Developments of auxiliary mass flow method

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Based on: 1711.09572, 2107.01864, 2201.11636, 2201.11637, 2201.11669

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第二届微扰量子场论研讨会

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Outline

I. Introduction

II. Auxiliary mass flow(AMF) method

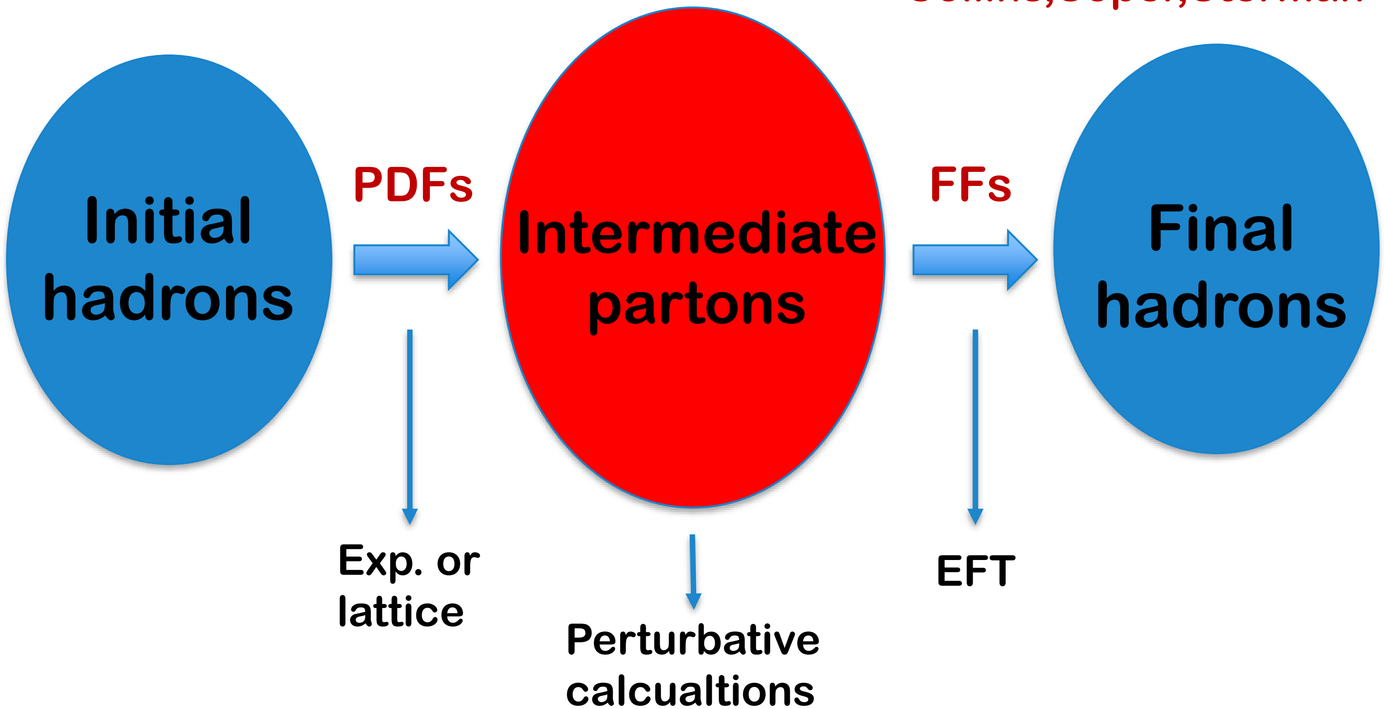
III. Iterative AMF

IV. More developments

V. Summary and outlook

Hadrons scattering process

Collins,Soper,Sterman 1989



$$\sigma(\text{hadrons})=\text{PDFs}\otimes\sigma(\text{partons})\otimes\text{FFs}$$

- σ (hadrons) measured by experiment
- σ (partons) calculated by perturbative theory
- Requirement: uncertainty of perturbative calculations<=uncertainty of experiments
- Test standard model or find new physics

Perturbative calculation process

- Generate scattering amplitude expressed by Feynman integrals(FIs) Relatively easier

Feynman diagram method

- Reduce the FIs to master integrals (MIs) by integration-by-part(IBP) identities Much harder

$$F\left(\mathcal{O}(10^4)\right) \xrightarrow{\text{IBP}} \sum_{i=1}^{\mathcal{O}(10^2)} c_i \times I_i$$

Laporta's algorithm
[Laporta, hep-ph/0102033]

Syzygy equations
[Gluza, et al, 1009.0472]

Block trianglular
[Guan,Liu,Ma, 1912.09294]

- Calculate master integrals Hard

Sector decomposition
[Binoth,Heinrich, hep-ph/0004013]

Canonical form
[Henn, 1304.1806]

