Hydrodynamics in non-equilbrium systems

严力

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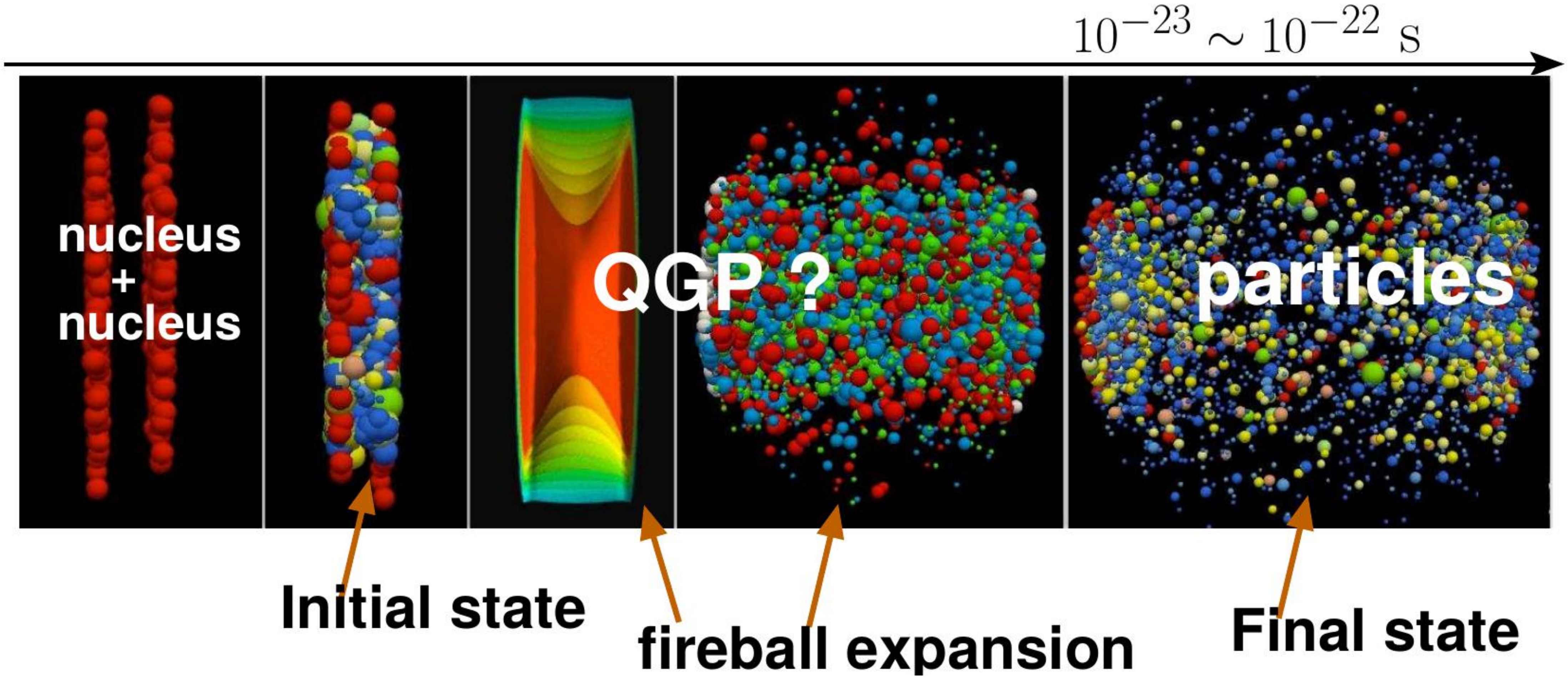
Outline

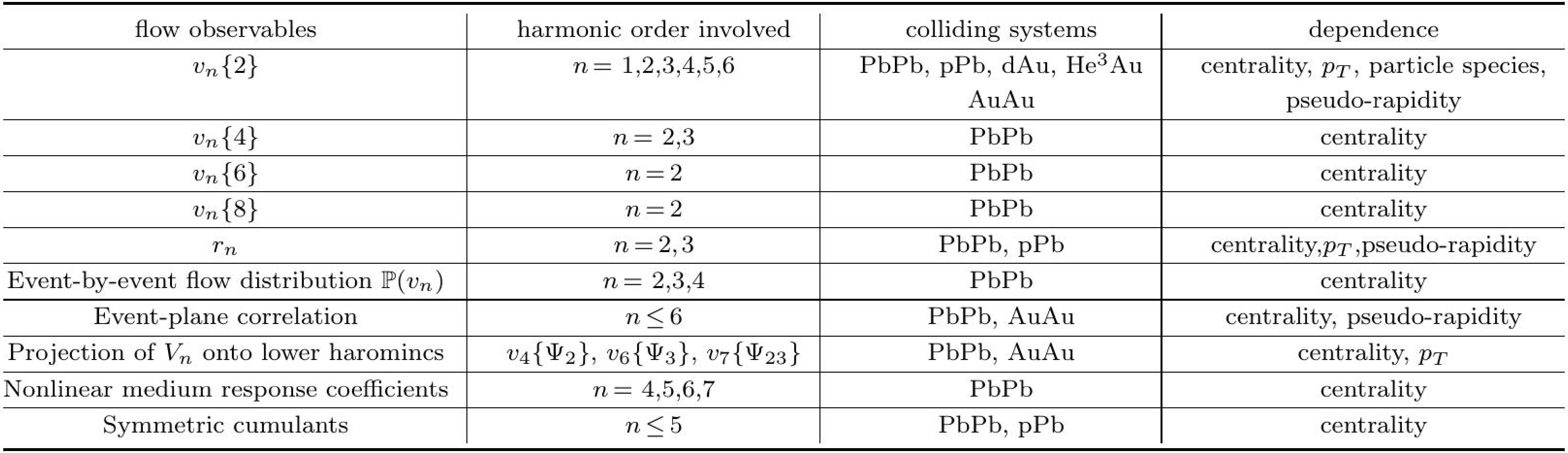
- Theoretical formulation of hydrodynamics, hydro modes.
- Evolution toward hydro in heavy-ion: emergence of attractor
- Hydrodynamic fluctuations

Evolution stages in heavy-ion collisions



Hydro modeling is unreasonably successful

- Hydro successfully characterizes all flow observables in HIC,



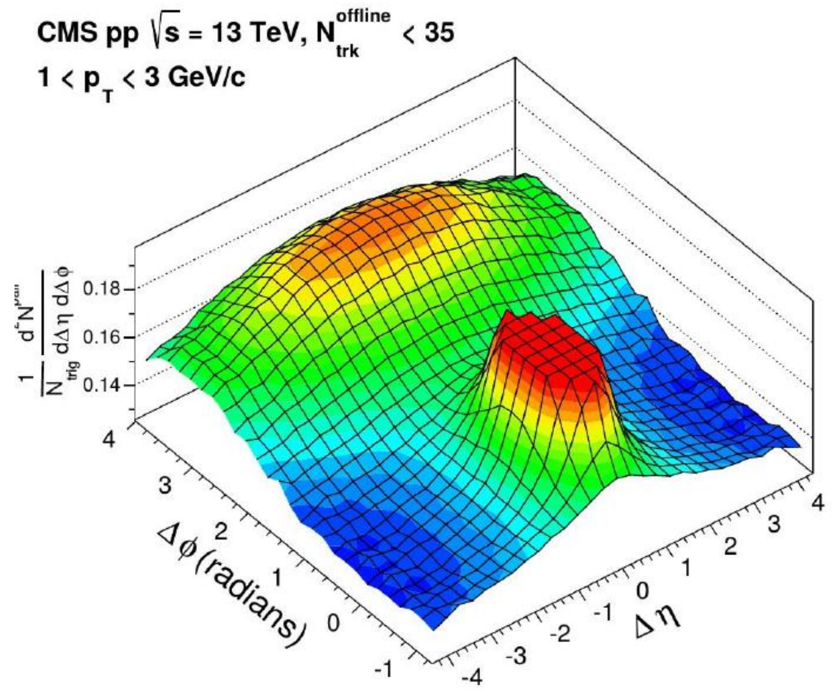
flow observables	harmonic order involved	colliding systems	dependence
$v_n\{2\}$	$n=$ 1,2,3,4,5,6	PbPb, pPb, dAu, He^{3}Au AuAu	centrality, p_T, particle species, pseudo-rapidity
$v_n\{4\}$	$n=$ 2,3	PbPb	centrality
$v_n\{6\}$	$n=2$	PbPb	centrality
$v_n\{8\}$	$n=2$	PbPb	centrality
r_n	$n=2,3$	PbPb, pPb	centrality,p_T,pseudo-rapidity
Event-by-event flow distribution $\mathbb{P}(v_n)$	$n=$ 2,3,4	PbPb	centrality
Event-plane correlation	$n\leq 6$	PbPb, AuAu	centrality, pseudo-rapidity
Projection of V_n onto lower haromincs	$v_4\{\Psi_2\}$, $v_6\{\Psi_3\}$, $v_7\{\Psi_{23}\}$	PbPb, AuAu	centrality, p_T
Nonlinear medium response coefficients	$n=$ 4,5,6,7	PbPb	centrality
Symmetric cumulants	$n\leq 5$	PbPb, pPb	centrality

- New flow observable from v_0-v_n correlations, ...

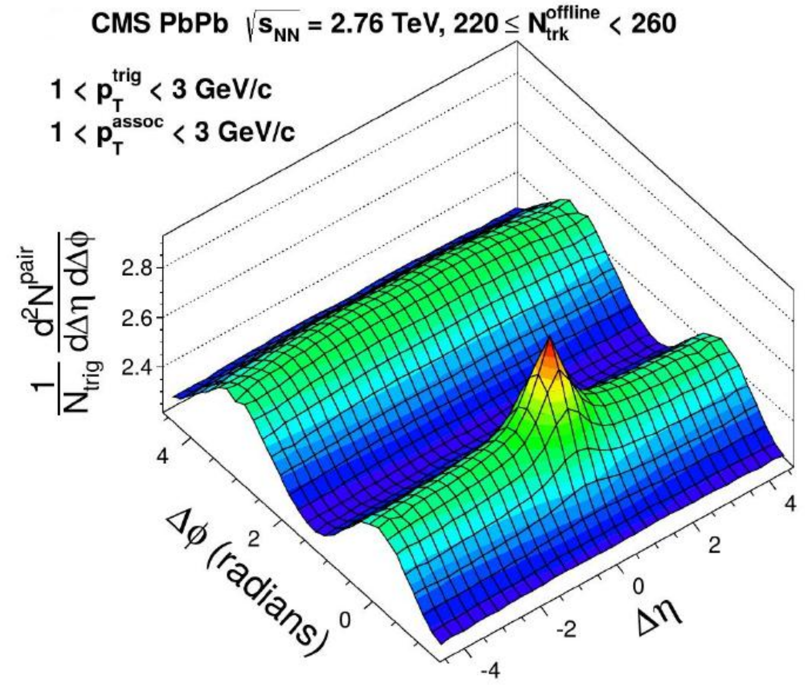
Hydro modeling is unreasonably successful

- Long-range correlation in small colliding systems,

without long-range correlations

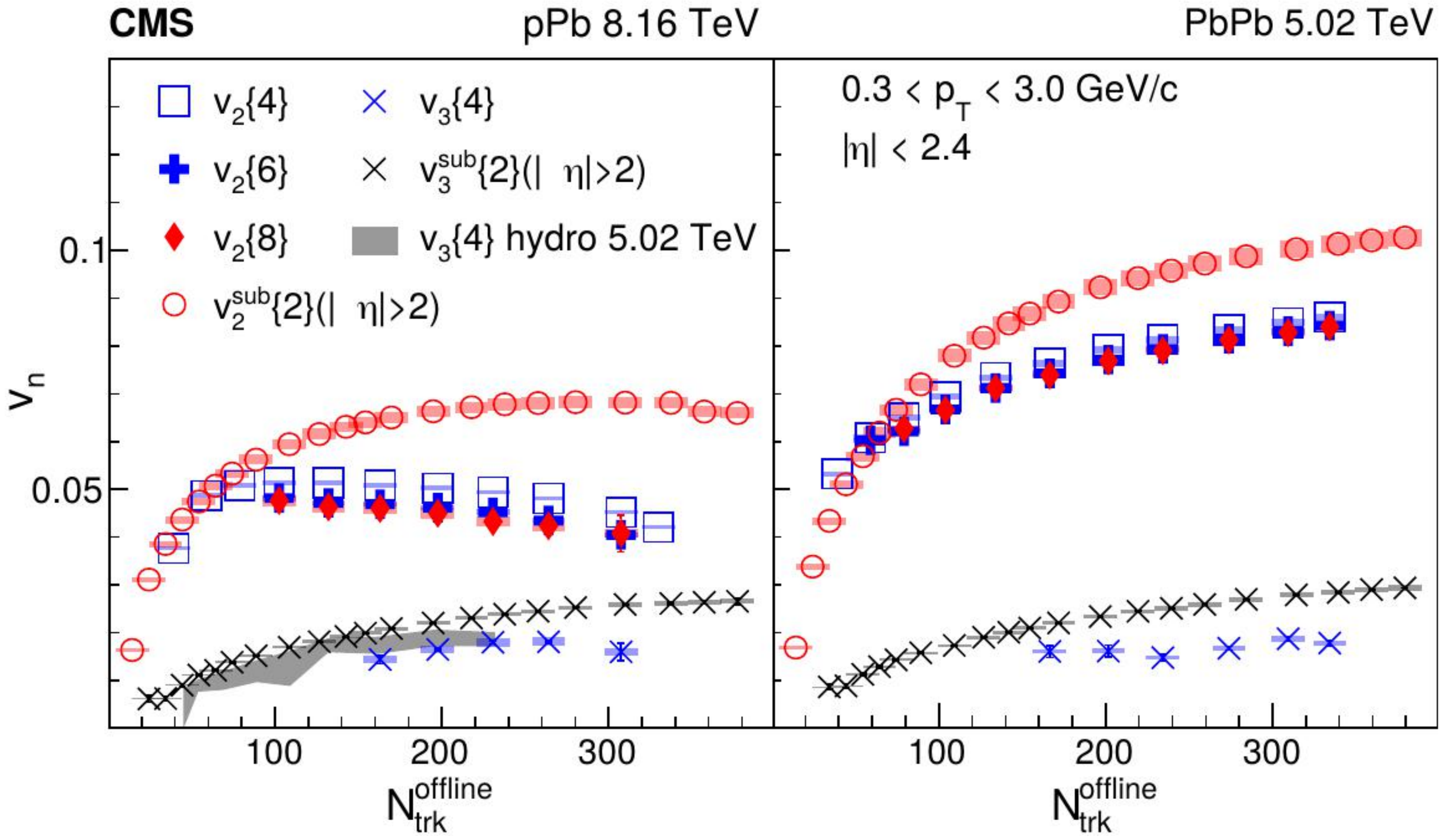


with long-range correlation



Hydro modeling is unreasonably successful

- Multi-particle correlation and collectivite flow in small colliding systems,



[CMS, 1904.11519]

