The symbology of Feynman integrals from twistor geometries I

杨清霖
中国科学院理论物理研究所

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w. 何颂, 刘家昊, 唐一朝;
M.Wilhelm, A. Spiering, M.Roger, 张驰

Symbology for Amplitudes & Integrals

- For multi-polylogarithmic (MPL) functions, symbol letters are most basic building blocks

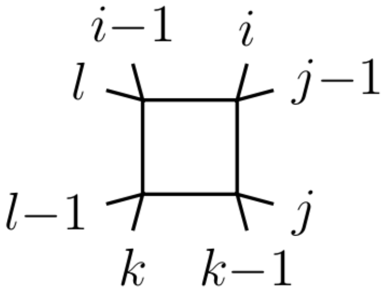
- Canonical DE method for general integrals (very efficient; mathematical structure?)

- Possible mathematical structures of symbol letters help us make prediction for the alphabet → bootstrap amplitudes/individual integrals

- In this talk (1^{st} part: N=4 SYM, 2^{nd} part: general cases), we introduce a geometric way, which recovers the symbol letters from twistor geometries

One-loop integrals and Schubert problems [N. Arkani-Hamed, 21']

- For dual conformal invariant integrals, symbol letters are DCI invariants → geometrical invariants in momentum twistor space
- E.g.1:one-loop four-mass box (weight-2 MPL)



$$= \int_{AB} \frac{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle}{\langle ABi{-}1i\rangle\langle ABj{-}1j\rangle\langle ABk{-}1k\rangle\langle ABl{-}1l\rangle}$$

$$\frac{1}{2\Delta_{i,j,k,l}}\left(v \otimes \frac{z_{i,j,k,l}}{\bar{z}_{i,j,k,l}} + u \otimes \frac{1-\bar{z}_{i,j,k,l}}{1-z_{i,j,k,l}}\right)$$

$$u{=}\frac{\langle i{-}1ij{-}1j\rangle\langle k{-}1kl{-}1l\rangle}{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle},\quad v{=}\frac{\langle i{-}1il{-}1l\rangle\langle j{-}1jk{-}1k\rangle}{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle}$$

$$z_{i,j,k,l}\bar{z}_{i,j,k,l} = u,\ \ (1{-}z_{i,j,k,l})(1{-}\bar{z}_{i,j,k,l}) = v.$$

$$\Delta_{i,j,k,l} = \sqrt{(1{-}u{-}v)^2 - 4uv}$$

One-loop integrals and Schubert problems [N. Arkani-Hamed, 21']

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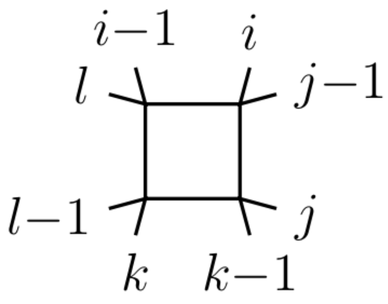
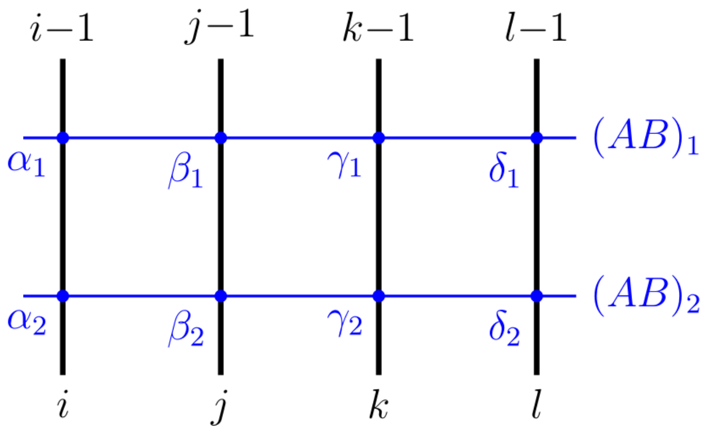


$$= \int_{AB} \frac{\langle i{-}1ik{-}1k\rangle\langle j{-}1jl{-}1l\rangle}{\langle ABi{-}1i\rangle\langle ABj{-}1j\rangle\langle ABk{-}1k\rangle\langle ABl{-}1l\rangle}$$

$$\frac{1}{2\Delta_{i,j,k,l}}\left(v\otimes\frac{z_{i,j,k,l}}{\bar{z}_{i,j,k,l}}+u\otimes\frac{1-\bar{z}_{i,j,k,l}}{1-z_{i,j,k,l}}\right)$$

- LS of box: AB intersects with 4 external lines simultaneously [Schubert, 19th century; N. Arkani-Hamed et al, 10']



$$\frac{(\alpha_1,\beta_1)(\gamma_1,\delta_1)}{(\alpha_1,\gamma_1)(\beta_1,\delta_1)}=z_{i,j,k,l},\ \frac{(\alpha_1,\delta_1)(\gamma_1,\beta_1)}{(\alpha_1,\gamma_1)(\beta_1,\delta_1)}=1{-}z_{i,j,k,l}$$

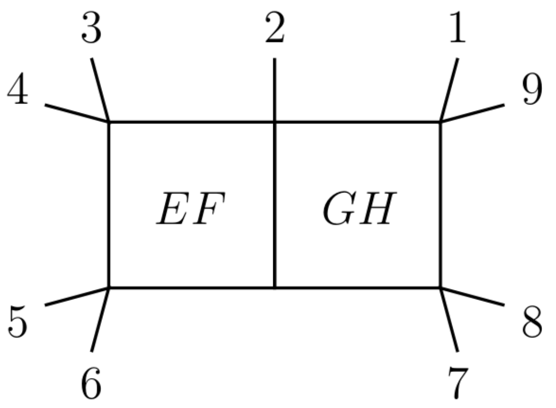
$$\frac{(X_1,X_3)(X_2,X_4)}{(X_1,X_4)(X_2,X_3)}:=\frac{\langle X_1X_3I\rangle\langle X_2X_4I\rangle}{\langle X_1X_4I\rangle\langle X_2X_3I\rangle}$$

$$\frac{(\alpha_2,\beta_2)(\gamma_2,\delta_2)}{(\alpha_2,\gamma_2)(\beta_2,\delta_2)}=\bar{z}_{i,j,k,l},\ \frac{(\alpha_2,\delta_2)(\gamma_2,\beta_2)}{(\alpha_2,\gamma_2)(\beta_2,\delta_2)}=1{-}\bar{z}_{i,j,k,l}$$

I: arbitrary reference line

Two-loop Symbology and twistor geometries

9-point double-box integral (weight-4 MPL)



$$S(I_9) = \sum A_1 \otimes A_2 \otimes B \otimes C$$

$A_1 \otimes A_2$ (first-two-entries) : letters from one-loop intersections on internal lines

B (third entries): ? ? (rational letters & algebraic letters containing four-mass square roots)

C (last entries): ? ? (4 letters containing two-loop square roots)

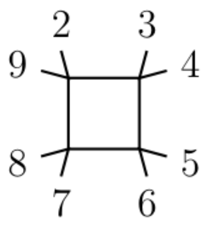
Two-loop Symbology and twistor geometries

- <u>B: cross-ratios from external lines</u>

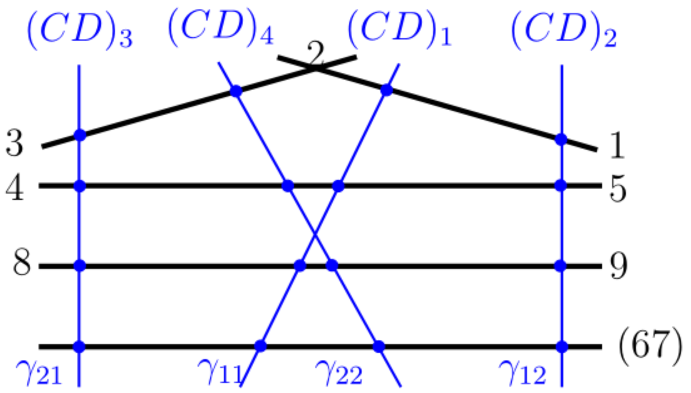
After two one-loop Schubert problems are solved, each shared external lines have four intersections as well

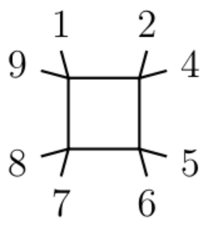
- Two four-mass boxes

$$\frac{(z_{2,5,7,9}-z_{3,5,7,9})(\bar{z}_{2,5,7,9}-\bar{z}_{3,5,7,9})}{(z_{2,5,7,9}-\bar{z}_{3,5,7,9})(\bar{z}_{2,5,7,9}-z_{3,5,7,9})}$$



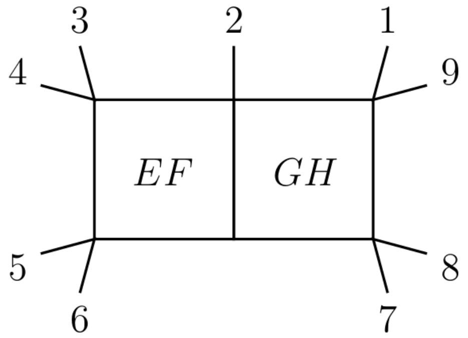
$F(3,5,7,9)$





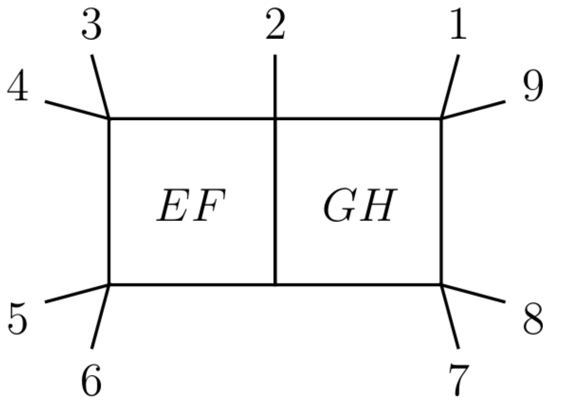
$F(2,5,7,9)$

degeneration: algebraic letters & rational letters
ALL 3rd entry of 9-point double-box integral can be generated by this way
(n<9 letters of MHV/NMHV amplitudes in N=4 SYM
6pt: 9 rational; 7pt: 42 rational, 8pt: 272 rational+18 algebraic)



Two-loop Symbology and twistor geometries

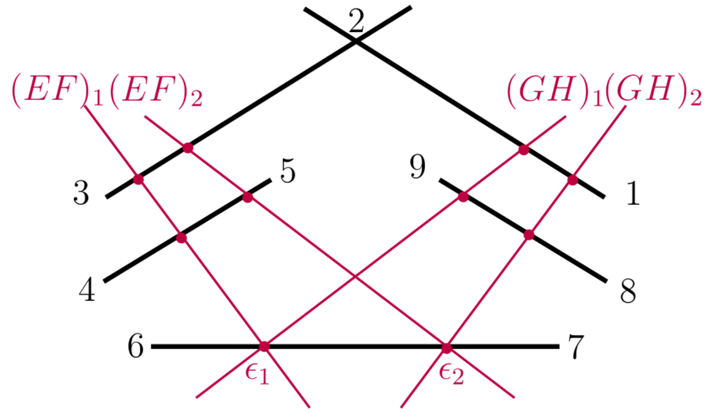
- <u>C: At 2 loops, new topologies lead to new intersections and correspondingly new letters</u>



$$E = \alpha_1 Z_5 + \beta_1 Z_6 + Z_4, \;\; F = \alpha_2 Z_5 + \beta_2 Z_6 + Z_7$$
$$G = \gamma_1 Z_7 + \delta_1 Z_8 + Z_9, \;\; H = \gamma_2 Z_7 + \delta_2 Z_8 + Z_6$$

$$\alpha_2 = \delta_2 = 0, \beta_2 = \frac{1}{\gamma_2}$$

- Solving 7 on-shell conditions

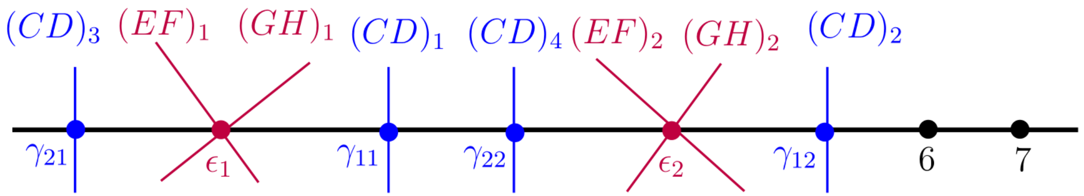


Jacobian factor

$$\langle 7\ (89)\cap(612)\ (645)\cap(623)\rangle - \langle 6\ (89)\cap(712)\ (745)\cap(723)\rangle\gamma_2^2$$
$$+(\langle 6\ (45)\cap(236)\ (789)\cap(712)\rangle - \langle 7\ (45)\cap(237)\ (689)\cap(612)\rangle)\gamma_2$$

