微扰QCD简介

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前沿系列讲座

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• Why (Perturbative) QCD ?

- $\alpha \sim \alpha_W \sim 10^{-2} <<< \alpha_s(M_W) \sim 0.1$

  $\Rightarrow$ More QCD activities at a Collider.

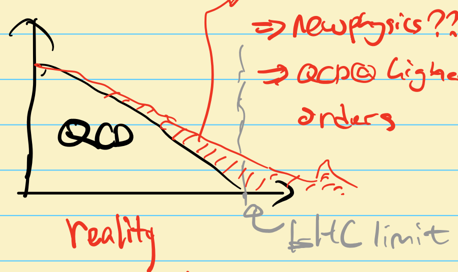
- dominant background to new physics Searches,

Saw a bump

$\Rightarrow$ new physics!!

$\Rightarrow$

ideal



excess ???

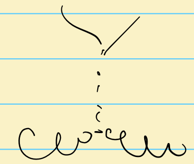
$\Rightarrow$ new physics???

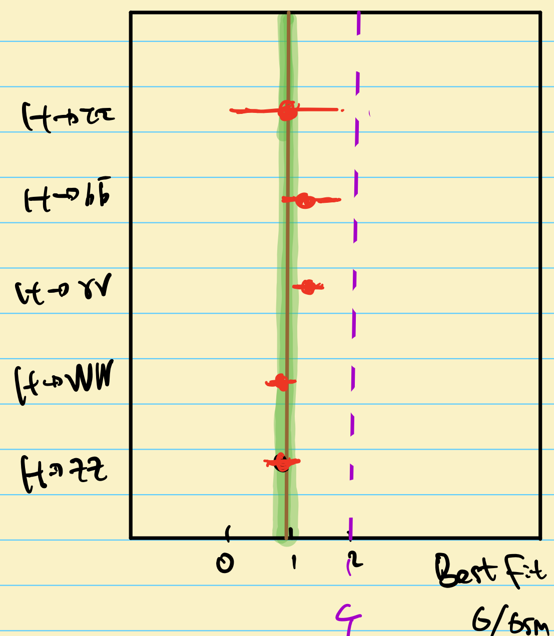
$\Rightarrow$ QCD @ higher orders

reality

New physics ~ Data − QCD prediction.

- Gigantic corrections.





if QCD corrections are not known!!

8TeV

LO: $9.6 \pm 25\%$ pb

N^3LO: $19.47 \pm^{0.32\%}_{2.91\%}$ pb

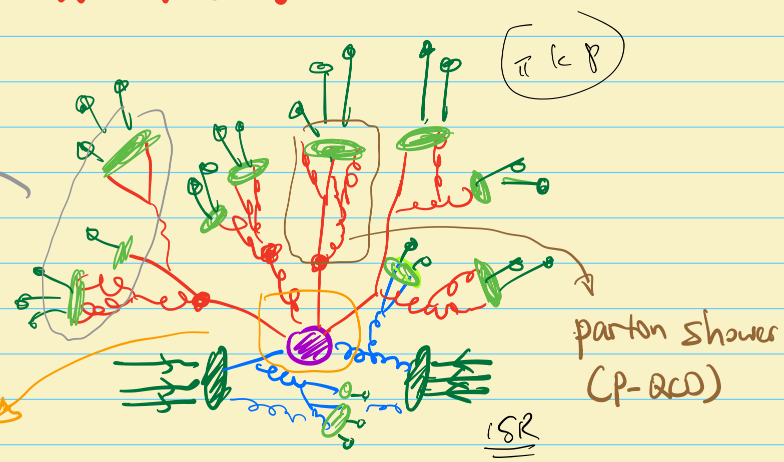
- Fundaments to the Monte-Carlo tools

e.g. Madgraph, pythia, Sherpa, ...

- ...

How QCD works ?

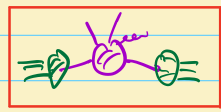
hadronization
by non-perturbative
QCD



hard Interaction
by perturbative QCD

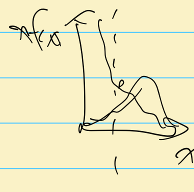
a typical event @ LHC

- Hard Interaction @ $\mathcal{O}(10 \sim 100)$ GeV $\gg \Lambda_{QCD}$
  - gluons + quarks

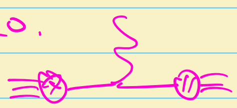


  - Fixed-order calculation

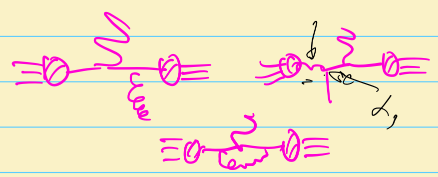
$$\frac{d\sigma}{d\Phi_F} \propto \int d\Phi_{UN} \frac{d\hat{\sigma}_{ij}}{d\Phi_F} f_i f_j \, \Theta(\Phi_F)$$



LO.



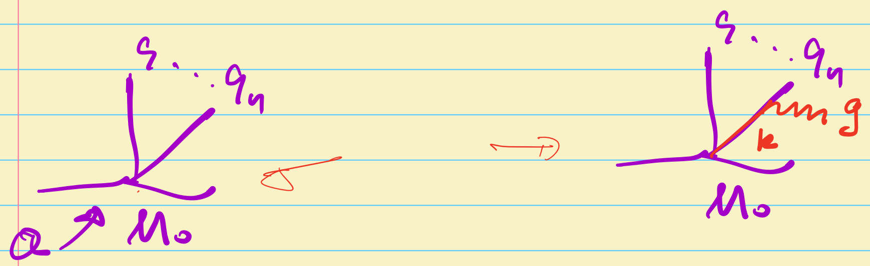
NLO



See 李剑's talk

  - good Approx. for Inclusive enough observables, limited multiplicity

Why Low multiplicity works?



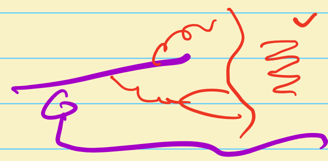
$$|M|^2\,\theta(q_1+\cdots+q_n-V) \simeq |M_0|^2\, k\,dk\, \frac{\alpha_s}{k^2}\,\theta(q_1 \cdots g - V)$$

$$\simeq |M_0|^2\, \alpha_s \ln v$$

if $v \sim Q$ inclusive enough α_s suppressed for high multiplicity.

$\ln v \sim 1$

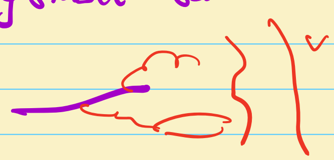
$\alpha_s(\frac{Q}{q})$



good representation

if $\ln V \sim \alpha_s^{-1}$ high multiplicity is important. $v \lll Q$

V very small sensitive to the soft & coll.

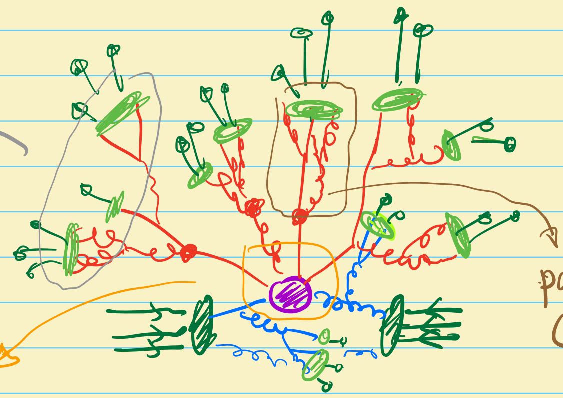


⊕ resolve the structure

$T^2|q\rangle = C_F|q\rangle$ $\quad$ $T_i|q\rangle \to t^a|q\rangle$ $\quad$ $T|g\rangle \to f_{abc}|g\rangle$

$T^2|g\rangle = C_A|g\rangle$

hadronization by non-perturbative QCD



soft+coll.

parton shower (P-QCD)

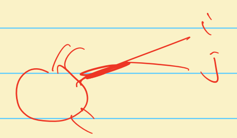
hard Interaction by perturbative QCD

a typical event @ LHC

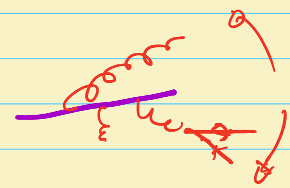
$|M|^2$

- Parton Shower / Resummation

- gluon + Quarks. still perturbative



$Q\theta \ll Q$



$Q\theta$

- Coll. + soft radiations

$$\lim_{|q|\to 0} |M|^2_{tree}(p_1\cdots p_N; q) \simeq -g_s^2 \sum_{ij} \vec{T}_i\cdot\vec{T}_j \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} |M|^2(p_1\cdots p_N)$$

$$\lim_{i\parallel j} |M|^2_{tree}(p_1\cdots p_i, p_j\cdots) \simeq \frac{2}{s_{ij}} g_s^2 P^{ss'}_{\tilde{ij}\to ij}(z) |M|^2_{ss'}(p_1\cdots \hat{p}_{\tilde{ij}},\cdots)$$

- Fixed order not reliable. need all order results

- Outline

- thrust as an example
  - NLO calculation
  - infrared safe
  - break-down of fixed order

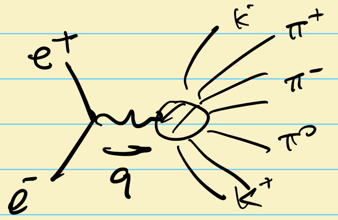
- Infrared behaviour in QCD / QED
  - Coherent branching
  - DL Resummation

    double log

- All particles are massless.
- dim. reg. $d = 4 - 2\epsilon$. to deal with UV & IR if any.

