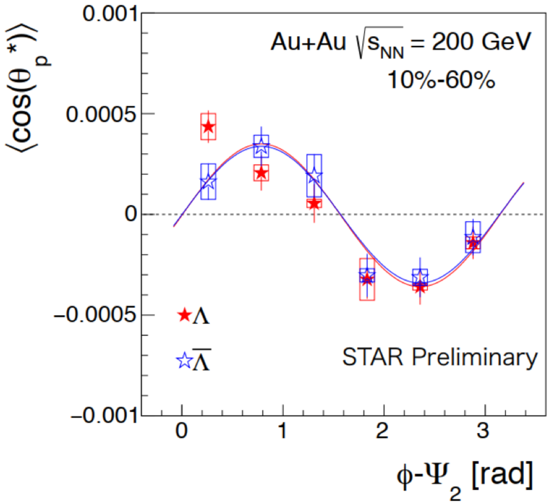
Freezeout surface



Karpenko, F. Becattini, Eur. Phys. J. C 77 (2017) 213
R.-H. Fang, L.-G. Pang, Q. Wang, X.-N. Wang, Phys. Rev. C94, 024904 (2016)

Local polarization from different vorticities





Kinematic vorticity:

$$\omega_{\mu\nu}^{(K)}=-\frac{1}{2}(\partial_\mu u_\nu-\partial_\nu u_\mu)$$

T-vorticity:

$$\omega_{\mu\nu}^{(T)}=-\frac{1}{2}[\partial_\mu(Tu_\nu)-\partial_\nu(Tu_\mu)]$$

Non-Relativistic vorticity

$$\omega_{\mu\nu}^{(\mathrm{NR})}=\epsilon_{\nu\mu\rho\eta}u^\rho\omega^\eta$$

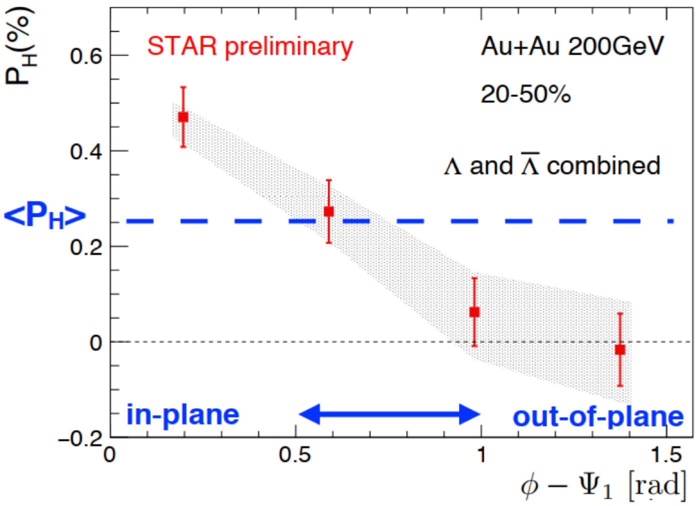
Thermal vorticity:

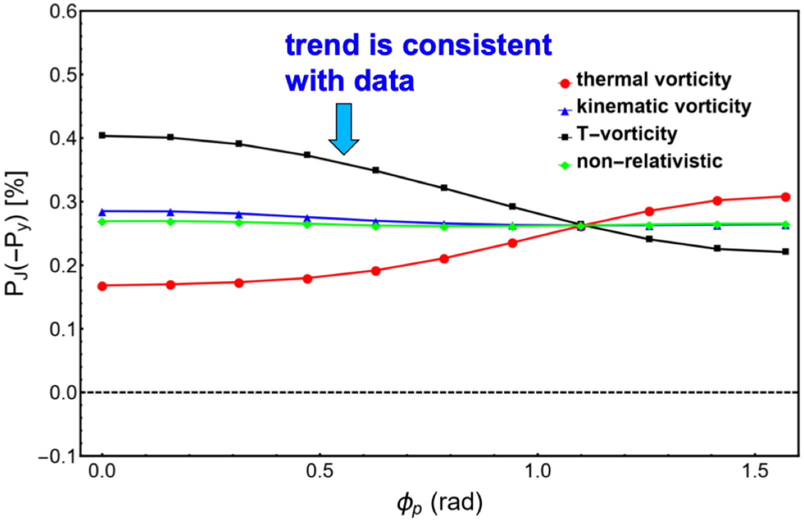
$$\omega_{\rho\sigma}^{\mathrm{th}}=\frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right)-\partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Wu, Pang, Huang, Wang, PRR 1, 033058(2019)

- Only T-vorticity gives the right trend for both Pz and Py
- Why T-vorticity? Out-of-equilibrium effects?

Local polarization from different vorticities





Kinematic vorticity:

$$\omega_{\mu\nu}^{(K)} = -\frac{1}{2}(\partial_\mu u_\nu - \partial_\nu u_\mu)$$

T-vorticity:

$$\omega_{\mu\nu}^{(T)} = -\frac{1}{2}[\partial_\mu(Tu_\nu) - \partial_\nu(Tu_\mu)]$$

Non-Relativistic vorticity

$$\omega_{\mu\nu}^{(\mathrm{NR})} = \epsilon_{\nu\mu\rho\eta}u^\rho\omega^\eta$$

Thermal vorticity:

$$\omega_{\rho\sigma}^{\mathrm{th}} = \frac{1}{2}\left[\partial_\rho\left(\frac{u_\sigma}{T}\right) - \partial_\sigma\left(\frac{u_\rho}{T}\right)\right]$$

Wu, Pang, Huang, Wang, PRR 1, 033058(2019)

- Only T-vorticity gives the right trend for both Pz and Py
- Why T-vorticity? Out-of-equilibrium effects?