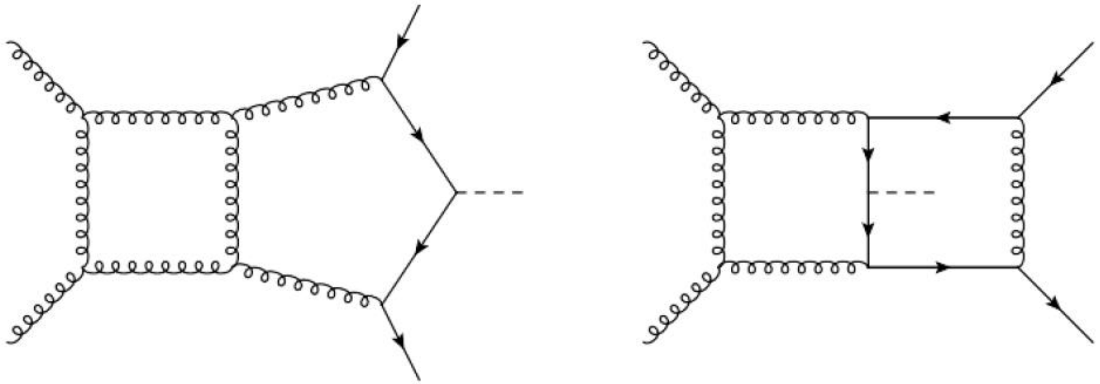
Auxiliary mass flow
[Liu,Ma,Wang, 1711.09572]

- One-loop order: solved problem
- Higher-loop order?

['t Hooft,Veltman 1979]
[Oldenborgh,Vermaseren 1990]

Challenges

➢ Difficulties of master integrals calculation



Two-loop $gg \to t\bar{t}H$

- (Semi-)analytical: “basis” of special functions not fully know
- Numerical: unsatisfactory efficiency

Auxiliary mass flow method

➢ Overview

- Original auxiliary mass flow method [Liu,Ma,Wang, 1711.09572]
- * auxiliary mass expansion [Liu,Ma, 1801.10523]
- * block triangular system [Guan,Liu,Ma, 1912.09294] see report of Xin Guan
- * extension to phase space integrations [Liu,Ma,Tao,Zhang 2009.07987]
- Iterative auxiliary mass flow [Liu,Ma, 2107.01864]
- Automatic computation of boundary conditions [ZFL,Ma, 2201.11637]
- Extension to linear propagators [ZFL,Ma, 2201.11636]
- Mathematica package AMFlow [Liu,Ma, 2201.11669]

Original auxiliary mass flow

➢ Introduce an auxiliary mass to all propagators

$$I_{\text{mod}}(\eta) = \int \left(\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}\eta)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}\eta)^{\nu_K}}$$

Liu,Ma,Wang, 1711.09572

- **Physical integral**

$$I_{\text{phy}} = \lim_{\eta \to 0^+} I_{\text{mod}}(\eta)$$

- **The limit is taken through following three steps:**

(1) Set up η −DEs of modified integral

(2) Determination of boundary conditions(BCs) at $\eta = \infty$

(3) Evolution of η from ∞ to 0^+

Boundary conditions

➢ Systematic and simple

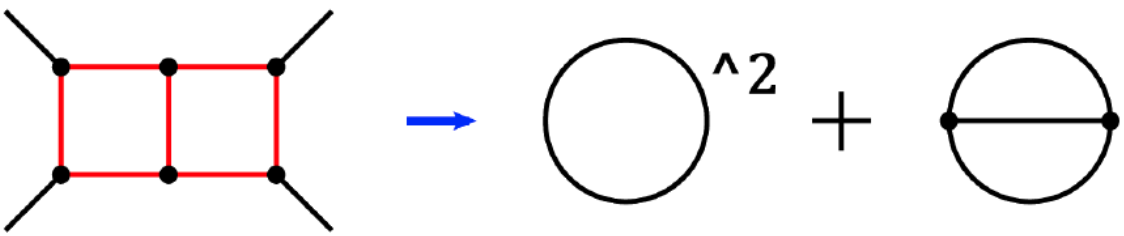
- **Method of region** [Beneke,Smirnov hep-ph/9711391]

$$\ell_i^\mu \sim \sqrt{\eta} \quad \Rightarrow \quad \frac{1}{((\ell+p)^2 - m^2 + \mathrm{i}\eta)^\nu} = \frac{1}{(\ell^2 + \mathrm{i}\eta)^\nu} \sum_{i=0}^{\infty} \frac{(\nu)_i}{i!} \left(-\frac{2\ell\cdot p + p^2 - m^2}{\ell^2 + \mathrm{i}\eta} \right)^i$$

