Axial current renormalization and non-decoupling mass logarithms

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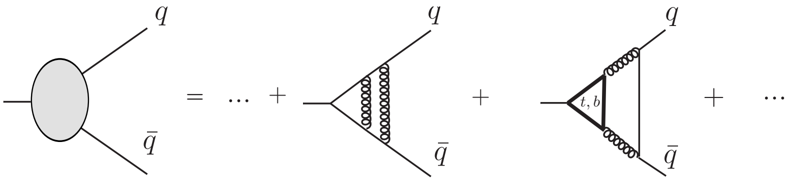
Based on: [**LC**, M.Czakon 2201.01797, 2112.03795]
[**LC**, M.Czakon, M.Niggetiedt 2109.01917]
[T.Ahmed, **LC**, M.Czakon 2101.09479]

Quark form factors in QCD

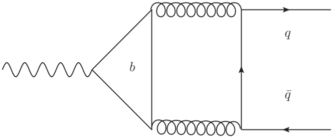


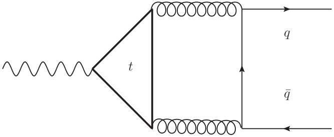
$$\bar{u}(p_1)\,\Gamma^\mu\, v(p_2) = \bar{u}(p_1)\left(v_q\, F^V \gamma^\mu + a_q\, F^A \gamma^\mu \gamma_5\right) v(p_2)$$

- Important objects in Precision Physics
  - $e^+e^- \to Z/\gamma^* \to Q\bar{Q} + X,\;\; H/Z \to Q\bar{Q} + X$, Drell-Yan, DIS etc
  - simplest objects to extract certain universal QCD quantities
- Start-of-art results in QCD:
  - Massless: 4-loop analytically [Lee, Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser 22]
  - Massive: Truncated series-expansion at 3-loop [Fael, Lange, Schönwald, Steinhauser 22].
- The axial part is not yet complete ...

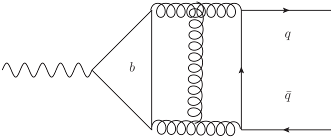
Singlet-type contributions

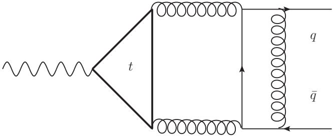
- Axial 2-loop:



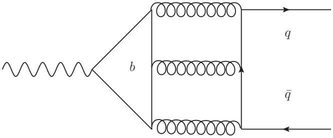


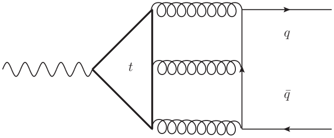
- Axial 3-loop:



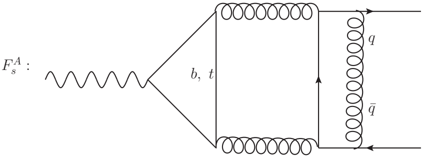


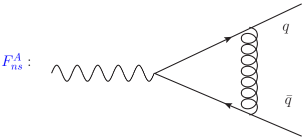
- Vector 3-loop:





UV-renormalization of singlet FFs





UV renormalization for individual singlet contributions [Chetyrkin, Kühn 93; LC, Czakon, Niggetiedt 21]

$$\mathbf{F}_{s,b}^A(a_s, m_t, \mu) = Z_{ns}\, Z_2\, F_{s,b}^A(\hat{a}_s, \hat{m}_t) + Z_s\, Z_2 \left(F_{ns}^A(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F_{s,i}^A(\hat{a}_s, \hat{m}_t) \right),$$

$$\mathbf{F}_{s,t}^A(a_s, m_t, \mu) = Z_{ns}\, Z_2\, F_{s,t}^A(\hat{a}_s, \hat{m}_t) + Z_s\, Z_2 \left(F_{ns}^A(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_f} F_{s,i}^A(\hat{a}_s, \hat{m}_t) \right),$$

- $Z_J\, \bar{\psi}\, \gamma^\mu \gamma_5\, \psi$ with a *non-anticommuting* γ_5: $Z_J = Z_{ns} + n_j\, Z_s$.