Polarization and axial current

- **The polarization tensor is connected to the axial current in phase space by modified Cooper-Frye formula**
  Karpenko, Becattini, EPJC. (2017); Fang, Pang, QW, Wang, PRC (2016)

$$\mathcal{S}^{\mu}(\mathbf{p}) = \frac{\int d\Sigma \cdot p \mathcal{J}_5^{\mu}(p,X)}{2m_{\Lambda} \int d\Sigma \cdot \mathcal{N}(p,X)},$$

- **For massless fermions, the left and right handed currents read**

$$\begin{aligned}\mathcal{J}^{\mu}_{\lambda}(p,X) &= 2\pi \mathrm{sign}(u\cdot p)\left\{p^{\mu} + \lambda\frac{\hbar}{2}\delta(p^2)\left[u^{\mu}(p\cdot\omega) - \omega^{\mu}(u\cdot p)\right.\right. \\ &\qquad \left.\left. -2S^{\mu\nu}_{(u)}\tilde{E}_{\nu}\right]\partial_{u\cdot p} + \lambda\frac{\hbar}{4}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\partial^{p}_{\nu}\delta(p^2)\right\} f^{(0)}_{\lambda},\end{aligned}$$

$\lambda = \pm$

+: right
-: left

$$\begin{aligned}S^{\mu\nu}_{(u)} &= \epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}/(2u\cdot p), \\ \tilde{E}_{\nu} &= E_{\nu} + T\partial_{\nu}\frac{\mu_{\lambda}}{T} + \frac{(u\cdot p)}{T}\partial_{\nu}T - p^{\sigma}[\partial_{<\sigma}u_{\nu>} + \frac{1}{3}\Delta_{\sigma\nu}(\partial\cdot u) + u_{\nu}Du_{\sigma}]. \\ f^{(0)}_{\lambda} &= 1/(e^{(u\cdot p-\mu_{\lambda})/T} + 1),\end{aligned}$$

Y. Hidaka, SP, and D.L. Yang, Phys. Rev. D97, 016004 (2018)

Polarization induced by different sources

- **Axial currents can be decomposed as**

$$\mathcal{J}_5^\mu = \mathcal{J}^\mu_{\rm thermal} + \mathcal{J}^\mu_{\rm shear} + \mathcal{J}^\mu_{\rm accT} + \mathcal{J}^\mu_{\rm chemical} + \mathcal{J}^\mu_{\rm EB},$$

where they are related to:

Thermal vorticity

$$\mathcal{J}^\mu_{\rm thermal} = a\frac{1}{2}\boxed{\epsilon^{\mu\nu\alpha\beta}p_\nu\partial_\alpha\frac{u_\beta}{T}},$$

Shear viscous tensor

$$\mathcal{J}^\mu_{\rm shear} = -a\frac{1}{(u\cdot p)T}\epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta p^\sigma\boxed{\partial_{<\sigma}u_{\nu>}}$$

Fluid acceleration

$$\mathcal{J}^\mu_{\rm accT} = -a\frac{1}{2T}\epsilon^{\mu\nu\alpha\beta}p_\nu u_\alpha\boxed{\left(Du_\beta - \frac{1}{T}\partial_\beta T\right)}.$$

Gradient of chemical potential

$$\mathcal{J}^\mu_{\rm chemical} = a\frac{1}{(u\cdot p)}\epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta\boxed{\partial_\nu\frac{\mu}{T};}$$

Electromagnetic fields

$$\mathcal{J}^\mu_{\rm EB} = a\frac{1}{(u\cdot p)T}\epsilon^{\mu\nu\alpha\beta}p_\alpha u_\beta\boxed{E_\nu + a\frac{B^\mu}{T},}$$

Y. Hidaka, SP, D.L. Yang, PRD97, 016004 (2018); C. Yi, SP, D.L. Yang, PRC 2021

Out-of-equilibrium corrections

- **Polarization vector**

$$
\begin{aligned}
\mathcal{P}^z(p) &= \int_{-1}^{+1} dY \mathcal{S}^z(p), \\
\mathcal{P}^y(p) &= \int_{-1}^{+1} dY \mathcal{S}^y(p),
\end{aligned}
$$

