Graphic Expansions of Amplitudes and BG Currents

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Based on the following work:

2207.02374, Wu, Du

JHEP 01 (2022) 162， Wu， Du

JHEP 04 (2021) 150， Tian， Gong， Xie， Du

JHEP 05 (2020) 008， Du, Hou

JHEP 05 (2019) 012， Hou, Du

JHEP 12 (2017) 038， Du, Feng, Teng

JHEP 09 (2017) 021, Fu, Du, Huang, Feng

Outline

YM and GR

Interactions in nature

Gravity	$\longleftrightarrow$	Strong, Weak, EM
	$\downarrow$	
General relativity	$\longleftrightarrow$	Gauge theory

Compaing Feynman rules of GR and YM

GR: Infinite number of vertices; Color singlets

YM: Three-point and four-point vertices; Color decomposition formulas

$$M_{\text{YM}} = \sum_{\boldsymbol{\sigma}\in S_{n-1}} \text{Tr}(T^{a_1}T^{a_{\sigma(2)}}\dots T^{a_{\sigma(n)}})A(1,\boldsymbol{\sigma})$$

$$M_{\text{YM}} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} if^{a_1 a_{\sigma(2)} e_1} if^{e_1 a_{\sigma(2)} e_2} \dots if^{e_{n-3} a_{\sigma(n-1)} a_n} A(1,\boldsymbol{\sigma},n)$$

GR as double copy of YM

$$\epsilon^{\mu\nu} \sim \epsilon^{\mu}\epsilon^{\nu}$$
$$\downarrow$$
$$\text{GR} \sim \text{YM} \times \text{YM}$$

GR as double-copy of YM:

- Kawai-Lewellen-Tye (1986)
- Bern-Carrasco-Johansson (2008)
- Cachazo-He-Yuan (2013)

Kawai-Lewellen-Tye double copy

KLT relation:

Closed string tree amplitudes	$\sim$	(Open string tree amplitudes)^2
$\Downarrow$		$\Downarrow$
GR tree amplitudes	$\sim$	(YM tree amplitudes)^2

KLT in field theory:

(Bern, De Freitas,Wong (1999); Bjerrum-Bohr,Damgaard,Feng,Sondergaard (2010))

$$M_n = \sum_{\sigma,\rho} A_n(\rho) S[\rho|\sigma] \widetilde{A}_n(\sigma)$$

M_n :	GR	color-dressed YM
A_n :	YM	ϕ^3 scalar
$\widetilde{A}_n$:	YM	YM

Bern-Carrasco-Johansson double copy

BCJ form of YM amplitudes:

$$M^{\mathrm{YM}} = \sum_{\mathcal{G}} \frac{c_{\mathcal{G}} n_{\mathcal{G}}}{\prod_i D^i_{\mathcal{G}}}$$

- $\mathcal{G}$: diagrams with cubic vertices
- $D^i_{\mathcal{G}}$: propagators $c_{\mathcal{G}}$: color factors
- $n_{\mathcal{G}}$: BCJ numerators satisfying

$$c_i = -c_j \quad \Rightarrow \quad n_i = -n_j$$
$$c_i + c_j + c_k = 0 \quad \Rightarrow \quad n_i + n_j + n_k = 0$$

BCJ form of GR amplitudes

$$M^{\mathrm{GR}} = \sum_{\mathcal{G}} \frac{n_{\mathcal{G}} \widetilde{n}_{\mathcal{G}}}{\prod_i D^i_{\mathcal{G}}}$$

Cachazo-He-Yuan double copy

$$M = \int d\mu\, \mathcal{I}_L \mathcal{I}_R$$

GR $\quad \mathcal{I}_L = \mathrm{Pf}'(\Psi), \quad \mathcal{I}_R = \mathrm{Pf}'(\Psi)$

Color-ordered YM $\quad \mathcal{I}_L = \frac{1}{z_{12}z_{23}\cdots z_{n1}}, \ \mathcal{I}_R = \mathrm{Pf}'(\Psi)$