Hydro modeling of system evolution

- Initial condition from effective models: IP-Glasma, MC-Glauber, ...

$$e(\tau_0, \vec{x}_\perp, \xi) \qquad u^\mu(\tau_0, \vec{x}_\perp, \xi)$$

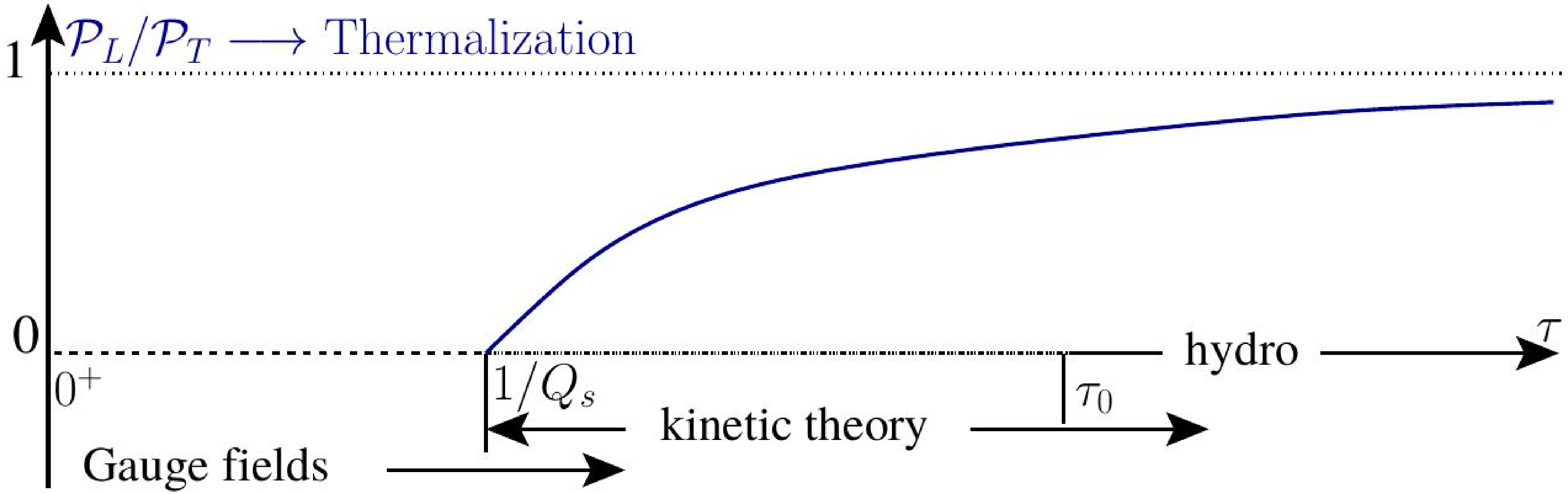
- Assuming onset of hydro at τ_0, after which system expansion is captured by hydrodynamics with respect to a proper equation of state,

Hydro EoM (2nd viscous hydro) + Lattice QCD EoS

- Convert to particles after freeze-out, and particle re-scatterings, resonance decay, etc. (UrQMD)
- One crucial requirement is (close to) local equilibrium -- hydro applicability?

Onset of hydro from far-from-equilbrium

- Pre-equilbrium expansion -- simple picture: hydro starts at later times



what happens in small systems, where $\tau_{\text{life}} \sim \tau_0$?

Non-relativistic hydro

- Conservation of mass: continuity equation

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho dV = 0 \to \partial_t \rho + \rho \nabla \cdot \vec{v} = 0$$

- Conservation of momentum: Euler equation

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\int_V \rho \vec{v} dV = \int_S \vec{f} dS + \text{visc.} \to \partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla P}{\rho} + \text{visc.}$$

- In particular, Navier-Stokes equation $\text{visc.} \sim -\frac{\partial_i \Pi^{ij}}{\rho}$

Relativistic hydrodynamics

- Conservation of energy-momentum and (charge: baryon, strangeness, etc.)

$$\partial_\mu T^{\mu\nu} = 0 \qquad \mu, \nu = 0, 1, 2, 3$$

- Constitutive equations: $T^{\mu\nu} = eu^\mu u^\nu + P(u^\mu u^\nu + g^{\mu\nu}) + \Pi^{\mu\nu}$

- Hydro fields (variables): energy density, pressure, flow velocity, stress tensor.

$$e, \quad P, \quad u^\mu = \gamma(1, \vec{v}), \quad \Pi^{\mu\nu}$$

Relativistic hydrodynamics

- Continuity equation,

$$\partial_\mu T^{\mu 0} = 0 \quad \longrightarrow \quad De + (e+P)\nabla \cdot u + \nabla_{(\mu} u_{\nu)} \Pi^{\mu\nu} = 0$$

- Euler equation,

$$\partial_\mu T^{\mu i} = 0 \quad \longrightarrow \quad Du^\mu (e+P) + \nabla^\mu P - \Delta^\mu_\nu d_\alpha \Pi^{\alpha\nu} =0$$

- Note that

$$D = u^\mu \partial_\mu \xrightarrow{\text{LRF}} \partial_t \qquad \nabla^\mu = \Delta^{\mu\nu}\partial_\nu \xrightarrow{\text{LRF}} \partial_i$$

Theoretical formualtion of hydrodynamics

- As an EFT, theory of hydrodynamics emerges w.r.t. the dominance of long wavelength formulation and small frequency modes -- hydro modes

- In practice, long-wavelength limit allows for gradient expansion:

$$\Pi^{\mu\nu} = O(\nabla) + O(\nabla^2) + \dots$$

up to $O(\nabla^n)$: nth order viscous hydro.

Navier-Stokes hydrodynamics

- Leading order in gradient is well understood -- NS hydro

$$\Pi^{\mu\nu} = -2\eta\langle\nabla^{\mu}u^{\nu}\rangle - \zeta\Delta^{\mu\nu}\partial\cdot u = -\eta\sigma^{\mu\nu} - \zeta\Delta^{\mu\nu}\partial\cdot u$$

1. 1st order transport coefficients appear, shear and bulk viscosities.
2. where ⟨ ⟩ stands for symmetric, traceless and transverse to u^{μ}

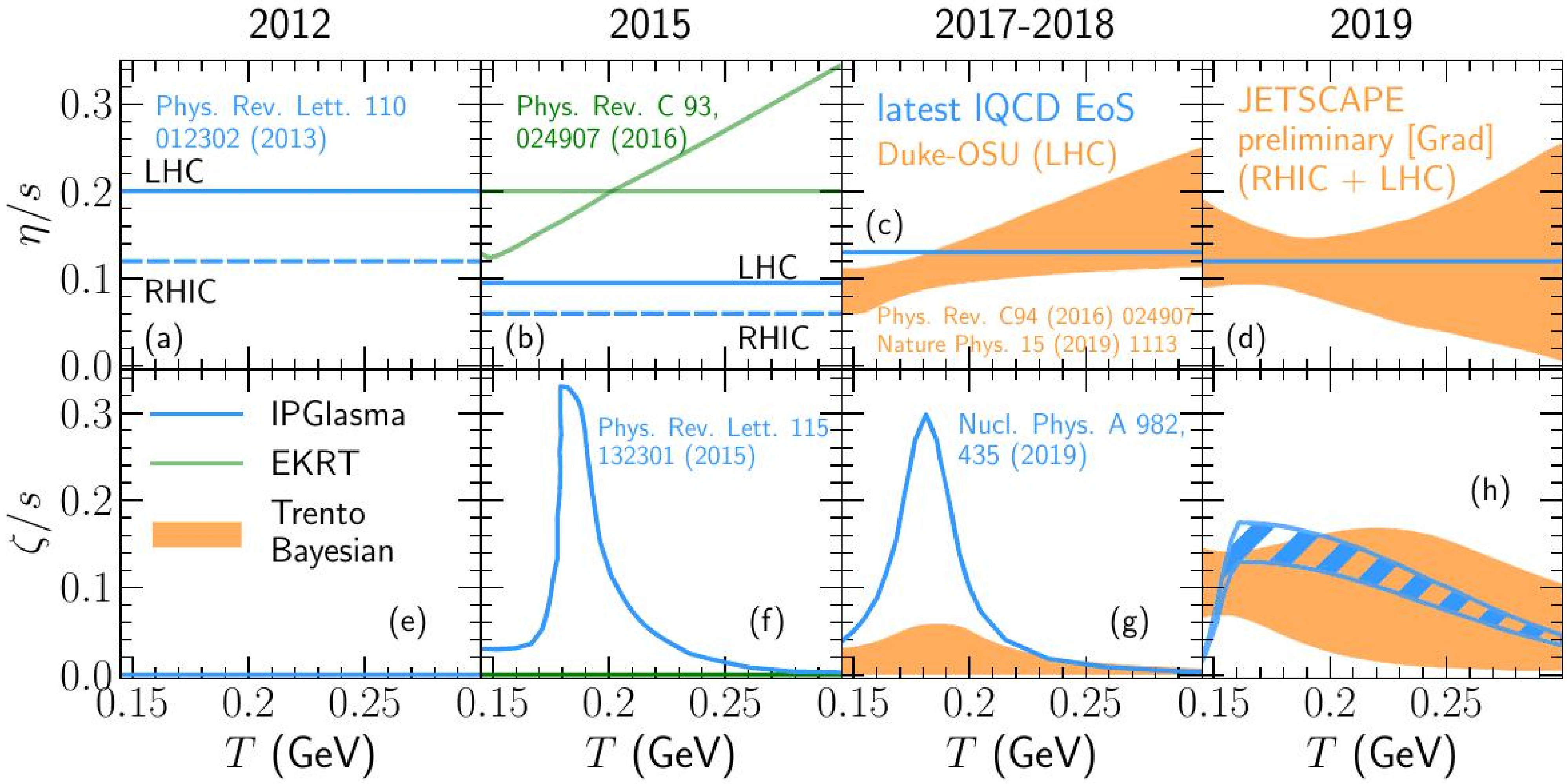
Entropy production

- Entropy production satisfies,

$$\partial_\mu(su^\mu) = -\frac{1}{T}\nabla_{(\mu}u_{\nu)}\Pi^{\mu\nu} = \frac{1}{T}\left[\eta\sigma^2 + \zeta\theta^2\right] \geq 0$$

1. Ideal fluid corresponds to equilbrium, entropy conserved.
2. It is also possible to identify the unkown form of $\Pi^{\mu\nu}$ from the condition of the production of entropy. [Landau&Lifshitz "Fluid dynamics"]
3. Positive entropy production applies to generalized hydro formulation with magnetic fields dof. (magnetohydrodynamics), spin dof. (spin-hydro), etc.

Constraints on shear and bulk viscosities



[S.Chen and LY 2010.12377]

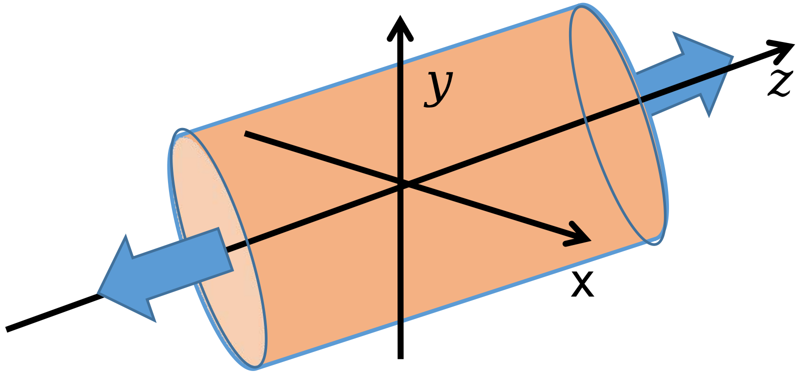
Analytical solution to NS hydro: Bjorken flow

[J. Bjorken, PRD27 (1983)140]

- 0+1 D Bjorken expansion: Boost invariant symmetry along space-time rapidity ξ and translational invariant symmetry in transverse plane.

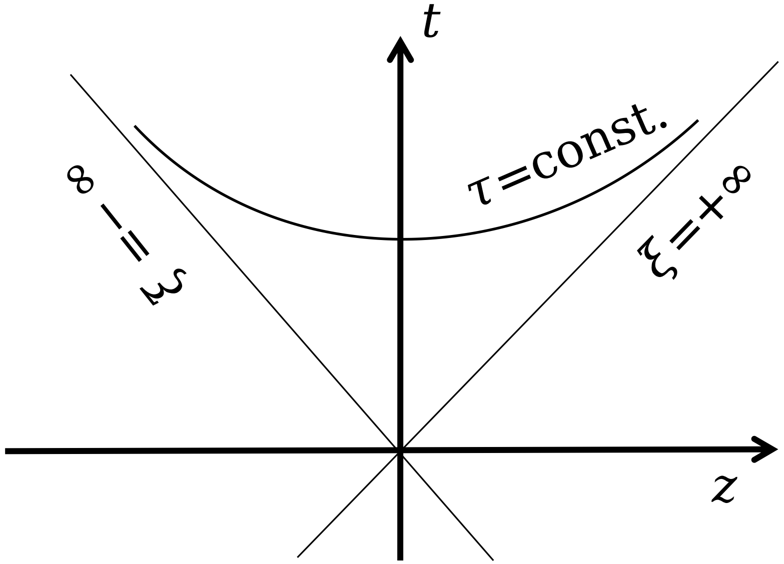
- Applies to very early stages of very high energy nuclear collisions

1. dominated by longitudinal expansion.
2. expansion in transverse plane negligible.
3. boost invariance approximated.