Thrust in e^+e^--annihilation

Consider



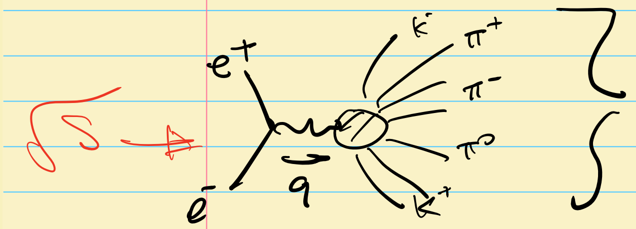
measurement

Incl. (1)

~~th~~ thrust (see later)

scalar for simplicity

$$\sigma = \frac{1}{2s} \int d\Phi_N \overline{|M|}^2 \, \Theta(\Phi_N)$$



Measurement

Incl. (1)

thrust (see later)

scalar for simplicity

$$\sigma = \underbrace{\frac{1}{4}}_{\text{spin ave.}}\underbrace{\frac{1}{2s}}_{\text{Flux}}\int \underbrace{d\Phi_N}_{\text{Phase space of N final particles.}} L_{\mu\nu}(l,\bar{l})\frac{1}{s^2}H^{\mu\nu}(q,\Phi_N)\,\Theta(\Phi_N)$$

$q^\mu H_{\mu\nu}=0$

$q^\mu L_{\mu\nu}=0$

gauge

scalar $\longrightarrow \frac{1}{4}\frac{1}{2s^3}L_{\mu\nu}(l,\bar{l})\int d\Phi_N\left(g_{\mu\nu}-\cancel{\frac{q^\mu q^\nu}{q^2}}\right)H(q,\Phi_N)\,\Theta(\Phi_N)$

$$= \frac{1}{4}\frac{1}{2s^3}L^\mu_\mu(l\bar{l})\int d\Phi_N\, H(q,\Phi_N)\,\Theta(\Phi_N)$$

where

$$H = \frac{1}{d-1}\,g_{\mu\nu}H^{\mu\nu}$$

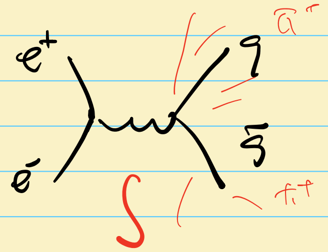
$S_{\mu\nu}H^{\mu\nu} = g_{\mu\nu}HA$

$d-1$

$$L^\mu_\mu = 4(2l\bar{l}-d\,l\bar{l}) = -2(d-2)s$$

For large $s = Q^2 \gg \Lambda_{QCD}$, it is well modeled by

perturbative calculation



$\left(\frac{1}{Q^2}\right)^2 \frac{1}{s^2}$

$L^{\mu\nu}$, $H^{\mu\nu\,(1)} = 4\left(q^\mu \bar{q}^\nu + q^\nu \bar{q}^\mu - g^{\mu\nu} q \cdot \bar{q}\right)$

$H^{(0)} = -\frac{1}{d-1} 2(d-2)s$

$L^\mu_\mu = -2(d-2)s$

$$= \sum_f Q_f^2 N_c e^4 \frac{1}{4} \frac{1}{2s^3} \frac{[4(1-\epsilon)s]^2}{d-1} \int d\Phi_2^{\circledast}\, \Theta(\Phi_2) = 1$$

Inclusive

First

$$\circledast\ \int d\Phi_2 = \int \frac{d^d q}{(2\pi)^{d-1}}\,\delta(q^2)\,\frac{d^d\bar{q}}{(2\pi)^{d-1}}\,\delta(\bar{q}^2)\,(2\pi)^d\,\delta^{(d)}(Q-q-\bar{q})$$

$$= (2\pi)\int \frac{d^d q}{(2\pi)^{d-1}}\,\delta(q^2)\,\delta\big((Q-q)^2\big) \quad \rightsquigarrow Q^2 - 2Q\cdot q = Q^2 - 2\ell\cdot q - 2\bar{\ell}\cdot q = s+u+t$$

$$= (2\pi)\int \frac{d^d q}{(2\pi)^{d-1}}\,\delta(q^2)\,\delta(s+u+t)$$

$$\boxed{\begin{aligned}\ell &= C(1,0,0,1)\\ \bar{\ell} &= C(1,0,0,-1)\end{aligned}}$$

Sudakov decomp.

$$\text{let } q^\mu = \frac{2q\cdot\bar{\ell}}{2\ell\cdot\bar{\ell}}\,\ell^\mu + \frac{2q\cdot\ell}{2\ell\cdot\bar{\ell}}\,\bar{\ell}^\mu + q_\perp^\mu$$

$$= -\frac{u}{s}\,\ell^\mu + \frac{-t}{s}\,\bar{\ell}^\mu + q_\perp^\mu$$

$$\text{Hence } q^2 = \frac{ut}{s} - \vec{q}_\perp^{\,2} = 0$$

$$\text{and } d^d q = \frac{1}{2}\,d\frac{(-u)}{s}\,d\frac{(-t)}{s}\,d^{d-2}q_\perp$$

$$= \frac{1}{4}\,d\frac{(-u)}{s}\,d\frac{(-t)}{s}\,(q_\perp^2)^{\frac{d-2}{2}}\,dq_\perp\,d\Omega_{d-2}$$

$$\text{let } \frac{-t}{s} = x,\quad \frac{-u}{s} = 1-x,\quad q_\perp^2 = x(1-x)s$$

$$\Rightarrow \int d\Phi_2 = \frac{(2\pi)}{(2\pi)^{d-1}}\,\frac{1}{4}\,d\Omega_{d-2}\,s^{-\epsilon}\int_0^1 dx\, x^{-\epsilon}(1-x)^{-\epsilon}$$

$$\sum_f Q_f^2 N_c e^4 \ \frac{1}{4}\frac{1}{2S^3}\frac{[4(1-\epsilon)S]^2}{d-1}\int d\Phi_2\ \Theta(\Phi_2)$$

$$= \sum_f Q_f^2 N_c e^4 \frac{1}{4}\frac{1}{2S}\frac{16(1-\epsilon)^2}{3-2\epsilon}(2\pi)\frac{1}{4}\frac{\Omega_{d-2}}{(2\pi)^{d-1}}S^{-\epsilon}\int_0^1 dx\, x^{-\epsilon}(1-x)^{-\epsilon}$$

where $x \equiv \frac{-t}{s}$, $\Omega_{d-2} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$ is the solid angle

$$\Rightarrow \sigma_0 = \sum_f Q_f^2 N_c \frac{4\pi\alpha^2}{S}\frac{1-\epsilon}{3-2\epsilon}\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}\left(\frac{4\pi\mu^2}{S}\right)^{\epsilon}$$

$$= \sum_f Q_f^2 N_c \frac{4\pi\alpha^2}{3S} + \mathcal{O}(\epsilon)$$

For NLO, we have both virtual and real corrections

Virtual:

$+$C.T.

scaleless $\frac{1}{\cancel{\epsilon_{UV}}} - \frac{1}{\epsilon_{IR}}$

renorm.

$$\Box = \mathcal{M}_0 \frac{\alpha_s}{4\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \underbrace{\left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon}\right.}_{\text{IR Poles}}\left.-2\right) \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} e^{\gamma_E \epsilon}$$

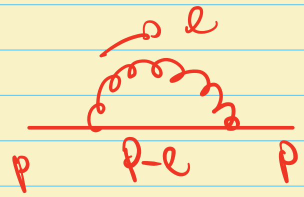
$\Rightarrow \sigma_{virt}$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} e^{\gamma_E \epsilon} \cos\pi\epsilon \left(-\frac{2}{\epsilon^2} + \frac{1}{\epsilon} - 2\right) \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)}$$

$$= \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2\right)$$

Counterterms Included

dim-reg.



$p^2 = 0$ massless

$$\propto \int d^d l \, \frac{1}{l^2 (p-l)^2} \sim (p^2)^{\frac{d-4}{2}} = 0$$

by dim analysis

all scaleless integrals vanishes in dim reg.