Color-ordered BS $\quad \mathcal{I}_L = \frac{1}{z_{12}z_{23}\cdots z_{n1}}, \ \mathcal{I}_R = \frac{1}{z_{\sigma_1\sigma_2}z_{\sigma_2\sigma_3}\cdots z_{\sigma_n\sigma_1}}$

$$\left(\mathrm{Pf}'[\Psi] \equiv \frac{perm(ij)}{z_{ij}}\mathrm{Pf}[\Psi^{ij}_{ij}]\right)$$

Cachazo-He-Yuan double copy

Ψ is a $2n \times 2n$ skew-symmetric matrix:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}$$

The $n \times n$ submatrices A, B and C are

	A_{ab}	B_{ab}	C_{ab}
$a \neq b$	$\frac{s_{ab}}{z_{ab}}$	$\frac{2\epsilon_a \cdot \epsilon_b}{z_{ab}}$	$\frac{2\epsilon_a \cdot k_b}{z_{ab}}$
$a = b$	0	0	$-\sum_{c \neq a} \frac{2\epsilon_a \cdot k_c}{z_{ac}}$

Kinematic decomposition

KLT

BCJ $\rightarrow$

CHY

$$M^{\mathrm{GR}} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{YM}}(1,\boldsymbol{\sigma},n)$$

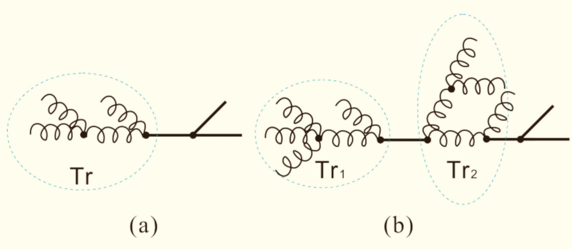
$$A^{\mathrm{YM}} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{BS}}(1,\ldots,n|1,\boldsymbol{\sigma},n)$$

Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove (2010); Bern and Dennen (2011); Du Feng Fu (2012); Cachazo, He, Yuan (2013)

$n_{1,\boldsymbol{\sigma},n} \rightarrow$ BCJ numerator in DDM basis

（Del Duca，Dixon，Maltoni （1999））

EYM single-trace and multi-trace



EYM recursive expansion: an approach to local numerators

EYM recursive expansion for single-trace amplitudes

(Stieberger and T. R. Taylor(2016),Nandan, Plefka, Schlotterer, Wen (2016),de la Cruz, Kniss,Weinzierl (2016),Schlotterer (2016),Du, Feng, Fu, Huang (2017); Chiodaroli,Gunaydin,Johansson (2017); Teng, Feng (2017))

$$A(1,2,\ldots,r\|\mathrm{H}) = \sum_{\mathrm{H}\setminus\{h_a\}\to \mathrm{h}|\boldsymbol{h}} \epsilon_{h_a}\cdot F_{\rho_1}\cdots F_{\rho_i}\cdot Y_{\rho_i} \times A(1,\{2,\ldots,r-1\}\sqcup\{\rho_i,\ldots,\rho_1,h_a\},r\|\boldsymbol{h})$$

- $\mathrm{H}\setminus\{h_a\}\to \mathrm{h}|\boldsymbol{h}$: splittings of elements in $\mathrm{H}\setminus\{h_a\}$; Permutations $\boldsymbol{\rho}$ are summed over
- h_a: fiducial graviton; $Y_h = \sum_{j\in\{1,\ldots,r-1\}\text{s.t.} j\prec h} k_j$; $F_i^{\mu\nu} = k_i^\mu\epsilon_i^\nu - k_i^\nu\epsilon_i^\mu$
- $\boldsymbol{A}\sqcup\boldsymbol{B}$: permutations with keeping the relative orders in each set

Recursive expansion of multi-trace amplitudes

EYM recursive expansion of multi-trace amplitudes (Du, Feng, Teng, (17))

Single-trace EYM $\xrightarrow{\text{Gravitons}\to\text{Gluon traces}}$ Multi-trace EYM

Type-I: Fiducial graviton $\rightarrow$ Graviton

Type-II: Fiducial graviton $\rightarrow$ Trace

Several problems

How to derive local numerators at on-shell level?