- **Feynman parametric representation**

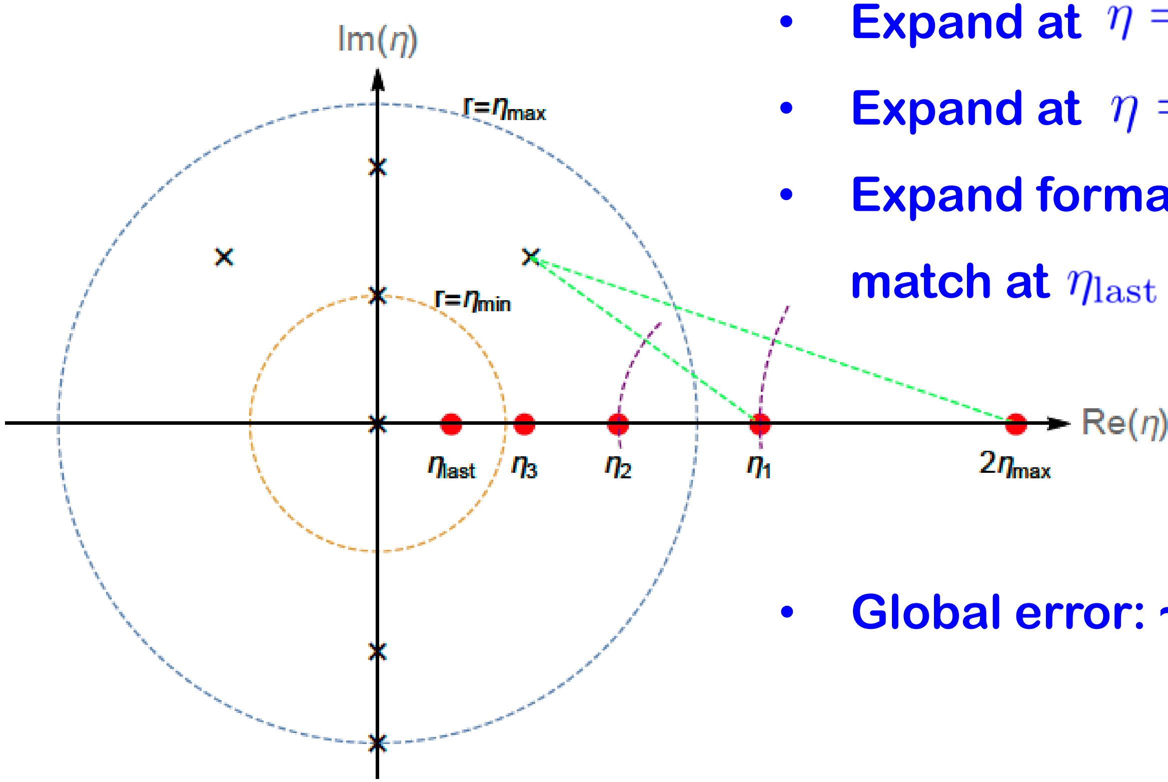
$$I_{\mathrm{mod}}(\vec{\nu};\eta) \propto \int \mathfrak{D}\vec{x} \frac{\mathcal{U}^{-D/2}}{(\mathcal{F}/\mathcal{U} - \mathrm{i}\eta)^{N_\nu - LD/2}} \sim (-\mathrm{i}\eta)^{LD/2-N_\nu} \int \mathfrak{D}\vec{x}\, \mathcal{U}^{-D/2}$$

- **Equal-mass vacuum integrals at the same loop**



The flow of auxiliary mass

- Solve differential equations through power series expansion



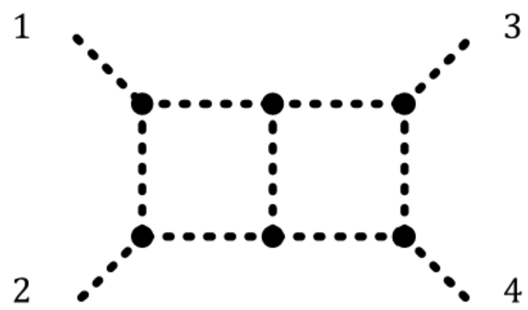
- Expand at $\eta=\infty$ to estimate $I(2\eta_{\max})$
- Expand at $\eta=2\eta_{\max}$ to estimate $I(\eta_1)$
- Expand at $\eta=\eta_i$ to estimate $I(\eta_{i+1})$
- Expand formally at $\eta=0$ and match at η_{last}

- Global error: ~ $\left(\frac{1}{2}\right)^n$

Liu,Ma,Wang, 1711.09572

An example

➢ Two-loop massless double-box



- $s = (p_1 + p_2)^2 = 10,\ t = (p_1 + p_3)^2 = -3$
- **Number of MIs:** $8 \to 39$
- **Result of corner integral at top sector:**

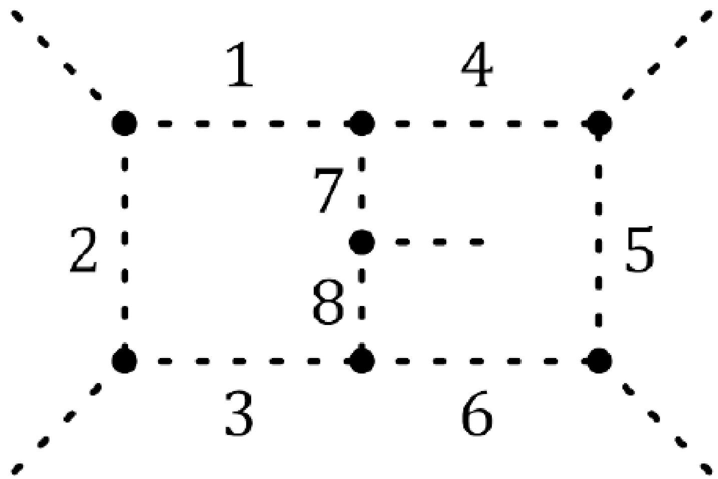
$$-\frac{1.067500000000000}{\mathrm{eps}^4} + \frac{1.611523974528965 - 0.235619449019234\, i}{\mathrm{eps}^3} - \frac{4.626024805160620 - 1.343010053160560\, i}{\mathrm{eps}^2} + \frac{10.41474820545243 - 8.216856662286993\, i}{\mathrm{eps}} + (7.80650469131764 + 22.84149768939458\, i) + O(\mathrm{eps}^1)$$

For more complicated problems, the growth of number of MIs may greatly affect the efficiency. What's the solution?