For the **total** *non-anomalous* combination:

$$\mathbf{F}_{s,b}^A(a_s, m_t) - \mathbf{F}_{s,t}^A(a_s, m_t) = Z_{ns}\, Z_2 \left(F_{s,b}^A(\hat{a}_s, \hat{m}_t) - F_{s,t}^A(\hat{a}_s, \hat{m}_t) \right),$$

Practical applications of singlet $\bar{\psi}\gamma^{\mu}\gamma_5\psi$

- Quantities related to the *global/external* anomalies
  e.g. $\pi \to \gamma\gamma$ decay [Steinberger 49; Sutherland, Veltman 67; Adler 69; Bell, Jackiw 69]

- Treatment of the singlet axial-current operator in heavy-top EFT [Chetyrkin, Kühn 91 93; **LC**, Czakon, Niggetiedt 21]

- Appearance of the *non-decoupling* heavy-quark-mass logarithms in pertubation theory [Collins, Wilczek, Zee 78; Chetyrkin, Kühn 93; **LC**, Czakon 22]

- Polarized structure and splitting functions [Matiounine, Smith, Neerven; Moch, Vermaseren, Vogt; Blümlein, Marquard, Schneider, Schönwald; Tarasov, Venugopalan...]

- ......

The γ_5 prescription in use

The HV/BM [72,79] prescription of γ_5 in dimensional regularization:

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\gamma_\mu\gamma_5 \to \frac{1}{2}\Big(\gamma_\mu\gamma_5 - \gamma_5\gamma_\mu\Big) = \frac{i}{6}\varepsilon_{\mu\nu\rho\sigma}\gamma^\nu\gamma^\rho\gamma^\sigma\,,$$

where the $\epsilon^{\mu\nu\rho\sigma}$ is treated outside the R-operation formally in D dimensions [Larin, Vermaseren 91; Zijlstra, Neerven 92] (Larin's prescription).

The properly renormalized singlet axial current reads

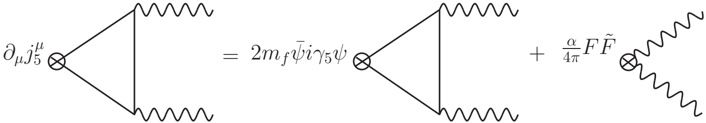
$$\begin{aligned} [J_5^\mu]_R &= Z_J\,\bar\psi_B\,\gamma^\mu\gamma_5\,\psi_B \\ &= Z_5^f\,Z_5^{ms}\,\bar\psi_B\,\frac{-i}{3!}\epsilon^{\mu\nu\rho\sigma}\gamma_\nu\gamma_\rho\gamma_\sigma\,\psi_B \end{aligned}$$

The Adler-Bell-Jackiw anomaly

The anomalous axial-vector divergence equation [Adler 69; Bell, Jackiw 69]

$$\partial_\mu \bar{\psi} \gamma^\mu \gamma_5 \psi = 2m_f \bar{\psi} i\gamma_5 \psi - \frac{\alpha}{4\pi} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} .$$

Diagrammatically,



The Adler-Bardeen theorem [Adler, Bardeen 69] : “one-loop” exact

Operator mixing under renormalization

The all-order axial-anomaly equation [Adler 69; Adler, Bardeen 69]

$$\left[\partial_\mu J_5^\mu\right]_R = a_s\, n_f\, \mathrm{T}_F\, \left[F\tilde{F}\right]_R$$

with $F\tilde{F} \equiv -\epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma}$ in QCD with n_f massless quarks.

- The renormalization of the operators involved: [Adler 69; Espriu,Tarrach 82; Breitenlohner,Maison,Stelle 84; Bos 92; Larin 93 ...]

$$\begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_R \\ \left[F\tilde{F}\right]_R \end{pmatrix} = \begin{pmatrix} Z_J & 0 \\ Z_{FJ} & Z_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_B \\ \left[F\tilde{F}\right]_B \end{pmatrix}$$

- with the matrix of *anomalous dimensions*:

$$\frac{\mathrm{d}}{\mathrm{d}\,\ln\mu^2} \begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_R \\ \left[F\tilde{F}\right]_R \end{pmatrix} = \begin{pmatrix} \gamma_S & 0 \\ \gamma_{FJ} & \gamma_{F\tilde{F}} \end{pmatrix} \cdot \begin{pmatrix} \left[\partial_\mu J_5^\mu\right]_R \\ \left[F\tilde{F}\right]_R \end{pmatrix}$$

Vacuum-Gluon matrix element (AVV diagram)