How to derive local numerators at off-shell (Feynman diagram) level?

What we can learn from the expansions?

- New on-shell/off-shell relations
- Amplitudes in 4d and interplay with CHY, Hodges formulas

$\rightarrow$ Graphic expansion induced by EYM expansion

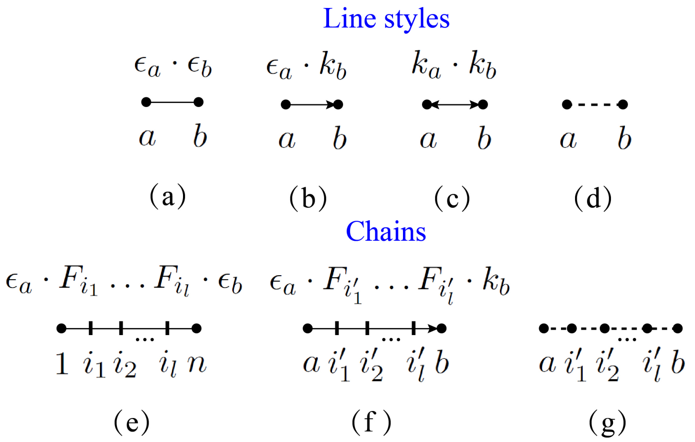
Graphic expansions

- Expanding amplitudes recursively: GR$\rightarrow$EYM $\rightarrow$YM

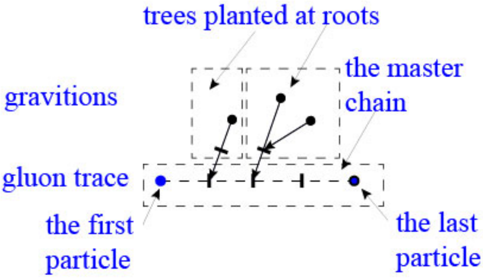
$\Rightarrow$Graphic expansions$\Rightarrow$ local numerators

- Properties of graphs $\Rightarrow$ Properties of amplitudes

Lines and chains



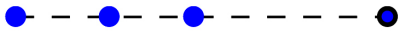
Single-trace EYM



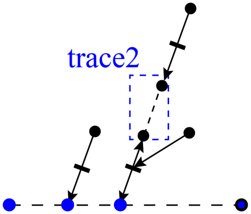
- Reference order: $\mathrm{R} = \{h_{\rho(1)}, \ldots, h_{\rho(s)}\}$
- Root set: $\mathcal{R} = \{1, \ldots, r-1\}$
- Permutations: Shuffling the branches
- $C_{1,\boldsymbol{\sigma},n}$: Summing all graphs contributing $\boldsymbol{\sigma}$

$$A^{\mathrm{EYM}}_{\mathrm{SingleTrace}} = \sum_{\boldsymbol{\sigma}} C_{1,\boldsymbol{\sigma},n} A^{\mathrm{YM}}(1, \boldsymbol{\sigma}, n)$$

Multi-trace EYM

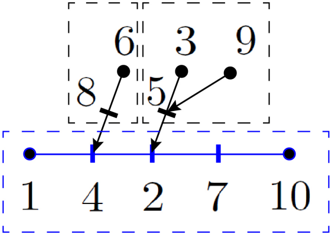
graviton → gluon trace → 

$\boldsymbol{\beta}, a, \boldsymbol{\alpha}, b \qquad (-1)^{N_{\boldsymbol{\beta}}}\; a, \boldsymbol{\alpha} \shuffle \boldsymbol{\beta}^T, b$



$$A^{\text{EYM}}_{\text{MultiTrace}} = \sum_{\boldsymbol{\sigma}} C_{1,\boldsymbol{\sigma},n} A^{\text{YM}}(1,\boldsymbol{\sigma},n)$$