Analytical solution to NS hydro: Bjorken flow

- Bjorken symmetry fixes flow velocity: $\vec{v} = (0, 0, z/t)$
- In Milne coordinates: $\tau = \sqrt{t^2 - z^2}$, $\xi = \tanh^{-1}(z/t)$



$$\mu = (\tau, \vec{x}_\perp, \xi) \to \begin{cases} u^\mu = (1, \vec{0}) \\ e = e(\tau), P = P(\tau) \end{cases}$$

- Hydro EoM becomes: (continuity eq.)

$$\partial_\tau e + \frac{e+P}{\tau} = \frac{1}{\tau^2}\left(\frac{4}{3}\eta + \zeta\right)$$

Euler eq. is trival.

Analytical solution to NS hydro: Bjorken flow

- Consider conformal fluid: $(\partial P/\partial e) = c_s^2 = 1/3$ and $\zeta = 0$
- Reparameterize shear viscosity:

1. AdS/CFT: $\eta/s = 1/4\pi \rightarrow \eta = H_0 e^{3/4}$

$$e(\tau) = \left(\frac{\tau_0}{\tau}\right)^{4/3}\left[e_0 - \frac{H_0}{2\tau}\left(\frac{\tau}{\tau_0}\right)^{1/3}\right]^4$$

2. Kinetic theory of massless particles: $\eta/s = T\tau_R/5 \rightarrow \eta = 4e\tau_R/15$

$$e(\tau) = e(\tau_0)\left(\frac{\tau_0}{\tau}\right)^{4/3}\exp\left[\frac{16\tau_R}{45}\left(\frac{1}{\tau_0} - \frac{1}{\tau}\right)\right]$$

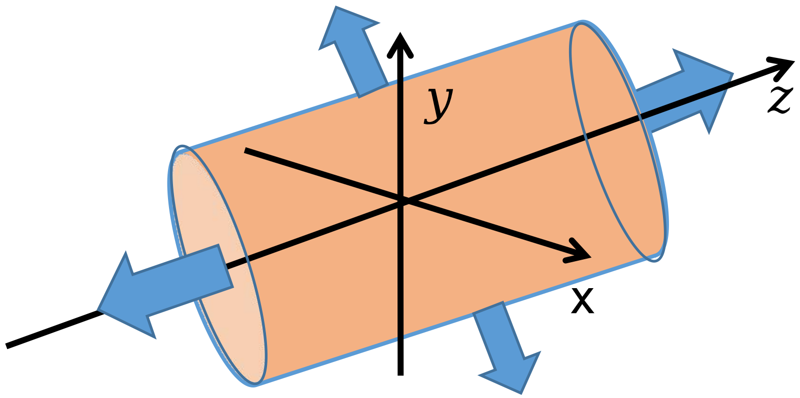
Analytical solution to NS hydro: Gubser flow

- 1+1 D Gubser expansion: Boost invariant symmetry along space-time rapidity ξ and rotational symmetry in transverse plane.

- Applies to ultra-central high energy nuclear collisions

1. isotropic expansion in transverse plane.
2. boost invariance approximated.

[S. Gubser, 1006.0006, S. Gubser and A. Yoram, 1012.1314]



Analytical solution to NS hydro: Gubser flow

- Symmtry is obvious in new coordinates: (ρ, θ)

$$d\hat{s}^2 = -d\rho^2 + d\xi^2 + \cosh^2\rho(d\theta^2 + \sin^2\theta d\phi) \to \begin{cases} \sinh\rho = -\frac{1-q^2\tau^2+q^2r^2}{2q\tau} \\ \tan\theta = \frac{2qr}{1+q^2\tau^2-q^2r^2} \end{cases}$$

- Flow velocity is fixed by the symmetry, $\hat{u}^\mu = (1, \vec{0})$
- NS hydro EoM reduces to $\partial_\rho \hat{e} + 2(\hat{e} + \hat{P})\tanh\rho - \frac{4}{3}\hat{\eta}\tanh^2\rho = 0\,.$
- For conformal flow, $\hat{\eta} = H_0 \hat{T}^3$

$$\hat{e}^{1/4} = \hat{T}(\rho) = (\cosh\rho)^{-\frac{2}{3}}\left[\hat{T}_0 + \frac{1}{3}H_0 F_d(\rho)\right] \to e(\tau, r)$$

Hydro modes in NS hydro (neglect bulk)

- Consider static fluid with perturbations and $\nabla \sim ik$ and $D \sim i\omega$,

$$\delta T^{00} = \delta e \qquad \delta T^{0i} = (e+P)u^i \qquad \delta T^{ij} = c_s^2 \delta e \delta^{ij} + \Pi^{ij}$$

- Sound mode propogation,

$$\omega^2 = -i\frac{4\eta}{3(e+P)}\omega k^2 + c_s^2 k^2 \rightarrow \omega = \pm c_s k - i\frac{2\eta}{3(e+P)}k^2 + O(k^3)$$

- Shear mode propogation, (large k modes are more damped)

$$\omega = i\frac{\eta}{e+P}k^2$$

$$\text{sound velocity: } c_s^2 = \frac{\partial P}{\partial e}$$

Causal Israel-Stwart hydrodynamics

[W. Israel, Annl Phys. 100 (1976), W. Israel and J. Stewart, Annl Phys. 118 (1979)]

- Acausal propogation,

$$\omega = i\frac{\eta}{e+P}k^2 \quad \longrightarrow \quad v_g = \frac{\partial w}{\partial k} \sim \frac{\eta k}{e+P} \xrightarrow{k\to\infty} \gg 1$$

- UV regulation of acausality,

$$\omega = i\frac{\eta}{e+P}k^2 \to i\frac{\eta}{e+P}\frac{k^2}{1+\alpha k^2} = \begin{cases} k \to \infty : v_g < 1 \\ k \to 0 : w \to i\frac{\eta}{e+P}k^2 \end{cases}$$

- Israed-Stewart hydro: $\alpha k^2 = -\tau_\pi \omega$

IS hydrodynamics

- Stress tensor expands to 2nd order in gradient,

$$\Pi^{\mu\nu} = \underbrace{-\eta\sigma^{\mu\nu}}_{O(\nabla)} \underbrace{-\tau_\Pi \left[\langle D\Pi^{\mu\nu} \rangle + \frac{4}{3}\Pi^{\mu\nu}\partial\cdot u \right]}_{O(\nabla^2)}$$

with a new transport coefficient: (shear) relaxation time

- For a N=4 SYM system,

$$\tau_\Pi = \frac{2 - \ln 2}{2\pi T}$$

Hydro and non-hydro modes in IS hydro

- Sound mode propogation,

$$-\tau_\Pi\omega^3 - i\omega^2 + \tau_\Pi\omega k^2 c_s^2 + \frac{4\eta}{3(e+P)}\omega k^2 + ic_s^2 k^2 = 0$$

$$\rightarrow \begin{cases} w_\pm = \pm c_s k - i\frac{2\eta}{3(e+P)}k^2 \pm \frac{2\eta}{3(e+P)c_s}\left(\tau_\Pi c_s^2 - \frac{\eta}{3(e+P)}\right)k^3 \\ \omega = -\frac{i}{\tau_\Pi} \end{cases}$$

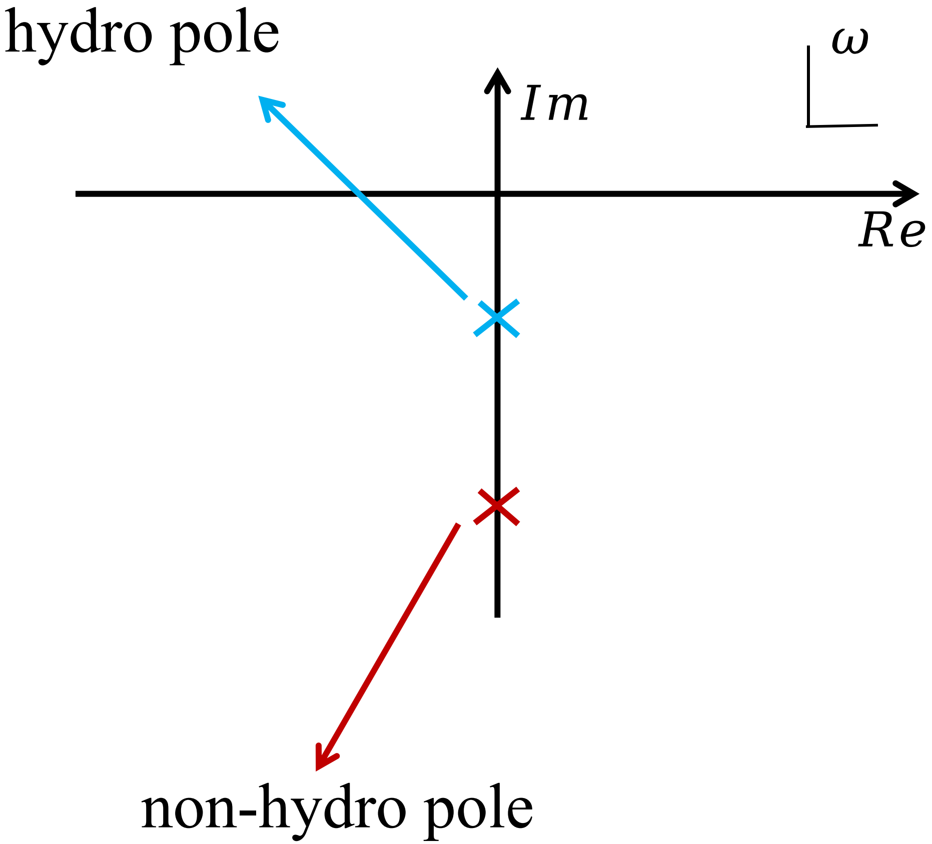
- Hydro modes go to 0 in long wavelength limit

Hydro and non-hydro modes in IS hydro

- Shear mode propogation,

$$\omega = \frac{i\eta}{e+P}\frac{k^2}{1-\tau_\Pi w}$$

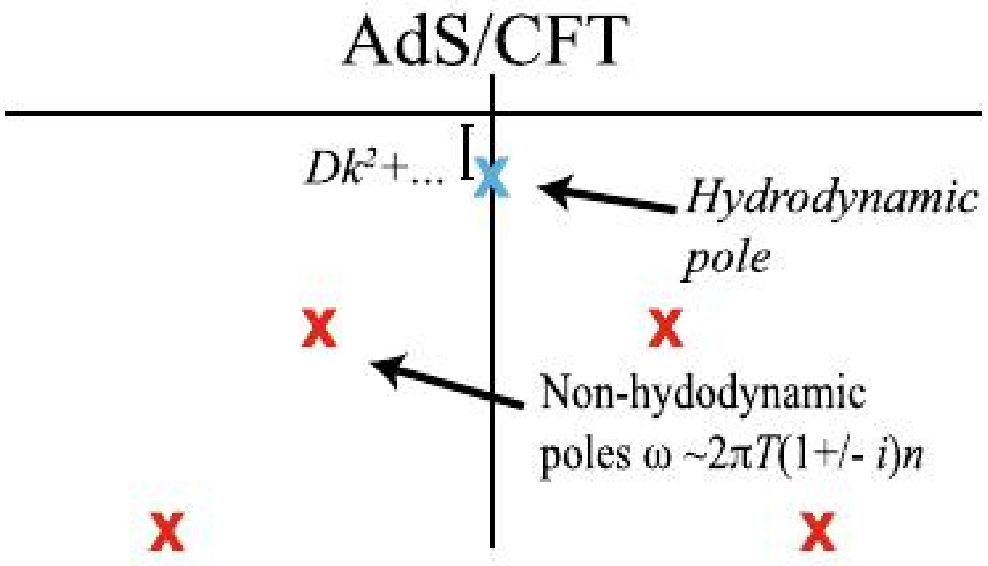
$$\to \begin{cases} w = -i\frac{\eta}{e+P}k^2 - i\left(\frac{\eta}{e+P}\right)^2 \tau_\Pi k^4 \\ \omega = -\frac{i}{\tau_\Pi} \end{cases}$$

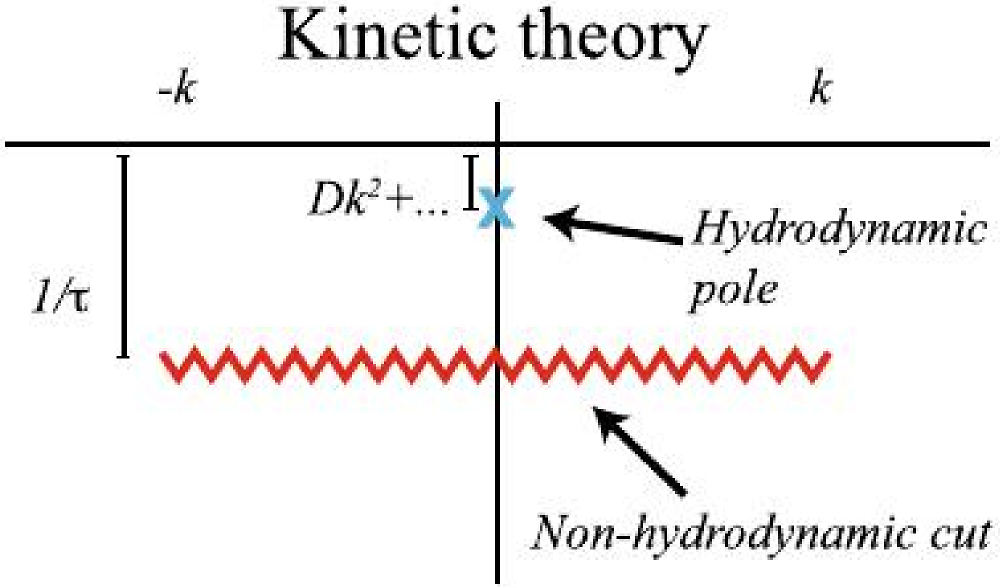


- Non-hydro mode evolution: $\delta \sim e^{-t/\tau_\Pi}$
- These pole structure arises also in retarded Green funcion G_R

Hydro and non-hydro modes in other theories

- AdS/CFT: strongly coupled system without quasi-particle excitation
- Kinetic theory: weakly coupled system with quasi-particle excitation





[A. Kurkela et al., EPJC 79, 965(2019),
P. Romatschke, EPJC 76:352 (2016)]

Hydro with more gradient corrections

- BRSSS hydro (2nd order conformal viscous hydro) [Baier et al., JHEP04(2008)100]

$$
\begin{aligned}
\Pi^{\mu\nu} &= -\eta\sigma^{\mu\nu} \\
&\quad + \eta\tau_\Pi\left[{}^{\langle}D\sigma^{\mu\nu\rangle} + \frac{1}{d-1}\sigma^{\mu\nu}(\nabla\cdot u)\right] + \kappa\left[R^{\langle\mu\nu\rangle} - (d-2)u_\alpha R^{\alpha\langle\mu\nu\rangle\beta}u_\beta\right] \\
&\quad + \lambda_1\sigma^{\langle\mu}{}_\lambda\sigma^{\nu\rangle\lambda} + \lambda_2\sigma^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda} + \lambda_3\Omega^{\langle\mu}{}_\lambda\Omega^{\nu\rangle\lambda}\,.
\end{aligned}
$$

Note the nonlinear couplings of gradients and new transport coefficients.