- **Polarization induced by thermal vorticity, shear viscous tensor and residual part of fluid acceleration**

$$
\begin{aligned}
\mathcal{S}^{\mu}_{\text{thermal}}(\mathbf{p}) &= \frac{\hbar}{8m_{\Lambda}N}\int d\Sigma^{\sigma}p_{\sigma}f_V^{(0)}(1-f_V^{(0)})\epsilon^{\mu\nu\alpha\beta}p_{\nu}\partial_{\alpha}\frac{u_{\beta}}{T}, \\
\mathcal{S}^{\mu}_{\text{shear}}(\mathbf{p}) &= -\frac{\hbar}{4m_{\Lambda}N}\int d\Sigma\cdot p f_V^{(0)}(1-f_V^{(0)})\frac{\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}}{(u\cdot p)T}\frac{1}{2}\{p^{\sigma}(\partial_{\sigma}u_{\nu}+\partial_{\nu}u_{\sigma})-Du_{\nu}\} \\
\mathcal{S}^{\mu}_{\text{accT}}(\mathbf{p}) &= -\frac{\hbar}{8m_{\Lambda}N}\int d\Sigma\cdot p f_V^{(0)}(1-f_V^{(0)})\frac{1}{T}\epsilon^{\mu\nu\alpha\beta}p_{\nu}u_{\alpha}(Du_{\beta}-\frac{1}{T}\partial_{\beta}T),
\end{aligned}
$$

C. Yi, SP, D.L. Yang, PRC 2021

Shear induced polarization

- Shear induced polarization draws some attentions.
- Shear induced Polarization from massless fermions (Theory):

  Y. Hidaka, SP, and D.L. Yang, PRD97, 016004 (2018);
- Shear induced Polarization from massive fermions:
  - Theory:

    S. Y. F. Liu, Y. Yin, 2103.09200

    F. Becattini, M. Buzzegoli, A. Palermo, 2103.10917
  - Hydrodynamic simulations:

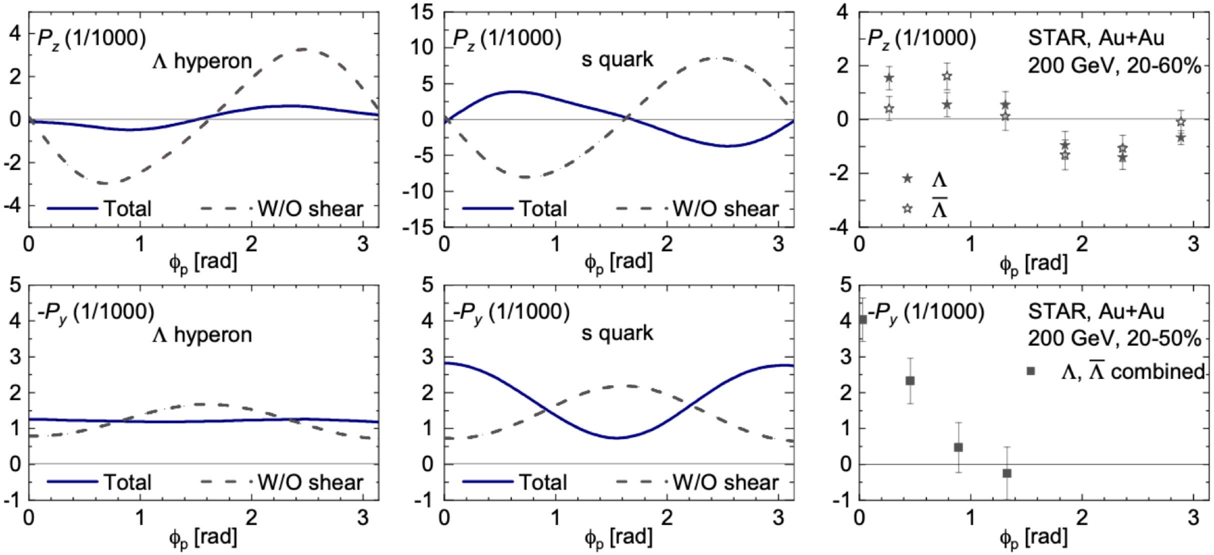
    B. Fu, S. Y. F. Liu, L. Pang, H. Song, Y. Yin, 2103.10403

    F. Becattini, M. Buzzegoli, A. Palermo, G. Inghirami, I. Karpenko, 2103.14621

    C. Yi, SP, D.L. Yang, PRC 2021
- Global polarization induced by shear and gradient of chemical potential

  S. Ryu, V. Jupic, C. Shen, arXiv:2106.08125

s quark scenario: why it works?



B. Fu, S. Y. F. Liu, L. Pang, H. Song, Y. Yin, PRL 2021

$$\mathcal{S}^{\mu}_{\text{thermal}}(\mathbf{p}) = \frac{\hbar}{8\boxed{m_\Lambda}N}\int d\Sigma^{\sigma}p_{\sigma}f_V^{(0)}(1-f_V^{(0)})\epsilon^{\mu\nu\alpha\beta}p_{\nu}\partial_{\alpha}\frac{u_{\beta}}{T},$$

$$\mathcal{S}^{\mu}_{\text{shear}}(\mathbf{p}) = -\frac{\hbar}{4\boxed{m_\Lambda}N}\int d\Sigma\cdot pf_V^{(0)}(1-f_V^{(0)})\frac{\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}}{\boxed{(u\cdot p)}T}\frac{1}{2}\{p^{\sigma}(\partial_{\sigma}u_{\nu}+\partial_{\nu}u_{\sigma})-Du_{\nu}\}$$