YM and GR



$$\boldsymbol{\sigma} \in \{1, 4, \{8, 6\} \shuffle \{2, \{5, \{3\} \shuffle \{9\}\} \shuffle \{7\}\}, 10\}$$

$$M^{\text{GR}} = \sum_{\boldsymbol{\sigma} \in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\text{YM}}(1, \boldsymbol{\sigma}, n) \quad A^{\text{YM}} = \sum_{\boldsymbol{\sigma} \in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\text{BS}}(1, \ldots, n | 1, \boldsymbol{\sigma}, n)$$

Summing over permutations v.s. summing over graphs

$$M = \underbrace{\sum_{\boldsymbol{\sigma} \in S_{n-2}}}_{\text{summing over permutations}} n_{1,\boldsymbol{\sigma},n} A(1,\boldsymbol{\sigma},n)$$

$$\Leftrightarrow M = \underbrace{\sum_{\mathcal{F}}}_{\text{summing over graphs}} C^{\mathcal{F}} \left[\underbrace{\sum_{\boldsymbol{\sigma}^{\mathcal{F}}}}_{\substack{\text{permutations} \\ \text{corresponding to a graph}}} A(1,\boldsymbol{\sigma}^{\mathcal{F}},n) \right]$$

Graphic expansions of Berends-Giele currents

Off-shell level: Berends-Giele currents

- A recursive approach to summing Feynman diagrams
- Solution to classical equation of motion

(Lee, Mafra,Schlotterer 2015;E. Bridges, Mafra 2019)

- B-G recursion for YM and BS

B-G currents for YM

Berends-Giele (1987) currents for YM

$$
\begin{aligned}
& J^{\rho}(1,...,n-1) \\
= \; & \frac{1}{s_{1...n-1}}\Bigg[\sum_{1\leq i<n-1} V_3^{\mu\nu\rho} J_{\mu}(1,...,i) J_{\nu}(i+1,...,n-1) \\
& + \sum_{1\leq i<j<n-1} V_4^{\mu\nu\tau\rho} J_{\mu}(1,...,i) J_{\nu}(i+1,...,j) J_{\tau}(j+1,...,n-1)\Bigg]
\end{aligned}
$$

Starting point: $J^{\mu}(a)=\epsilon_a$

B-G currents for BS

B-G currents for BS (Mafra, 2016)

$$\begin{aligned}
&\phi(1,...,n-1|\sigma_1,...,\sigma_{n-1})\\
=\ &\frac{1}{s_{1\ldots n-1}}\sum_{i=1}^{n-2}\Big[\phi(1,...,i|\sigma_1,...,\sigma_i)\phi(i+1,...,n-1|\sigma_{i+1},...,\sigma_{n-1})\\
&-\phi(1,...,i|\sigma_{n-i},...,\sigma_{n-1})\phi(i+1,...,n-1|\sigma_1,...,\sigma_{n-i})\Big]
\end{aligned}$$

Starting point $\phi(a|a)=1$, $\phi(a|b)$ $(a\neq b)$

From on-shell to off-shell

$$A^{\mathrm{YM}} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} n_{1,\boldsymbol{\sigma},n} A^{\mathrm{BS}}(1,...,n|1,\boldsymbol{\sigma},n)$$

$$\downarrow \ ?$$

$$J^{\rho} = \sum_{\boldsymbol{\sigma}\in S_{n-2}} N^{\rho}_{1,\boldsymbol{\sigma},n} \phi^{\mathrm{BS}}(1,...,n|1,\boldsymbol{\sigma},n)$$

In a proper choice of gauge, the second line holds.