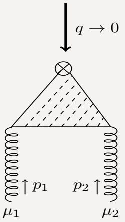
Determine Z_5 via computing the 2-gluon matrix elements of $[\partial_\mu J_5^\mu]_R = a_s\, n_f\, \mathrm{T}_F\, [F\tilde{F}]_R$

$$\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y}\, \langle 0|\hat{\mathrm{T}}\left[J_5^\mu(y)\, A_a^{\mu_1}(x)\, A_a^{\mu_2}(0)\right]|0\rangle|_{\mathrm{amp}}$$

Form factor decomposition:

$$\begin{aligned}\Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) &= F_1\, \epsilon^{\mu\,\mu_1\,\mu_2\,(p_2-p_1)} \\ &+ F_2\left(p_1^{\mu_1}\epsilon^{\mu\,\mu_2\,p_1\,p_2} - p_2^{\mu_2}\epsilon^{\mu\,\mu_1\,p_1\,p_2}\right) \\ &+ F_3\left(p_1^{\mu_2}\epsilon^{\mu\,\mu_1\,p_1\,p_2} - p_2^{\mu_1}\epsilon^{\mu\,\mu_2\,p_1\,p_2}\right)\end{aligned}$$

$$q_\mu \Gamma^{\mu\mu_1\mu_2}_{lhs}(p_1,p_2) = 2F_1\, \epsilon^{\mu_1\,\mu_2\,p_1\,p_2}$$



taking into account the odd parity and Bose symmetry w.r.t gluons ($p_1 \leftrightarrow p_2\,, \mu_1 \leftrightarrow \mu_2$).

The non-Abelian Adler-Bardeen theorem

The equality verified to 4-loop in QCD [Ahmed,**LC**,Czakon 21]:

$$Z_{F\tilde{F}} = Z_{a_s}$$

The ABJ equation in QCD in terms of the *bare* fields:

$$\left(Z_J - n_f \, \mathrm{T}_F \, a_s \, Z_{FJ} \right) \left[\partial_\mu J_5^\mu\right]_B = \hat{a}_s \, n_f \, \mathrm{T}_F \, \left[F\tilde{F}\right]_B$$

- In an Abelian theory with Pauli-Villar regularization (with an AC γ_5), the coefficient is 1 to all orders [Adler 69; Adler, Bardeen 69]
- The coefficient is not 1 with a NAC γ_5 in DR in QCD, but the LHS current remains ***RG-invariant*** (albeit in D=4 limit):

$$\gamma_{F\tilde{F}} = -\mu^2 \frac{\mathrm{d} \ln a_s}{\mathrm{d}\mu^2} = -\beta \,, \;\; \gamma_S|_{\epsilon=0} = n_f \, \mathrm{T}_F \, a_s \, \gamma_{FJ} \,.$$

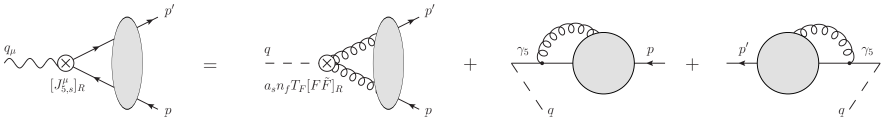
- An all-order argument of the non-Abelian extension was sketched [Breitenlohner, Maison, Stelle 84]; A proof is completed only recently [Lüscher, Weisz 21]
- However, $Z_J = Z_5^f \, Z_5^{ms}$ needs to be computed order by order ... Z_5^f at $\mathcal{O}(a_s^3)$ from 4-loop AVV-amplitude [Ahmed,**LC**,Czakon 21]

Vacuum-Quark matrix element and AWTI

Much more efficient to extract Z_J by using the off-shell Ward-Takahashi identity for an axial current with a non-anticommuting γ_5 [LC,Czakon 21]

The anomalous Ward-Takahashi identity:

$$q_\mu \, \Gamma^{\mu}_{5,s}(p',p) = -a_s \, n_f \, \mathrm{T}_F \, \Lambda(p',p) \, + \, \gamma_5 \, \hat{S}^{-1}(p) \, + \, \hat{S}^{-1}(p') \, \gamma_5 \, ,$$



- q can not be 0 to have a non-zero anomaly
- Either p' or p' should be 0 to reduce to the propagator-type integrals
- γ_5 on the RHS does not require any renormalization!