Hydro with more gradient corrections

- Third order hydro [A. Jaiswal, 1305.3480]

$$\begin{aligned}\dot{\pi}^{\langle\mu\nu\rangle} = &-\frac{\pi^{\mu\nu}}{\tau_\pi} + 2\beta_\pi\sigma^{\mu\nu} + 2\pi_\gamma^{\langle\mu}\omega^{\nu\rangle\gamma} - \frac{10}{7}\pi_\gamma^{\langle\mu}\sigma^{\nu\rangle\gamma} \\ &- \frac{4}{3}\pi^{\mu\nu}\theta + \frac{25}{7\beta_\pi}\pi^{\rho\langle\mu}\omega^{\nu\rangle\gamma}\pi_{\rho\gamma} - \frac{1}{3\beta_\pi}\pi_\gamma^{\langle\mu}\pi^{\nu\rangle\gamma}\theta \\ &- \frac{38}{245\beta_\pi}\pi^{\mu\nu}\pi^{\rho\gamma}\sigma_{\rho\gamma} - \frac{22}{49\beta_\pi}\pi^{\rho\langle\mu}\pi^{\nu\rangle\gamma}\sigma_{\rho\gamma} + \dots\end{aligned}$$

- Note the nonlinear couplings of gradients and new transport coefficients.
- Formulation becomes much more complicated: n! growth?

Nature of hydro gradient expansion

- (Global) gradient expansion of stress tensor w.r.t. Knudsen number,

$$\Pi^{\mu\nu} = O(\nabla) + O(\nabla^2) + \ldots = \sum_{n=1}^{\infty} \alpha_n \mathrm{Kn}^n$$

- Knudsen number is a dimensionless parameter,

$$\mathrm{Kn} \sim \frac{\text{mean free path}}{\text{system size}} \Rightarrow \text{how far away a system is from equilibrium}$$

- In fluids, there exist also dimensionless Reynolds number Re and Mach number Ma, satisfying Re×Kn~Ma. In the rapid expansion of QGP, Ma~1, which leaves only one parameter necessary for analysis.
- Relevant to the early-time expansion of QGP in heavy-ion collisions.

Nature of hydro gradient expansion

- Kn must be small so that hydro applies, namely, when Kn<<1, system is close to equilibrium, hydro can be truncated

$$\Pi^{\mu\nu} = O(\nabla) + O(\nabla^2) + \ldots = \sum_{n=1}^{\infty} \alpha_n \mathrm{Kn}^n$$

- E.g., space shuttle entering atmosphere at 20 km experiences Kn~10^{-8}
- However, it is known that expansion in Kn is asymptotic.

[H. Grad, Phys. Fluids 6 (1963) 147, S. Groot and W. Leeuwen and C van Weert, G. Denicol and J. Noronha, 1608.07869, J. Blaizot and LY, 1703.10694]

Example, solving BRSSS hydro in Bjorken flow

- BRSSS hydro becomes coupled ODE in Milne coordinates, with $\mathrm{Kn} = \tau_\pi / \tau$

$$\partial_\tau e = -\frac{4}{3}\frac{e}{\tau} - \frac{\pi}{\tau}$$

$$\pi = -\frac{4}{3}\frac{\eta}{\tau} - \tau_\pi \left(\partial_\tau \pi + \frac{4}{3}\frac{\pi}{\tau} \right) + \frac{\lambda_1}{2\eta^2}\pi^2$$

- To solve the equation, one may reparameterize the transport coefficients

$$\eta = C_\eta s \qquad \tau_\pi = \frac{C_\tau C_\eta}{T} \qquad \lambda_1 = C_\lambda \frac{s}{T}$$

with the dimensionless parameters determined via underlying theory. E.g.,

kinetic theory: $C_\eta = \frac{1}{4\pi} \qquad C_\tau = 5 \qquad C_\lambda = \frac{5}{7} C_\eta C_\tau$

- The coupled ODE can be recast into nonlinear ODE

$$w\frac{\mathrm{d}g}{\mathrm{d}w}\left(1+\frac{g}{4}\right)+\left(g+\frac{4}{3}\right)^2\left[1+\frac{3w}{8}\frac{C_\lambda}{C_\eta}\right]+w\left(g+\frac{4}{3}\right)-\frac{16}{9}\frac{C_\eta}{C_\tau}=0 \qquad (*)$$

where we define

$$g(w)\equiv\frac{\mathrm{d}\ln e}{\mathrm{d}\ln\tau}=-1-\frac{\mathcal{P}_L}{e}\longrightarrow\begin{cases}-1 & \text{anisotropic}\\ \\ -4/3 & \text{isotropic/thermalized}\end{cases}$$

and $\mathrm{Kn}^{-1}=w\equiv\dfrac{\tau}{\tau_\pi}$

- Note that $g(w)=-4/3\leftrightarrow e(\tau)\sim\tau^{-4/3}$ corresponds to ideal fluid.

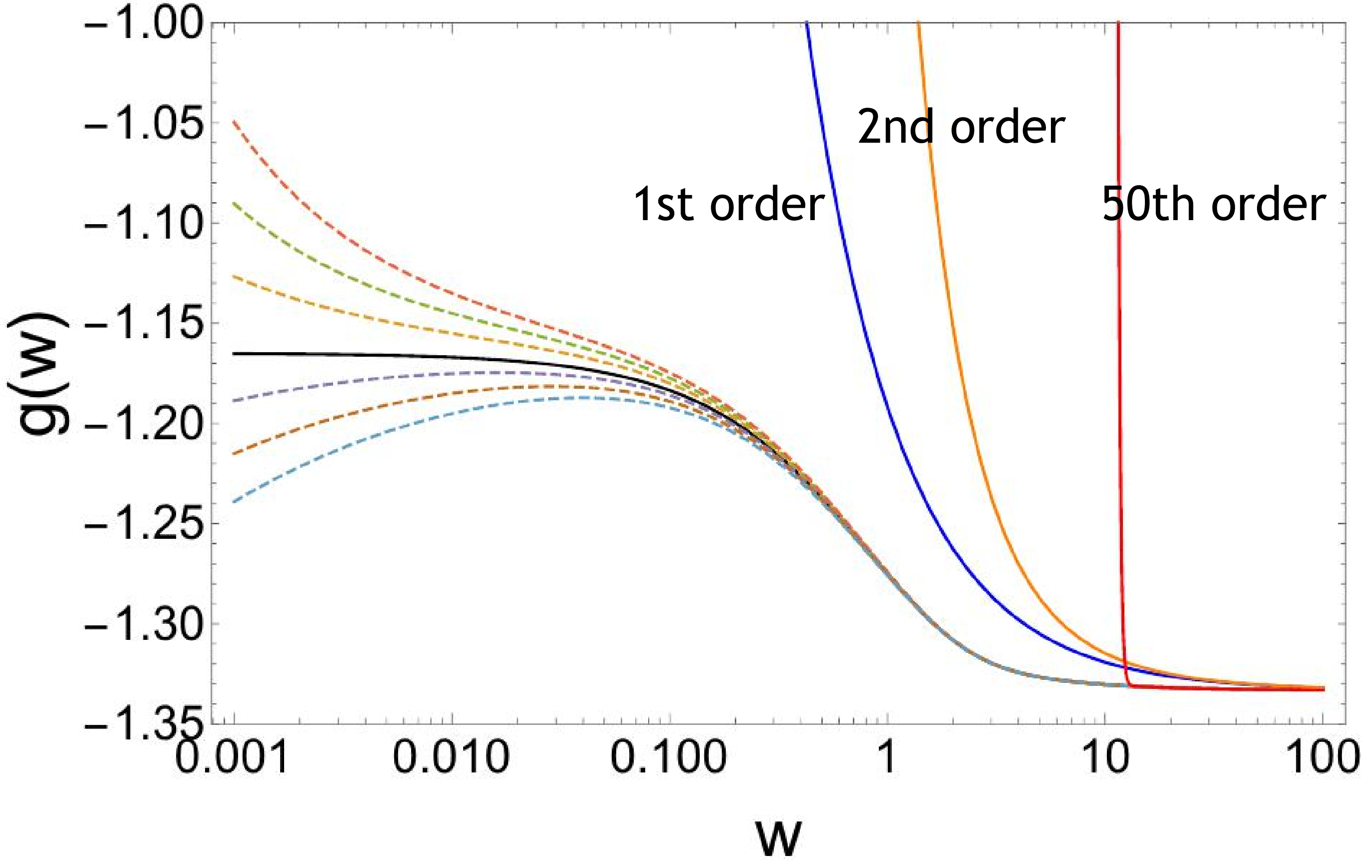
Solution to eq.(*)

- Numerical solution.
- Semi-analytical solution w.r.t. expansion in Kn,

$$g(w) = g^{\rm hydro}(w) \equiv \sum_{n=0}^{\infty} f_n^{(0)} w^{-n} = \underbrace{-\frac{4}{3}}_{\text{ideal hydro}} \underbrace{- \frac{16}{9} \frac{C_\eta}{C_\tau} \frac{1}{w}}_{\text{1st order viscous hydro}} + \dots$$

Solution to eq.(*)

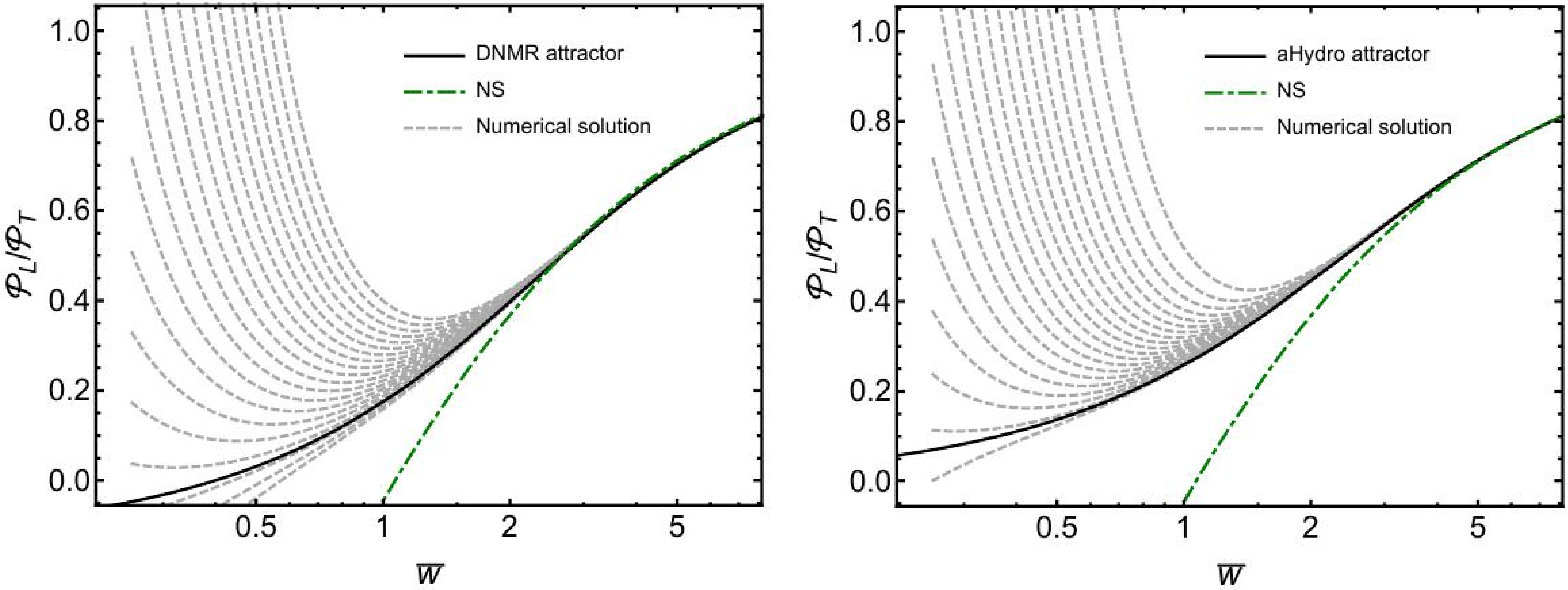
- Hydro gradients summation does not improve solution.