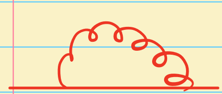
$d = 4 - 2\epsilon$

$f(p^2)$

$$\boxed{\int \frac{d^d l}{l^4}}$$

$$= \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}}$$

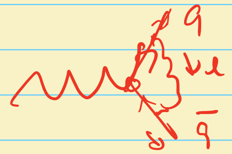
UV will be cancelled by the counter term



$$= -\frac{\alpha_s}{2\pi} \left(\frac{1}{4\epsilon} - \frac{1}{4\epsilon} \right)$$

UV IR

cancelled by Counter term



$$[d\ell] \equiv \frac{d^d\ell}{(2\pi)^d}$$

$$= \int d\ell \;\; \bar{u}_i \, i g_s \gamma^\alpha t^a_{ik} \; i\frac{\not q+\not \ell}{(q+\ell)^2} \; ie\gamma^\mu \; (-i)\frac{\not{\bar q}-\not \ell}{(\bar q-\ell)^2} \; ig_s t^a_{kj}\gamma_\alpha v_j \; \frac{-i}{\ell^2}$$

$$= +ieg_s^2 (t^a t^a)_{ij} \int [d\ell] \boxed{\bar{u}_i \gamma^\alpha (\not \ell+\not q)\gamma^\mu (\not \ell-\not{\bar q})\gamma_\alpha v_j} \boxed{\frac{1}{(q+\ell)^2}\frac{1}{(\ell-\bar q)^2}\frac{1}{\ell^2}}$$

$$\int \ell \to dL$$

Now using the Feyn. Param. we find

$$\boxed{\frac{1}{(\ell+q)^2}\frac{1}{(\ell-\bar q)^2}\frac{1}{\ell^2}} = \Gamma(3)\int d\alpha_1 d\alpha_2 d\alpha_3 \frac{\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(L^2-\alpha_2\alpha_3 s)^3}$$

$$\alpha_1\ell^2 + \alpha_2\ell^2 + 2\alpha_2 \ell\cdot q + \alpha_3\ell^2 - 2\alpha_3 \ell\cdot\bar q$$

$$= \ell^2 + 2(\alpha_2 q - \alpha_3\bar q)\cdot\ell + (\alpha_2 q-\alpha_3\bar q)^2 + \alpha_2\alpha_3 s$$

$$= \underbrace{(\ell+\alpha_2 q-\alpha_3\bar q)}_{\equiv L}{}^2 + \alpha_2\alpha_3 s$$

We simplify the numerator

$$\boxed{} = \bar{u}_i \gamma^\alpha (\not{t} + \bar{\alpha}_2 \not{q} + \alpha_3 \not{\bar{q}}) \gamma^\mu (\not{t} - \alpha_2 \not{q} - \bar{\alpha}_3 \not{\bar{q}}) \gamma_\alpha v_j$$

$$= -2 \bar{u}_i (\not{t} - \cancel{\alpha_2 \not{q}} - \bar{\alpha}_3 \not{\bar{q}}) \gamma^\mu (\not{t} + \bar{\alpha}_2 \not{q} + \cancel{\alpha_3 \not{\bar{q}}}) v_j$$

$$+2\epsilon \bar{u}_i (\not{t} + \cancel{\bar{\alpha}_2 \not{q}} + \alpha_3 \not{\bar{q}}) \gamma^\mu (\not{t} - \alpha_2 \not{q} - \cancel{\bar{\alpha}_3 \not{\bar{q}}}) v_j$$

$\propto L^{2n} \rightarrow$ odd power integrated to zero

$$-2 \bar{u}_i \not{t} \gamma^\mu \not{t} - \bar{\alpha}_3 \bar{\alpha}_2 \not{\bar{q}} \gamma^\mu \not{q} v_j$$

$$+2\epsilon \bar{u}_i \not{t} \gamma^\mu \not{t} - \alpha_2 \alpha_3 \not{\bar{q}} \gamma^\mu \not{q} v_j$$

$$= -2(1-\epsilon) \bar{u}_i \not{t} \gamma^\mu \not{t} v_j$$

$$+ (2\bar{\alpha}_3 \bar{\alpha}_2 - 2\epsilon \alpha_3 \alpha_2) \bar{u}_i \not{\bar{q}} \gamma^\mu \not{q} v_j$$

$L^\mu L^\nu \rightarrow \frac{1}{d} L^2 g^{\mu\nu}$

$$\longrightarrow \boxed{\frac{(d-2)^2}{d} L^2 \bar{u}_i \gamma^\mu v_j + 2(\bar{\alpha}_2 \bar{\alpha}_3 - \epsilon \alpha_2 \alpha_3)(-s) \bar{u}_i \gamma^\mu v_j}$$

$\frac{d\ell^4}{\ell^6}$

$d/2 = 2-\epsilon$

Now we use

$$\int [d\ell] \frac{L^2}{(L^2 + \alpha_2 \alpha_3 s)^3} = \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-\epsilon} (-s)^{-\epsilon} \quad ①$$

UV

$$\int [d\ell] \frac{1}{(\ell^2 + \alpha_2 \alpha_3 s)^3} = \frac{-i}{(4\pi)^{d/2}} \frac{\Gamma(1+\epsilon)}{\Gamma(3)} (\alpha_2 \alpha_3)^{-1-\epsilon} (-s)^{-1-\epsilon} \quad ②$$

$\Rightarrow$

① $\Rightarrow$

$$g_s^2 \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(\epsilon)}{\cancel{\Gamma(3)}} (-s)^{-\epsilon} \frac{(d-2)^2}{d} \int_0^1 d\alpha_2 \int_0^{\bar{\alpha}_2} d\alpha_3 \, (\alpha_2\alpha_3)^{-\epsilon} \, \cancel{\Gamma(3)}$$

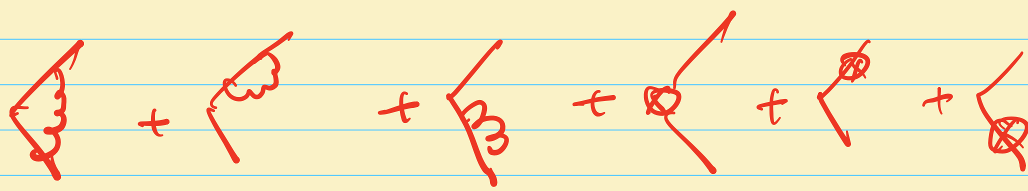
$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{+\epsilon} \left[\frac{1}{2\epsilon} + \frac{1}{2}\right]$$

UV cancelled by counter term

② $\Rightarrow$

$$\frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{+\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{2}{\epsilon} - \frac{9}{2} + \frac{\pi^2}{12}\right]$$

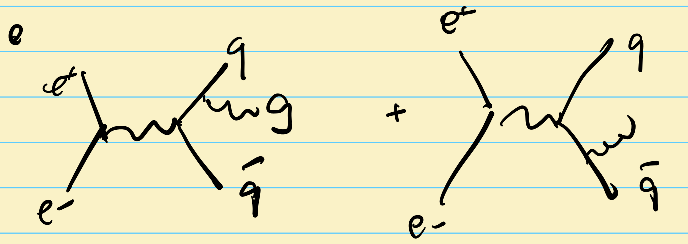
coll IR



$$= \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{-s}\right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{3}{2}\frac{1}{\epsilon} - 4 + \frac{\pi^2}{12}\right]$$

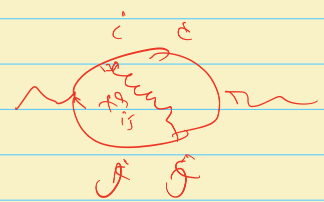
$i\pi$

$$H \sim \frac{g_{\mu\nu} H^{\mu\nu}}{(d-1)}$$



NLO real

$$g^{\mu\nu} H^{(1)}_{\mu\nu} = \Big| \bar{u}_q \not\epsilon_a \, i g_s t^a \, i\frac{\not q + \not g}{2 q\cdot g} \gamma^\mu v_{\bar q} + \bar{u}_q \gamma^\mu (-i) \frac{\not{\bar q} + \not g}{2\bar q \cdot g} \, i g_s t^b \not\epsilon_b v_{\bar q} \Big|^2 g_{\mu\nu}$$



$$= g_s^2 \, \mathrm{Tr}(t^a t^a)$$

$$\Big\{ \left(\frac{1}{2q\cdot g}\right)^2 \mathrm{Tr}\left[\not q \gamma^\alpha (\not q + \not g) \gamma^\mu \not{\bar q} \gamma_\mu (\not q + \not g) \gamma_\alpha\right]$$

$$+ \left(\frac{1}{2\bar q g}\right)^2 \mathrm{Tr}\left[\not q \gamma^\mu (\not{\bar q} + \not g) \gamma_\alpha \not{\bar q} \gamma^\alpha (\not{\bar q} + \not g) \gamma_\mu\right]$$

$$- 2 \frac{1}{2qg} \frac{1}{2\bar q g} \mathrm{Tr}\left[\not q \gamma^\alpha (\not q + \not g) \gamma^\mu \not{\bar q} \gamma_\alpha (\not{\bar q} + \not g) \gamma_\mu\right] \Big\}$$

$$\sigma \propto \int \underbrace{d\Phi_3}_{\epsilon} H \; \underbrace{\Theta(\Phi_n)}_{\to 1}$$

$$H^{(1)} = - C_F N_c \, g_s^2 \, \delta(1-\epsilon) \frac{1}{d-1}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2 + y_3 - \epsilon y_3}{y_2} + \frac{-2 + y_2 - \epsilon y_2}{y_3} - 2\epsilon \right\}$$