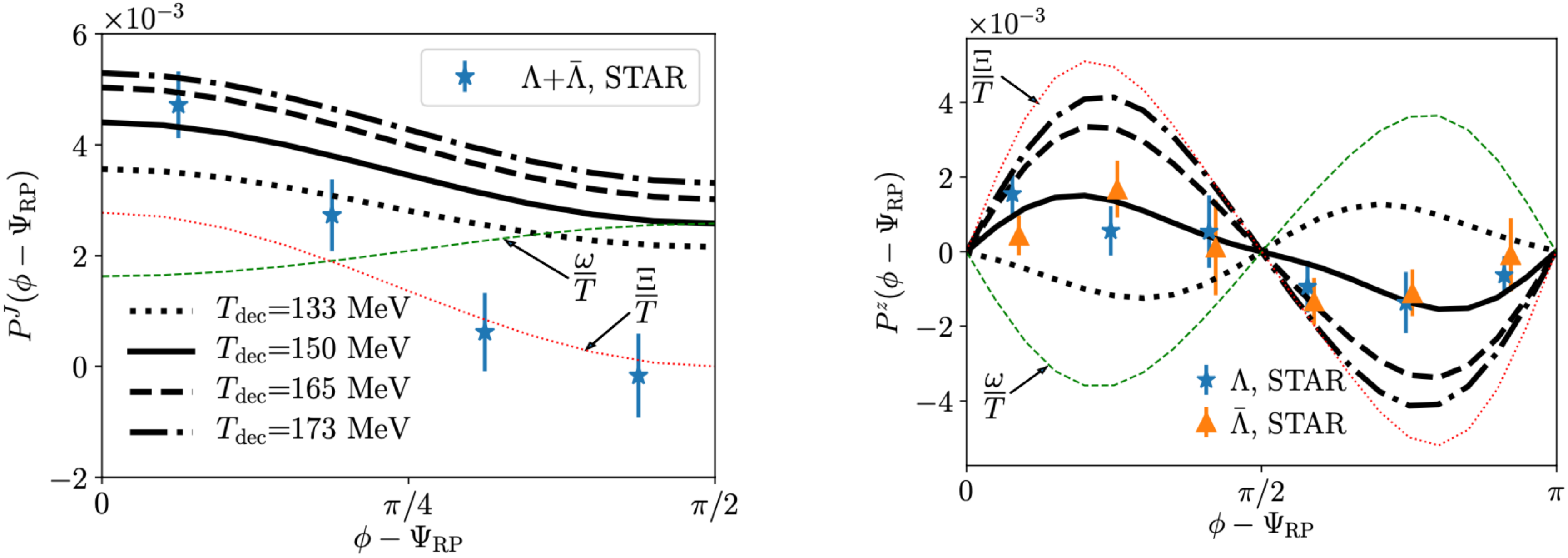
$$m_\Lambda \to m_s$$

$$(u\cdot p)\sim m$$

$$m_s \simeq 0.3\text{GeV}$$

$$m_\Lambda \simeq 1.116\text{GeV}$$

Isothermal local equilibrium



$$S^\mu_{\rm ILE}(p) = -\epsilon^{\mu\rho\sigma\tau} p_\tau \frac{\int_\Sigma {\rm d}\Sigma\cdot p\, n_F(1-n_F)\left[\omega_{\rho\sigma} + 2\,\hat{t}_\rho \frac{p^\lambda}{\varepsilon}\Xi_{\lambda\sigma}\right]}{8mT_{\rm dec}\int_\Sigma {\rm d}\Sigma\cdot p\, n_F}$$

$$\omega_{\rho\sigma} = \frac{1}{2}(\partial_\sigma u_\rho - \partial_\rho u_\sigma)$$

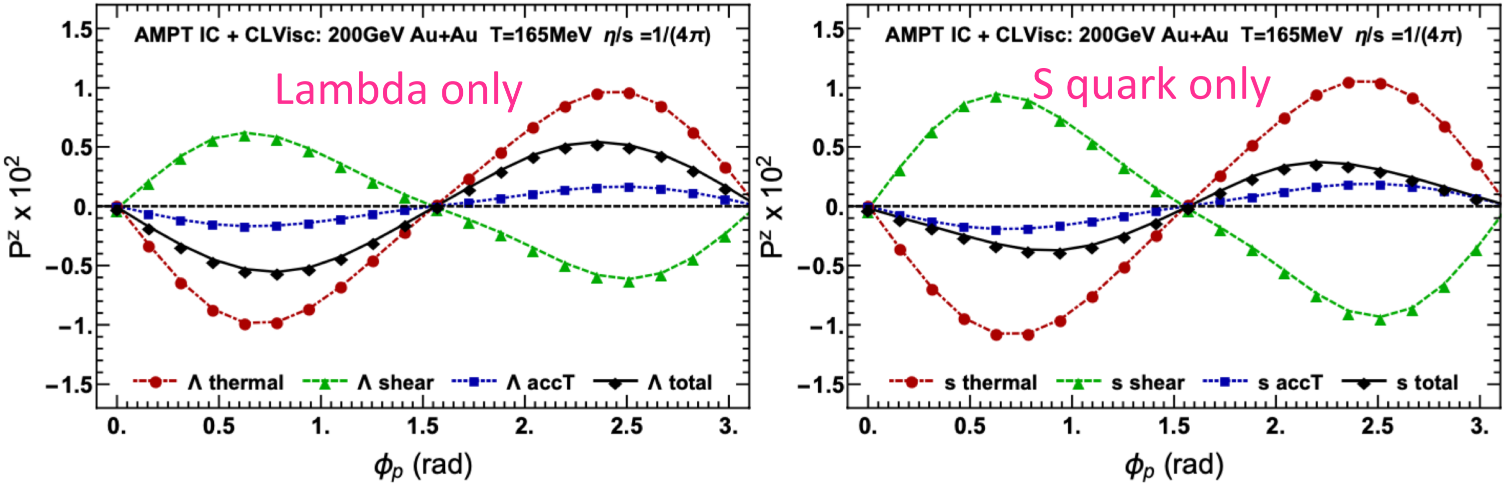
All gradient of temperature are neglected!