A decomposition formula for B-G currents

BG current in Feynman gauge

$$\underbrace{J^{\rho}(1,2,...,n-1)}_{\text{Feynman gauge}} = \underbrace{\widetilde{J}^{\rho}(1,2,...,n-1)}_{\text{BCJ gauge}} + \underbrace{K^{\rho}(1,2,...,n-1) + L^{\rho}(1,2,...,n-1)}_{\text{Vanish in the on-shell limit}}$$

K, L terms

Terms vanishing in the on-shell limit

$$K^{\rho}(1,2,...,n-1)=\frac{1}{s_{12\ldots n-1}}\,k^{\rho}_{1,n-1}\sum_{i=1}^{n-2}\widetilde{J}(1,...,i)\cdot\widetilde{J}(i+1,...,n-1)$$

$$L^{\rho}(1,2,...,n-1)=\sum_{\{a_i,b_i\}\subset\{1,\ldots,n-1\}}(-1)^{l+1}J^{\rho}\Big(S_{1,a_1-1},K_{(a_1,b_1)},S_{b_1+1,a_2-1},K_{(a_2,b_2)},...,K_{(a_l,b_l)},S_{b_l+1,n-1}\Big)$$

$$\epsilon\cdot[K(1,2,...,n-1)+L(1,2,...,n-1)]=0$$

$\widetilde{J}^\rho$ term

Effective current:

$$\widetilde{J}^\rho(1,2,\ldots,n-1)=\sum_{\boldsymbol{\sigma}\in P(2,n-1)} N_A^\rho(1,\boldsymbol{\sigma})\phi(1,2,\ldots,n-1|1,\boldsymbol{\sigma})$$

- $N_A^\rho(1,\boldsymbol{\sigma})$: numerators with 1 off-shell line, constructed by graphs
- On-shell limit: $\epsilon_n\cdot N_A(1,\boldsymbol{\sigma})=n_A(1,\boldsymbol{\sigma},n)$

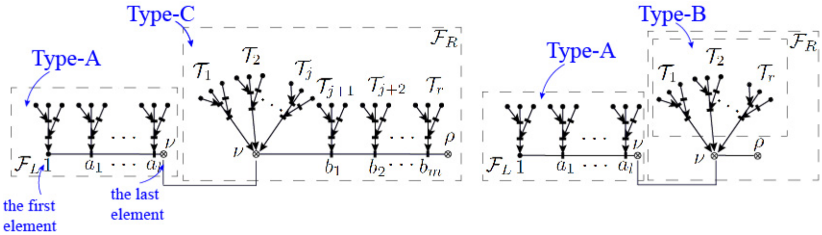
Generalized Strength tensor $\widetilde{F}^{\nu\rho}$

Generalized strength tensor: $\widetilde{F}^{\nu\rho}(A) \equiv 2k_A^{\nu}\widetilde{J}^{\rho}(A) - 2k_A^{\rho}\widetilde{J}^{\nu}(A)$

$$\widetilde{F}^{\nu\rho}_{(1,n-1)} = \sum_{\boldsymbol{\sigma}\in P(1,n-1)} N_C^{\nu\rho}(\boldsymbol{\sigma})\phi(1,...,n-1|\boldsymbol{\sigma}) + \sum_{1\leq i<n-1} 2\left[\widetilde{J}^{\nu}_{(1,i)}\widetilde{J}^{\rho}_{(i+1,n-1)} - \widetilde{J}^{\rho}_{(i+1,n-1)}\widetilde{J}^{\nu}_{(1,i)}\right]$$

- $N_C^{\rho}(1,\boldsymbol{\sigma})$: numerators with 2 off-shell lines, constructed by graphs

Three types of off-shell numerators



$$\boldsymbol{\sigma} = \{\sigma_2, ..., \sigma_{n-1}\} \in P(2, n-1),\ \boldsymbol{\sigma}_L = \{\sigma_2, ..., \sigma_{i-1}\},\ \boldsymbol{\sigma}_R = \{\sigma_i, ..., \sigma_{n-1}\}$$

$$N_A^{\rho}(1,\boldsymbol{\sigma}) = \left[N_A(1,\boldsymbol{\sigma}_L)\cdot N_C(\boldsymbol{\sigma}_R) - N_A(1,\boldsymbol{\sigma}_L)N_B(\boldsymbol{\sigma}_R)\cdot 2k_{1,i-1}\right]^{\rho}$$