Z_5^{ms} up to $\mathcal{O}(a_s^5)$ from 4-loop calculations

- The anomalous dimension of $[J_5^\mu]_R$:

$$\gamma_S \equiv \frac{d\, \ln Z_J}{d\, \ln \mu^2} = \frac{d\, \ln Z_s^{ms}}{d\, \ln \mu^2} + \frac{d\, \ln Z_s^f}{d\, \ln \mu^2} = \gamma_s^{ms} + \beta \frac{d\, \ln Z_s^f}{d\, \ln a_s} - \epsilon \frac{d\, \ln Z_s^f}{d\, \ln a_s}.$$

- Z_5^{ms} at $\mathcal{O}(a_s^5)$ using ABJ equation with $Z_{F\tilde{F}} = Z_{a_s}$

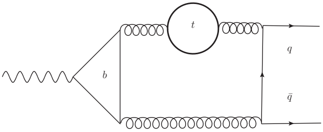
$$\gamma_s^{ms} = a_s\, n_f\, \mathrm{T}_F\, \gamma_{FJ} - \beta \frac{d\, \ln Z_s^f}{d\, \ln a_s}.$$

  - γ_s^{ms} at $\mathcal{O}(a_s^5)$ requires only γ_{FJ} and Z_s^f up to 4-loop (from AWTI) [LC, Czakon 21, 22]
  - Z_5^f at $\mathcal{O}(a_s^5)$ not known yet

In this way, the $\mathcal{O}(a_s^4)$ result for Z_s^{ms} could have been given in early 90s (along with Larin's $\mathcal{O}(a_s^3)$ result).

Light quark form factors in the heavy top limit

Appearance of the non-decoupling m_t-logarithms [Collins, Wilczek, Zee 78; Chetyrkin, Kühn 93]



$$\bar{\mathcal{F}}^{A,3}_{s,b}(m_t \to \infty) = \bar{\mathcal{F}}^{\bar{A},3}_{s,b}(\mu) + \bar{\mathcal{F}}^{A_{\text{nDC}},3}_{s,b}(m_t,\mu)\,,$$
$$= \bar{\mathcal{F}}^{\bar{A},3}_{s,b}(\mu) - \frac{85}{9}C_F + \frac{4}{3}C_F L_\mu - \frac{1}{4}C_F L_\mu^2 + \mathcal{O}(1/m_t^2)$$

where $L_\mu \equiv \ln\frac{\mu^2}{m_t^2}$.

A **Wilson coefficient** $C_w(\bar{a}_s, \mu/m_t)$

$$\mathcal{F}^A_{s,b} - \mathcal{F}^A_{s,t}\Big|_{m_t\to\infty} = \bar{\mathcal{F}}^{\bar{A}}_{s,b}(\bar{a}_s,\mu) - C_w(\bar{a}_s,\mu/m_t)\left(\bar{\mathcal{F}}^A_{ns}(\bar{a}_s,\mu) + \sum_{i=1}^{n_l}\bar{\mathcal{F}}^{\bar{A}}_{s,i}(\bar{a}_s,\mu)\right) + \mathcal{O}(1/m_t^2)$$

The renormalized low-energy effective Lagrangian [Chetyrkin, Kühn 93; LC, Czakon, Niggetiedt 21]

$$\delta\mathcal{L}^R_{\text{eff}} = \Big(Z_{ns}\sum_{i=1}^{n_l} a_i\,\bar{\psi}^B_i\,\gamma^\mu\gamma_5\,\psi^B_i + a_b\,Z_s\,[J^\mu_5]_B \quad \longrightarrow n_l\text{-flavor massless part}$$
$$+ a_t\,C_w(a_s,\mu/m_t)\,(Z_{ns} + n_l\,Z_s)\,[J^\mu_5]_B\Big)Z_\mu\,,$$

with $J^\mu_5 = \sum_{i=1}^{n_l}\bar{\psi}_i\gamma^\mu\gamma_5\psi_i$ and $\mu^2\frac{\mathrm{d}Z_{ns}}{\mathrm{d}\mu^2} = 0$ while $\mu^2\frac{\mathrm{d}Z_s}{\mathrm{d}\mu^2} = \bar{\gamma}_s(Z_{ns} + n_l\,Z_s)$