Improved auxiliary mass flow

- **A simple observation**

Liu,Ma, 2107.01864



108 master integrals

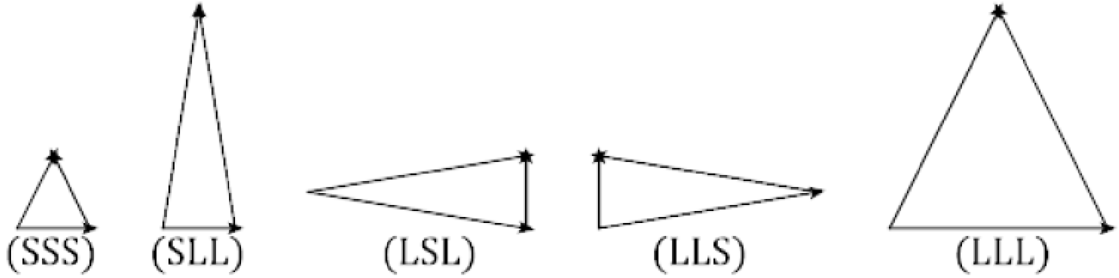
Mode	Propagators	#MIs
all	{1,2,3,4,5,6,7,8}	476
loop	{4,5,6,7,8}	305
	{1,2,3,4,5,6}	319
branch	{4,5,6}	233
	{7,8}	234
propagator	{4}	178
	{5}	176
	{7}	220

Integration regions at the boundary

➢ Region analysis [Beneke,Smirnov hep-ph/9711391]

- principles: loop momentum of each branch can be either of $O(1)$ or $O(\sqrt{\eta})$
- regions for one-loop:

- regions for two-loop:



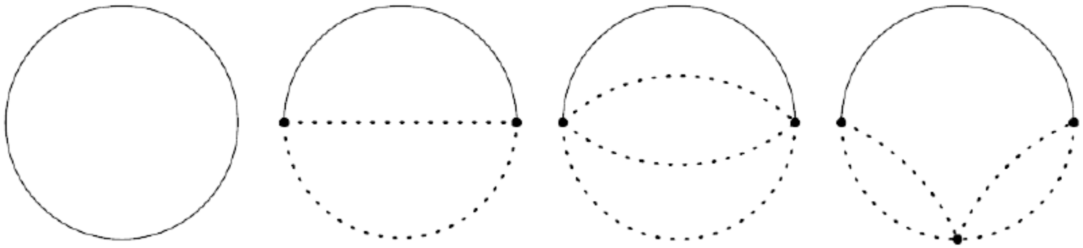
  - (LSS), (SLS), (SSL) excluded by momentum conservation
- $R_1 = 2, R_2 = 5, R_3 = 15, R_4 = 47, \ldots$

Contributions of each region

➢ Expansion in each region

- (L … L): $$\frac{1}{(\ell+p)^2 - m^2 + \kappa \times \mathrm{i}\eta} \sim \frac{1}{\ell^2 + \kappa \times \mathrm{i}\eta}$$ ⟹ vacuum
- mixed: $$\frac{1}{(\ell_{\mathrm{L}}+\ell_{\mathrm{S}}+p)^2 - m^2 + \kappa \times \mathrm{i}\eta} \sim \frac{1}{\ell_{\mathrm{L}}^2 + \kappa \times \mathrm{i}\eta}$$ ⟹ factorized
- (S … S): $$\frac{1}{(\ell+p)^2 - m^2 + \mathrm{i}\eta} \sim \frac{1}{\mathrm{i}\eta}$$ ⟹ sub-family