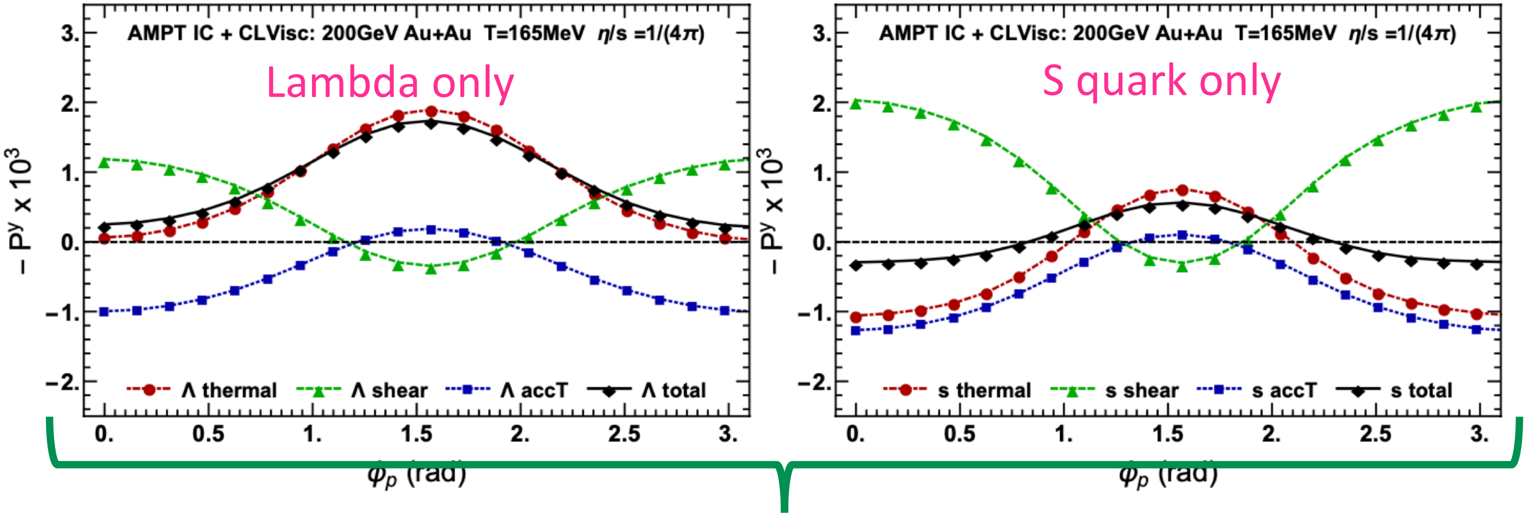
$$\Xi_{\rho\sigma} = \frac{1}{2}(\partial_\sigma u_\rho + \partial_\rho u_\sigma)$$

F. Becattini, M. Buzzegoli, A. Palermo, G. Inghirami, I. Karpenko, 2103.14621

Local spin polarization induced by shear tensor

Polarization along beam direction





Polarization along out-of-plane direction

Main result for shear induced polarization

We found that

- Shear induced polarization always give a “correct” sign.
- Total local polarization is sensitive to mass of s quark, EoS, freeze out temperature and eta / s.

- The local spin polarization is still an open question. We still need to consider the out-of-equilibrium effects carefully through the spin hydrodynamics and the quantum kinetic theory with collisions.

Yi, Pu, Yang, PRC (2021)

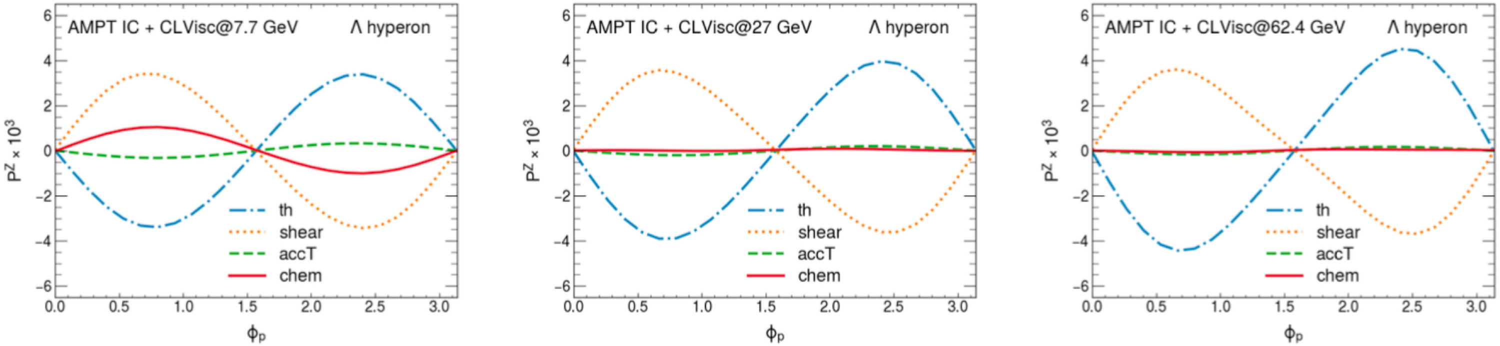
Simulations for spin Hall effects

- **“Spin Hall effect”: polarization induced by the gradient of chemical potential**