- Numerical solution insensitive to initial condition -- attractor

- *In the mathematical field of dynamical systems, an attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system. System values that get close enough to the attractor values remain close even if slightly disturbed.*
  – Wikipedia

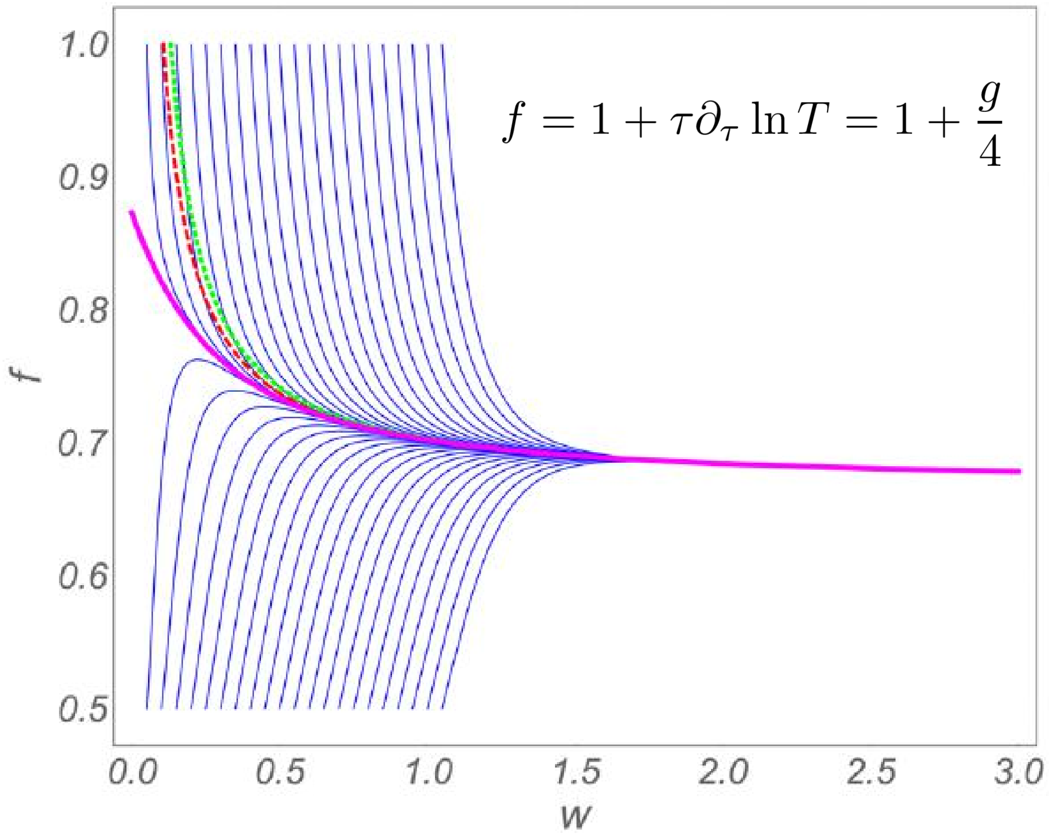
Attractor from different theories



[G. Denicol et al., 1709.06644]

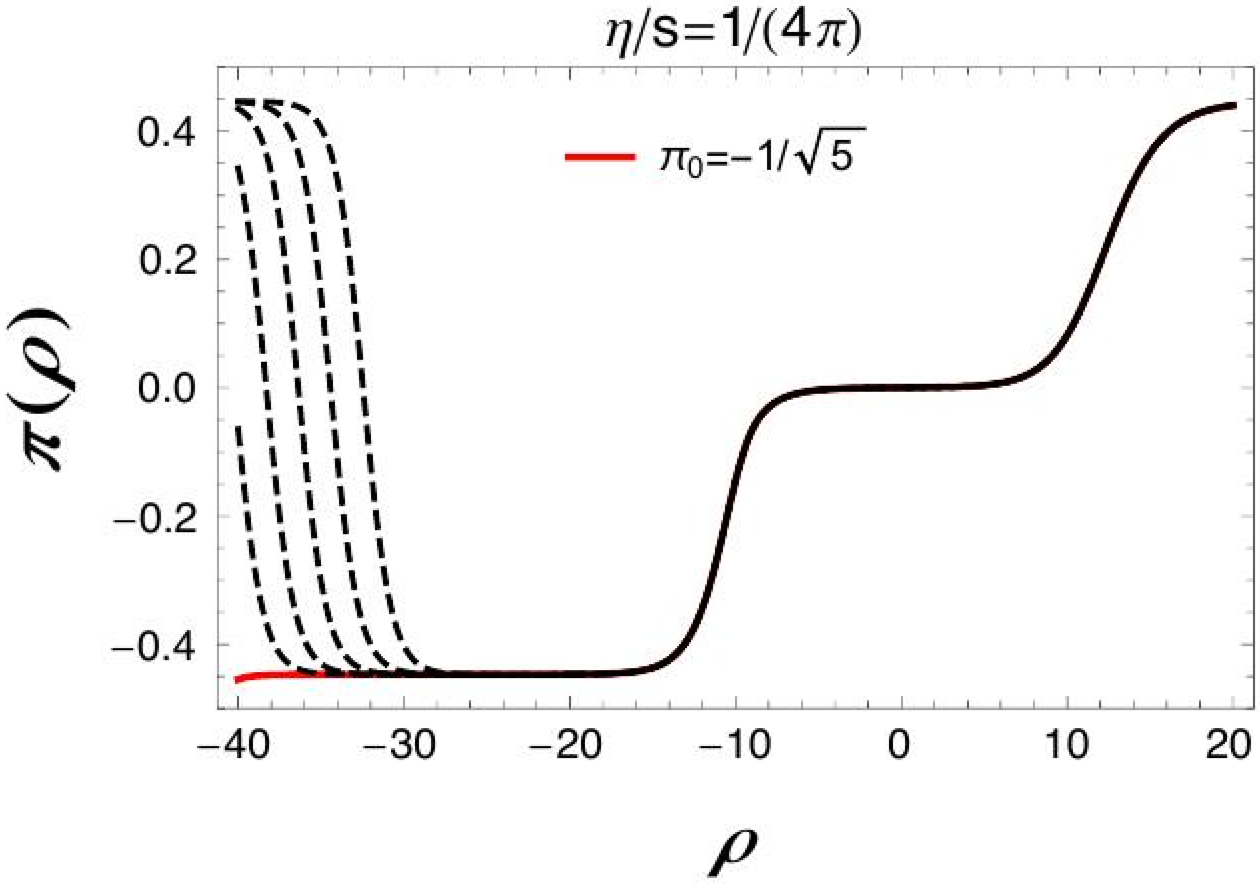
Attractor from different theories

[G. Denicol et al., 1804.04771]



Muller-Israel-Stewart Hydro

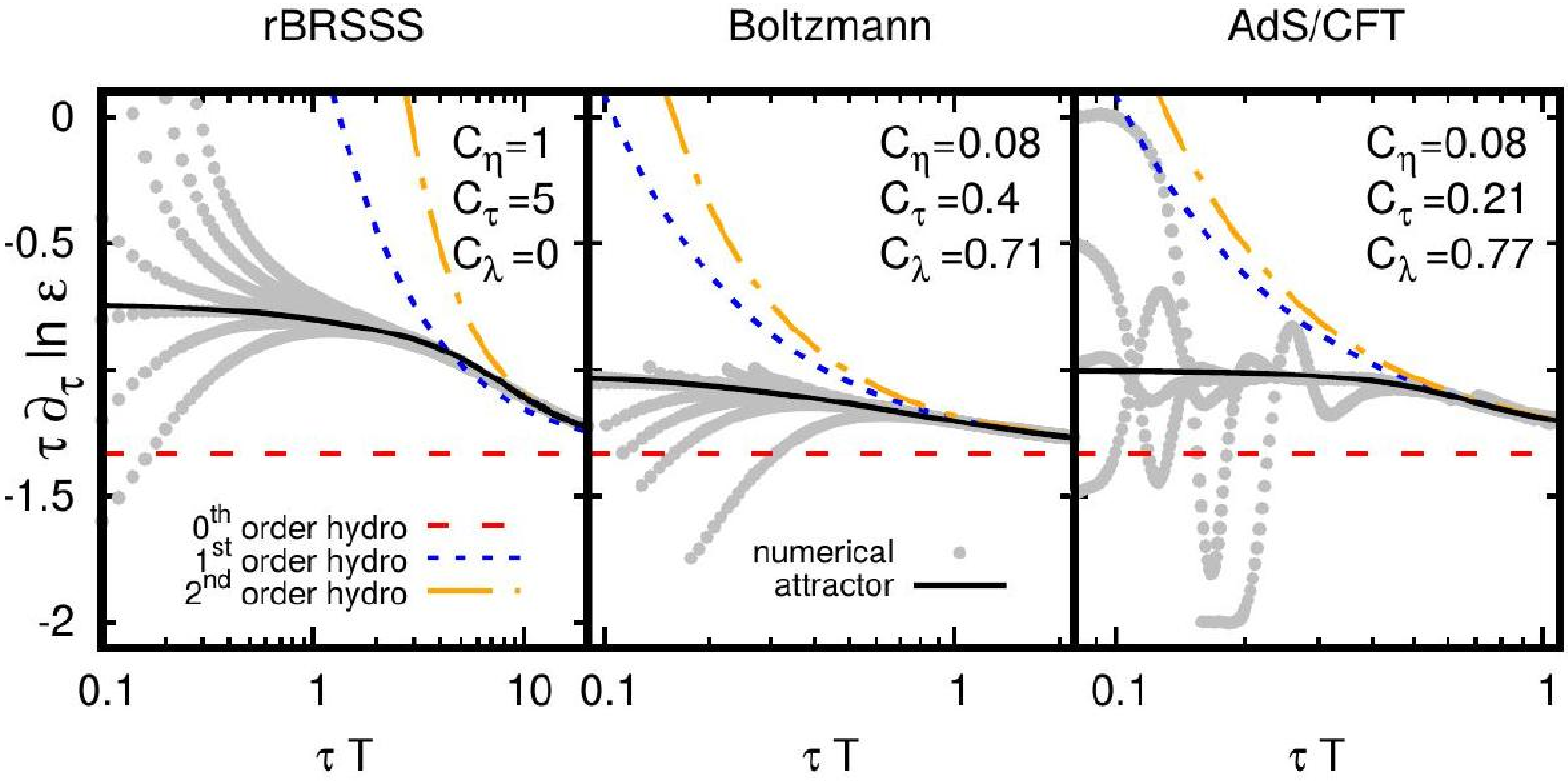
[M. Heller and M. Splinski, PRL 115, 072501 (2015)]



attractor in Gubser flow

(attractor in Hubble flow [Z.Du et al. 2104.12534])

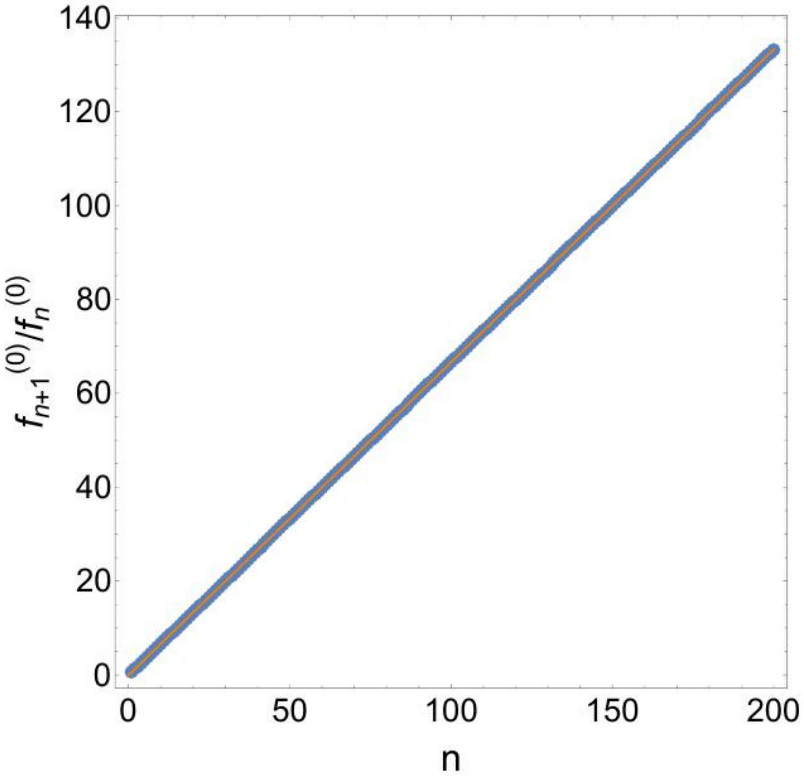
Attractor from different theories



[P. Romatschke, 1074.08699]

Hydro gradient expansion in Kn is aymptotic

- One can solve the expansion coefficient order by order, that



$$f^{(0)}_{n+1}/f^{(0)}_{n} \sim S^{-1}(n+\beta) + O(1/n)$$

$$\rightarrow \text{for large } n: \quad f_n^{0)} \sim \frac{\Gamma(n+\beta)}{S^{n+\beta}} \sim n!$$

- S and beta depend on transport coefficients.

- n! leads to zero radius of convergence, similar to perturbative expansion in QFT.

Borel resum of asymptotic series

For asymptotic series, Borel resummation technique can be applied.

1. Borel transform of the hydro gradient expansion leads to a convergent series,

$$g^{\mathrm{hydro}}(w) = \sum_{n=0}^{\infty} f_n^{(0)} w^{-n} \quad \rightarrow \quad \mathcal{B}[g](z) \equiv \sum_{n=0}^{\infty} \frac{f_n^{(0)}}{n!} z^n$$

2. Borel resum (Laplace transform)

$$\tilde{g}^{\mathrm{hydro}}(w) \equiv w \int_0^{\infty} dz e^{-zw} \mathcal{B}[g](z) \quad \rightarrow \quad w \int_C dz e^{-zw} \mathcal{B}[g](z)$$

3. Borel resum effective represents the asymptotic series, if it is Borel summable, i.e., there is no singularity on R^+ of $\mathcal{B}[g](z)$

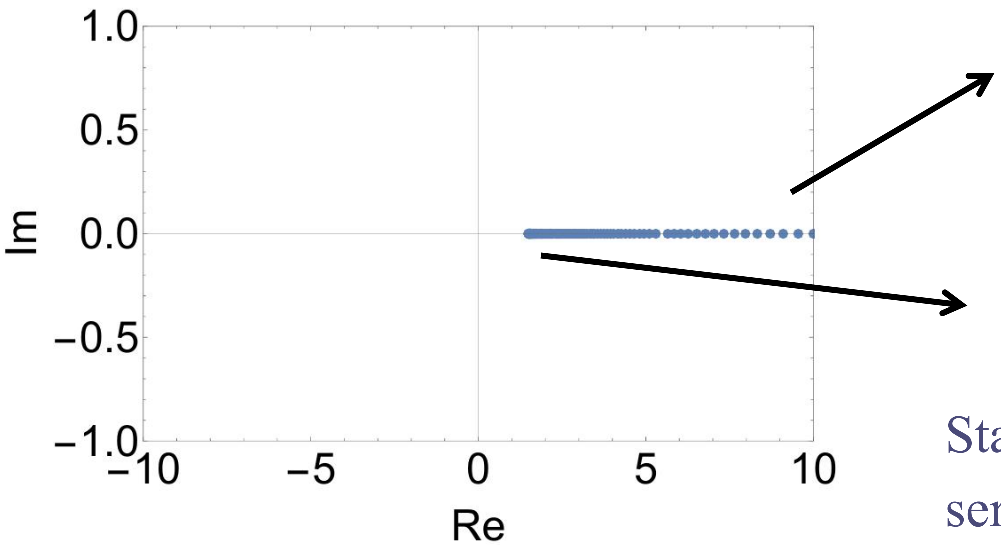
Hydro gradient is not Borel summable

- There exists singularity of the Borel transform of hydro gradient expansion,

[Basar and Dunne, PRD92,125011]

analytically: $\mathcal{B}[g](z) \sim \dfrac{1}{(S-z)^{\beta}}$

numerically: (also in practice) using a Padé approximation of $\mathcal{B}[g](z)$

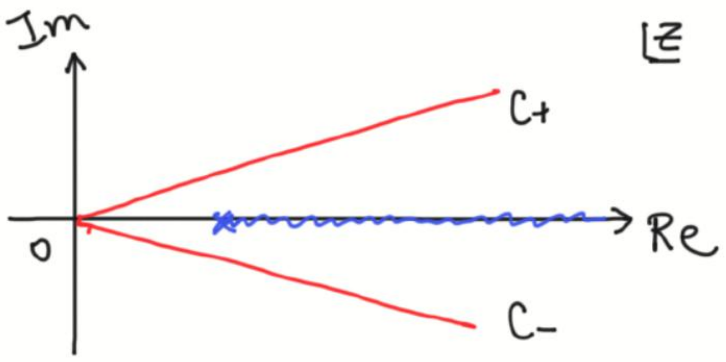


Pade approximation of $\mathcal{B}[g](z)$ gives a branch cut on the real axis.