Renormalization of $\bar{\psi}_q \gamma^\mu \gamma_5 \psi_q$

What is $\left[J^{\mu}_{5,q}\right]_R$ as in $\left[J^{\mu}_{5}\right]_R = \sum_{i=1}^{n_l} \left[J^{\mu}_{5,i}\right]_R$?

$$\left[J^{\mu}_{5}\right]_R = Z_J\, J^{\mu}_5 = \left(Z_{ns} + n_l\, Z_s\right) \sum_{i=1}^{n_l} \bar{\psi}^B_i\, \gamma^\mu \gamma_5\, \psi^B_i$$

$$\Downarrow$$

$$\left[J^{\mu}_{5,q}\right]_R \neq \cancel{\left(Z_{ns} + n_l\, Z_s\right) \bar{\psi}^B_q\, \gamma^\mu \gamma_5\, \psi^B_q}$$

$$= \left(Z_{ns}\, \bar{\psi}^B_q\, \gamma^\mu \gamma_5\, \psi^B_q + Z_s \sum_{i=1}^{n_l} \bar{\psi}^B_i\, \gamma^\mu \gamma_5\, \psi^B_i\right)$$

The renormalized singlet contribution featuring EW a_b:

$$\mathbf{F}^A_{s,b}(a_s, m_t, \mu) = \langle 0 | \left[J^{\mu}_{5,b}\right]_R | b\bar{b} \rangle |_{singlet}$$

$$= Z_{ns}\, Z_2\, F^A_{s,b}(\hat{a}_s, \hat{m}_t) + Z_s\, Z_2 \left(F^A_{ns}(\hat{a}_s, \hat{m}_t) + \sum_{i=1}^{n_l} F^A_{s,i}(\hat{a}_s, \hat{m}_t)\right)$$

Note $\mu^2 \frac{\mathrm{d}Z_{ns}}{\mathrm{d}\mu^2} = 0$ while $\mu^2 \frac{\mathrm{d}Z_s}{\mathrm{d}\mu^2} = \bar{\gamma}_s \left(Z_{ns} + n_l\, Z_s\right)$.

Resuming the non-decoupling m_t logarithms

RG equation of the Wilson coefficient $C_w(\bar{a}_s, \mu/m_t)$ [Chetyrkin, Kühn 93; LC, Czakon, Niggetiedt 21]

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} C_w(\bar{a}_s, \mu/m_t) = \bar{\gamma}_s - n_l\, \bar{\gamma}_s\, C_w(\bar{a}_s, \mu/m_t)\,,$$

$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} \left[J^{\mu}_{5,q}\right]_R = \bar{\gamma}_s \left[J^{\mu}_{5}\right]_R$ with $J^{\mu}_5 = \sum_{i=1}^{n_l} = \bar{\psi}_i \gamma^{\mu} \gamma_5 \psi_i$.

The solution for $C_t \equiv -1/n_l + C_w$:

$$\mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2} C_t(\bar{a}_s, \mu/m_t) = n_l\, \bar{\gamma}_s\, C_t(\bar{a}_s, \mu/m_t)\,,$$

$$C_t(\bar{a}_s(\mu), \mu/m_t) = C_t(\bar{a}_s(m_t), 1) \exp\left(\int_{\bar{a}_s(m_t)}^{\bar{a}_s(\mu)} \frac{-n_l\, \bar{\gamma}_s(a_s)}{\beta(a_s)} \frac{\mathrm{d}a_s}{a_s} \right)$$

in Larin's scheme.