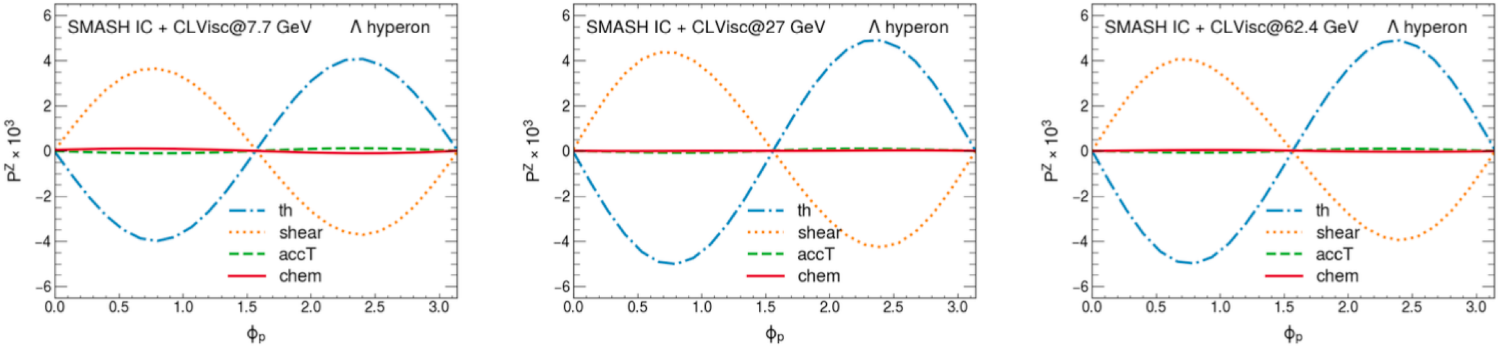
$$\mathcal{S}^{\mu}_{\text{chemical}}(\mathbf{p}) \;=\; 2\int d\Sigma^{\sigma}F_{\sigma}\frac{1}{(u\cdot p)}\epsilon^{\mu\nu\alpha\beta}p_{\alpha}u_{\beta}\partial_{\nu}\frac{\mu}{T},$$

- **We study the polarization at RHIC beam energy scan energies via the (3+1)-dimensional CLVisc hydrodynamics model with AMPT and SMASH initial conditions. The results depend on initial condition and baryon diffusion.**

From AMPT Initial condition



From SMASH Initial condition



X.Y. Wu, C. Yi, G.Y. Qin, SP, arXiv:2204.02218, accepted by PRC

Other possible corrections

- Electromagnetic fields make the differences between Λ and Λ_bar

- Collisional effects to modified Cooper-Frye formula

  Fang, SP, Yang, PRD (2022)
  Z.Y. Wang, arXiv:2205.09334
  Lin, Wang, arXiv:2206.12573