The leading pole corresponds to S, i.e., the radis of convergence of $\mathcal{B}[g](z)$

Standard procedure to determine radius of convergence of a series, e.g., exp. of LQCD w.r.t. μ_B/T

Imaginary ambiguity from Borel resum



- To avoid branch cut in the resum, one needs analytic continuation to complex plane, with a complex ambiguity arises w.r.t. integration contour,

$$\int_{C_+} - \int_{C_-} \sim i e^{-Sw} w^{\beta}$$

accordingly, Borel sum leads to $\tilde{g}^{\text{hydro}}(w) = \text{Re}\left(\tilde{g}^{\text{hydro}}(w)\right) + (\propto i e^{-Sw} w^{\beta})$

Extends to trans-series and resurgence

- Borel resum must give rise to real solution, which implies trans-series solution,

$$g(w) = \sum_{m=0} (\sigma e^{-Sw} w^{\beta})^m g^{(m)}(w) = \sum_{m=0} (\sigma e^{-Sw} w^{\beta})^m \sum_{n=0} f_n^{(m)} w^{-n}$$

- Each $g^{(m)}(w) = \sum_{n=0} f_n^{(m)} w^{-n}$ is an asymptotic series, to be resummed.
- Especially, $g^{(0)}(w) = g^{\mathrm{hydro}}(w)$
- The complex constant σ is to be fixed via i.c. and resurgence relations:

$$e^{-Sw} w^{\beta} \mathrm{Im}(\sigma^{(m+1)}) \mathrm{Re}(\tilde{g}^{(m+1)}) + \mathrm{Re}(\sigma^{(m)}) \mathrm{Im}(\tilde{g}^{(m)}) = 0$$

very commonly, this has to be done numerically.

Brief summary (1)

- Hydro gradient expansion w.r.t. Kn diverges.
- Borel resum applies to the divergent hydro gradient expansion.
- After resum, hydro series expansion extends to trans-series,

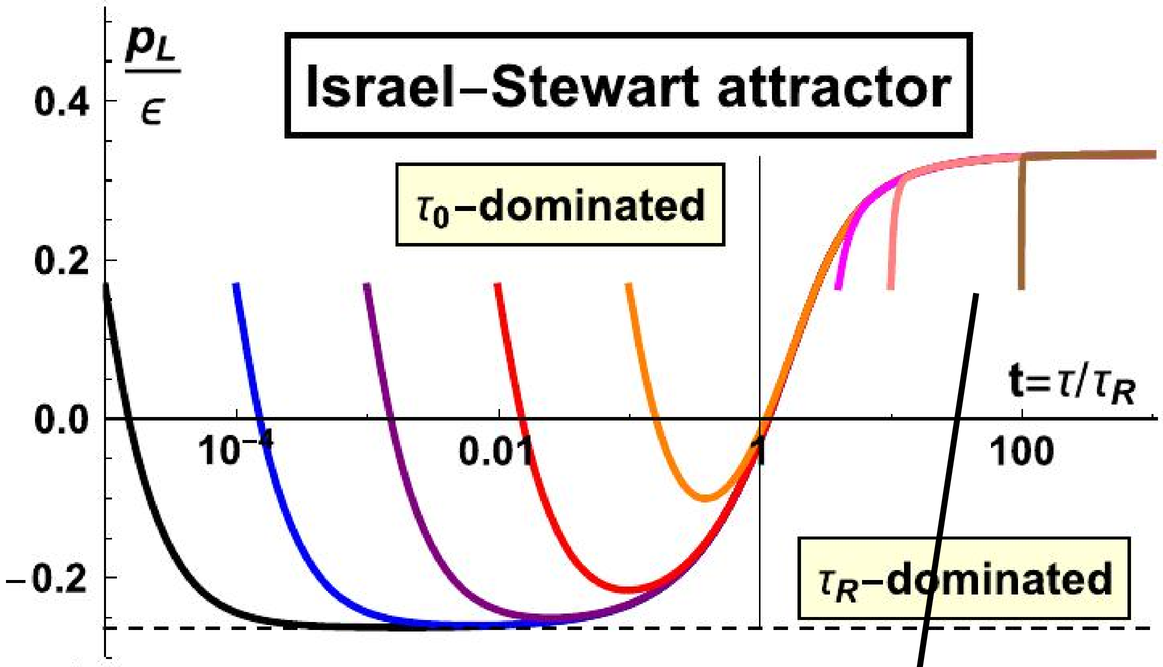
$$g^{(0)}(w) \to \sum_{m=0} (\sigma e^{-Sw} w^{\beta})^m g^{(m)}(w)$$

- Note that the factor $e^{-Sw} w^{\beta}$ emerges naturally along with i.c.: initial condition dependent evolution decays! -- attractor behavior
- This is how attractor emerges mathematically.

[M. Heller and M. Splinski, PRL 115, 072501 (2015), J. Blaizot and LY, 2006.08815]

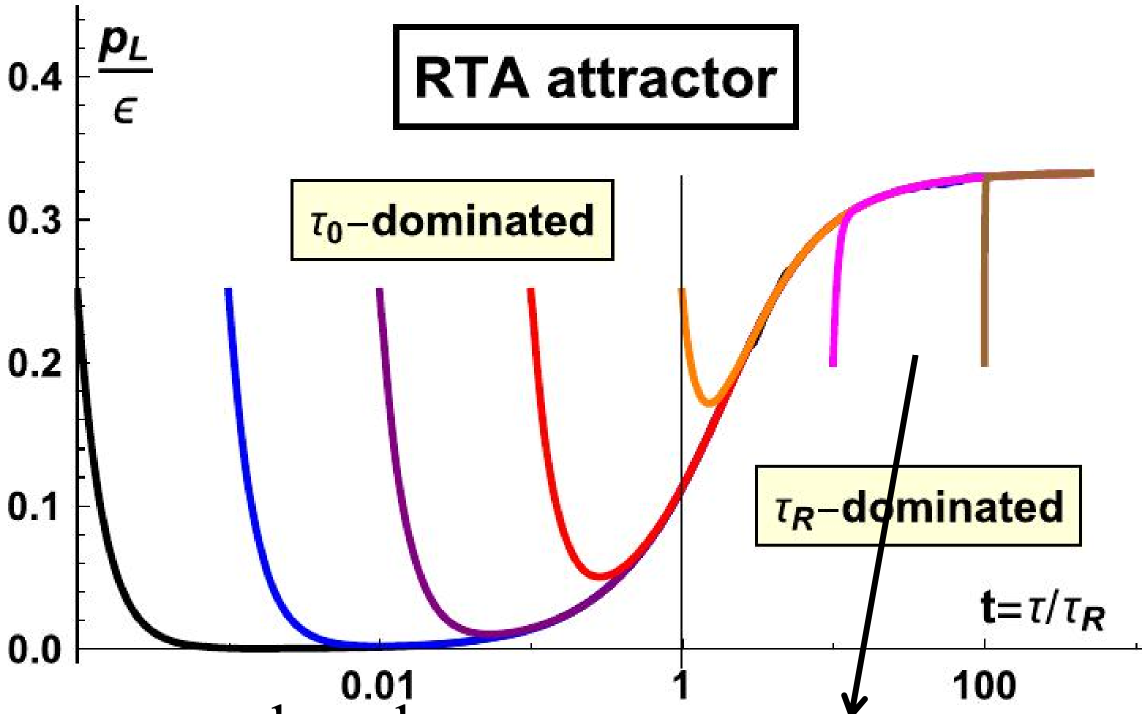
Observed attractor [A.Kurkela et. al., PRL124 (102301)]