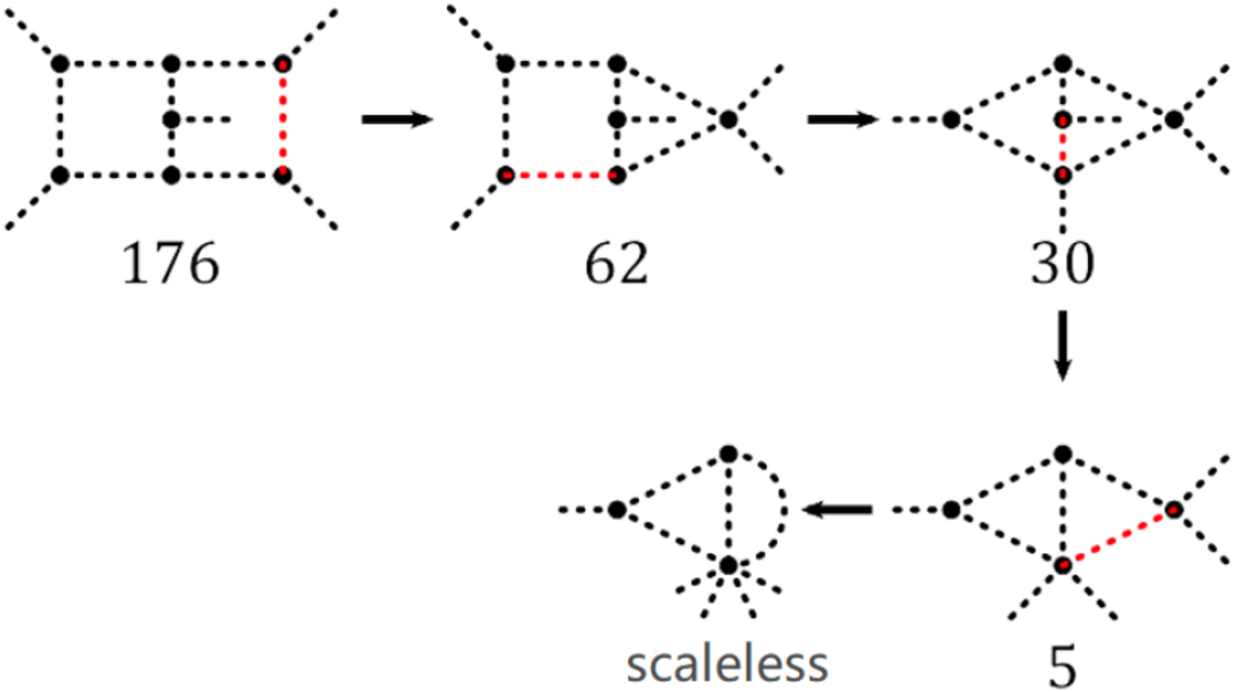
Integrals can be solved by iteration!



Example

➢ All-small region iteration

Liu,Ma, 2107.01864

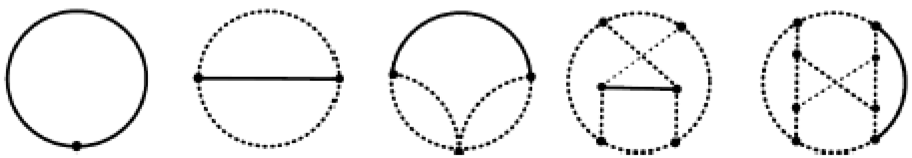


- number of propagators decreases
- end at single-scale integrals or scaleless integrals

Merits and demerits

➢ Advantages

- Systematic and efficient for any loop FIs(given BCs)
- Simple BCs input: single mass vacuum Feynman integrals
- Arbitrary accuracy

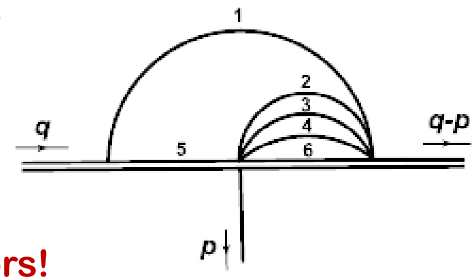


Some single mass vacuum FIs up to 5 loops
(known analytic expressions up to 3 loops)

➢ Deficiencies

- No arbitrary accuracy at more than three loops
- Can't handle FIs containing linear propagators (often appear in many physical problems)

Region analysis is not easy when there are linear propagators!

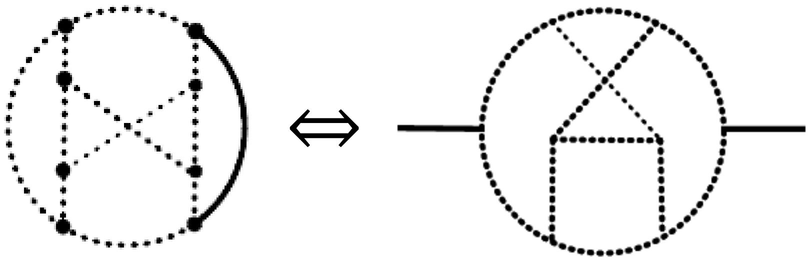


Arbitrary accuracy for any loops

ZFL,Ma, 2201.11637

- Loop-drop technique

A L-loop single mass vacuum FI is equivalent to
a (L-1)-loop massless propagator integral(p-integral).



- Use AMF and Loop-drop by turns

(L-1)-loop p-integrals

$\Downarrow$ AMF

(L-1)-loop single mass tadpoles

$\Downarrow$ Loop-drop

(L-2)-loop p-integrals

$\Downarrow$ (AMF+Loop-drop)$^{L-2}$

0-loop p-integrals(=1)

$\Rightarrow$ Loop integrations totally bypassed!