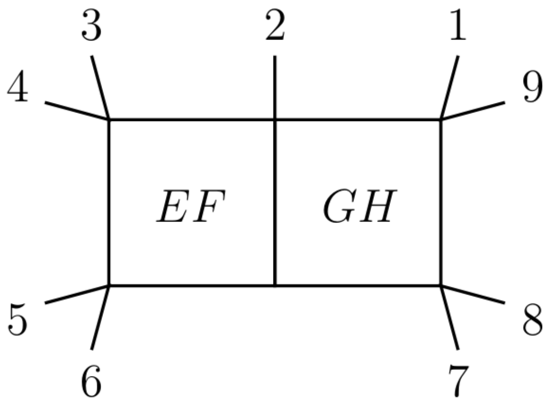
- (composite) LS: jacobian=0, two solutions for γ_2, two intersections ϵ_1 and ϵ_2

a new square root Δ_9



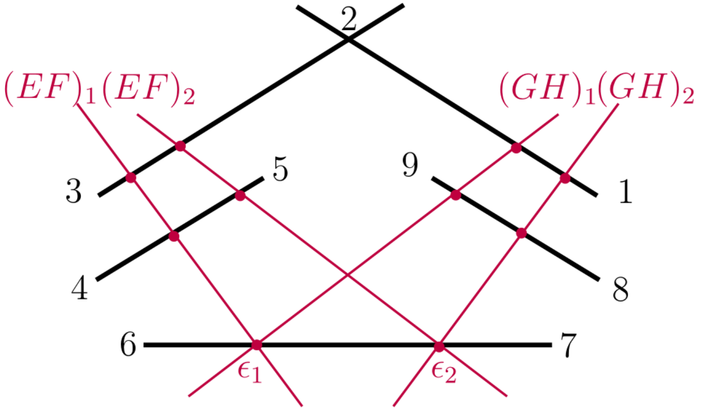
Two-loop Symbology and twistor geometries

- C: At 2 loops, new topologies lead to new intersections and correspondingly new letters



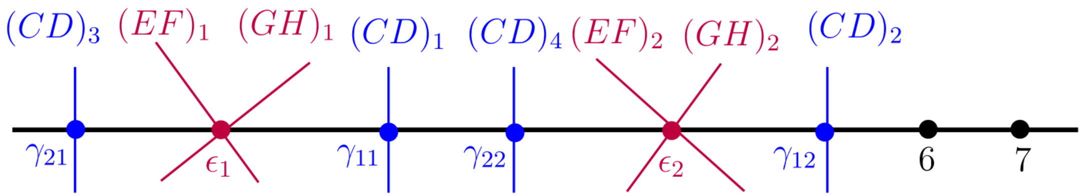
Using two intersections, we can construct:

$$\frac{(\epsilon_1,\gamma_{11})(\epsilon_2,\gamma_{12})}{(\epsilon_2,\gamma_{11})(\epsilon_1,\gamma_{12})}=\frac{(1+az_{2,5,7,9})(1+b\bar{z}_{2,5,7,9})}{(1+a\bar{z}_{2,5,7,9})(1+bz_{2,5,7,9})},\ \frac{(\epsilon_1,\gamma_{21})(\epsilon_2,\gamma_{22})}{(\epsilon_2,\gamma_{21})(\epsilon_1,\gamma_{22})}=\frac{(\epsilon_1,\gamma_{11})(\epsilon_2,\gamma_{12})}{(\epsilon_2,\gamma_{11})(\epsilon_1,\gamma_{12})}\Big|_{reflection}$$

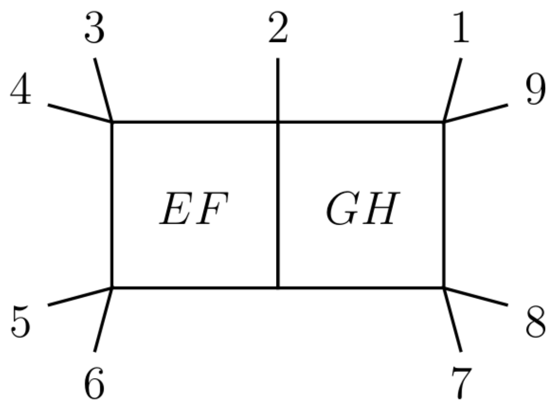


a & b: combinations involving Δ_9

last entries of the db integral!



Two-loop Symbology and twistor geometries



9-point double-box integral (weight-4 MPL)

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B (third entries): letters from one-loop intersections on external lines

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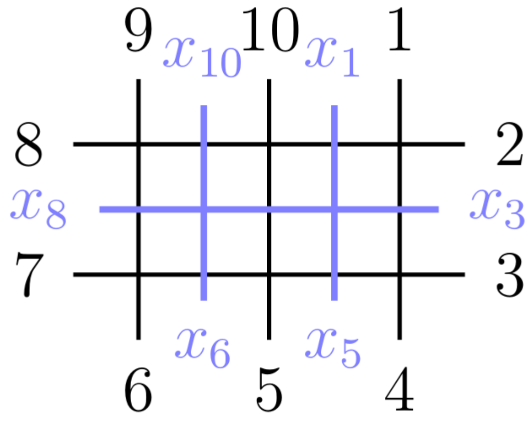
More general integrals?...

Elliptic Schubert problems and letters [in progress w. M.Wilhelm et al]



- Cannot take residues in all 4L variables→ rigidity

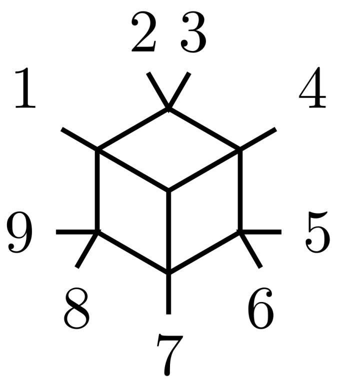
Rigidity=1: elliptic curve, Rigidity=2, K3 surface ···

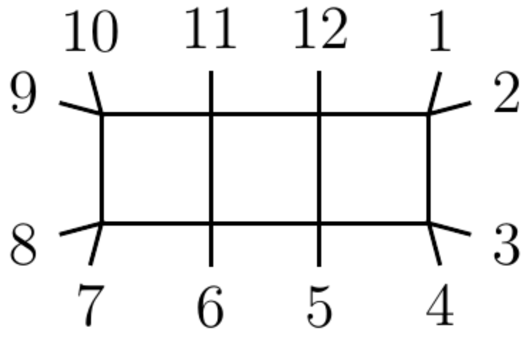
e.g. 10-pt double-box integral

$$E = \alpha_1 Z_6 + \beta_1 Z_7 + Z_5, \;\; F = \gamma_1 Z_6 + \delta_1 Z_7 + Z_8$$
$$G = \alpha_2 Z_{10} + \beta_2 Z_1 + Z_2, \;\; H = \gamma_2 Z_{10} + \delta_2 Z_1 + Z_3$$

Solving 7 on-shell conditions, the jacobian reads $\sqrt{P(\alpha)}$ with P a polynomial to the 4^{th} power (soft limit→perfect square)

Rigidity 2:
$K(\alpha, \beta)$





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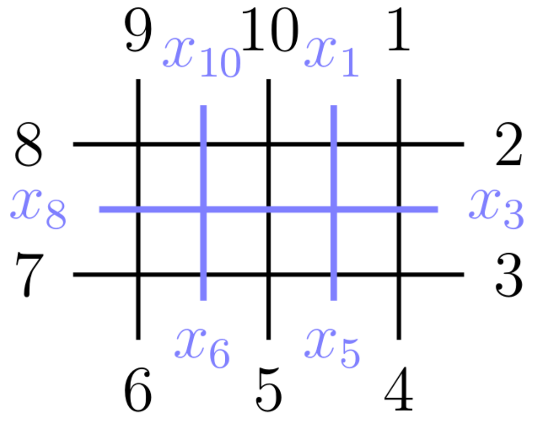
Elliptic Schubert problems and letters [in progress w. M.Wilhelm et al]



e.g. 10-pt double-box integral[2106.14902]

Elliptic "cross-ratios:"

$$\frac{1}{\omega_1}\int_a^b \frac{\mathrm{d}x}{\sqrt{P(x)}}$$

a & b: two intersections from another Schubert problem

ω_1: elliptic period

$$S(I_{10}) = \sum A_1 \otimes A_2 \otimes B \otimes \mathrm{D}$$

Successfully reproduce the last entries of 10-point db integrals and give a predictions of 12-point db integrals→ bootstrapping

More general cases: K3 last entries?...

Summary

- Twistor geometries provide a geometric explanations for the symbol letters, both for L-loop amplitudes and integrals in N=4 SYM

- Through this way, we can predict symbol letters for individual integrals and finally bootstrap them instead of direct computations

- This method can be generalized to elliptic cases, and proves to be useful for 10-/12-pt double-box integral

- Relations to cut integrals/diagrammatic coactions…
- K3 integrals?...

Thanks!

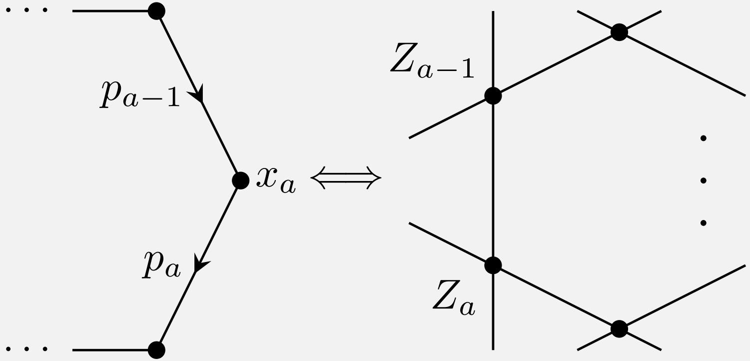
Symbology of Feynman Integrals from Twistor Geometries (II)

唐一朝 (Institute of Theoretical Physics, CAS)

August 24, 2022

Based on [2207.13482], also see [2203.16112] [2206.04609]

Momentum Twistors and Dual Conformal Invariance $SO(6) = SL(4)$



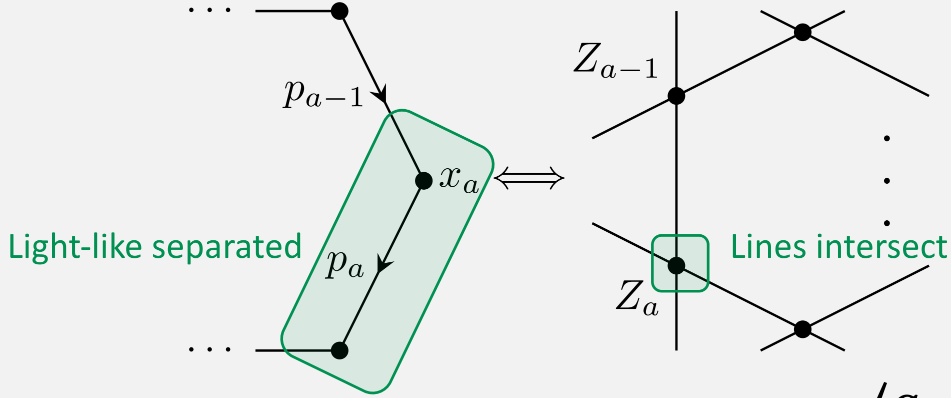
$$(p_a + p_{a+1} + \cdots + p_{b-1})^2 = (x_a - x_b)^2 = \frac{\langle a-1ab-1b\rangle}{\langle a-1aI_\infty\rangle\langle b-1bI_\infty\rangle}$$

Any well-defined expression must be projective-invariant.