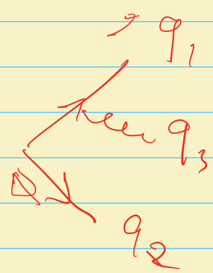
Where

$$y_1 = \frac{s_{12}}{Q^2}, \quad y_2 = \frac{s_{13}}{Q^2}, \quad y_3 = \frac{s_{23}}{Q^2}$$

and $s_{ij} = 2 q_i \cdot q_j$

$$d\Phi_3 = \frac{1}{(2\pi)^{2d-3}} \frac{1}{2^{d+1}} s^{d-3} \, d\Omega_{D-1} \, d\Omega_{D-2}$$

$$(y_1 y_2 y_3)^{-\epsilon} \int_0^1 dy_1 \, dy_2 \, dy_3 \, \delta(1 - y_1 - y_2 - y_3)$$



$+\overline{MS}$

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$$\times\, \Theta(\Phi_3)$$

$$\frac{1}{\epsilon} \quad \cancel{+\ln 4\pi} \quad +\gamma_E$$

$$\cancel{\gamma_E + \ln 4\pi}$$

Inclusive: $\Theta(\Phi_3) = 1$

let $y_1 = (1-z_1)$, $y_2 = z_1 z_2$, $y_3 = (1-z_2) z_1$

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

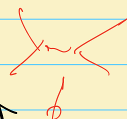
$\to$ Jacobian

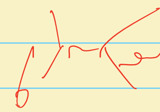
$$\times \int_0^1 dz_1\, dz_2 \quad \bar{z}_1^{-\epsilon}\, z_1^{-2\epsilon}\, z_2^{-\epsilon}\, \bar{z}_2^{-\epsilon} \quad z_1$$

$$\times \left\{ \frac{2}{z_1^2 z_2 \bar{z}_2} + \frac{-2+z_1\bar{z}_2-\epsilon z_1 \bar{z}_2}{z_1 z_2} + \frac{-2+z_1 z_2-\epsilon z_1 z_2}{z_1 \bar{z}_2} - 2\epsilon \right\}$$

$$\Rightarrow \sigma_{real} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{7}{6}\pi^2\right)$$

$$\sigma_{virt.} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2\right)$$





Inclusive X-sec

$\Theta = 1$

$$\sigma = \sigma_0 \left\{ 1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2} \right\}$$

- all IR poles cancelled, KLN Theorem
- Virt & Real almost completely cancelled with each other : $V + R \sim 1 \leftarrow$ Unity
- Origin of the IR poles :

  all IR poles from the soft & coll. limit

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dz_1 dz_2 \quad \bar{z}_1^{-\epsilon} \, z_1^{-2\epsilon} \, z_2^{-\epsilon} \, \bar{z}_2^{-\epsilon} \quad z_1$$

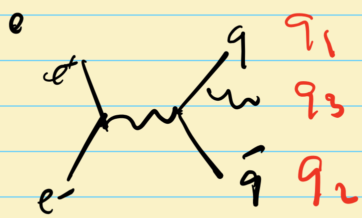
$$\times \left\{ \frac{2}{z_1^2 z_2 \bar{z}_2} + \frac{-2 + z_1\bar{z}_2 - \epsilon z_1 \bar{z}_2}{z_1 z_2} + \frac{-2 + z_1 z_2 - \epsilon z_1 z_2}{z_1 \bar{z}_2} - 2\epsilon \right\}$$

$\bar{z}_2 \equiv 1 - z_2$

$$y_1 = \frac{s_{12}}{Q^2}, \quad y_2 = \frac{s_{13}}{Q^2}, \quad y_3 = \frac{s_{23}}{Q^2}$$

and $s_{ij} = 2 q_i \cdot q_j$

let $y_1 = (1 - z_1)$, $y_2 = z_1 z_2$, $y_3 = (1 - z_2) z_1$

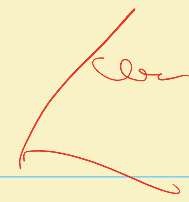


$z_1 \to 0$

$z_2 \to 0$

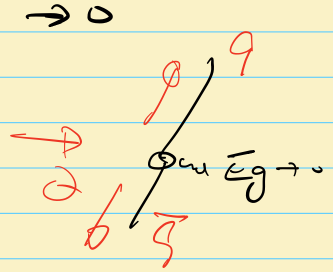
$1 - z_2 \to 0$

$\rightarrow \frac{1}{\epsilon^2}, \frac{1}{\epsilon}$



1 $z_1 \to 0. \Rightarrow y_2 \,\&\, y_3 \to 0$

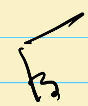
$\Rightarrow p_3 \to 0$ soft



2. z_2 or $\bar{z}_2 \to 0 \Rightarrow y_2$ or $y_3 \to 0$

$\Rightarrow p_3 \cdot p_1 \to 0,\ p_3 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_1$

or $p_3 \cdot p_2 \to 0,\ p_2 \cdot p_1 \neq 0 \Rightarrow p_3 \parallel p_2$



$z_1 \to 0$

$$\int_0^1 dz_1 dz_2\, z_1^{-2\epsilon} z_2^{-\epsilon} \bar{z}_2^{-\epsilon} \times \left\{ \frac{2}{\underbrace{z_1}_{\text{soft}} \underbrace{z_2 \bar{z}_2}_{\text{collinear}}} \right\} = \underbrace{\frac{2}{\epsilon^2}}_{\text{soft+coll.}} + \cdots$$

$z_2 \to 0$

$$\int_0^1 dz_1 dz_2\, \bar{z}_1^{-\epsilon} z_1^{-2\epsilon} z_2^{-\epsilon}$$

$$\times \left\{ \cancel{\frac{2}{z_1 z_2 \bar{z}_2}} + \frac{-2 + z_1 - \epsilon z_1}{z_2} \right\} = \frac{3}{2}\frac{1}{\epsilon}$$

subtract out to avoid double counting

does reproduce all poles

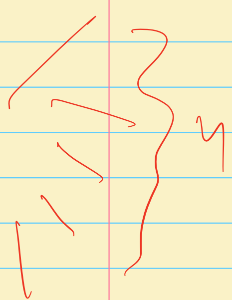
every order you will have

$$\Delta^{(n)}_{\text{real}} \propto \left(\frac{\alpha_s}{2\pi}\right)^{n} \left(\frac{\#_{2n}}{\epsilon^{2n}} + \cdots + \frac{\#_1}{\epsilon} + \text{Finite terms} \right)$$

n-emission

n soft + coll.

cancel against the IR poles in the virtual. For inclusive processes

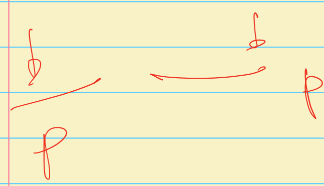


for exclusive processes.

e.g. $pp \to \ldots$

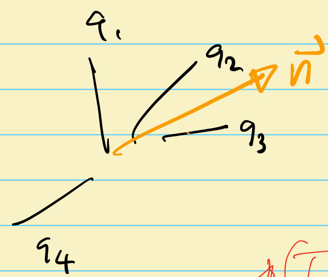
$e^+e^- \to \text{hadron} + \ldots$

remaining poles to be absorbed into PDFs or FFs.

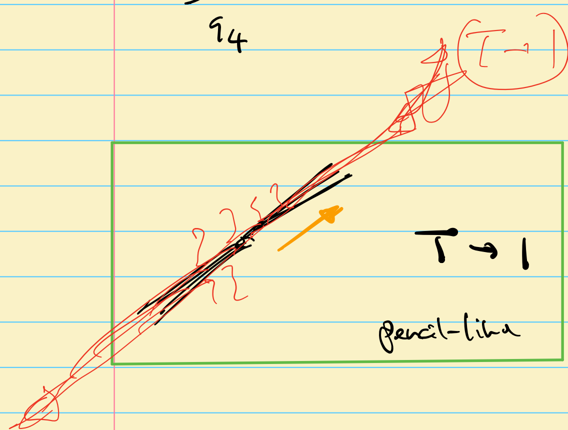


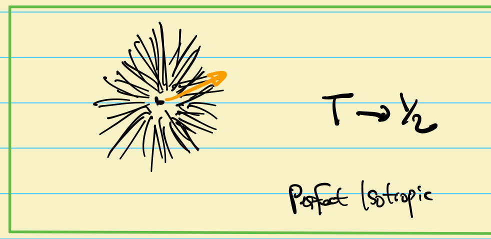
event-shape

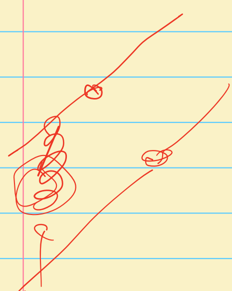
thrust: $T = \max_{\vec n} \dfrac{\sum_i |\vec q_i \cdot \vec n|}{\sum_i |\vec q_i|}$ $\qquad |\vec q_1 \cdot \vec n| + |\vec q_2 \cdot \vec n| + (\vec q_3 \cdot \vec q)$