New viewpoint

ZFL,Ma, 2201.11637

- Feynman integral is Linear algebra!

$$\text{FIs} \triangleq \text{Linear algebra} \oplus \text{Vacuum integrals}$$

(2017-2021)

$$\Downarrow$$

$$\text{FIs} \triangleq \text{Linear algebra}$$

(2022)

- Only input is integration-by-part(IBP) identities(to set up η $-$DEs)
- No other input
- No kinematics
- No loops

For linear propagators

- **Change linear to quadratic propagators**

ZFL, Ma, 2201.11636

$$\frac{1}{p_a \cdot \ell_a + \Delta_a + \mathrm{i}0^+} \to \frac{1}{x\,(\ell_a^2 + \mathrm{i}0^+) + p_a \cdot \ell_a + \Delta_a}$$

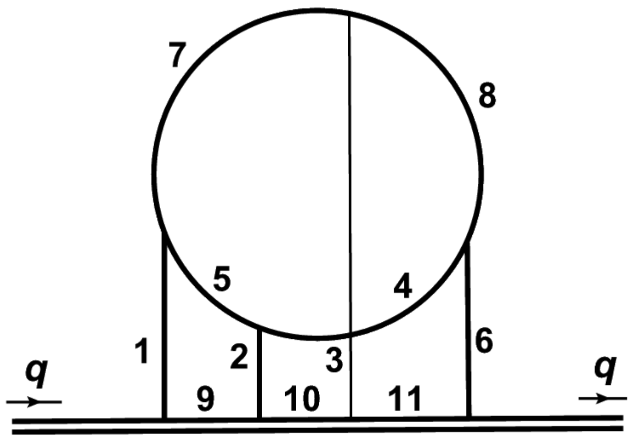
- Quadratic integrals at finite x_0 calculated by AMF for any loops
- Original linear integrals obtained by taking $x \to 0^+$:
  1) Set up $x-\mathrm{DEs}$; 2) BCs at x_0 by AMF; 3) Solve $x-\mathrm{DEs}$
- A new method to calculate linear integrals systematically and efficiently
- Useful in effective field theory and region expansion

Application to four loops

ZFL, Ma, 2201.11636

- Results of all four-loop MIs of vacuum integrals containing a gauge link for the first time

65MIs



The most complex diagram

$$
\begin{aligned}
I_{(1,\cdots,1,0,0,0)} &= 2.46740110027234 0\epsilon^{-4} \\
&- (2.107259492497 74 - 31.006276680299 82i)\epsilon^{-3} \\
&- (142.406163111538 4 + 26.480603763353 1i)\epsilon^{-2} \\
&+ (258.910450629448 6 - 157.423637906974 2i)\epsilon^{-1} \\
&- (1638.22642672354 6 - 1859.68156682683 1i) \\
&- (4750.00666988440 7 + 11690.8960012112 23i)\epsilon \\
&+ (22176.4642151077 4 + 23525.1565537379 0i)\epsilon^{2} \\
&- (120064.474485779 1 + 165534.045050465 0i)\epsilon^{3} \\
&+ (855055.431012103 5 + 539390.203559220 9i)\epsilon^{4} \\
&- (3796471.43849419 7 + 1678278.56790650 8i)\epsilon^{5}.
\end{aligned}
$$

- Higher precision and higher orders in ϵ achieved easily
- Useful for studying parton distribution functions

The package: AMFlow

https://gitlab.com/multiloop-pku/amflow

➢ Contents

Liu,Ma, 2201.11669

- Main package: AMFlow.m
  - Provides the main functions to realize auxiliary mass flow: AMFSetupSystem, SolveIntegrals, SolveIntegralsGaugeLink,…
- Differential equation solver: diffeq_solver/DESolver.m
  - Provides functions to solve differential equations numerically: CalcInf, AMFlow, …
- Interface to reducer: ibp_interface
  - FiniteFlow+LiteRed, Fire+LiteRed, Kira
- Pedagogical demonstration: examples
  - automatic_loop, automatic_phasespace, linear_propagator,…

Usage

➢ Installation

- Download a reducer and give its path to ibp_interface
  - Install Kira-2.2 or later: https://kira.hepforge.org/
  - Reset variables defined in ibp_interface/Kira/install.m
    - $KiraExecutable, $FermatExecutable

➢ Usage

- examples/box
  - SetReductionOptions["IBPReducer" -> "Kira"]
  - SetAMFOptions[“AMFMode" -> …]
  - AMFlowInfo[…]
  - SolveIntegrals[integrals_, precision_, epsorder_]

Summary and outlook

➢ Summary

- Auxiliary mass flow method is a systematic and efficient method for numerical computation of Feynman integrals
- Cutting edge problems are now reachable in this framework

➢ Outlook

- Further phenomenology applications: $t\bar{t}j, t\bar{t}H, \cdots$
- Deeper understanding of FIs calculation from Linear algebra
- A wide range of applications of AMF for all purposes(any loops, any dimension, linear propagators, …)

Thank you!