Expressions independent of I_∞ are DCI, e.g., cross-ratios.

$$\frac{(x_a - x_b)^2(x_c - x_d)^2}{(x_a - x_d)^2(x_b - x_c)^2} = \frac{\langle a-1ab-1b\rangle\langle c-1cd-1d\rangle}{\langle a-1ad-1d\rangle\langle b-1bc-1c\rangle}$$

Momentum Twistors and Dual Conformal Invariance $SO(6)=SL(4)$



$$(p_a+p_{a+1}+\cdots+p_{b-1})^2=(x_a-x_b)^2=\frac{\langle a-1ab-1b\rangle}{\langle a-1aI_\infty\rangle\langle b-1bI_\infty\rangle}$$

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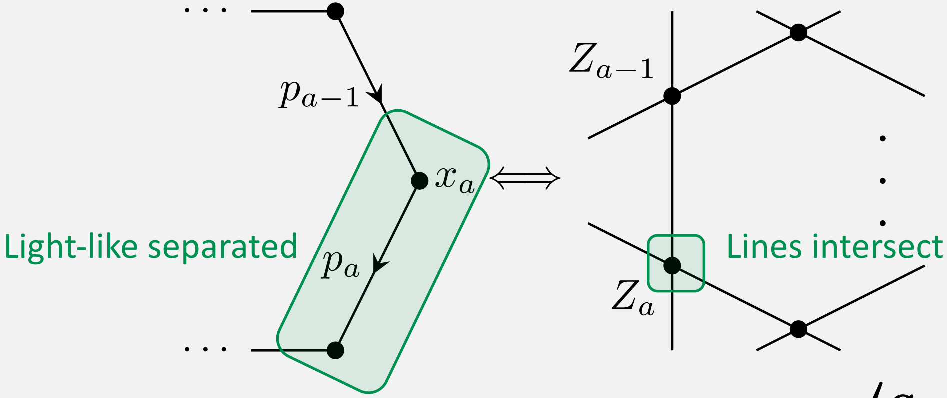
Momentum Twistors and Dual Conformal Invariance $SO(6) = SL(4)$



Space-like infinity

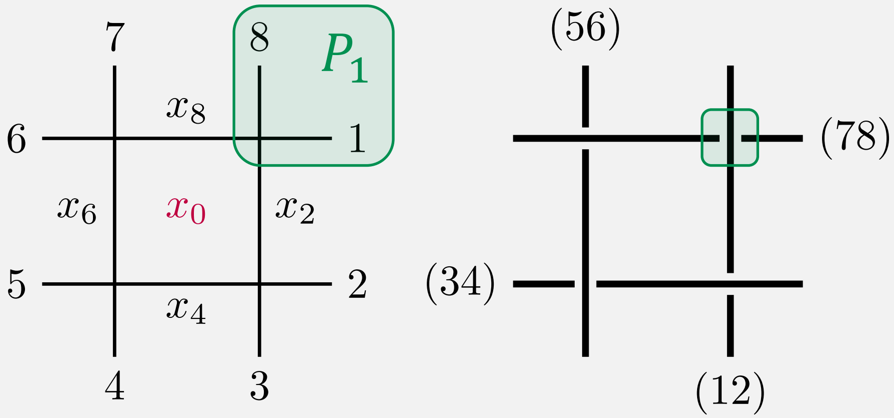
$$(p_a + p_{a+1} + \cdots + p_{b-1})^2 = (x_a - x_b)^2 = \frac{\langle a-1ab-1b\rangle}{\langle a-1aI_\infty\rangle\langle b-1bI_\infty\rangle}$$

Any well-defined expression must be projective-invariant.

Expressions independent of I_∞ are DCI, e.g., cross-ratios.

$$\frac{(x_a - x_b)^2(x_c - x_d)^2}{(x_a - x_d)^2(x_b - x_c)^2} = \frac{\langle a-1ab-1b\rangle\langle c-1cd-1d\rangle}{\langle a-1ad-1d\rangle\langle b-1bc-1c\rangle}$$

Planar Integrals (DCI Kinematics)



$$\frac{1}{(x_0 - x_a)^2} = \frac{\langle AB I_\infty \rangle \langle a-1\, a I_\infty \rangle}{\langle AB\, a-1\, a \rangle}$$

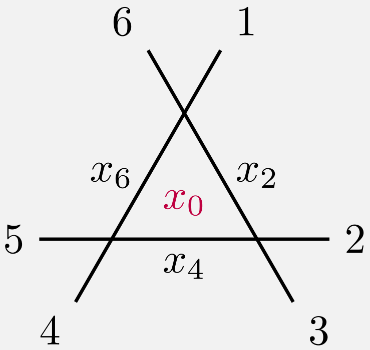
$$\mathrm{d}^4 x_0 = \frac{\mathrm{d}^4 A\, \mathrm{d}^4 B}{GL(2)} \frac{1}{\langle AB I_\infty \rangle^4} =: \frac{\mathrm{d}\mu_{AB}}{\langle AB I_\infty \rangle^4}$$

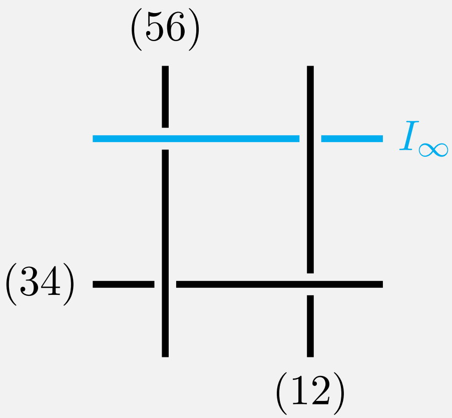
$$\int \frac{\mathrm{d}^4 x_0}{(x_0 - x_2)^2 (x_0 - x_4)^2 (x_0 - x_6)^2 (x_0 - x_8)^2} = \int \frac{\mathrm{d}\mu_{AB}\, \langle 12 I_\infty \rangle \langle 34 I_\infty \rangle \langle 56 I_\infty \rangle \langle 78 I_\infty \rangle}{\langle AB12 \rangle \langle AB34 \rangle \langle AB56 \rangle \langle AB78 \rangle}$$

$$\propto \int \frac{\mathrm{d}\mu_{AB}\, \langle 1256 \rangle \langle 3478 \rangle}{\langle AB12 \rangle \langle AB34 \rangle \langle AB56 \rangle \langle AB78 \rangle}$$

Planar Integrals (Generic Kinematics)

Generically, an integral may depend on $x_\infty \sim I_\infty$.





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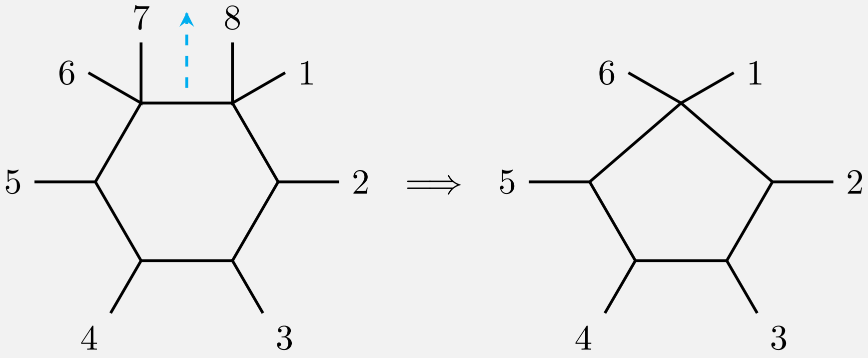
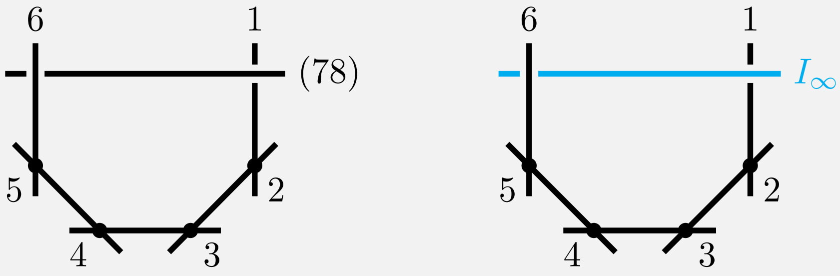
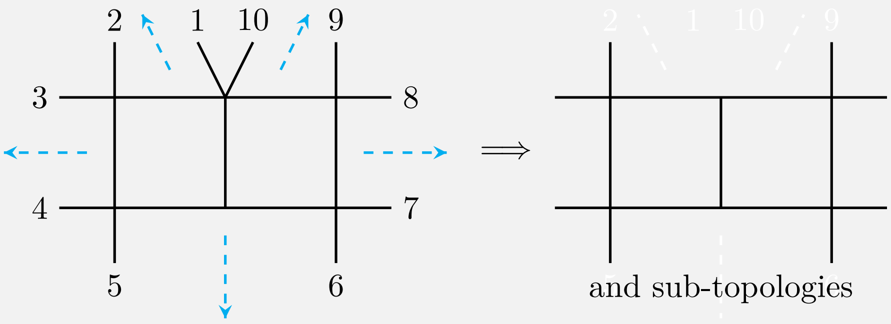
Non-DCI Alphabets from DCI



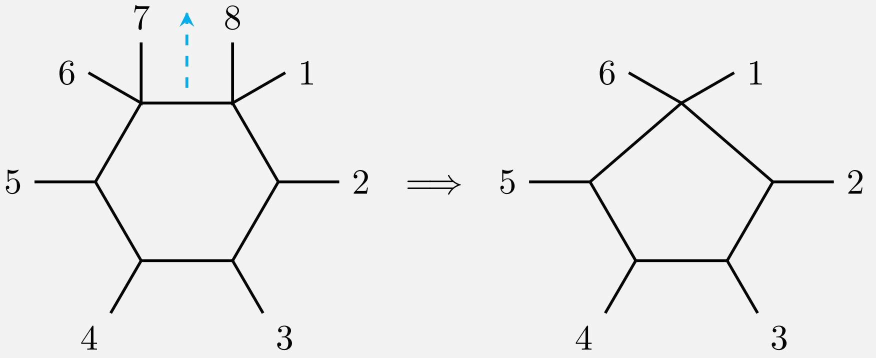


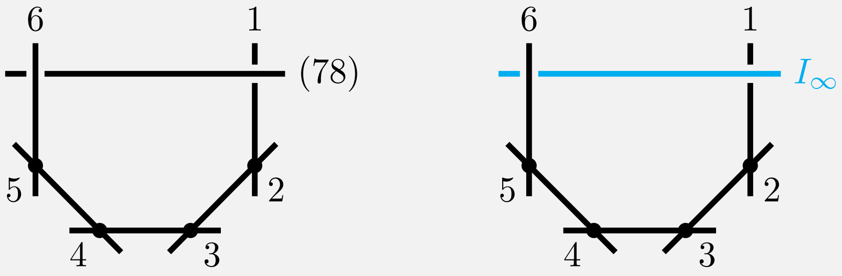
DCI alphabet from kinematic cluster algebras, missing only a few letters. [Henn et al., 2012.12285]