- IS hydro
- RTA kinetic theory



power-law decay

Exponential decay

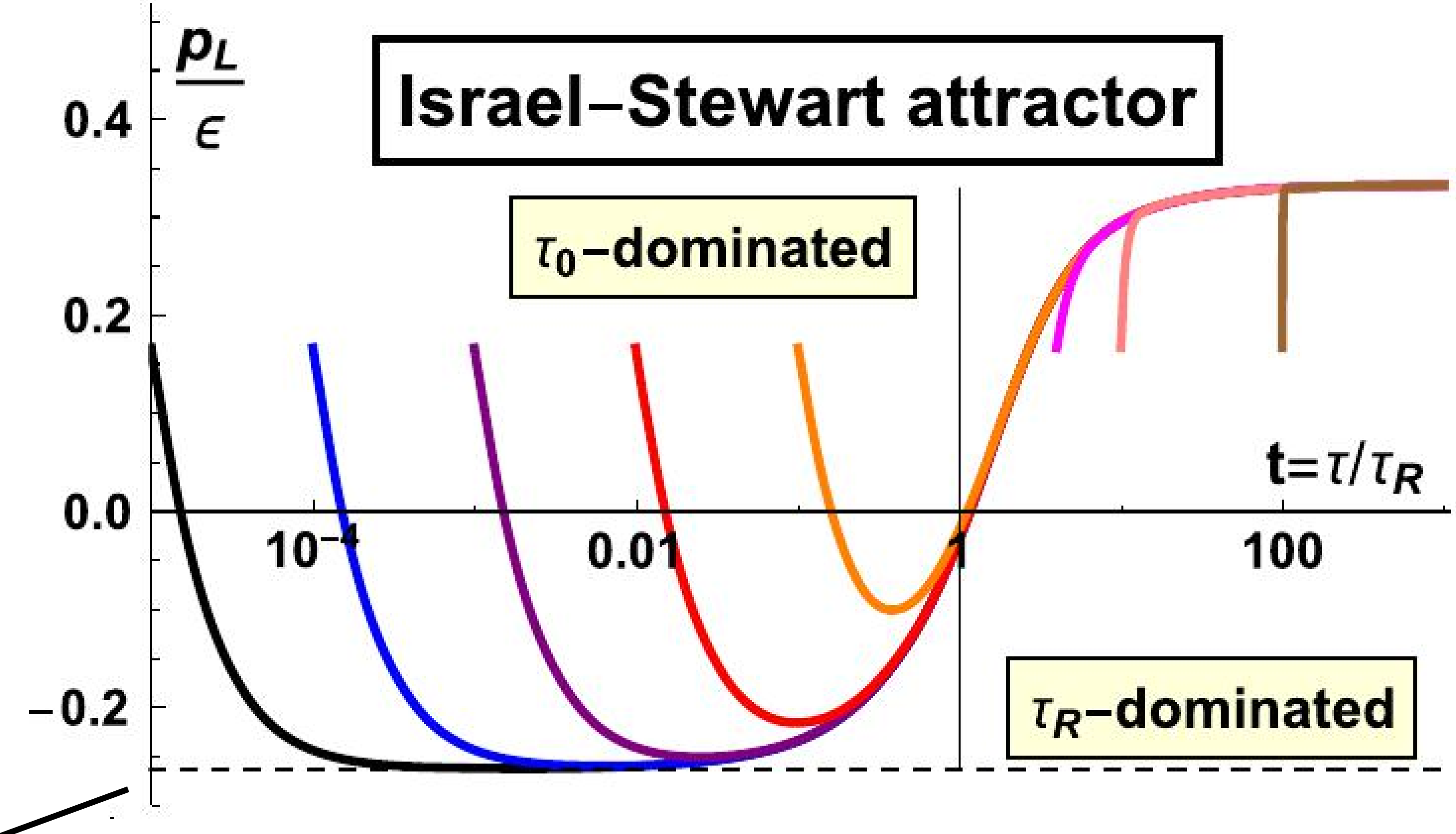


power-law decay

Exponential decay

Attractor and fixed point analysis

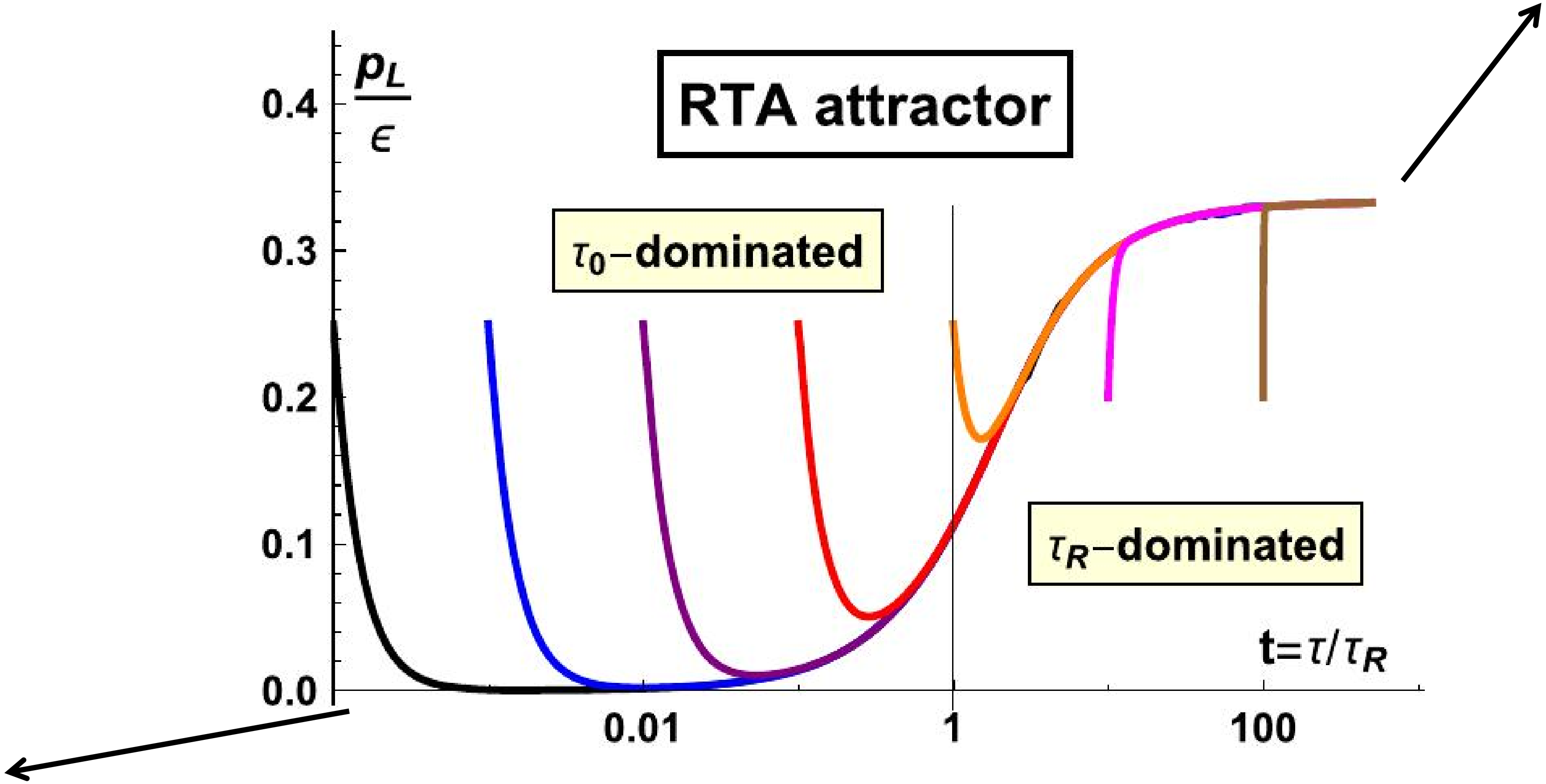
hydro fixed point at late times



free streaming fixed point at early times

Attractor and fixed point analysis

hydro fixed point at late times



free streaming fixed point at early times

Fixed points in the pre-equilibrium (e.g., IS hdyro)

- EoM of IS hydro w.r.t. Bjorken flow can be written in a matrix form,

$$\begin{cases} \tau\partial_\tau e = -\dfrac{4}{3}e + \pi \\ \tau\partial_\tau \pi = \dfrac{4}{3}\dfrac{\eta}{\tau_\pi} - \left(\dfrac{4}{3} + \dfrac{\tau}{\tau_\pi}\right)\pi \end{cases} \longleftrightarrow \quad \tau\partial_\tau V = -\underset{\text{free streaming}}{\underset{\swarrow}{H_F}} V - \underset{\text{collision}}{\underset{\searrow}{w H_w}} V$$

where,

$$V = \begin{pmatrix} e \\ \pi \end{pmatrix} \qquad H_F = \begin{pmatrix} \frac{4}{3} & -1 \\ \frac{16}{9C_\tau} & -\frac{4}{3} \end{pmatrix} \qquad H_w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- In late time limit, EoM reduces to hdyro fixed point solution: $\tau\partial_\tau e = -\dfrac{4}{3}e$

Fixed points in the pre-equilibrium (e.g., IS hydro)

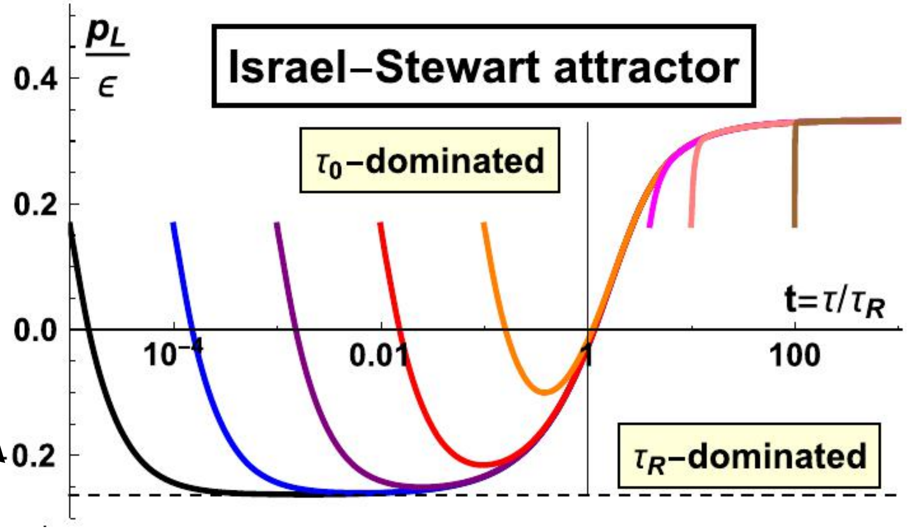
- At early times, eigenvalues of the free-streaming matrix detemine free-streaming fixed points,

$$H_F \phi_\pm = \lambda_\pm \phi_\pm \quad \rightarrow \quad \lambda_\pm = 1.92962,\ 0.737049$$

which implies decay of energy density in free streaming,

$$e(\tau) \sim \tau^{-0.737} \rightarrow \frac{p_L}{e} = -1 - g \sim 0.263$$

[J. Blaizot and LY, 1712.03856]



Fixed points in the pre-equilibrium (e.g., IS hydro)

- Alternatively, one may derive the beta function defined with respect to the evolution of g(w),

$$\beta(w) \equiv \partial_w g(w) = \Phi(w)\left[\frac{16}{9C_\tau} - \left(\frac{4}{3} + g(w)\right)\left(\frac{4}{3} + g(w) + w\right)\right]$$

[J. Blaizot and LY, 1904.08677]

fixed points can be solved accordingly via

$$\beta(w) = 0 \quad \rightarrow \begin{cases} \tau \rightarrow 0^+ : g \rightarrow -1.92962,\ -0.737049 \\ \tau \rightarrow \infty : g \rightarrow -\frac{4}{3} \end{cases}$$

Slow mode evolution and attractor (e.g., IS hydro)

$$\tau\partial_\tau V = -H(w)V\,, \quad \text{with} \quad H(w) = H_F + wH_w$$

- Solving the coupled EoM of IS hydro effectively via the time-dependent eigenvalue problem of matrix, in analogy to QM,

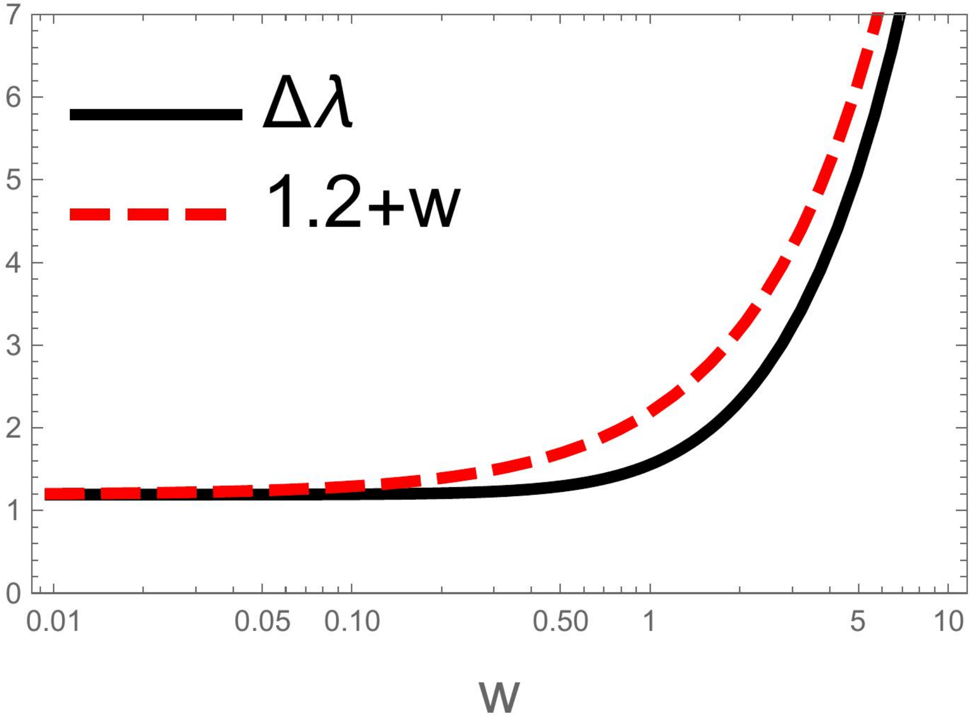
$$H(w)\phi_\pm(w) = \lambda_\pm(w)\phi_\pm(w)$$

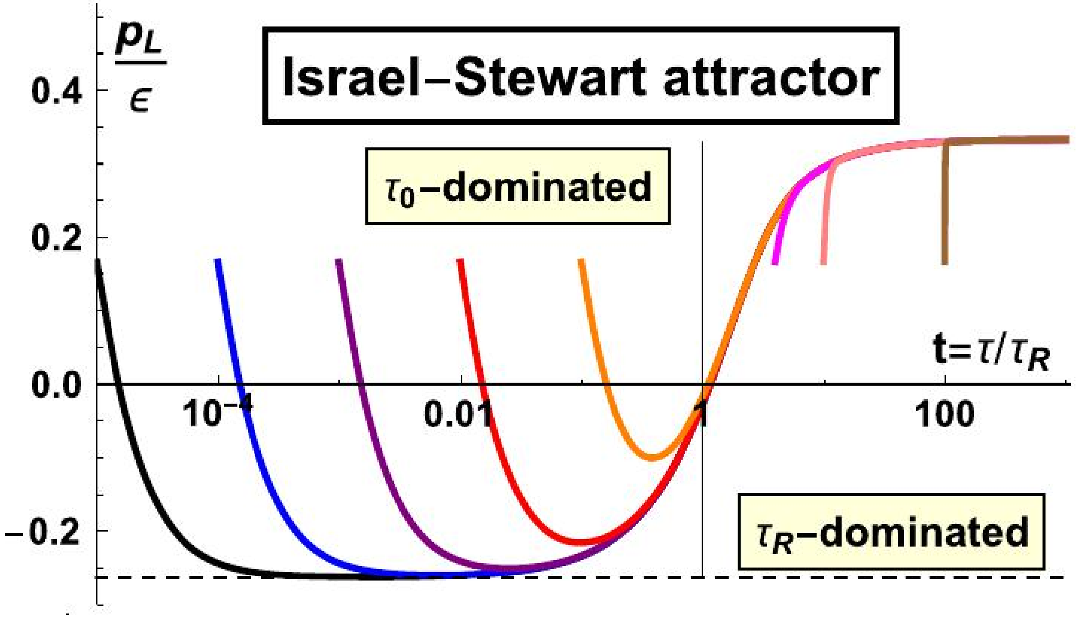
arbitrary evolution should be dominated by the slowest mode,

$$V(w) \sim \underbrace{\phi_-(w)}_{\text{attractor}} + \underbrace{\# e^{-\Delta\lambda(w)}\phi_+(w)}_{\text{perturbations}} \quad \text{if} \quad \Delta\lambda = \lambda_+ - \lambda_- > 0$$

[J. Brewer et al., 1910.00021]

Slow mode evolution and attractor (e.g., IS hydro)





- Gap of eigenvalues: $\Delta\lambda(w) \sim 1.2 + w \quad \rightarrow \quad \tau^{-1.2} e^{-\tau/\tau_\pi}$

  explains early-time power-law decay and late-time exponential decay.

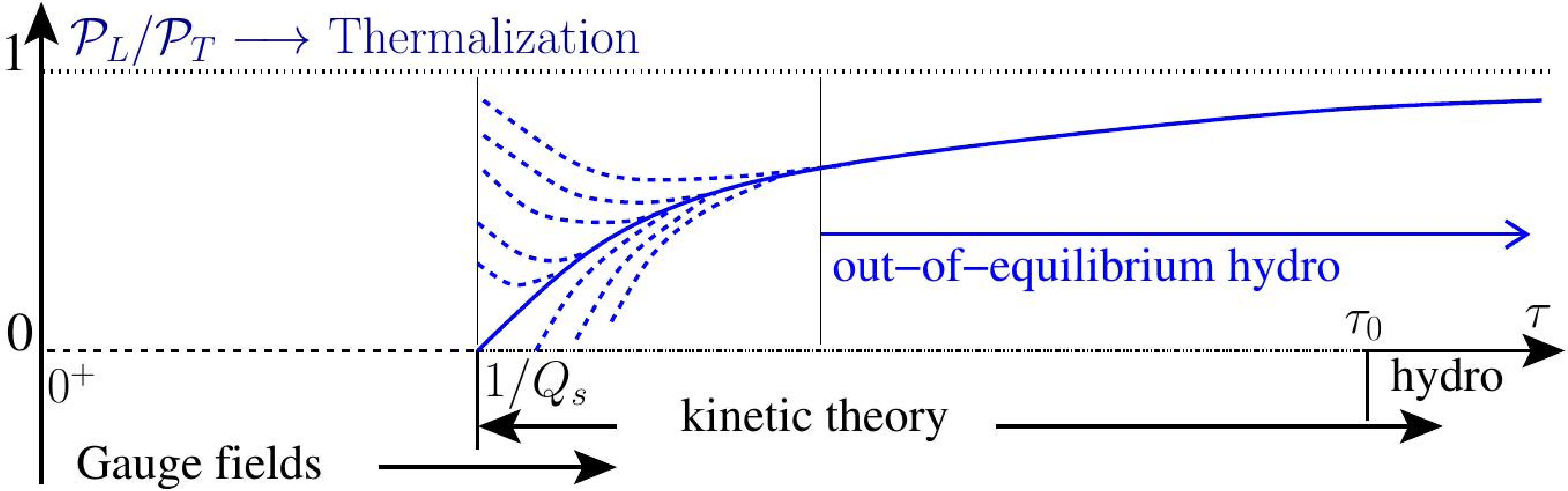
Brief summary (2)

- Hydro attractor solution can be as well understood in terms of

1. Fixed point analysis: Free-streaming fixed point corresponds to 1D expansion. Hydro fixed point corresponds to collisions among excitations. Therefore, emergence of hydro attractor due to competing effects of expasion and collision.
2. Slow mode (adiabatic) evolution: Slow mode dominates system evolution throughout all stages, if the gap is perserved. In particular, the fast mode plays the role of non-hydro mode, which at late times, behavior as $\delta \sim e^{-t/\tau_\Pi}$

Onset of hydro from far-from-equilibrium

- Pre-equilbrium expansion -- attractor picture: hydro starts much earlier



Fluctuating hydrodynamic

- Hydro fluctuation: thermal fluctuation in fluids, universal in nature.
- Classical hydro (w/o fluc.) ==> hydro with thermal fluctuations

$$\partial_\mu T^{\mu\nu} = 0 \quad \Rightarrow T^{\mu\nu} = T^{\mu\nu}_{\mathrm{cl}} + \delta T^{\mu\nu} + S^{\mu\nu}$$

where $S^{\mu\nu}$ characterizes random noise, and $\delta T^{\mu\nu} \sim \delta e,\ \delta u^\mu$ are induced thermal perturbations accordingly.