Definition of FIs

➢ A family of Feynman integrals

$$I_{\vec{\nu}}(D,\vec{s}) = \int \prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{\mathrm{i}\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + \mathrm{i}0^+)^{\nu_1} \cdots (\mathcal{D}_K + \mathrm{i}0^+)^{\nu_K}}$$

$$\mathcal{D}_\alpha = A_{\alpha ij} \ell_i \cdot \ell_j + B_{\alpha ij} \ell_i \cdot p_j + C_\alpha$$

- $\ell_1, \ldots, \ell_L$: loop momenta; $p_1, \ldots, p_E$: external momenta;
- A, B: integers; C: linear combination of $\vec{s}$ (including masses)
- $\mathcal{D}_1, \ldots, \mathcal{D}_K$: inverse propagators; $\nu_1, \ldots, \nu_K$: integers
- $\mathcal{D}_{K+1}, \ldots, \mathcal{D}_N$: irreducible scalar products; $\nu_{K+1}, \ldots, \nu_N$: non-positive integers

➢ Difficulties of calculating FIs

- Analytical: known special functions are insufficient to express FIs
- Numerical: UV or IR divergence at $D \to 4$

Calculate p-integrals

Liu, YQM, 2201.11637

- **Apply AMF method on $(L-1)$-loop p-integral**

  1) IBP to setup η-DEs

  2) Single-mass vacuum integrals no more than $(L-1)$ loops as input

  Single-mass vacuum integrals with L loops are determined by that with no more than $(L-1)$ loops (besides IBP)

  - Boundary: 0-loop p-integrals equal 1

- **Only IBPs are needed to determine FIs!**

  - IBPs: linear algebra, exact (in $D, \vec{s}$) relations between FIs
  - Loop integrations are completely avoided!

New workflow

- The 'FICalc' to calculate FIs can be defined as (any given nonsingular D and $\vec{s}$):

LZF,YQM,2201.11637

① If it is a 0-loop p-integral, return 1;

② If it is a single-mass vacuum integral, express it by a p-integral, and call 'FICalc' to calculate the p-integral;

③ Otherwise:

a) Introduce η to one propagator (if the mass mode is not possible)

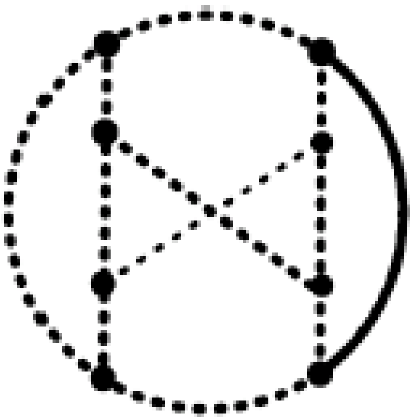
b) Setup η-DEs using IBP as input

c) Call 'FICalc' to calculate boundary FIs at $\eta \to \infty$

d) Numerically solve η-DEs with given BCs to obtain $\eta \to i0^-$

A five-loop example

LZF,YQM,2201.11637



$$= -2.07385551028674 0\epsilon^{-2} - 7.812755312590133\epsilon^{-1} - 17.25882864945875 + 717.6808845492140\epsilon + 8190.876448160049\epsilon^{2} + 78840.29598046500\epsilon^{3} + 566649.1116484678\epsilon^{4} + 3901713.802716081\epsilon^{5} + 23702384.71086095\epsilon^{6} + 142142936.8205112\epsilon^{7},$$

- IBP relations are the only input!
- Terms up to $\mathcal{O}(\epsilon^4)$ agree with literature; Others are new ($D = 4 - 2\epsilon$)

  Lee, Smirnov, Smirnov, 1108.0732
- An arbitrary dimension D = 4/7, no other known method can do it

-9.793112097048649321808795980069111646428182547465428330614694726443 1

516031830610056668242341877309401032293901004574319494017206091158244

708224654193885680661950372372090211196168499966402592016363 21*10^7

with about 130 significant digits

2203.08271

TABLE IX: Values of $\alpha_S(m_Z^2)$ determined at N^{3}LO accuracy from $\Gamma_Z^{\rm tot}$, R_Z, and $\sigma_Z^{\rm had}$ individually, combined, as well as from a global SM fit, with propagated experimental, parametric, and theoretical uncertainties broken down. The last two rows list the expected values at the FCC-ee from all Z pseudoobservables combined and from the corresponding SM fit.