DCI alphabet from DCI Schubert [see YQL's talk], covering the entire alphabet. [He et al., 2206.04609]

Non-DCI Alphabets from DCI





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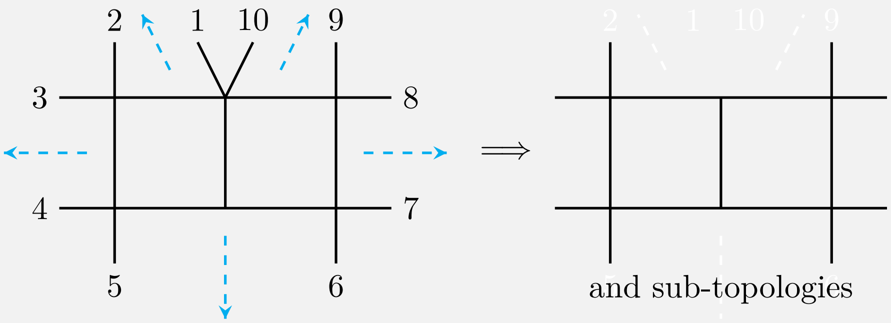
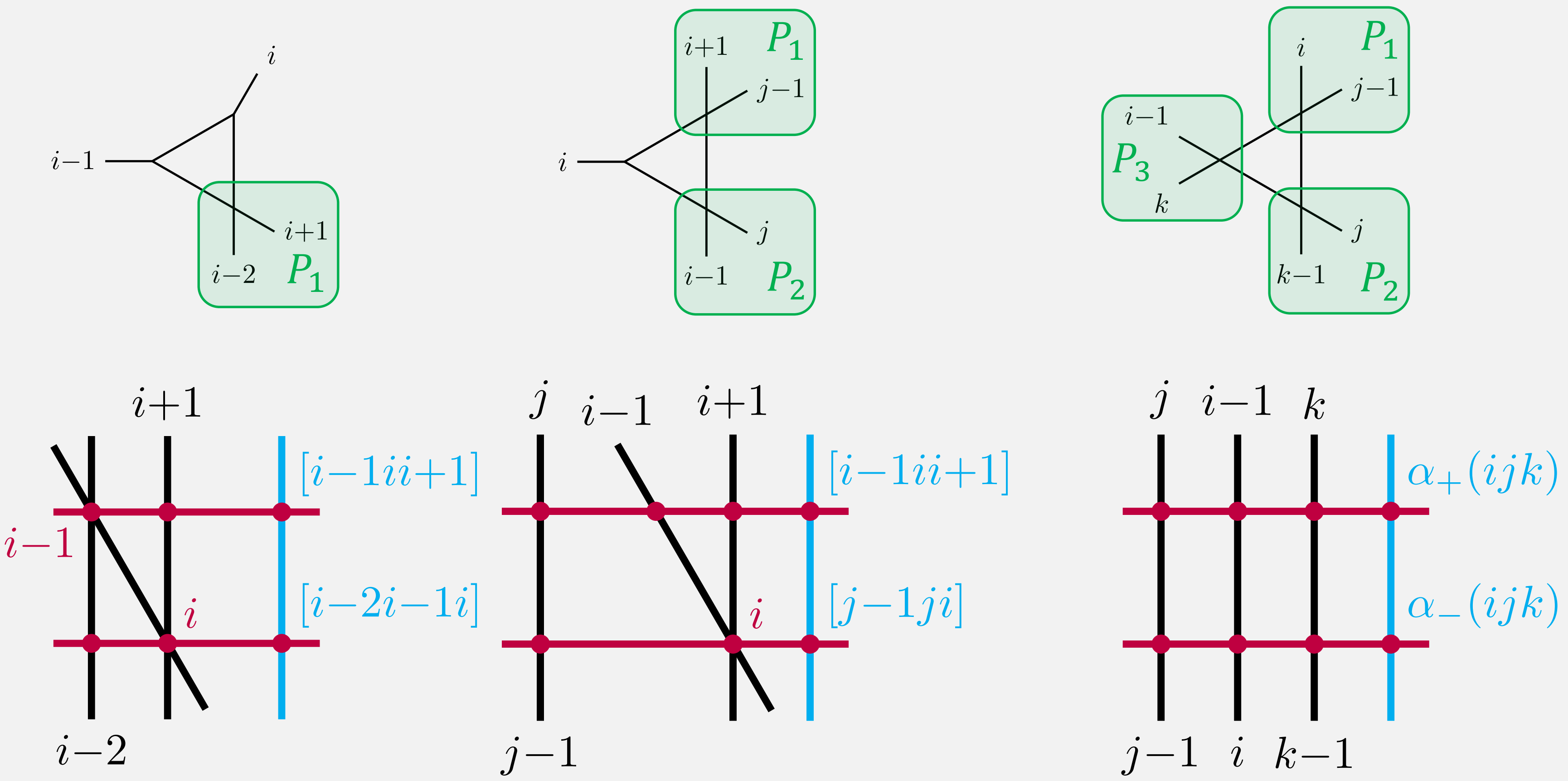


DCI alphabet from DCI Schubert [see YQL's talk], covering the entire alphabet. [He et al., 2206.04609]

Massive non-DCI equivalent to DCI

Study non-DCI integrals on their own?

One-Loop Building Blocks: Triangles



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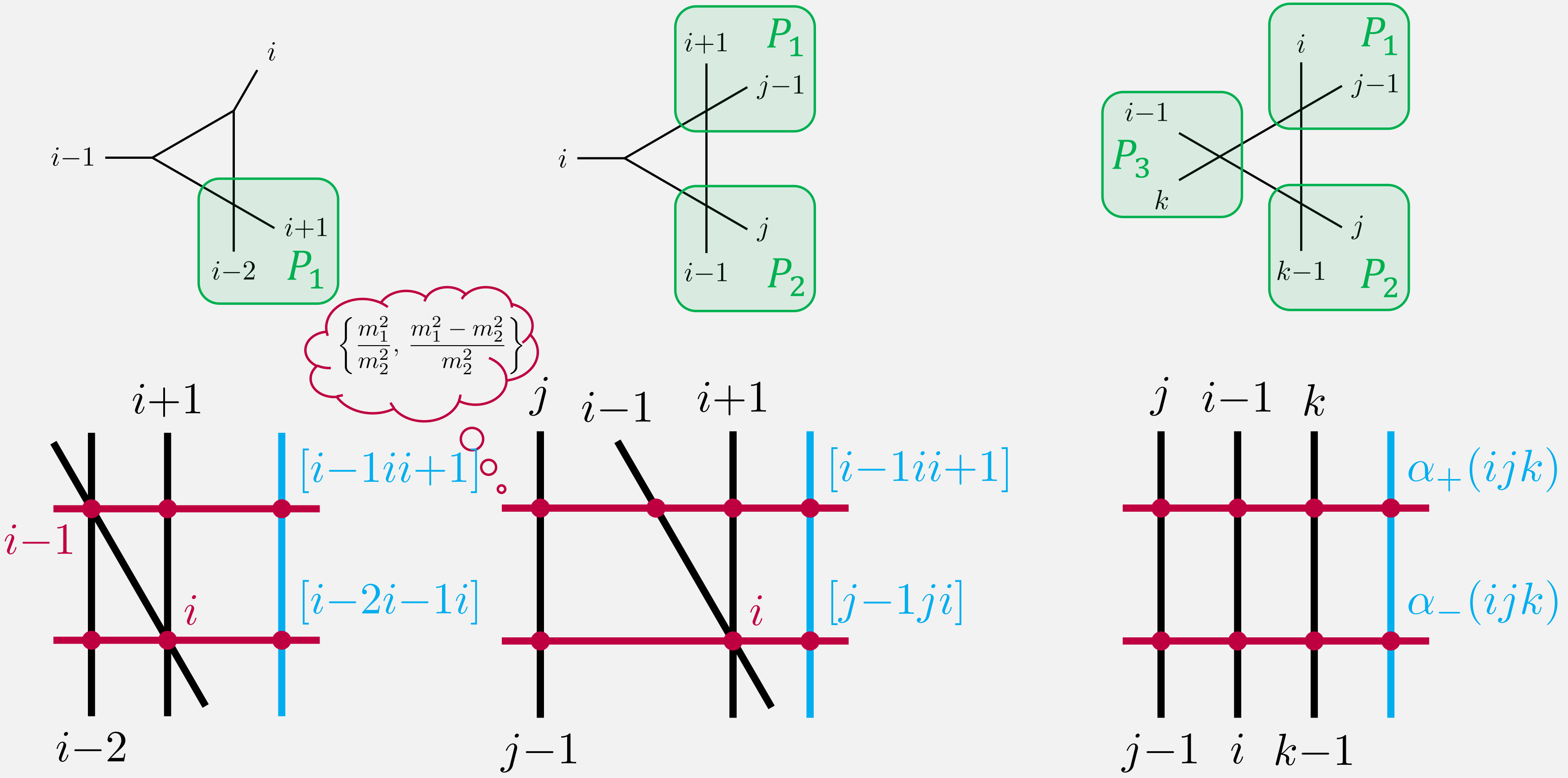
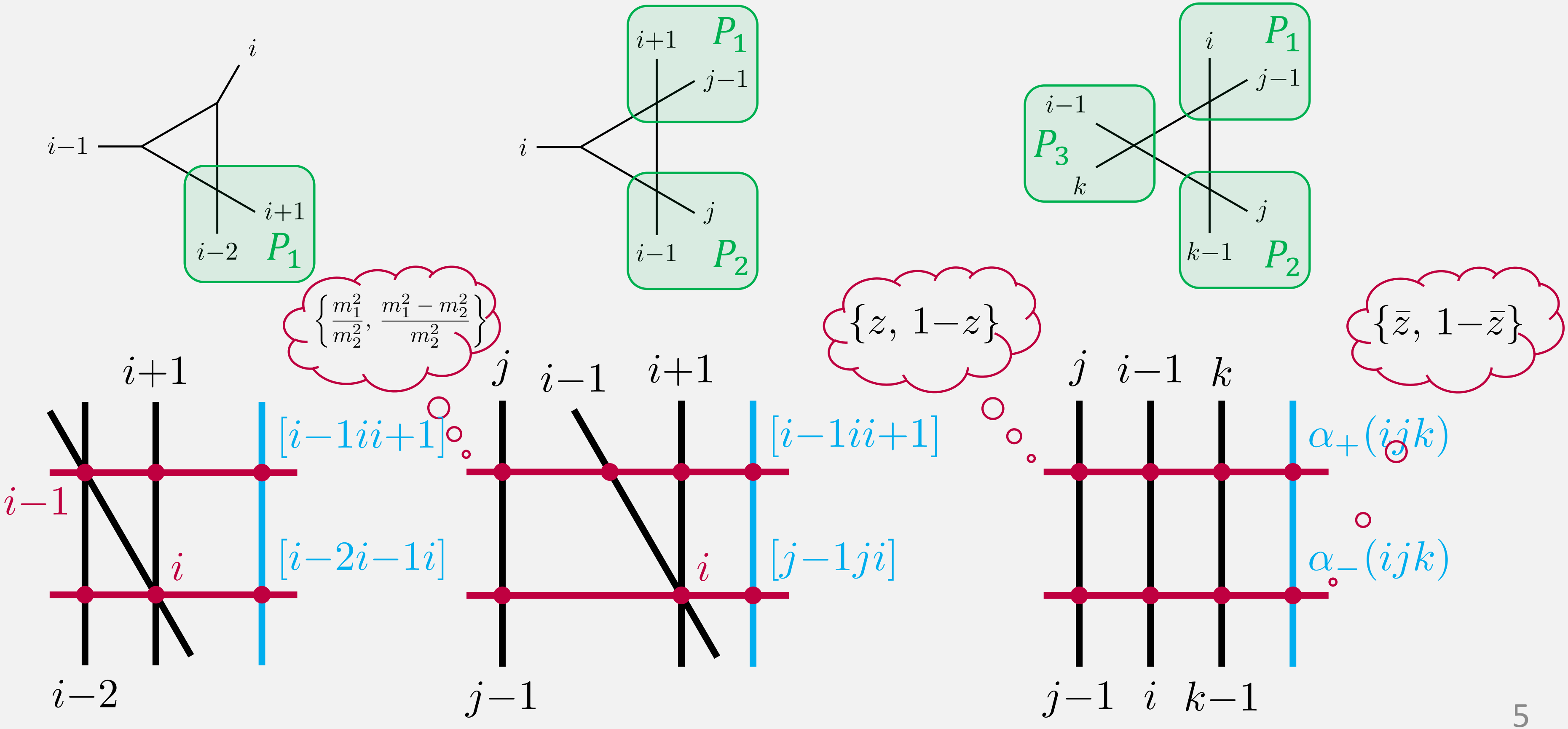
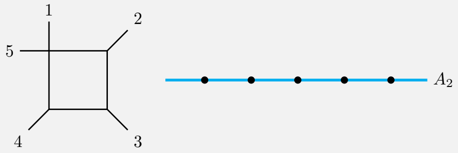
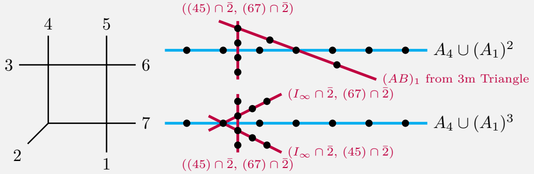