Sketching the proof: Decomposing vertices

YM vertices in Feynman gauge

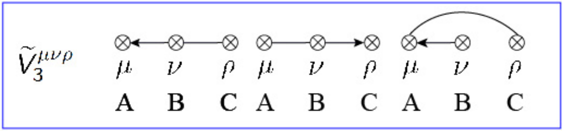
$$V_3^{\mu\nu\rho} = \eta^{\mu\nu}(k_A - k_B)^\rho + \eta^{\nu\rho}(k_B - k_C)^\mu + \eta^{\rho\mu}(k_C - k_A)^\nu$$

$$V_4^{\mu\nu\tau\rho} = 2\eta^{\mu\rho}\eta^{\nu\lambda} - \eta^{\mu\nu}\eta^{\rho\lambda} - \eta^{\mu\lambda}\eta^{\nu\rho}$$

Effective 3-point vertex

$$\widetilde{V}_3^{\mu\nu\rho} \equiv \left[\left(2k_B^\mu\eta^{\nu\rho} - \eta^{\mu\nu}2k_B^\rho\right) - \eta^{\mu\rho}2k_A^\nu\right]$$

$$V_3^{\mu\nu\rho} = \widetilde{V}_3^{\mu\nu\rho} + \eta^{\nu\rho}k_A^\mu - \eta^{\mu\rho}k_B^\nu + \eta^{\mu\nu}(k_A + k_B)^\rho$$



B-G recursion

B-G recursion:

$$J^{\rho}(1,...,n-1) = \frac{1}{s_{1...n-1}}\Big[\sum_{1\leq i<n-1} V_3^{\mu\nu\rho}J_{\mu}^{(1)}J_{\nu}^{(2)} + \sum_{1\leq i<j<n-1} V_4^{\mu\nu\tau\rho}J_{\mu}^{(1)}J_{\nu}^{(2)}J_{\tau}^{(3)}\Big]$$

- lower-point $J \to \widetilde{J} + K + L$
- $V_3 = \widetilde{V}_3 + (V_3 - \widetilde{V}_3)$
- $\widetilde{V}_3^{\mu\nu\rho}\widetilde{J}_{\mu}^{(1)}\widetilde{J}_{\nu}^{(2)} = \widetilde{J}_{\mu}^{(1)}\widetilde{F}_{(2)}^{\mu\rho} - \widetilde{J}_{\rho}^{(1)}(\widetilde{J}^{(2)}\cdot 2k_{(1)}) \to \widetilde{J}$ term
- $\widetilde{J}_{\mu}^{(2)} \to N_B$; $F_{(2)}^{\mu\rho} \to N_C$, J
- Others terms $\to K, L$ terms

Symmetric BCJ numerators

Symmetrization of the numerators

(Fu, Du, Feng 2014)

Numerators indepednent of reference order:

$$\bar{N}^{\rho}(1,\sigma_1,...,\sigma_{n-2})=\frac{1}{(n-2)!}\sum_{\gamma\in P(2,n-1)}N_A^{\{\rho,\gamma,1\},\rho}(1,\sigma_1,...,\sigma_{n-2})$$

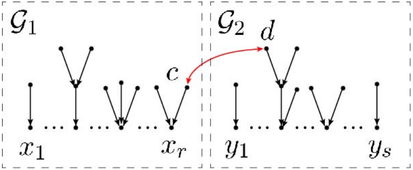
Numerators with Lie symmetry:

$$N^{\rho}_{\text{sym}}(a_1,a_2,...,a_{n-1})\equiv\frac{1}{n-1}\bar{N}^{\rho}([[[a_1,a_2],a_3],...,a_{n-1}])$$