Alternative renormalization prescriptions

- $\overline{\text{MS}}$ scheme: $Z_5^{ms}\,[J_5^\mu]_B$
  - The ABJ (AWTI) is not respected
  - $\mathbf{F}^A_{s,b}(a_s,m_t) - \mathbf{F}^A_{s,t}(a_s,m_t)$ is still anomalous
- Chetyrkin scheme: $Z^f_{ns}\,Z_5^{ms}\,[J_5^\mu]_B$
  - The ABJ (AWTI) is not respected
  - $\mathbf{F}^A_{s,b}(a_s,m_t) - \mathbf{F}^A_{s,t}(a_s,m_t)$ is non-anomalous (correct)
- Larin scheme: $Z^f_s\,Z_5^{ms}\,\partial_\mu\,[J_5^\mu]_B = a_s\,n_f\,T_F\left(Z_{F\tilde{F}}\,[J_5^\mu]_B + Z_{FJ}\partial_\mu\,[J_5^\mu]_B\right)$
  - The ABJ (AWTI) is respected and $\gamma_s \neq 0$
  - $\mathbf{F}^A_{s,b}(a_s,m_t) - \mathbf{F}^A_{s,t}(a_s,m_t)$ is non-anomalous (correct)
- Renormalization-group invariant (RGI) scheme: $Z_{ext}(a_s) \equiv \hat{P}\exp\left(\int_0^{a_s} \frac{-\bar{\gamma}_S(a)}{\beta(a)}\,\frac{\mathrm{d}a}{a}\right)$
  - The ABJ (AWTI) is respected and $\gamma_s = 0$ (no more running!)
  - $Z^{\text{RGI}}_{F\tilde{F}} \neq Z_{\alpha_s}$
  - No more explicit $\ln(\mu^2/m_t^2)$ when expressed w.r.t $\alpha_s(\mu = m_t)$ [LC, Czakon 22].

Summary and Outlook

- We have described a very efficient way for computing the renormalization constants of axial-vector currents in QCD with a non-anticommuting γ_5 in dimensional regularization.

- We have verified explicitly up to 4-loop order $Z_{F\tilde{F}} = Z_{\alpha_s}$ (in the $\overline{\mathrm{MS}}$ scheme), from which follows $\gamma_s = a_s\, n_f\, \mathrm{T}_F\, \gamma_{FJ}$ valid to $\mathcal{O}(\alpha_s^4)$.

- A proof of $Z_{F\tilde{F}} = Z_{\alpha_s}$ in dimensionally regularized QCD to all orders is recently completed [Lüscher, Weisz 21].

- We have extended the result of Z_J of the flavor-singlet axial-vector current to $\mathcal{O}(\alpha_s^4)$, with its $\overline{\mathrm{MS}}$ part to $\mathcal{O}(\alpha_s^5)$ by the virtue of ABJ equation

- Needed for resumming the non-decoupling heavy-quark-mass logarithms and in the calculation of polarized splitting functions at high orders...

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谢谢!

Backup Slides

Treatment of the operator $F\tilde{F}$

The axial-anomaly (topological-charge density) operator $F\tilde{F}$ with the *Chern-Simons current* K^μ

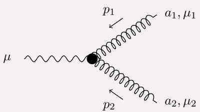
$$
\begin{aligned}
F\tilde{F} &= \partial_\mu K^\mu \\
&= \partial_\mu \left(-4\,\epsilon^{\mu\nu\rho\sigma} \left(A^a_\nu \partial_\rho A^a_\sigma + g_s \frac{1}{3} f^{abc} A^a_\nu A^b_\rho A^c_\sigma \right) \right)
\end{aligned}
$$

by the virtue of total antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$ [Bardeen 74].
Unlike J_5^μ, the current K^μ is not gauge-invariant.

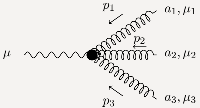
$$
\Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1,p_2) \equiv \int d^4x d^4y\, e^{-ip_1\cdot x - iq\cdot y} \langle 0|\hat{T}\left[K^\mu(y)\, A^{\mu_1}_a(x)\, A^{\mu_2}_a(0)\right]|0\rangle|_{amp}
$$

$$
\mathcal{M}_{rhs} = \mathcal{P}_{\mu\mu_1\mu_2}\, \Gamma^{\mu\mu_1\mu_2}_{rhs}(p_1,-p_1)\,.
$$

The *Feynman Rules* in use:



$= 4\, \delta_{a_1 a_2}\, \varepsilon_{\mu_1\mu_2\mu(p_2-p_1)}$



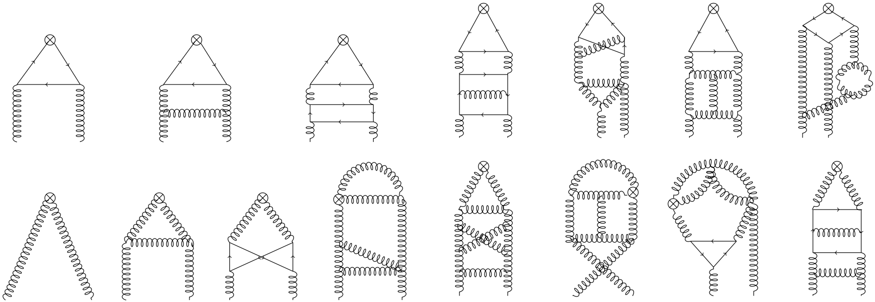
$= 8\, i\, g_s\, f^{a_1a_2a_3}\, \varepsilon_{\mu_1\mu_2\mu_3\mu}$