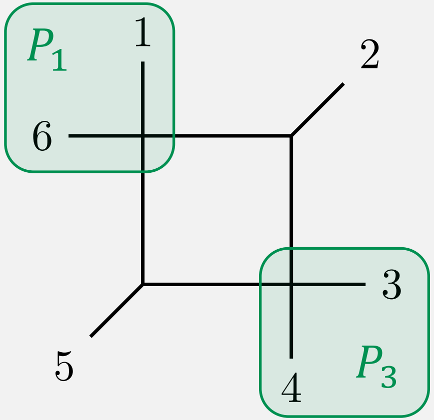
One-Loop Building Blocks: Triangles

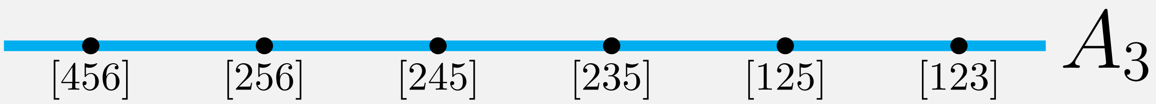


One-Loop: Boxes

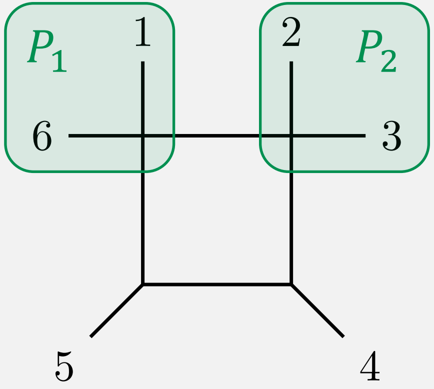


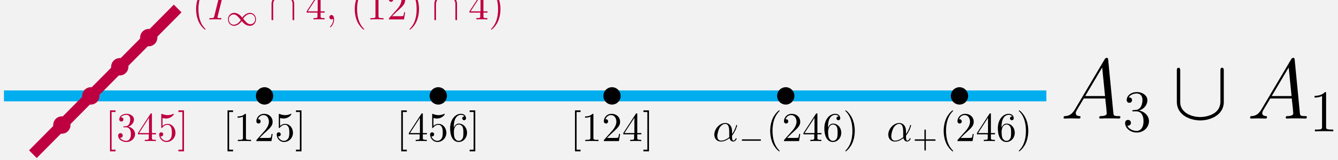






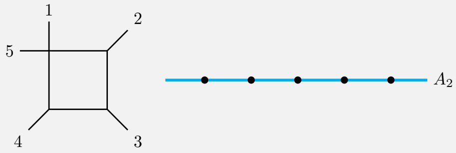
$\{m_1^2, m_3^2, s, t, m_1^2-s, m_3^2-s, m_1^2-t, m_3^2-t, m_1^2+m_3^3-s-t, m_1^2m_3^2-st\}$

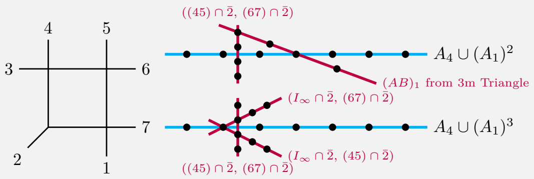


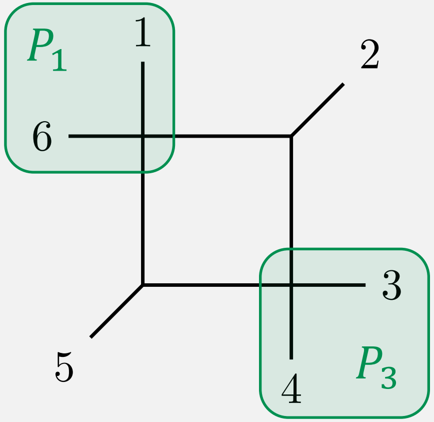


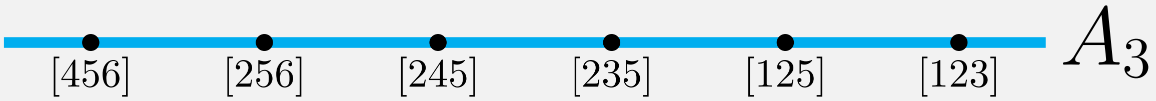
$\{s, t, m_1^2, m_2^2, m_1^2-t, m_2^2-t, \Delta_{2,4,6}, \frac{z}{\bar{z}}, \frac{1-z}{1-\bar{z}}, \frac{sz(1-\bar{z})+t}{s\bar{z}(1-z)+t}, m_1^2m_2^2-m_1^2t-m_2^2t+st+t^2\}$

One-Loop: Boxes



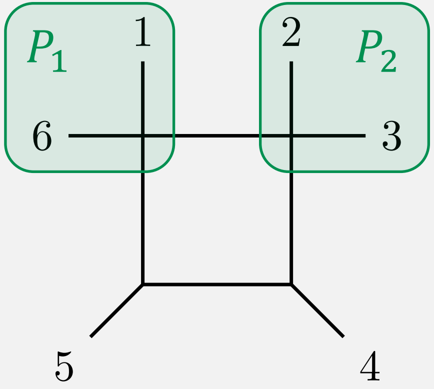


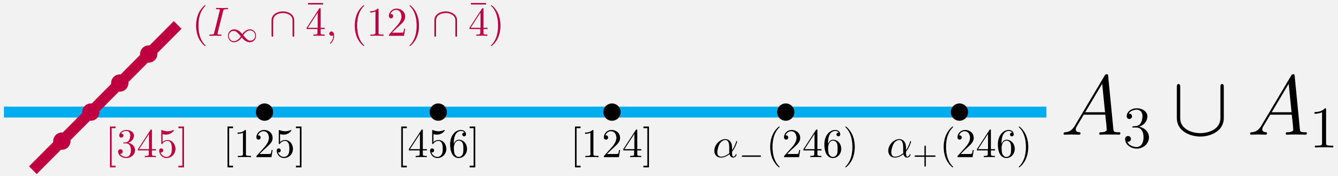




$$\{m_1^2, m_3^2, s, t, m_1^2-s, m_3^2-s, m_1^2-t, m_3^2-t, m_1^2+m_3^3-s-t, m_1^2m_3^2-st\}$$

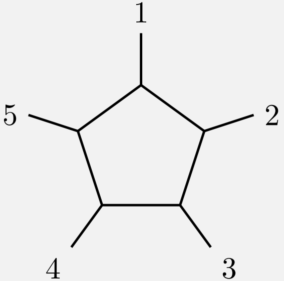
CA from the bottom up



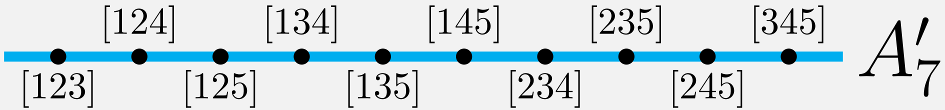


$$\{s, t, m_1^2, m_2^2, m_1^2-t, m_2^2-t, \Delta_{2,4,6}, \tfrac{z}{\bar{z}}, \tfrac{1-z}{1-\bar{z}}, \tfrac{sz(1-\bar{z})+t}{s\bar{z}(1-z)+t}, m_1^2m_2^2-m_1^2t-m_2^2t+st+t^2\}$$

One-Loop: 0m Pentagon



$$\mathrm{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma}P_1^\mu P_2^\nu P_3^\rho P_4^\sigma = \sqrt{\det(2P_i \cdot P_j)_{i,j=1,\cdots,4}} \xrightarrow{\text{parity}} -\mathrm{tr}_5$$



Box sub-topologies
Parity-even

Mix different boxes
Parity-odd

Not parity-invariant

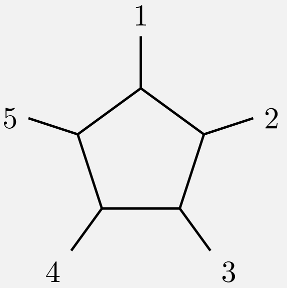
Parity-invariant Subspace

$$s_{12},\ \ s_{12} - s_{45},\ \ s_{12} + s_{23} - s_{45}$$

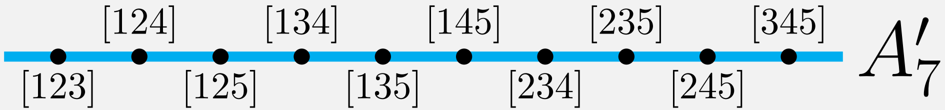
$$\frac{s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} - s_{51}s_{12} + \mathrm{tr}_5}{s_{12}s_{23} - s_{23}s_{34} + s_{34}s_{45} - s_{45}s_{51} - s_{51}s_{12} - \mathrm{tr}_5}$$

$$-s_{12}s_{23} + s_{23}s_{34} + s_{34}s_{45} + s_{45}s_{51} + s_{51}s_{12} + \mathrm{tr}_5$$

One-Loop: 0m Pentagon



$$\mathrm{tr}_5 = 4i\epsilon_{\mu\nu\rho\sigma}P_1^\mu P_2^\nu P_3^\rho P_4^\sigma = \sqrt{\det(2P_i \cdot P_j)_{i,j=1,\cdots,4}} \xrightarrow{\text{parity}} -\mathrm{tr}_5$$



Box sub-topologies
Parity-even

$s_{12},\ s_{12}-s_{45},\ s_{12}+s_{23}-s_{45}$

Parity-invariant
Subspace
= Correct Alphabet

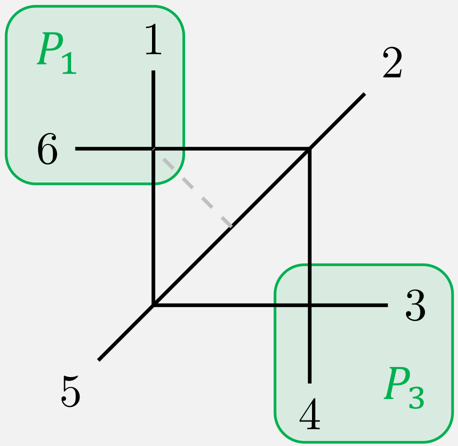
Mix different boxes
Parity-odd

$$\frac{s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}-s_{51}s_{12}+\mathrm{tr}_5}{s_{12}s_{23}-s_{23}s_{34}+s_{34}s_{45}-s_{45}s_{51}-s_{51}s_{12}-\mathrm{tr}_5}$$