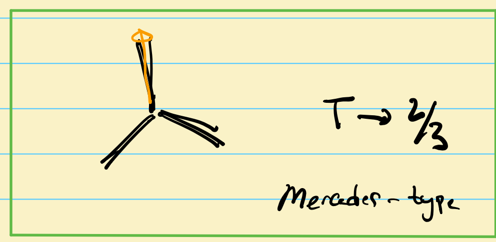


massless case $\sum_i |\vec q_i| = Q$



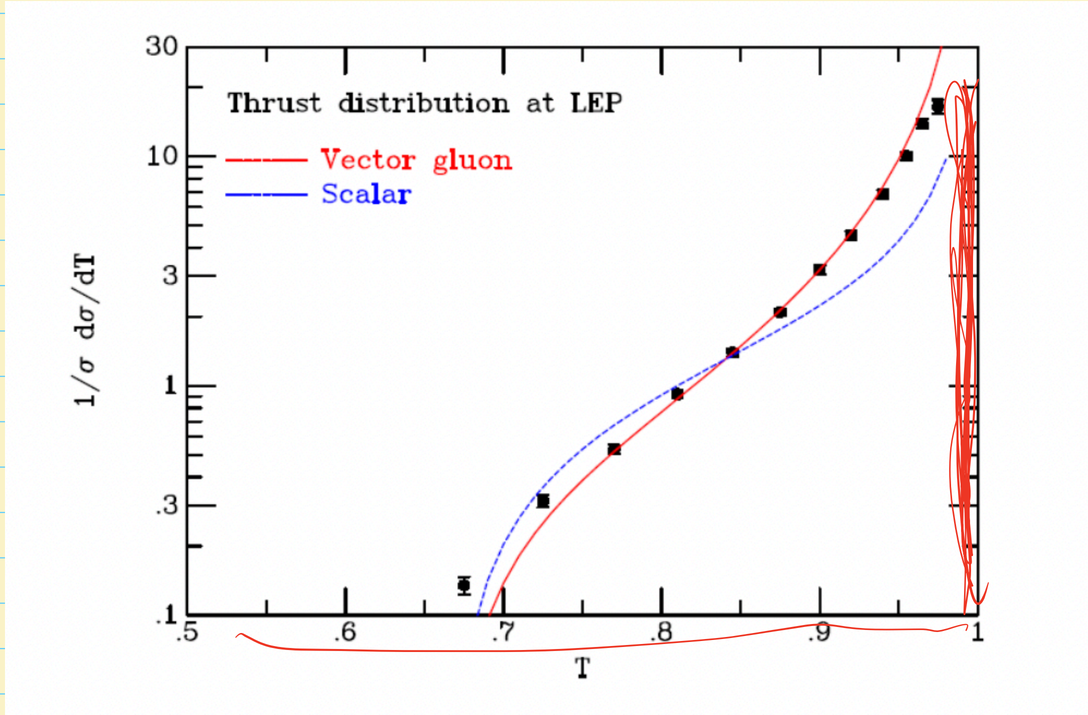






measure the shape of an event

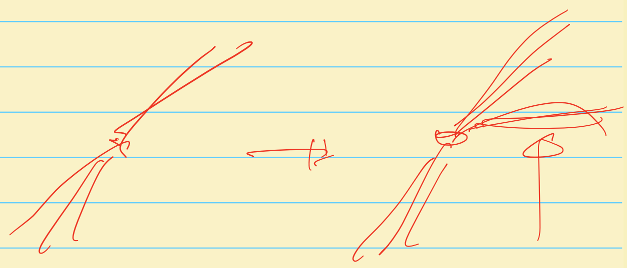
as $T \to 1$, only soft or coll. radiations are allowed

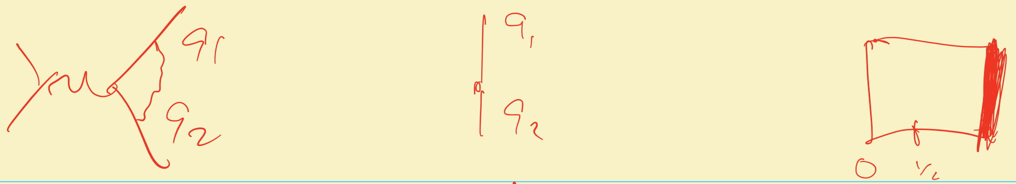


Used for discovery of gluon

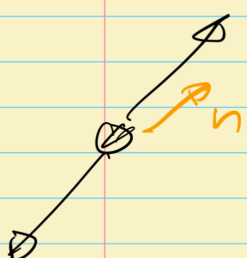
+ gluon spin determination

+ precision determination of α_s





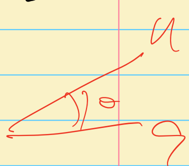
restrictions on the phase space:



$$\Theta(\Phi_2) = \delta\left(T - \max_{\vec n} \frac{|\vec q_1\cdot\vec n| + |\vec q_2\cdot\vec n|}{Q}\right)$$

$$= \delta\left(T - \max_{\vec n} \frac{|\vec q_1\cdot\vec n| + |-\vec q_1\cdot\vec n|}{Q}\right)$$

$$= \delta\left(T - \max_{\vec n} \frac{Q/2\cos\theta + Q/2\cos\theta}{Q}\right)$$



$$= \delta\left(T - \max_{\vec n} \cos\theta\right)$$

$$= \delta(1-T) \quad \text{always.}$$

No restrictions

does not contribute to $T \neq 1$ case, which starts from 3-body final state

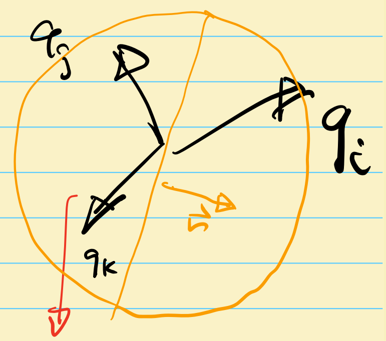
$$\sigma^{(0)} + \sigma^{(1)}_{\text{virt}} \to \left(\sigma^{(0)} + \sigma^{(1)}_{\text{virt}}\right)\delta(1-T)$$

Peer

$$\Theta(E_3) = \delta\left(T - \max_{\vec n} \frac{|\vec q_1\cdot\vec n| + |\vec q_2\cdot\vec n| + |\vec q_3\cdot\vec n|}{Q}\right)$$

too complicated for analytic calculation

- for an arbitary given $\vec n$, we can separate the phase space to 2 hemispheres, with one covers only one particle q_i



- the max of $\vec n$ for such config. is given by

$$\max \frac{|\vec q_i\cdot\vec n|}{Q} + \frac{|\vec q_j\cdot\vec n|}{Q} + \frac{|\vec q_k\cdot\vec n|}{Q}$$

$$= \max \frac{|\vec q_i\cdot\vec n|}{Q} + \left|\frac{\vec q_j\cdot\vec n}{Q} + \frac{\vec q_k\cdot\vec n}{Q}\right|$$

$$= \max \frac{|\vec q_i\cdot\vec n|}{Q} + \frac{|-\vec q_i\cdot\vec n|}{Q} = \max \frac{2|\vec q_i\cdot\vec n|}{Q} = \frac{2E_i}{Q} = x_i$$

$$= \frac{2q_i\cdot Q}{Q^2} = 1 - y_i$$

- we need to maximize all possible configs

$\Rightarrow T = \max(x_1, x_2, x_3) = \max(1-y_1, 1-y_2, 1-y_3)$

$\Rightarrow 1 - T = \min(y_1, y_2, y_3) \equiv \tau$

$\delta(\tau - \min(y_1, y_2, y_3))$ $\quad V_{ER} = 1 + \alpha_s$

Which leads to the phase space restriction

$$\Theta(\Phi_3) = \sum_i \delta(\tau - y_i)\,\theta(y_i < y_j)\,\theta(y_i < y_k)$$
$$= \sum_i \delta(\tau - y_i)\,\theta(\tau < y_k < 1 - 2\tau)\,\theta(\tau < 1/3)$$

Plug into the real-emission cross section

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$
$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1 - y_1 - y_2 - y_3)$$
$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2 + y_3 - \epsilon y_3}{y_2} + \frac{-2 + y_2 - \epsilon y_2}{y_3} - 2\epsilon \right\}$$
$$\Theta(\Phi_3)$$

You can work out the integration with ε kept using mathematica

$\delta(1-T) = \delta(\tau)$ 2G

Lo + Virt.