Feynman diagrams

The Work Flow:

- Generating Feynman diagrams DiaGen/IdSolver [Czakon]
- Applying Feynman Rules, Dirac/Lorentz algebra, Color algebra
- IBP reduction of loop integrals [Tkachov 81; Chetyrkin,Tkachov 81]
- Inserting Master integrals

Loop order \ Generator	DiaGen		Qgraf	
	l.h.s.	r.h.s.	l.h.s.	r.h.s.
1	2	3	2	4
2	20	57	21	64
3	429	1361	447	1488
4	11302	37730	11714	40564



IBP reduction and master integrals

Loop integrals in diagrams, reduced by IBP [Tkachov 81; Chetyrkin,Tkachov 81];

Analytic results of p-master integrals, up to 4 loop [Baikov, Chetyrkin 10; Lee, Smirnov,Smirnov 12].

- DiaGen/IdSolver [Czakon] + Forcer [Ruijl,Ueda,Vermaseren]
  - **Amplitude projection**: about $3+6$ days @ 24 cores (Intel® Xeon® Silver 4116)
  - **Forcer (pre-solved IBP)**: about $12+24$ hours @ 8 cores (Intel® Xeon® E3-1275 V2)
- QGRAF [Nogueira] + FORM [Vermaseren] + Reduze 2 [Manteuffel, Studerus] + FIRE [Smirnov] combined with LiteRed [Lee]
  - **IBP (by Laporta)**: about one month @ 32 cores (Intel® Xeon® Silver 4216)
  - a few hundred GB RAM

At 4-loop: $\sim 10^5$ loop integrals in Feynman amplitudes reduced to **28** masters.

The analytical results were found to be identical between the two set-ups.

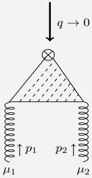
The zero momentum insertion limit

Evaluating $\mathcal{M}$ at $q = 0$ with off-shell gluon momenta $p_1^2 \neq 0$:

$$\Gamma_{lhs}^{\mu\mu_1\mu_2}(p_1, -p_1) = -2F_1\, \epsilon^{\mu\,\mu_1\,\mu_2\,p_1} ,$$

$$\mathcal{P}_{\mu\mu_1\mu_2} = -\frac{1}{6\,p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2\nu}\, p_1^{\nu} ,$$

[Ahmed,**LC**,Czakon 21]



$\frac{1}{(6-11D+6D^2-D^3)\,p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2 p_1} \longrightarrow \frac{-1}{6\,p_1 \cdot p_1}\, \epsilon_{\mu\mu_1\mu_2 p_1}$ albeit with indices in D [**LC** 19]

- possible IR divergences nullified owing to the *IR-rearrangement* [Vladimirov 79]
- 4-loop massless *propagator-type* master integrals available [Smirnov, Tentyukov 10, Baikov, Chetyrkin 10; Lee, Smirnov, Smirnov 11]
- gauge-dependent $\mathcal{M} \Longrightarrow$ UV renormalization of *gauge parameter* ξ !

From [Baikov, Chetyrkin, Kühn, Rittinger, arXiv:1201.5804]

The decay rate of the Z-boson into hadrons in massless QCD up to $\mathcal{O}(\alpha_s^4)$:

$$\begin{aligned}\Gamma_Z =& \Gamma_0 \, R^{\text{nc}} = \frac{G_F \, M_Z^3}{24\pi\sqrt{2}} \, R^{\text{nc}} \\ R^{\text{nc}} =& 20.1945 + 20.1945 \, \alpha_s \\ &+ (28.4587 - 13.0575 + 0) \, \alpha_s^2 \\ &+ (-257.825 - 52.8736 - 2.12068) \, \alpha_s^3 \\ &+ (-1615.17 + 262.656 - 25.5814) \, \alpha_s^4 \, ,\end{aligned}$$

The three terms in the brackets display separately non-singlet, axial singlet and vector singlet contributions.