Not parity-invariant

$-s_{12}s_{23}+s_{23}s_{34}+s_{34}s_{45}+s_{45}s_{51}+s_{51}s_{12}+\mathrm{tr}_5$

Two-Loop: 2me Slashed-Box

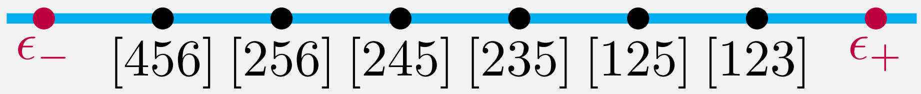


The minor $(\epsilon_+\epsilon_-)$ contributes the 2-loop root:

$$\Delta_{nc} = \sqrt{(s+t)^2 - 4m_1^2m_3^2}$$

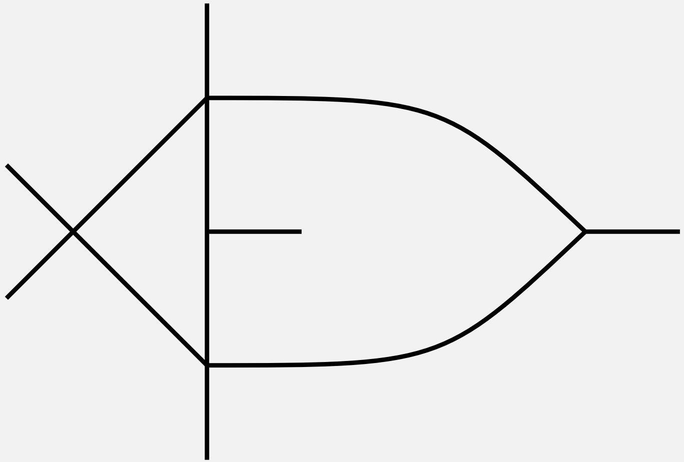
Cross-ratios $\frac{(X_1\epsilon_+)(X_2\epsilon_-)}{(X_1\epsilon_-)(X_2\epsilon_+)}$ yield algebraic letters.

These are precisely the last entries.

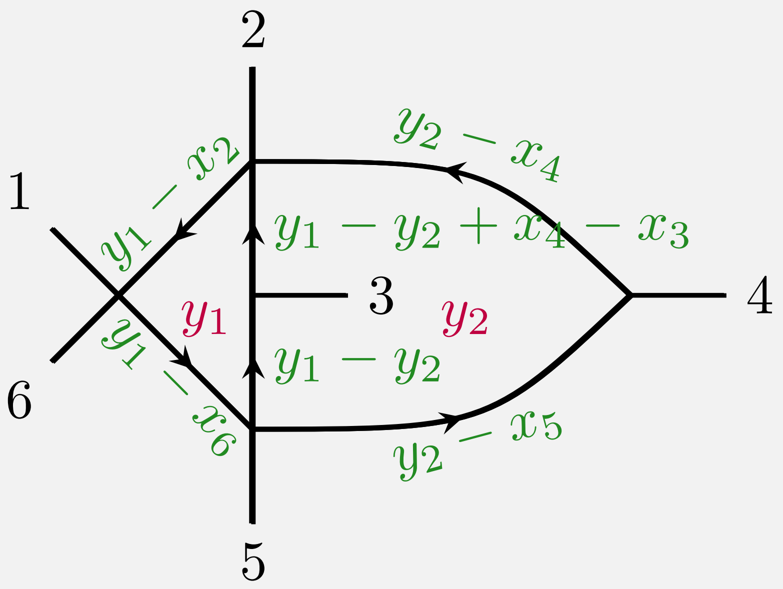


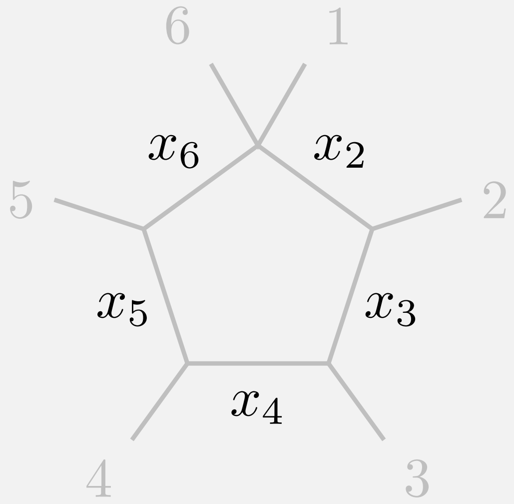
$$L_1 = \frac{s+t+\Delta_{nc}}{s+t-\Delta_{nc}}, L_2 = \frac{s-t+\Delta_{nc}}{s-t-\Delta_{nc}}, L_3 = \frac{-2m_1^2+s+t+\Delta_{nc}}{-2m_1^2+s+t-\Delta_{nc}}$$

Generalizing to Non-Planar Cases

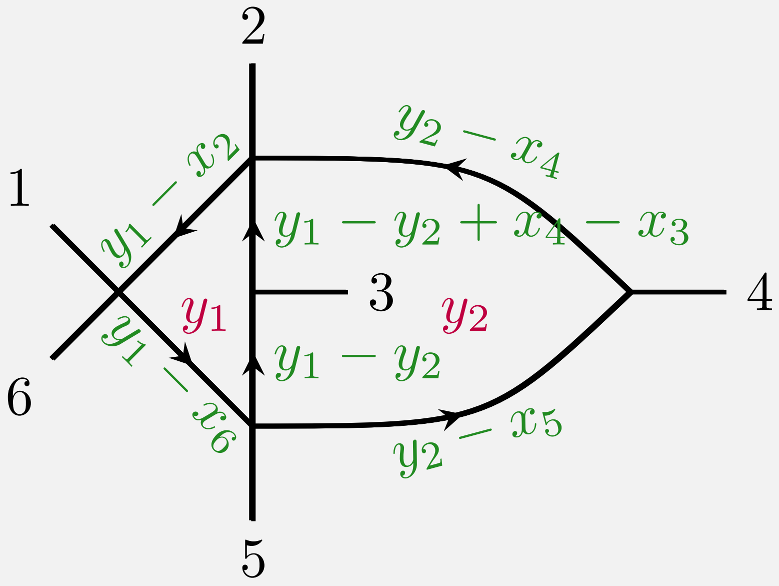


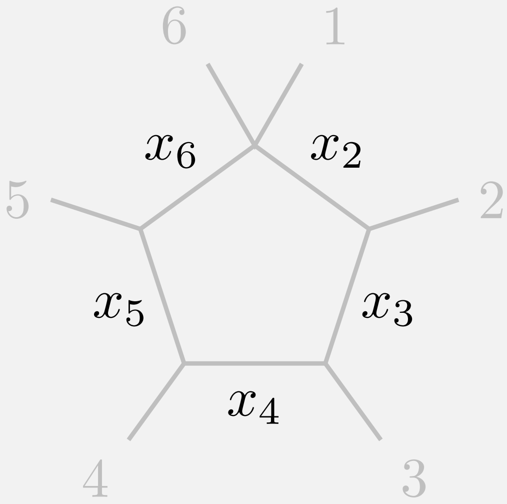
Generalizing to Non-Planar Cases





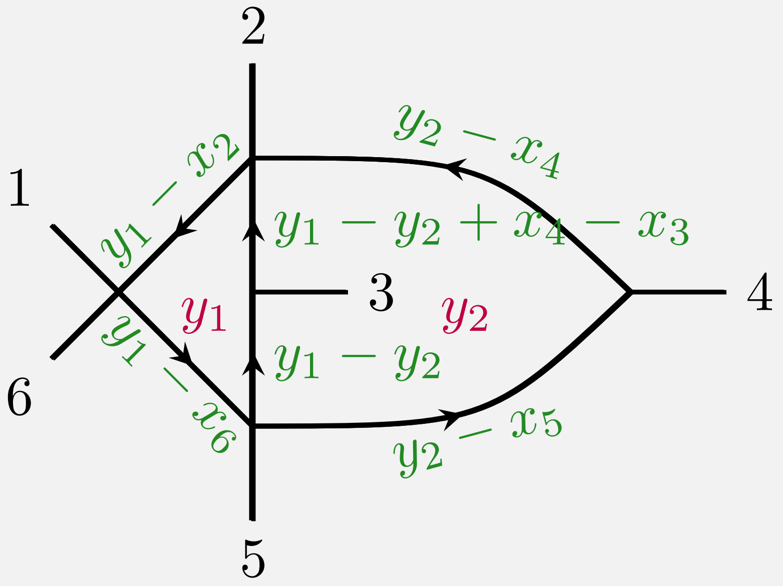
Generalizing to Non-Planar Cases

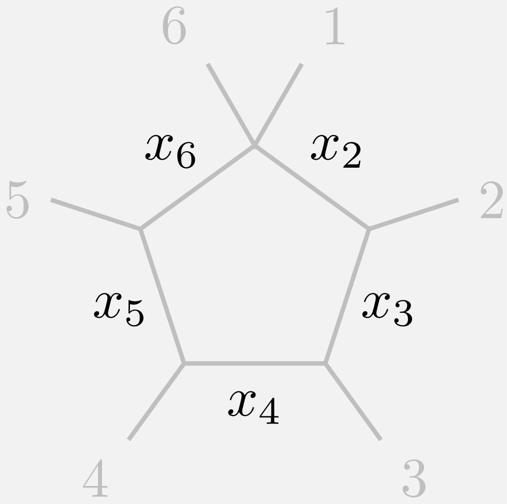




$$
\begin{aligned}
(y_1 - y_2 + x_4 - x_3)^2 &= (y_1 - y_2)^2 + (x_4 - x_3)^2 \\
&+ (y_1 - x_3)^2 - (y_1 - x_4)^2 \\
&- (y_2 - x_4)^2 + (y_2 - x_3)^2
\end{aligned}
$$

Generalizing to Non-Planar Cases

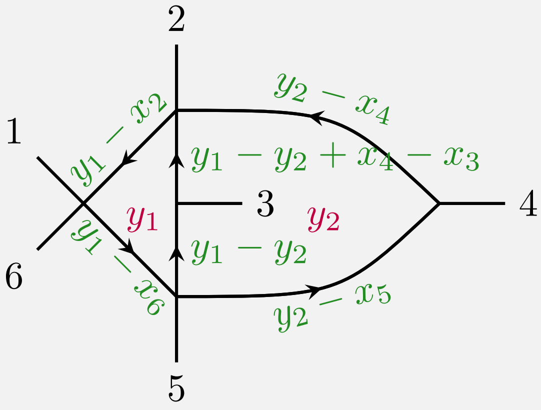




$$
\begin{aligned}
(y_1 - y_2 + x_4 - x_3)^2 = {} & (y_1 - y_2)^2 + (x_4 - x_3)^2 \\
& + (y_1 - x_3)^2 - (y_1 - x_4)^2 \\
& - (y_2 - x_4)^2 + (y_2 - x_3)^2
\end{aligned}
$$

Surprising! “Clever” routing only?