$\rightarrow$ Jacobi identity and antisymmetry are automatically satisfied

Interplay with Hodges formula and CHY in 4d

Graphic expansion of EYM → refined graphs (Du, Hou, 2019, 2020) →
Double-trace EYM in 4d, 2 negative gluons (Tian,Gong, Xie, Du,2021)



$$A^{(g_i,g_j)}(x_1,\dots,x_r|y_1,\dots,y_s||\mathrm{H})$$

$$=\frac{\langle g_i,g_j\rangle^4}{(x_1,\dots,x_r)\,(y_1,\dots,y_s)}\left[\sum_{\mathcal{G}=\mathcal{G}_1\oplus\mathcal{G}_2}\sum_{\substack{c\in\mathcal{G}_1\\ d\in\mathcal{G}_2}}(-k_c\cdot k_d)\frac{\langle c,\zeta\rangle\langle d,\chi\rangle}{\langle c,d\rangle\langle\zeta,\chi\rangle}\right.$$
$$\left.\times\prod_{e(x,y)\in\mathcal{E}(\mathcal{G})}\frac{\langle\xi,y\rangle\langle\lambda_e,y\rangle[y,x]}{\langle\xi,x\rangle\langle\lambda_e,x\rangle\langle y,x\rangle}\right]$$

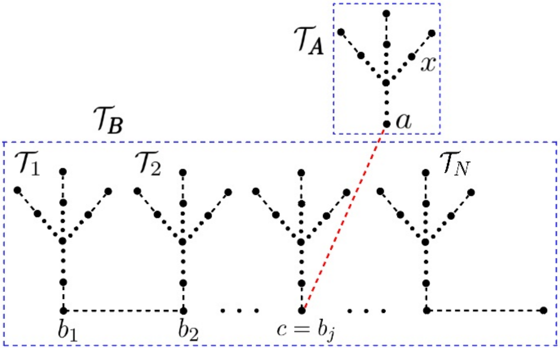
Interplay with Hodges formula and CHY in 4d

- The Hodges determinant for gravity (Hodges, 2012), single trace EYM (Hodges 2012), double-trace MHV with only gluons (Cachazo, He, Yuan, 2013), can also be reproduced. The graphs are closely related to spanning forests in 4d (Feng, He, 2012)

- The double-trace formula can be derived by substituting the MHV solution into CHY formula (Xie, Du, to appear)

- When $n_g^- \leq n_{\text{trace}} - 1$ or $n_g^+ \leq n_{\text{trace}} - 1$ is satisfied, EYM amplitude with an arbitrary number of negative-helicity particles has to vanish (Xie, Du, to appear)

Inducing new relations



Inducing new relations

- Graph-based relations for BG currents in BS:

  (Du, Wu, 2022)

$$\sum_{a\in\mathcal{T}_A}(-)^{|ax|}\sum_{c\in\mathcal{T}_{\boldsymbol{B}}}\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_a}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\left[\sum_{\boldsymbol{\gamma}\in\boldsymbol{\alpha}\sqcup\boldsymbol{\beta}|_{c\prec a}} s_{ac}\,\phi(\,\boldsymbol{\sigma}\,|\,\boldsymbol{\gamma}\,)\right]$$
$$=\sum_{\boldsymbol{\alpha}\in\mathcal{T}_{\boldsymbol{A}}|_x}\sum_{\boldsymbol{\beta}\in\mathcal{T}_{\boldsymbol{B}}|_b}\Big[\phi(\boldsymbol{\sigma}_{1,i}|\boldsymbol{\beta})\phi(\boldsymbol{\sigma}_{i+1,l}|\boldsymbol{\alpha})-\phi(\boldsymbol{\sigma}_{1,l-i}|\boldsymbol{\alpha})\phi(\boldsymbol{\sigma}_{l-i+1,l}|\boldsymbol{\beta})\Big]$$

- $\widetilde{J}=\sum N\phi\rightarrow$ Graph-based relations for BG currents YM

  (Du, Wu, 2022)

- On shell graph-based BCJ relation (Hou, Du, 2018)

Summary and further discussions

Summary

- Graphic expansion for on-shell amplitudes
- Graphic expansion for B-G currents
- Applications in 4d, inducing relations

Further discussions

- The interplay with 1-loop CHY (Zhou 2022)
- The relation with the free Lie algebraic approach (e.g. H. Frost, C. R. Mafra, and L. Mason, 2020) gluon trace $\rightarrow$ Lie brackets

End

谢 谢!