- Corrections from spin potential to modified Cooper-Frye formula

  Liu, Huang, arXiv: 2109.15301

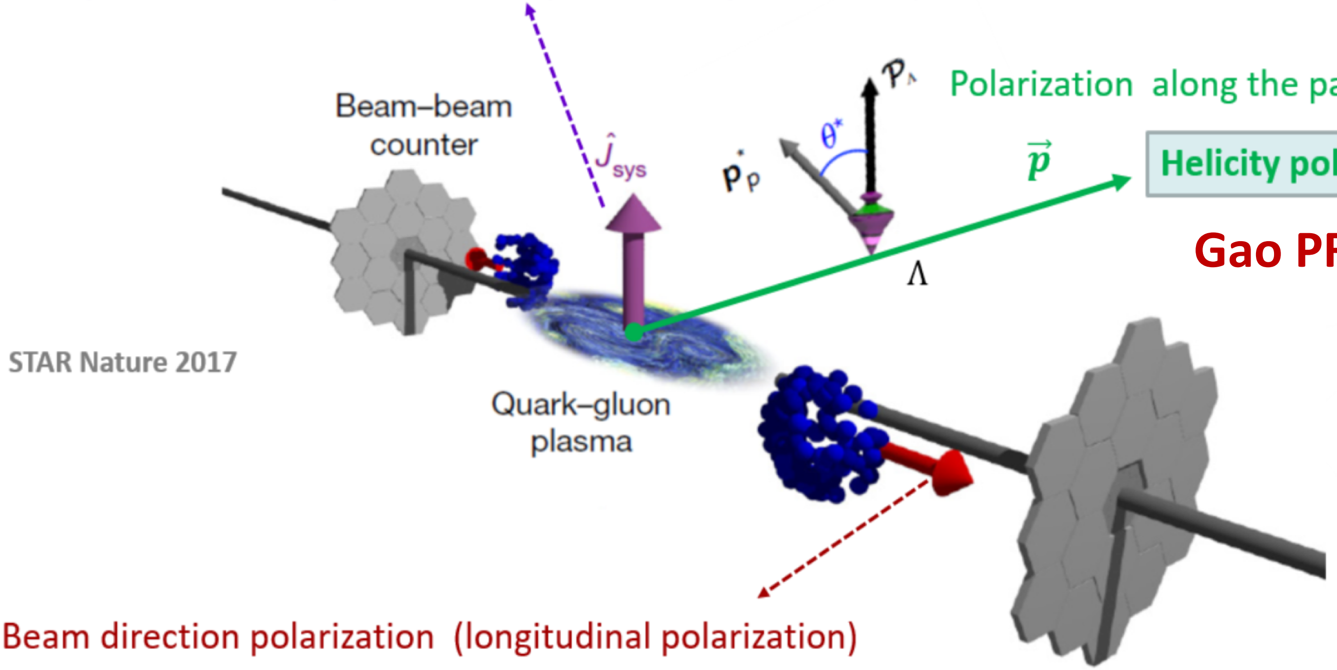
- Hadronization

- Hadronic interaction after chemical freezeout

Helicity polarization

$$S^h = \widehat{\mathbf{p}}\cdot\boldsymbol{S}(\mathbf{p}) = \widehat{p}^x \mathcal{S}^x + \widehat{p}^y \mathcal{S}^y + \widehat{p}^z \mathcal{S}^z,$$

Out-plane direction polarization (transverse polarization)



Polarization along the particle's momentum

Helicity polarization

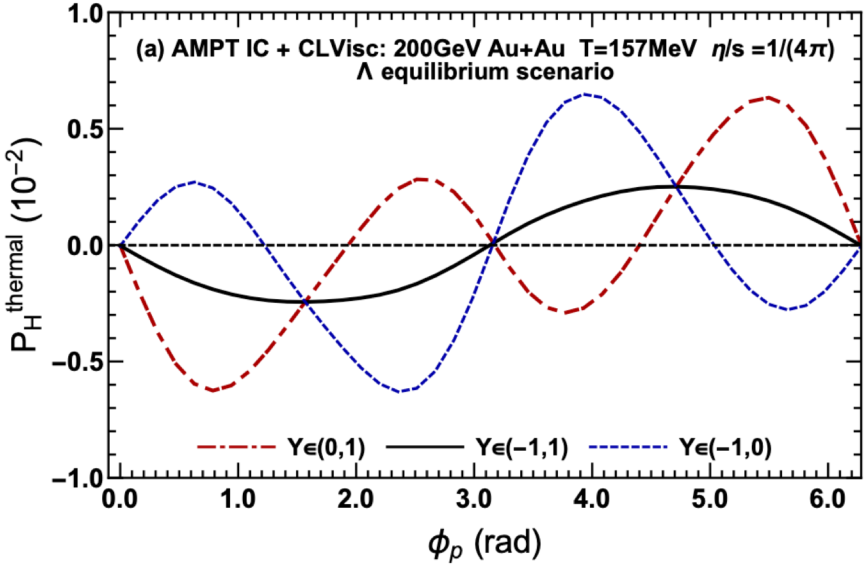
Gao PRD (2021)

Beam direction polarization (longitudinal polarization)

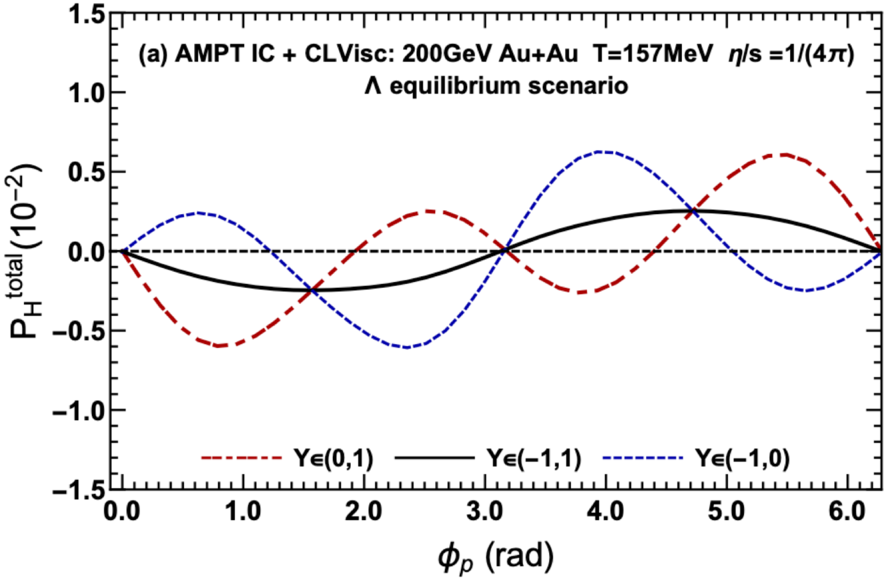
Helicity polarization from QKT

- Helicity polarization polarization can also induced by thermal, shear and fluid acceleration. Yi, Pu, Gao, Yang, PRC (2022)

Contribution from thermal vorticity



Total helicity polarization



- Our numerical simulation shows that the thermal vorticity dominates over other contributions in helicity polarization. The helicity polarization can be used to detect the vortical structure in the fireball.