For simplicity, we set $\epsilon = 0$ to find

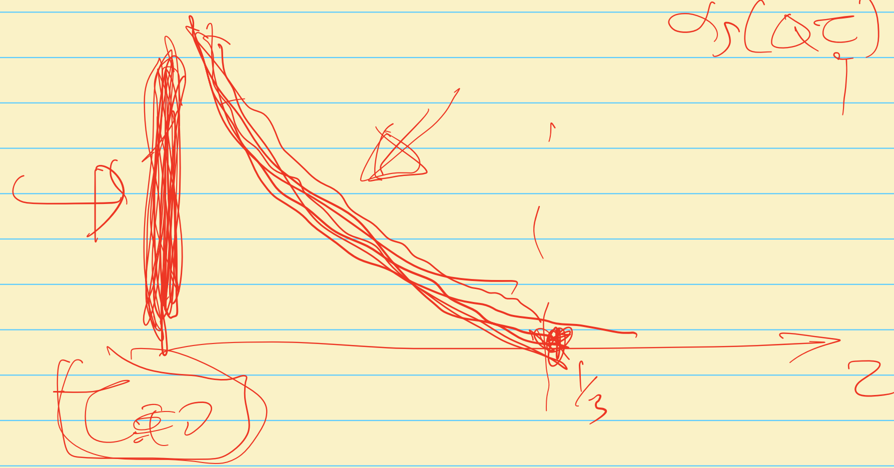
$$\frac{d\sigma_{real}}{d\tau} = \frac{\alpha_s}{2\pi} C_F \left\{ \frac{4}{\tau} \frac{\log\left(\frac{1-2\tau}{\tau}\right)}{1-\tau} - \frac{3(1+\tau)(1-3\tau)}{\tau} - 6\log\left(\frac{1-2\tau}{\tau}\right) \right\}$$

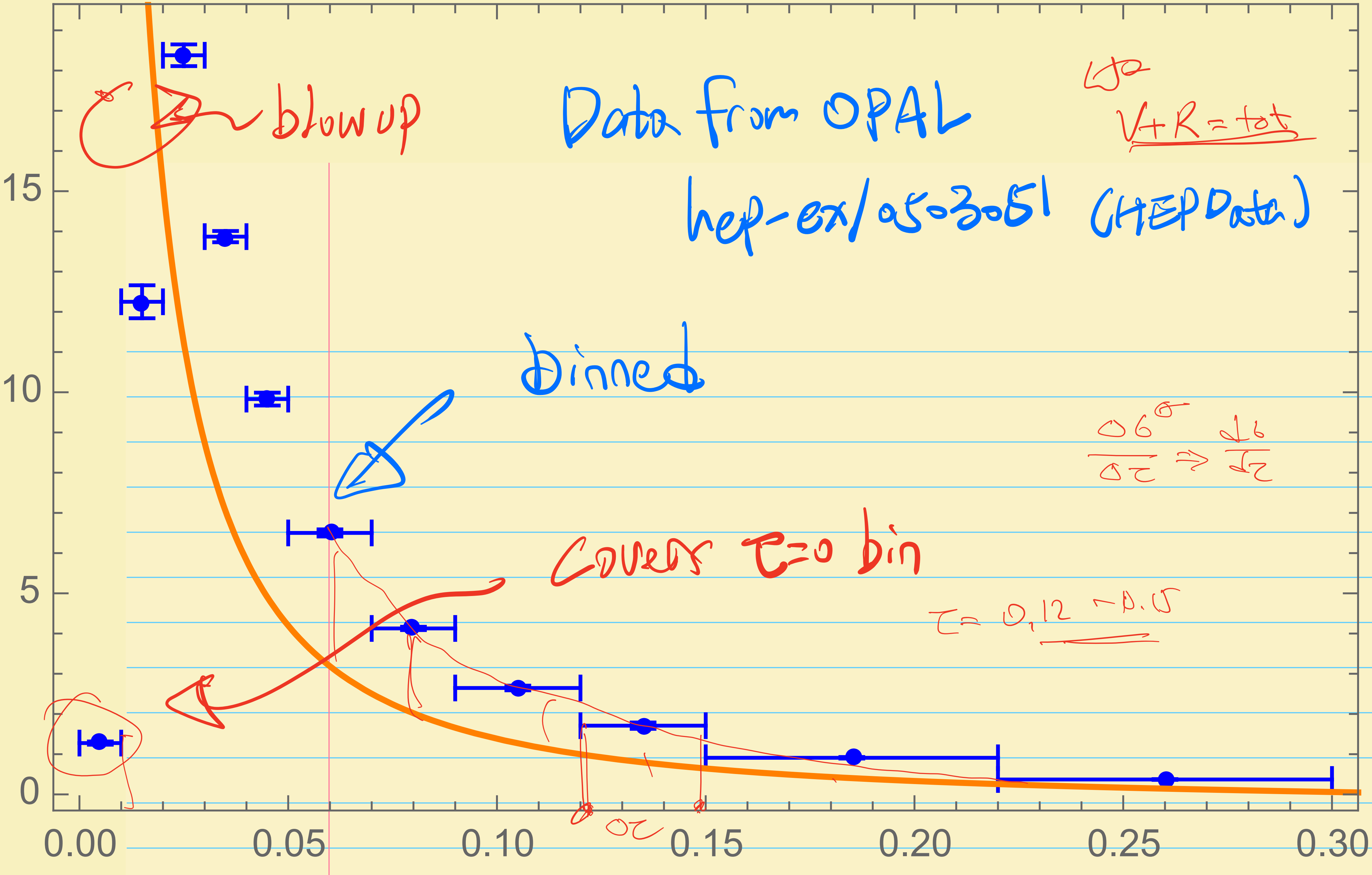
For $0 < \tau < 1/3$, otherwise vanishes.

$\alpha_s(Q\tau)$

$\frac{4}{\tau}\log\left(\frac{1}{\tau}\right)$

$-\frac{3}{2}$





15
10
5
0
0.00
0.05
0.10
0.15
0.20
0.25
0.30
blowup
Data from OPAL
hep-ex/0503051 (HEPData)
w/o
V+R=tot
Binned
Covers τ=0 bin
τ= 0.12 ~ 0.15

We can slightly improve our result to cover the "$\tau=0$" bin.

Since we know $\sigma_{tot} = \sigma_0 \left(1 + \frac{\alpha_s}{2\pi} C_F \frac{3}{2}\right)$

hence $\int_0^{1/3} \frac{d\sigma}{d\tau} = \sigma_{tot}$ ← finite

Fixed order

which gives

virt + real we missed when $\tau=0$

$$\int_0^{1/3} \boxed{\frac{d\sigma_{real}}{d\tau} + \text{"}\sigma_{virt.}\text{"}\,\delta(\tau)}\, d\tau = \frac{\alpha_s}{2\pi} C_F \frac{3}{2}$$

we know we did not get $\tau=0$ correct.

regular

singular

$$\Rightarrow \sigma^{\text{c}}_{\text{"virt"}} = \frac{\alpha_s}{2\pi} C_F \, 3/2 - \int_0^{1/3} \frac{d\sigma_{\text{real}}}{d\tau} d\tau$$

$$= \frac{\alpha_s}{2\pi} C_F \, 3/2 - \int_0^{1/3} \frac{d\sigma^{\text{reg.}}_{\text{real}}}{d\tau} d\tau - \int_0^{1/3} \frac{d\sigma^{\text{sing.}}_{\text{real}}}{d\tau} d\tau$$

$$= \frac{\alpha_s}{2\pi} C_F \, 3/2 - \int_0^{1/3} \frac{d\sigma^{\text{reg.}}_{\text{real}}}{d\tau} d\tau + \int_{\frac{1}{3}}^{1} \frac{d\sigma^{\text{sing.}}_{\text{real}}}{d\tau} d\tau$$

$$- \int_0^{1} \frac{d\sigma^{\text{sing.}}_{\text{real}}}{d\tau} d\tau$$

"+"-pre...

recall $\frac{d\sigma_{\text{real}}}{d\tau}$

$$= \frac{\alpha_s}{2\pi} C_F \left\{ \frac{4}{\tau} \frac{\log\left(\frac{1-2\tau}{\tau}\right)}{1-\tau} - \frac{3(1+\tau)(1-3\tau)}{\tau} - 6 \log\left(\frac{1-2\tau}{\tau}\right) \right\}$$

$\tau \to 0$

We define

$$\frac{d\sigma^{\text{sing}}_{\text{real}}}{d\tau} = \frac{\alpha_s}{2\pi} C_F \left\{ \frac{4}{\tau} \log\left(\frac{1}{\tau}\right) - \frac{3}{\tau} \right\}$$

$$\Rightarrow \sigma_{\text{"virt"}} = \frac{\alpha_s}{2\pi} C_F \left[3/2 - 5/2 + \pi^2/3 \right] - \int_0^1 \frac{d\sigma^{\text{sing.}}_{\text{real}}}{d\tau} d\tau$$

$\frac{d\sigma^{\text{real}}}{d\tau}$ $(\epsilon \neq 0)$

$$\Rightarrow \frac{d\sigma^{(1)}}{d\tau} = \boxed{\frac{\alpha_s}{2\pi} C_F \left\{ \frac{4}{\tau} \frac{\log\left(\frac{1-2\tau}{\tau}\right)}{1-\tau} - \frac{3(1+\tau)(1-3\tau)}{\tau} - 6\log\left(\frac{1-2\tau}{\tau}\right) \right\}}$$

$$+ \frac{\alpha_s}{2\pi} C_F \left[-1 + \pi^2/3\right] \delta(\tau) \boxed{- \frac{\alpha_s}{2\pi} C_F \int_0^1 \frac{4}{\tau}\log\left(\frac{1}{\tau}\right) - \frac{3}{\tau} \, d\tau \; \delta(\tau)}$$