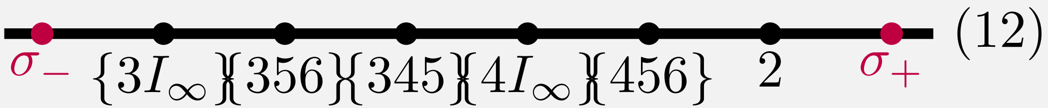
Generalizing to Non-Planar Cases



$$(y_\bullet - x_\bullet)^2 = 0 \implies \langle AB12\rangle = \langle AB56\rangle = \langle CD34\rangle = \langle CD45\rangle = 0$$

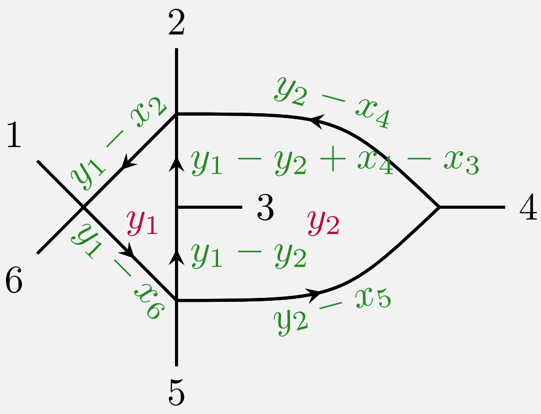
$$(y_1 - y_2)^2 = 0 \implies \langle ABCD\rangle = 0$$

$$(y_1 - y_2 + x_4 - x_3)^2 = 0 \implies \langle ABI_\infty\rangle\langle CD\bar{3}\cap(3I_\infty)\rangle - \langle CDI_\infty\rangle\langle AB\bar{3}\cap(3I_\infty)\rangle = 0$$



Non-planar root Σ_5 and 5 associated algebraic letters

Generalizing to Non-Planar Cases



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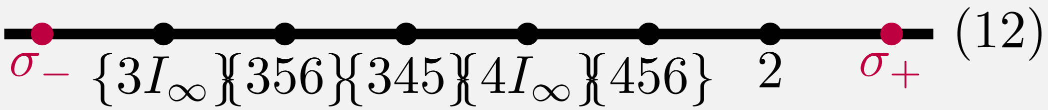
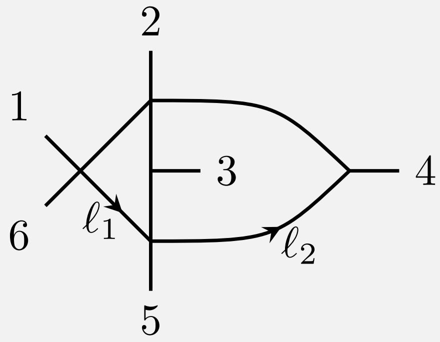
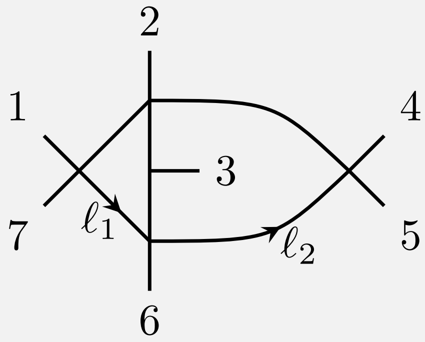
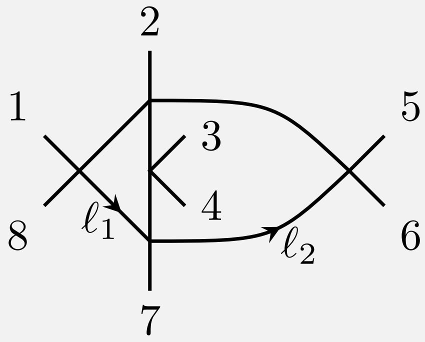


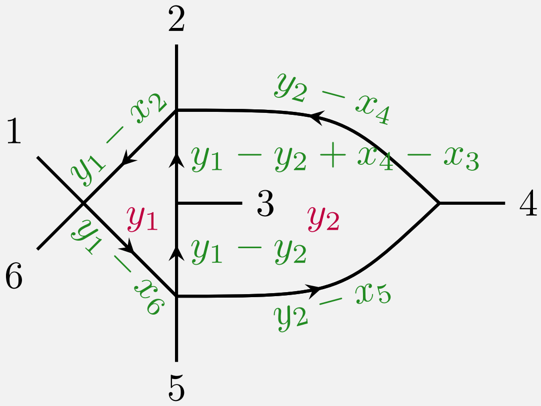
Non-planar root Σ_5 and 5 associated algebraic letters







Generalizing to Non-Planar Cases



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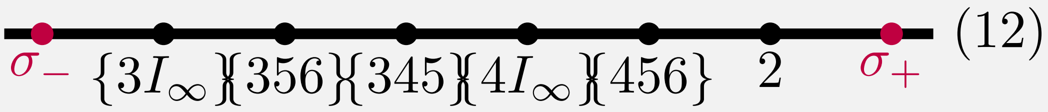
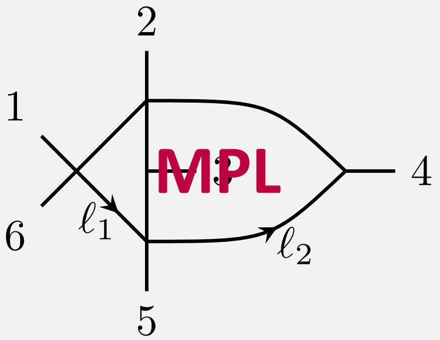
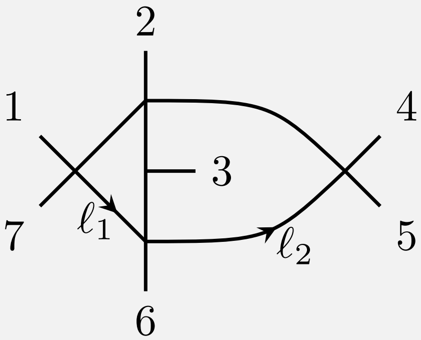
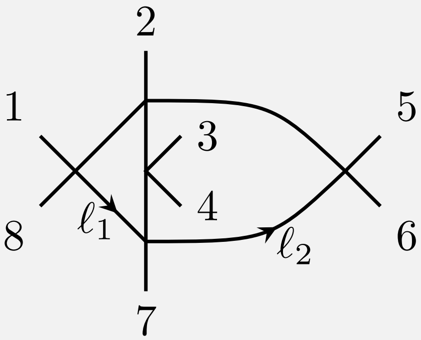


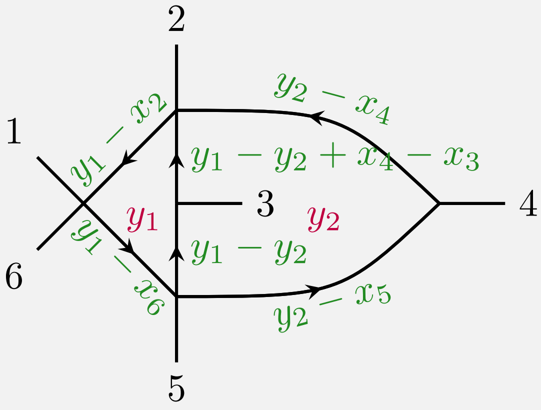
Non-planar root Σ_5 and 5 associated algebraic letters







Generalizing to Non-Planar Cases



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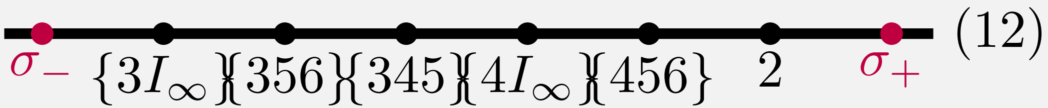
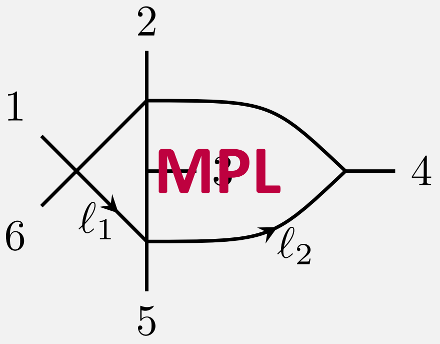
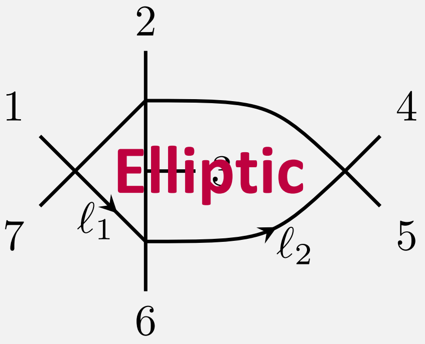
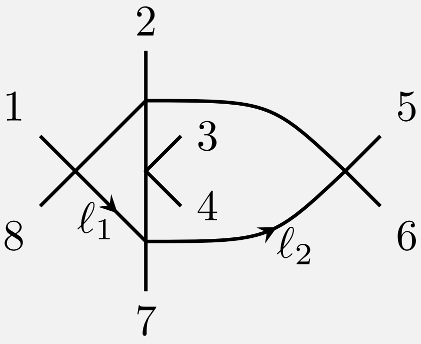


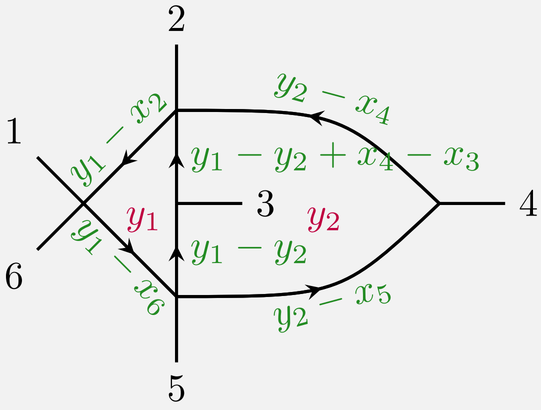
Non-planar root Σ_5 and 5 associated algebraic letters







Generalizing to Non-Planar Cases



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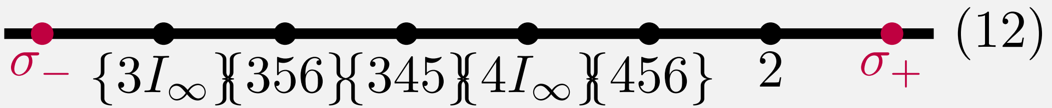
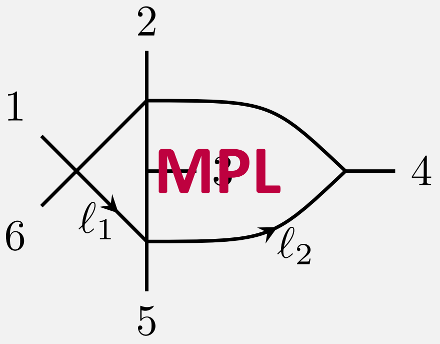
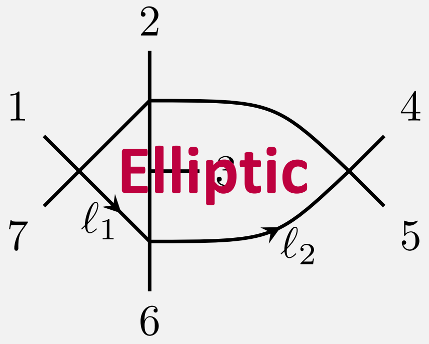
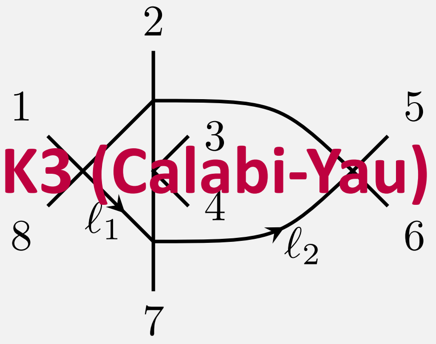


Non-planar root Σ_5 and 5 associated algebraic letters







Summary & Outlook

- Full coverage of alphabets – Geometric handle on symbology
- Exact matches – Predicting letters and bootstrapping integrals

- Systematic construction (no more & no less)
- Elliptic, K3, non-planar, non-planar elliptic, …
- Detailed structure of symbols / alphabets

- WHY ?