- Here classical hydro consists of classical hydro fields -- hydro fields without correction from thermal fluctuations, (what we have discussed so far)

$$T^{\mu\nu}_{cl} = e_{cl} u^\mu_{cl} u^\nu_{cl} + P_{cl} \Delta^{\mu\nu}_{cl} + \Pi^{\mu\nu}_{cl}$$

Fluctuating hydrodynamic

- Fluctuation-dissipation relation, (define {...} emsemble average)

$$\{S^{\mu\nu}(x_1)S^{\alpha\beta}(x_2)\} = 2T[\eta(\Delta^{\nu\alpha}\Delta^{\nu\beta} + \Delta^{\mu\beta}\Delta^{\nu\alpha}) + (\zeta - \frac{2}{3}\eta)\Delta^{\mu\nu}\Delta^{\alpha\beta}]\delta(x_1 - x_2)$$

[Landan and Lifshtz, "Fluid dynamics"]

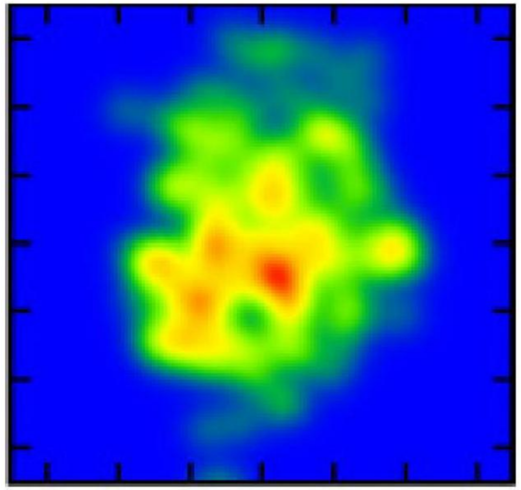
- Similarly for charge current: (σ charge conductivity)

$$J^\mu_B = J^\mu_{B\,cl} + \delta J^\mu_B + I^\nu\,, \qquad \{I^\mu(x_1)I^\nu(x_2)\} = 2\sigma T\Delta^{\mu\nu}\delta(x_1 - x_2)$$

[J. Kapusta, B. Muller and M. Stephanov, PRC85, 054906]

Properties of hydrodynamic fluctuations

- Qualitatively different from quantum fluctuations: initial state fluctuations

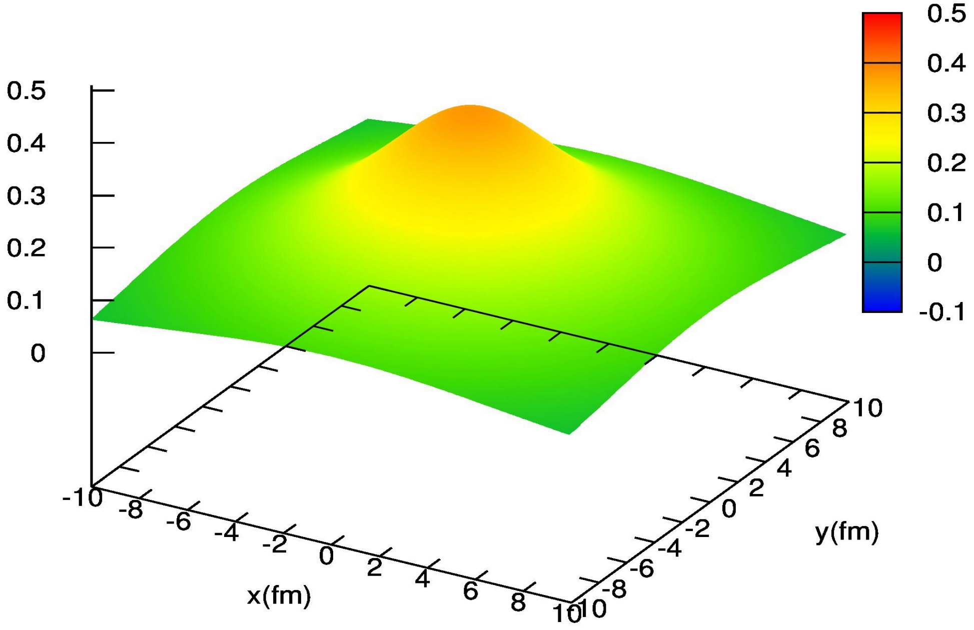


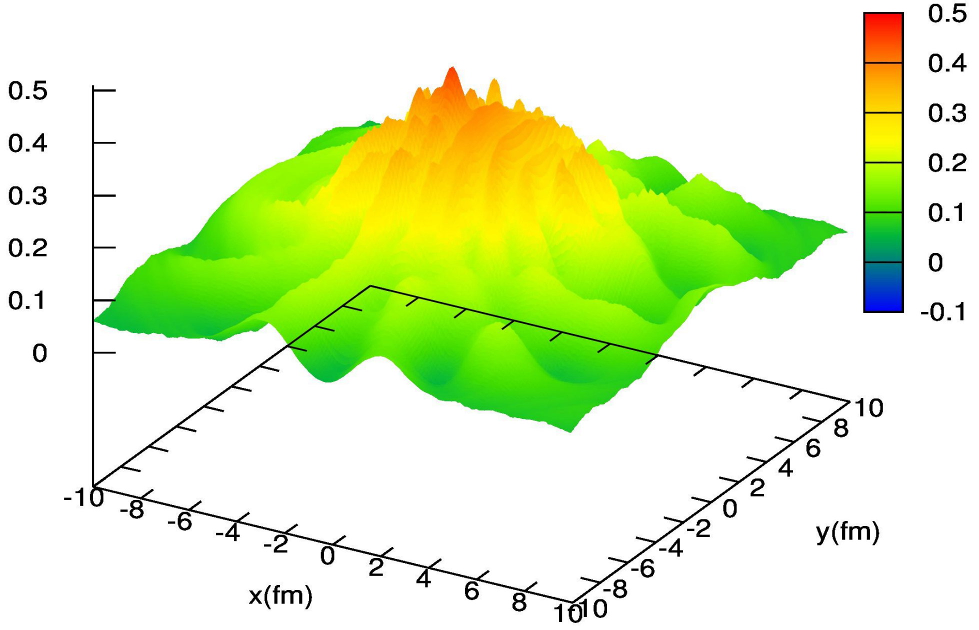
e.g.: event-by-event fluctuations of initial energy density $\delta e_i(\tau_0, \vec{x}_\perp)$

$$\{\delta e_i \delta e_i\} \propto \frac{1}{N_p}\,, \quad \{\delta e \delta e\} \propto \frac{Te}{V}\,, \quad \{\delta e_i \delta e\} = 0$$

- Hydro fluctuations are strong in dissipative and small systems.
- Hydro fluctuations are substantial near critical region.

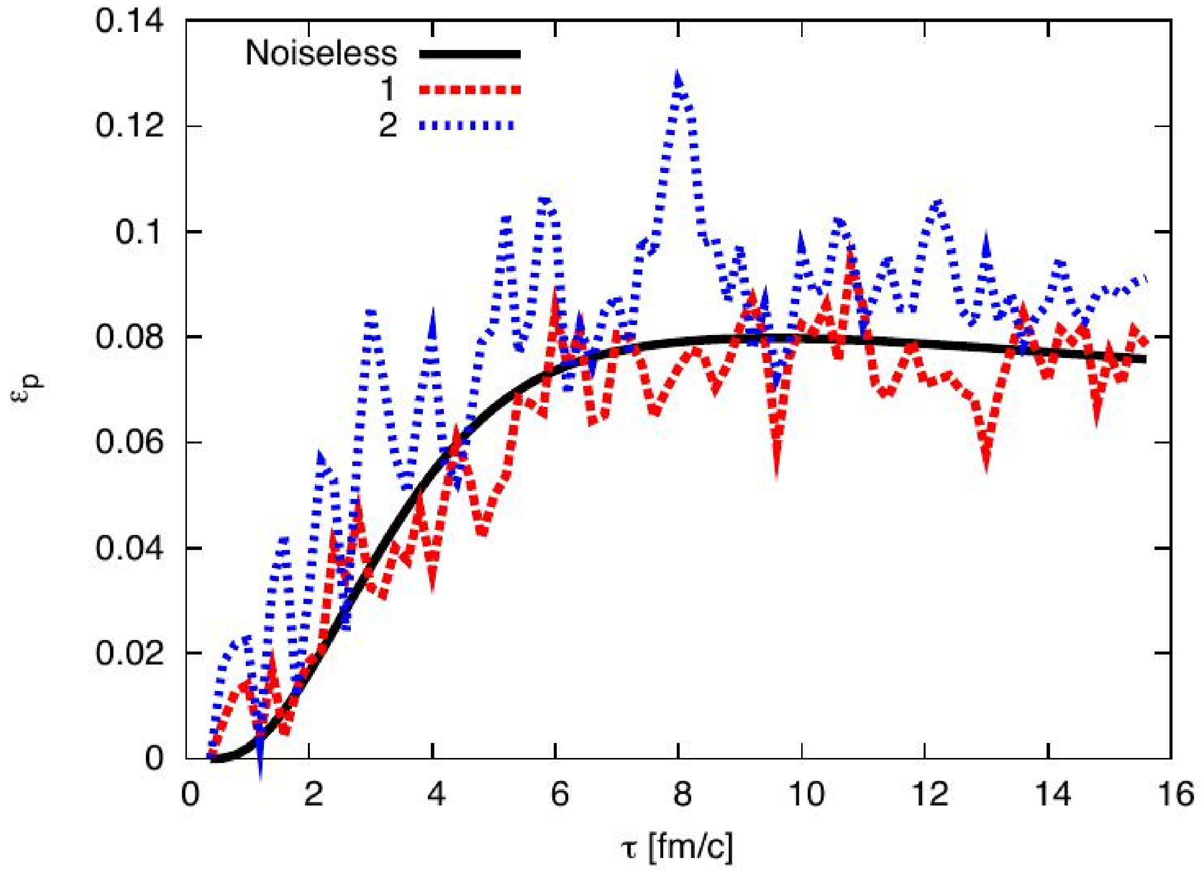
Hydrodynamic fluctuations in heavy-ion collisions





Hydrodynamic fluctuations in heavy-ion collisions

- Numerical simulations of fluctuating hydrodynamics.
- Solving stochastic partial differential equations.
- Numerical realization of Dirac delta function on grid.
- Needs average for obserbles.



[B. Schenke et. al., 2005.00621, see also C. Young, PRC89, 024913, A. Sakai et al., 2111.08963, A. De et al., 2203.02134]

Evolution of n-point correlation: EFT

- Fluctuating hydro implies deterministic equation for n-point correlations.
- Tree-level: classical hydro EoM
- One-point level: $\{\delta\} = 0$
- At two-point level (one-loop): $\{\delta\delta\}$ ==> hydro-kinetic equation,

$$\partial_t N_A = \underbrace{-\alpha_A(k^2, \eta(t))(N_A - \bar{N}_A)}_{\text{relaxation type}} \quad + \quad \underbrace{\beta_A(t, \vec{k})N_A}_{\text{expansion bkg flow}}$$

where $N_A \sim \{\delta T^{00}\delta T^{00}\}, \{\delta T^{0i}\delta T^{0i}\}$

- N-point cumulants. [X.An et al., 2009.10742]

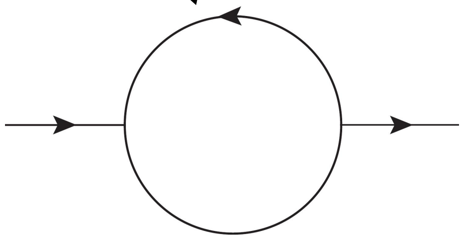
Renormalization from hydro fluctuations

- Taking into account of hydro fluctuations, averaged emergy-momentum tensor receives corrections from n-point correlations.

- For the Bjorken flow, for instance, for energy density, one has

$$\{\tau^2 T^{\xi\xi}\} = P - \frac{4}{3}\frac{\eta}{\tau} + \frac{\{\delta T^{0z}\delta^{0z}\}}{e+P} \quad = P_R - \frac{4}{3}\frac{\eta_R}{\tau} + \tau^2 \Delta T^{\xi\xi}$$

loop contains to k-integral



$$= \int d^3\vec{k} N_A = \int_0^\Lambda k^2 dk + \text{finite}$$

Renormalization from hydro fluctuations

- Renormalized shear viscosity, [P. Kovtun, G. Moore and P. Romatschke, PRD84, 025006(2016)]

$$\eta_R = \eta + \frac{17\Lambda T(e+P)}{120\pi^2\eta}$$

- Renormalized pressure, [Y.Aakamatsu et al., 1606.07742]

$$P_R = \frac{1}{3}e_R = P_{cl} + \frac{T\Lambda^3}{6\pi^2}$$

- Long-time tails $\tau^2 \Delta T^{\xi\xi} \propto \frac{1}{\tau^{3/2}} \to \tau^2\{T^{\xi\xi}\} = O(\nabla^0) + O(\nabla) + O(\nabla^{3/2}) + O(\nabla^2)$

Summary

- Hydro emerges as EFT in long wavelength limit: hydro gaped from non-hydro
- Hydro from far-from-equilibrium exhibits attractor behavior

1. Borel resum ==> attractor and non-hydro (decay of perturbations).
2. Fixed point analysis ==> attractor smoothly connects fixed points.
3. Slow mode evolution ==> attractor dominated by slow mode.

- Hydro fluctuations

1. Flutuating hydro can be solved via stochastic ODE.
2. n-point function as EFT