σ "virt" $\delta(\tau)$ $\qquad (\tau \to 0)$

$$= \frac{\alpha_s}{2\pi} C_F \left\{ \frac{4}{1-\tau}\left(\frac{\log\left(\frac{1}{\tau}\right)}{\tau}\right)_+ - 3(1+\tau)(1-3\tau)\left[\frac{1}{\tau}\right]_+ + \frac{4}{\tau}\frac{\log(1-2\tau)}{1-\tau} - 6\log\left(\frac{1-2\tau}{\tau}\right)\right.$$

$$\left. + \left(-1 + \pi^2/3\right)\delta(\tau) \right\}$$

$\int_0^1 \frac{4}{1-\tau}\left(\frac{\log\left(\frac{1}{\tau}\right)}{\tau}\right)_+ d\tau$

where

$$\int_0^1 f_+(\tau)\, g(\tau)\, d\tau = \int_0^1 f(\tau)\left(g(\tau) - g(0)\right) d\tau$$

$$= \int_0^1 f(\tau) g(\tau)\, d\tau - \int_0^1 f(\tau)\, d\tau \; g(0)$$

$\int_0^{1/3} \left[\left(\frac{4}{1-\tau}\right) - 4\right] \quad \left(\frac{\log\left(\frac{1}{\tau}\right)}{\tau}\right)$

$$\Rightarrow f_+(\tau) \triangleq f(\tau) - \int_0^1 f(\tau)\, d\tau \; \delta(\tau)$$

The result is equivalent to keep ϵ in real do the calc. $\Rightarrow$ Laurant expand in ϵ, + Virt.

$$\int_0^\delta \frac{d\sigma}{d\tau}$$

We can thus have the cumulant cross-section

$$\frac{\sigma(\tau<\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} C_F$$

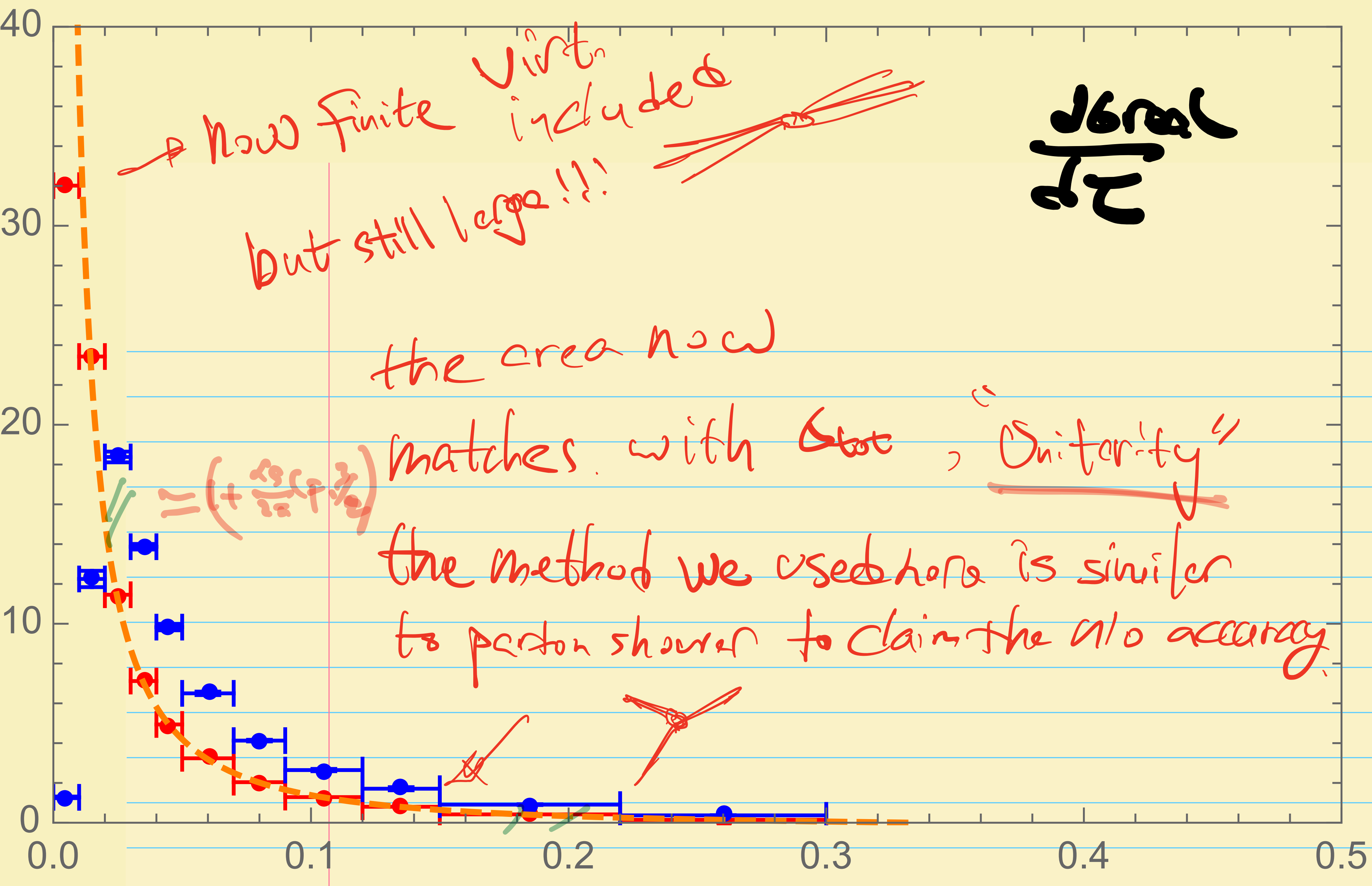
$$\Big\{ -2L^2 - 3L - 1 - 4\,\mathrm{Li}_2(-1+2\delta)$$

$$+ 6\delta\left[\log\left(\frac{\delta}{1-2\delta}\right) + 1\right] + \frac{9}{2}\delta^2 + 3\log(1-2\delta)$$

$$-4\Big[\log(1-2\delta)\log(2-2\delta) - \log(1-\delta)\log\delta$$

$$-\mathrm{Li}_2(\delta) + \mathrm{Li}_2(2\delta)\Big]\Big\}$$

For $\delta \le 1/3$, $L \equiv \log\delta$



40
30
20
10
0
0.0
0.1
0.2
0.3
0.4
0.5
now finite virt. included
but still large!!!
dσreal / dτ
the area now matches with σtot, "Unitarity"
the method we used here is similar to parton shower to claim the nlo accuracy.

- in the "$\tau = 0$"-bin, there is an exact cancellation between real & Virt. poles

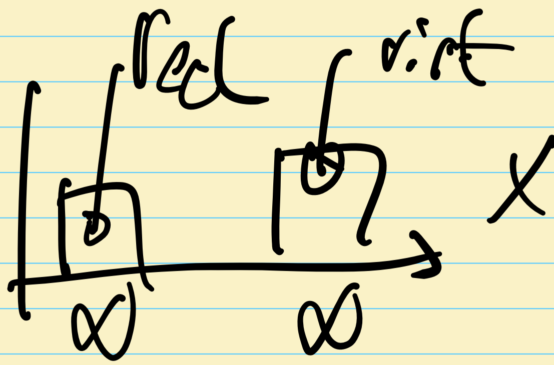
$$\frac{1}{\sigma_0}\frac{d\sigma_{virt}}{d\tau} = \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{7}{6}\pi^2\right) \delta(\tau)$$

$$\frac{1}{\sigma_0}\frac{d\sigma_{real}}{d\tau} = \frac{\alpha_s}{2\pi} C_F \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} \cdots\right) \delta(\tau) + \cdots$$

The cancellation ensures the predictive power of a F.O. calculation.

Not all observable has this feature !!)

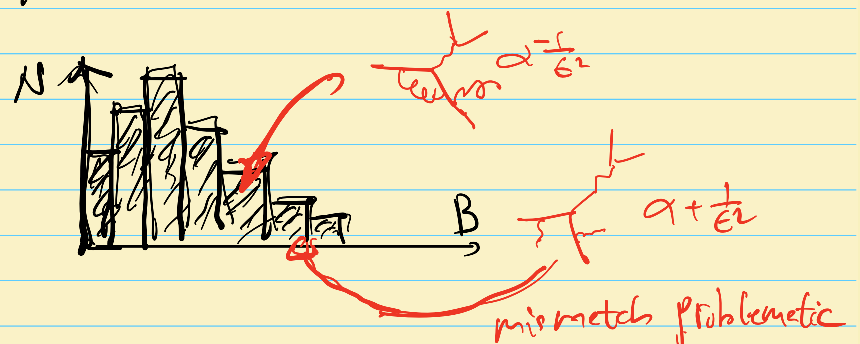
IR Safe



recall that

- all poles come from the soft & coll. limit of the emitted partons.
- virt correction does not change multiplicities

$\Rightarrow$ to cancel poles. soft & coll. of real should fall into the same bin of virt.