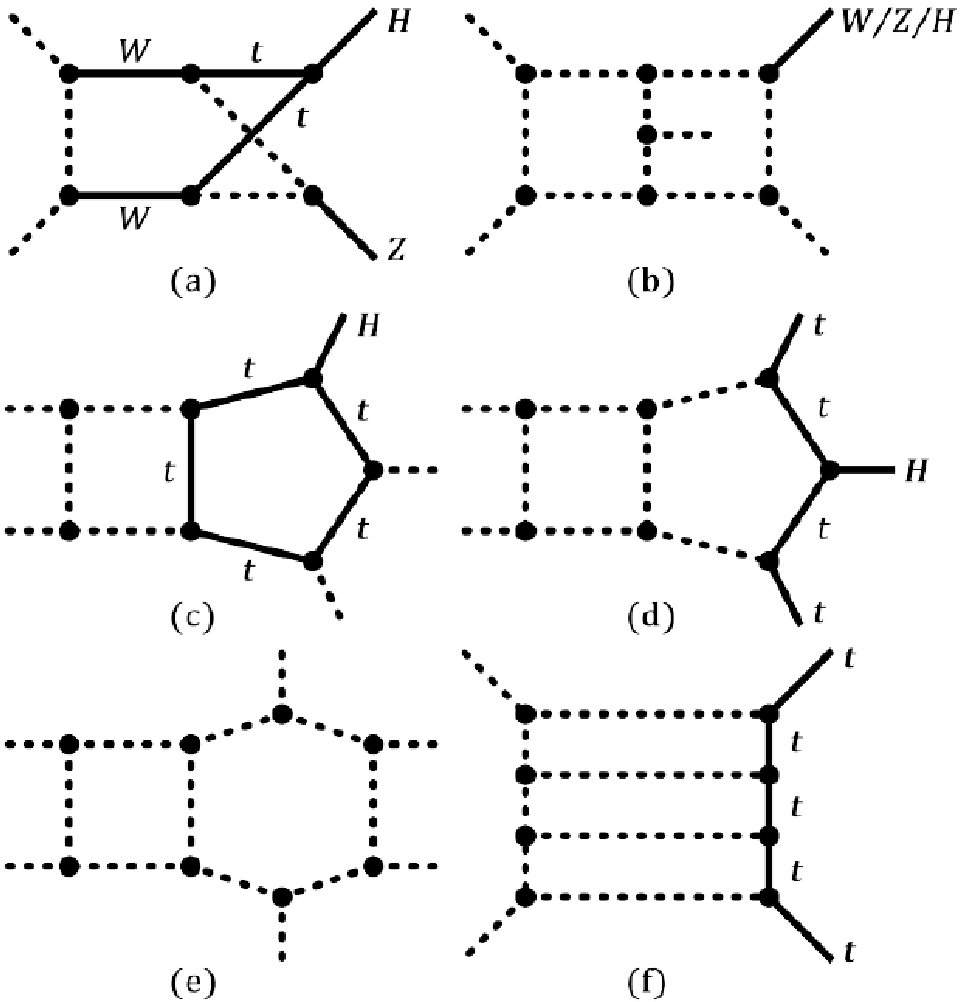
Observable	$\alpha_S(m_Z^2)$	uncertainties		
		exp.	param.	theor.
$\Gamma_Z^{\rm tot}$	0.1192 ± 0.0047	± 0.0046	± 0.0005	± 0.0008
R_Z	0.1207 ± 0.0041	± 0.0041	± 0.0001	± 0.0009
$\sigma_Z^{\rm had}$	0.1206 ± 0.0068	± 0.0067	± 0.0004	± 0.0012
All above combined	0.1203 ± 0.0029	± 0.0029	± 0.0002	± 0.0008
Global SM fit	0.1202 ± 0.0028	± 0.0028	± 0.0002	± 0.0008
All combined (FCC-ee)	0.12030 ± 0.00026	± 0.00013	± 0.00005	± 0.00022
Global SM fit (FCC-ee)	0.12020 ± 0.00026	± 0.00013	± 0.00005	± 0.00022

TABLE XII: PDG average of the categories of observables [March'22 update of the PDG'21 results]. These are the final input to the world average of $\alpha_S(m_Z^2)$.

category	$\alpha_S(m_Z^2)$	relative $\alpha_S(m_Z^2)$ uncertainty
τ decays and low Q^2	0.1178 ± 0.0019	1.6%
$Q\overline{Q}$ bound states	0.1181 ± 0.0037	3.1%
PDF fits	0.1162 ± 0.0020	1.7%
e^+e^- jets & shapes	0.1171 ± 0.0031	2.6%
electroweak	0.1208 ± 0.0028	2.3%
hadron colliders	0.1165 ± 0.0028	2.4%
lattice	0.1182 ± 0.0008	0.7%
world average (without lattice)	0.1176 ± 0.0010	0.9%
world average (with lattice)	0.1179 ± 0.0009	0.8%

Cutting-edge problems

➢ Cutting-edge examples



- two-loop EW corrections to $H+Z$ production at e^+e^- colliders
- two-loop QCD corrections to $H/W/Z+2j,\, t\bar{t}H,\, 4j$ production at hadron colliders
- three-loop QCD correction to $t\bar{t}$ production at hadron colliders

Family	dp	(a)	(b)	(c)	(d)	(e)	(f)
T_{setup}	6	20	18	8	1	25	30
T_{solve}	7	11	15	6	3	15	42
P_1	95%	99%	96%	99%	98%	94%	93%

- time: CPU core hour
- weight-8 functions with 16-digit precision

References

[1] Zhi-Feng Liu and Yan-Qing Ma, "Automatic computation of Feynman integrals containing linear propagators via auxiliary mass flow", Phys.Rev.D 105 (2022) 7 (Editors' Suggestion), [arXiv:2201.11636 [hep-ph]].

[2] Zhi-Feng Liu and Yan-Qing Ma, "Feynman integrals are completely determined by linear algebra", [arXiv:2201.11637 [hep-ph]].

[3] Xiao Liu and Yan-Qing Ma, "AMFlow: a Mathematica package for Feynman integrals computation via Auxiliary Mass Flow", [arXiv:2201.11669 [hep-ph]].