Back-up Slides

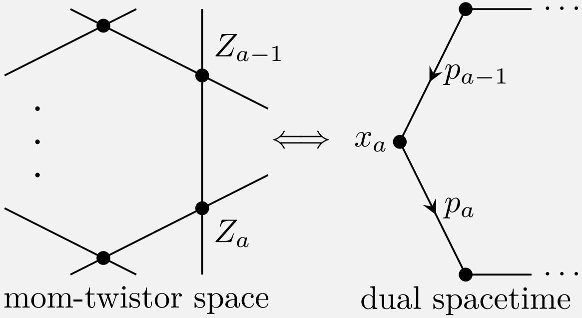
Momentum Twistors

	momentum conservation	$\Longrightarrow$	dual momenta
+	massless particles	$\Longrightarrow$	spinor-helicity variables
=	planar kinematics	$\Longrightarrow$	momentum twistors

Dual conformal transformations $SL(4)$ } $G(4,n)$

Projectivity { Overall scaling $GL(1)$ } $G(4,n)$] $G(4,n)/T$

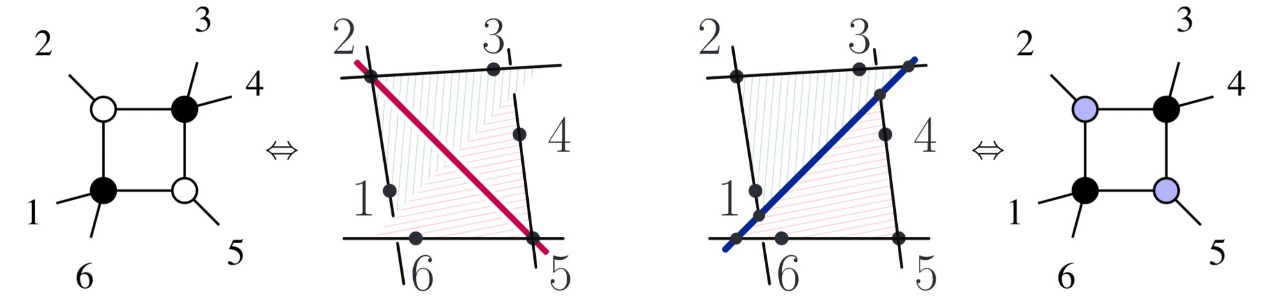
Relative scalings $[GL(1)]^{n-1}$] $G(4,n)/T$



Point $\in G(1,4)$, dof = 3

Line $\in G(2,4)$, dof = 4

Plane $\in G(3,4)$, dof = 3



$(AB) = (25)$

$(AB) = (123) \bigcap (456)$

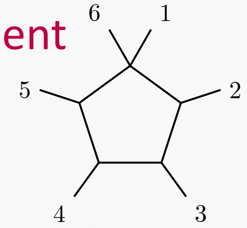
For reviews, see [1012.6032] [1204.3894]

Parametrization of Kinematics

"Gauge fixing": educated guess (trial and error)

In[16]:= $$\text{Zmat} = \begin{pmatrix} 0 & 1 & f1 & f2 \\ 1 & 0 & 0 & 0 \\ f5 & f6 & 1 & f7 \\ f8 & f9 & f10 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & f3 & f4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} //. \left\{f3 \to \frac{1 + f1\, f4}{f2},\ f8 \to 1,\ f6 \to 1 + f5\, f9,\ f10 \to 1,\ f9 \to 1\right\};$$

Kinematics "knows about" point Z_2 but only line (12), hence, Z_1 and $Z_1' = Z_1 + \alpha Z_2$ are equivalent



ruleKin =

$$\left\{m1 \to \frac{ab[5,6,1,2]}{ab[5,6,7,8] \times ab[1,2,7,8]},\ s12 \to \frac{ab[5,6,2,3]}{ab[5,6,7,8] \times ab[2,3,7,8]},\ s23 \to \frac{ab[1,2,3,4]}{ab[1,2,7,8] \times ab[3,4,7,8]},\right.$$
$$\left. s34 \to \frac{ab[2,3,4,5]}{ab[2,3,7,8] \times ab[4,5,7,8]},\ s45 \to \frac{ab[3,4,5,6]}{ab[3,4,7,8] \times ab[5,6,7,8]},\ s51 \to \frac{ab[4,5,1,2]}{ab[1,2,7,8] \times ab[4,5,7,8]}\right\};$$

$$\text{ruleF2M} = \text{First@Solve}\left[\left(\frac{\{s12, s23, s34, s45, s51\}}{m1} \text{ /. ruleKin /. } ab[sth\_\_] :\to \text{Det@Zmat}[[\{sth\}]]\right) == \frac{\{s12, s23, s34, s45, s51\}}{m1}, \text{Variables[Zmat]}\right] \text{ /.}$$
$$\sqrt{(s12^2\, s23^2 + 2\, m1\, s12\, s23\, s34 - 2\, s12\, s23^2\, s34 + m1^2\, s34^2 - 2\, m1\, s23\, s34^2 + s23^2\, s34^2 - 4\, m1\, s23\, s34\, s45 + 2\, s12\, s23\, s34\, s45 - 2\, m1\, s34^2\, s45 - 2\, s23\, s34^2\, s45 + s34^2\, s45^2 - 2\, s12^2\, s23\, s51 - 2\, m1\, s12\, s34\, s51 + 2\, s12\, s23\, s34\, s51 + 2\, s12\, s23\, s45\, s51 + 2\, m1\, s34\, s45\, s51 + 2\, s12\, s34\, s45\, s51 + 2\, s23\, s34\, s45\, s51 - 2\, s34\, s45^2\, s51 + s12^2\, s51^2 - 2\, s12\, s45\, s51^2 + s45^2\, s51^2)} \to +tr5 \text{ // Simplify}$$

Solve the parameters with Mandelstams

Out[18]= $$\left\{f1 \to -\frac{s23^2\, s34 - s23\, s34\, s45 + s23\, s34\, s51 + s23\, s45\, s51 + s34\, s45\, s51 - s45\, s51^2 - m1\, s34\, (s23 + s51) + s12\, (-s23^2 + s51^2) + s23\, tr5 + s51\, tr5}{2\, s23\, s34\, s45},\right.$$

$$f2 \to \frac{2\, s23\, s34\, s45\, s51 + m1^2\, s34\, (s23 + s51) - m1\, (s23^2\, s34 + s12\, (-s23^2 + s51^2) + s51\, (s34\, s45 - s45\, s51 + tr5) + s23\, (-s34\, s45 + s34\, s51 + s45\, s51 + tr5))}{2\, m1\, s23\, s34\, s45},$$

Chirality: choose either $\sqrt{\cdots} = +\text{tr}_5$ or $\sqrt{\cdots} = -\text{tr}_5$

$$f4 \to \frac{1}{2\, m1\, s23\, s34\, (s23 + s34 - s51)} (-m1^2\, s34\, (2\, s23 + s34 - s51) + s23\, s34\, (s23\, s34 - s34\, s45 + s45\, s51 + s12\, (-s23 + s51) - tr5) +$$
$$m1\, (s23\, s34\, (-2\, s34 + s51) + s12\, (2\, s23^2 + s23\, (3\, s34 - s51) + (s34 - s51)\, s51) + (s34 - s51)\, (s34\, s45 - s45\, s51 + tr5))),$$

$$f5 \to \frac{-m1\, s34 - s23\, s34 - 2\, s34^2 - s34\, s45 + s12\, (s23 + 2\, s34 - s51) + 2\, s34\, s51 + s45\, s51 + tr5}{2\, s34\, (m1 - s12 + s34 - s51)},$$

$$\left. f7 \to \frac{2\, m1^2 - s23\, s34 - s34\, s45 + m1\, (-2\, s12 + 3\, s34 - 2\, s51) + s12\, (s23 - s51) + s45\, s51 + tr5}{2\, m1\, (m1 - s12 + s34 - s51)}\right\}$$

3-dim Projective Geometry

No notion of “parallel”

- Lines and planes always intersect $(ab) \cap (cde) = Z_a\langle bcde\rangle - Z_b\langle acde\rangle$
- Planes and planes always intersect $(abc) \cap (def) = (ab)\langle cdef\rangle + (bc)\langle adef\rangle + (ca)\langle bdef\rangle$

$\frac{m(m-3)}{2}$ cross-ratios for m points on a line

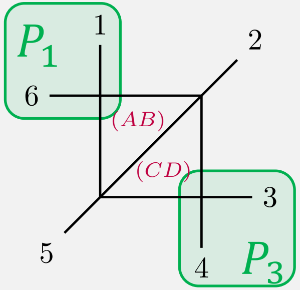
- Write $\{Z_1 \sim Z_m\}$ as linear combinations of any two points $\{P, Q\}$ on the line.
- Compute cross-ratios using minors of $\mathbf{C}$, or using any reference line I_{ref}.

$$(Z_1 \quad \cdots \quad Z_m)_{4\times m} = (P \quad Q)_{4\times 2}\mathbf{C}_{2\times m}$$

$$u_{ij} = \frac{(i,j-1)(i-1,j)}{(i,j)(i-1,j-1)} = \frac{\langle ij-1I_{\text{ref}}\rangle\langle i-1jI_{\text{ref}}\rangle}{\langle ijI_{\text{ref}}\rangle\langle i-1j-1I_{\text{ref}}\rangle}$$

2-loop Schubert

Maximal cut (with Jacobian) (7+1) equations, 8 unknowns



$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha_1 & \beta_1 \\ 0 & 1 & \gamma_1 & \delta_1 \end{pmatrix} \begin{pmatrix} Z_1 \\ P \\ Z_2 \\ Q \end{pmatrix}, \qquad \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 0 & \alpha_2 & \beta_2 \\ 0 & 1 & \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} Z_2 \\ P \\ Z_3 \\ Q \end{pmatrix}$$

1. Solve cutting conditions for any 7 unknowns

$$\langle AB12\rangle = \langle AB56\rangle = \langle ABPQ\rangle = \langle ABCD\rangle = \langle CD23\rangle = \langle CD45\rangle = \langle CDPQ\rangle = 0$$

$$\Longrightarrow \begin{cases} \beta_1 = \gamma_1 = \beta_2 = \gamma_2 = 0,\ \alpha_1 = \alpha_1^{(1)}(\delta_2),\ \delta_1 = \delta_1^{(1)}(\delta_2),\ \alpha_2 = \alpha_2^{(1)}(\delta_2) \\ \beta_1 = \gamma_1 = \beta_2 = \gamma_2 = 0,\ \alpha_1 = \alpha_1^{(2)}(\delta_2),\ \delta_1 = \delta_1^{(2)}(\delta_2),\ \alpha_2 = \alpha_2^{(2)}(\delta_2) \end{cases}$$

2. Solve the Jacobian for the remaining unknown

$$\left.\frac{\partial(\text{cutting eqns.})}{\partial(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2)}\right|_{\text{cutting sol.}^{(1)}} = 0 \implies \delta_2^{(1,1)} = \cdots \text{ or } \delta_2^{(1,2)} = \cdots$$

$$\left.\frac{\partial(\text{cutting eqns.})}{\partial(\alpha_1,\beta_1,\gamma_1,\delta_1,\alpha_2,\beta_2,\gamma_2)}\right|_{\text{cutting sol.}^{(2)}} = 0 \implies \delta_2^{(2,1)} = \cdots \text{ or } \delta_2^{(2,2)} = \cdots$$

Generically,
4 Schubert solutions.

In this example, it happens that $\delta_1 = \delta_2$.

2 intersections on I_∞, corresponding to $\delta_{1,2} = \frac{\cdots \pm \Delta_{nc}}{\cdots}$.