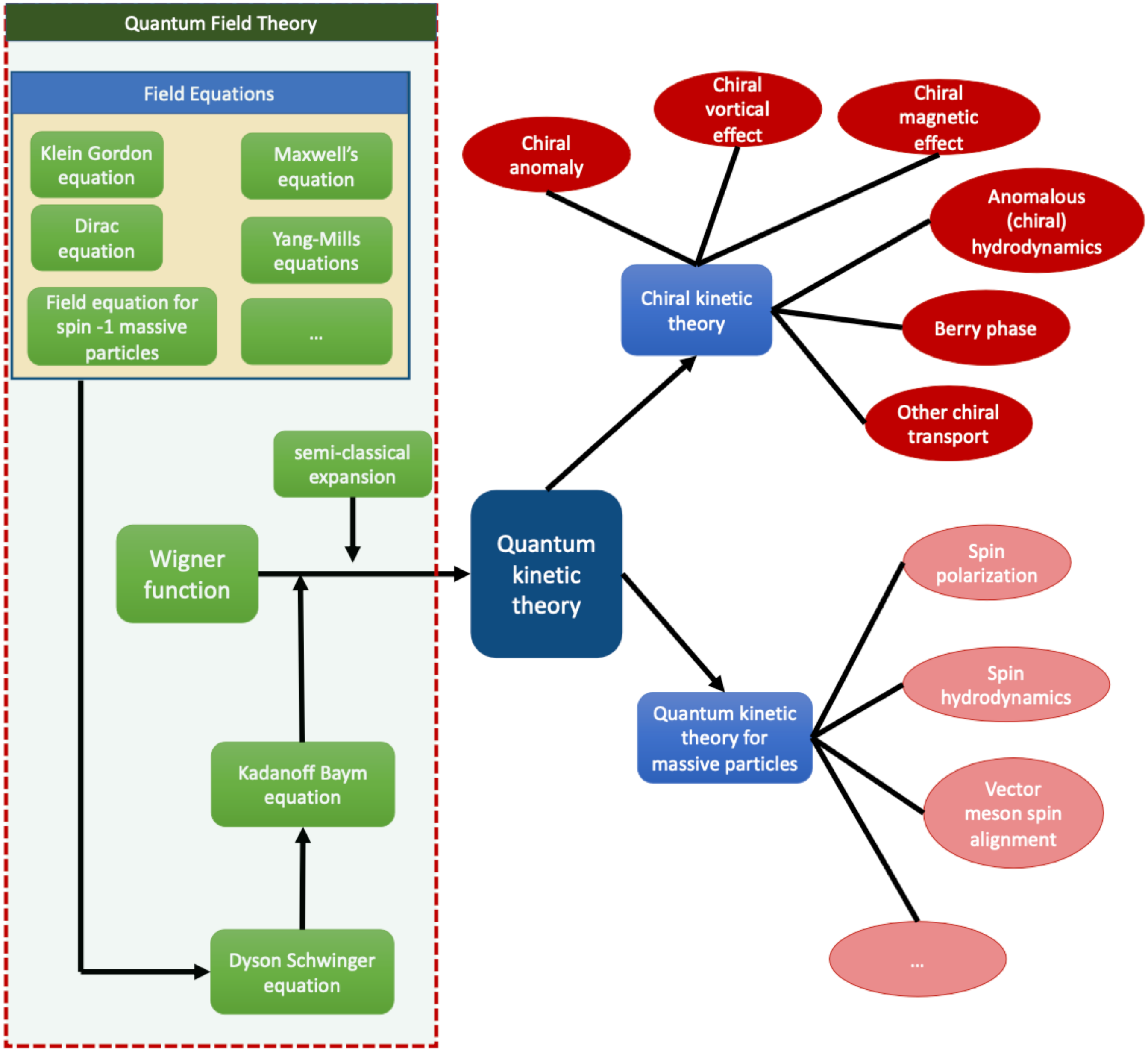
What we get for spin polarization?

- Shear induced polarization always give a “correct” sign.

- Total local polarization is very sensitive to EoS, freeze out temperature and eta / s.

- The local spin polarization is still an open question. We still need to consider the out-of-equilibrium effects carefully through the spin hydrodynamics and the quantum kinetic theory with collisions.

Summary


Quantum Field Theory
Field Equations
Klein Gordon equation
Maxwell's equation
Dirac equation
Yang-Mills equations
Field equation for spin -1 massive particles
...
semi-classical expansion
Wigner function
Kadanoff Baym equation
Dyson Schwinger equation
Quantum kinetic theory
Chiral kinetic theory
Chiral anomaly
Chiral vortical effect
Chiral magnetic effect
Anomalous (chiral) hydrodynamics
Berry phase
Other chiral transport
Quantum kinetic theory for massive particles
Spin polarization
Spin hydrodynamics
Vector meson spin alignment
...

Thank you for your time!

欢迎批评指正！

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