IRC safe observables:

$$\Theta_N(p_1, p_2, p_3 \cdots p_i \cdots p_n) \xrightarrow[p_i \to 0]{\lim} \Theta_{N-1}(p_1, p_2, p_3 \cdots p_{i-1}, p_{i+1} \cdots p_N)$$

$$p = p_i + p_j$$

$$\Theta_N(p_1, p_2, p_3 \cdots p_i \cdots p_j \cdots p_N) \xrightarrow[p_i \parallel p_j]{\lim} \Theta_{N-1}(p_1, p_2 \cdots p_i + p_j \cdots p_{j-1}, p_{j+1} \cdots p_N)$$

You can check that thrust is IR safe.

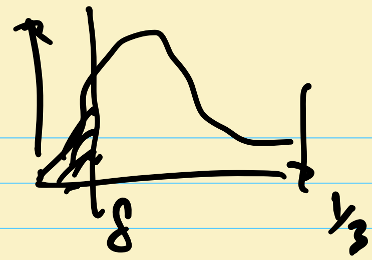
$$\underline{10} \neq \underline{9}$$

IRC unsafe:

e.g. particle numbers

$$\underline{\mathcal{O}_N(p_1 \cdots p_N) = N} \xrightarrow[p_1 \to 0]{\lim} N \neq \underline{\mathcal{O}_{N-1}}(p_2 \cdots p_N) = N-1$$

We need IRC safe observables to make F.O. predictive.



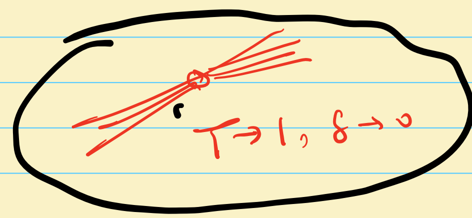
L indicates the incomplete cancellation between real. & Virt.

as $\delta \to 1/3$, $\sigma^{(1)} \to \frac{\alpha_s}{2\pi} C_F \, 3/2$, "complete"

as $\delta \to 0$, $L \to \infty$, recovers the divergence

recall only soft and coll. radiations are allowed

$(0, \delta) \to$



$\log \delta \to$

$$\frac{\sigma(\tau<\delta)}{\sigma_0} = 1 + \frac{\alpha_s}{2\pi} C_F \Big\{ -2L^2 - 3L - 1 - 4\,\mathrm{Li}_2(-1+2\delta)$$

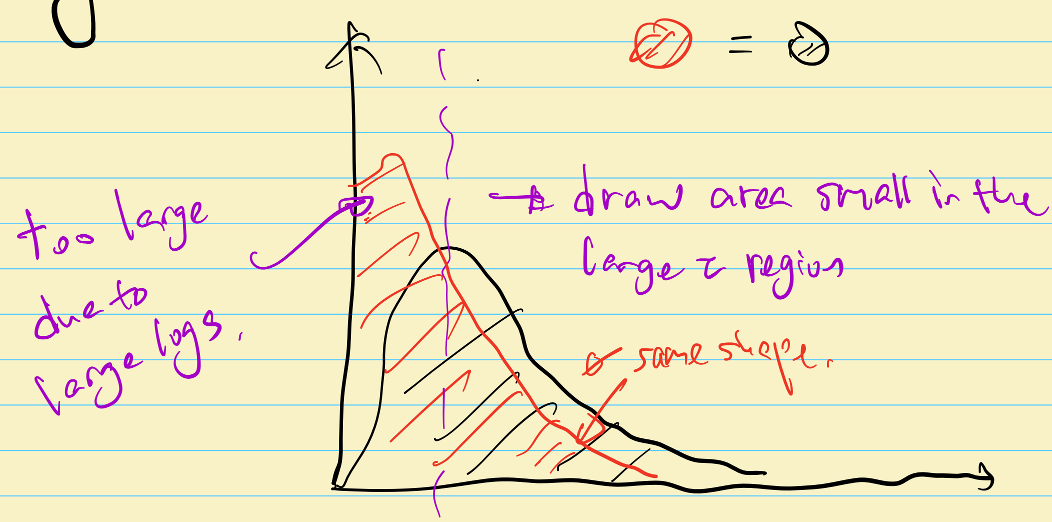
$$+ 6\delta\left[\log\left(\frac{\delta}{1-2\delta}\right) + 1\right] + \frac{9}{2}\delta^2 + 3\log(1-2\delta)$$

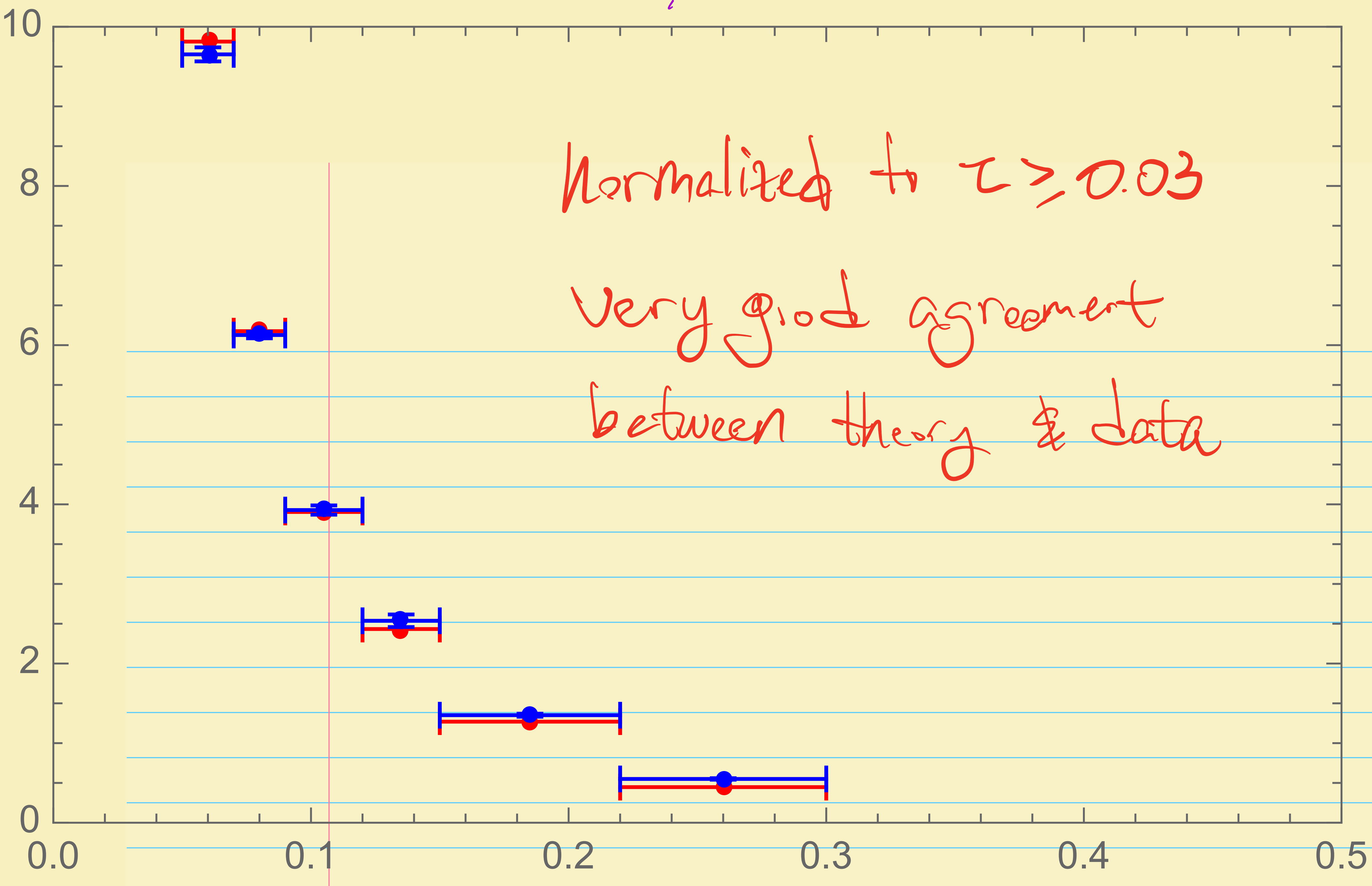
$$- 4\left[\log(1-2\delta)\log(1-2\delta) - \log(1-\delta)\log\delta\right]$$

$-\mathrm{Li}_2(\delta) + \mathrm{Li}_2(2\delta)\,]\;\}$

Break down of F.O.

The entire $\frac{1}{\sigma}\frac{d\sigma}{d\tau}$ spectrum does not agree with data. all due to large logs in the small τ region




Normalized to τ ≥ 0.03
Very good agreement
between theory & data
10
8
6
4
2
0
0.0
0.1
0.2
0.3
0.4
0.5

Now we focus on the $\delta \ll 1$ region,

When $\delta \ll 1$, only soft & coll. radiations are allowed, and we have

$$\frac{\sigma(\delta)}{\sigma_0} = 1 - \frac{\alpha_s}{2\pi} C_F \left(\frac{4}{2} L^2 + 3 \cdot L + 1 - \pi^2/3 \right)$$

recall the poles

$$\frac{\sigma_{rad}}{\sigma_0} \sim \frac{\alpha_s}{2\pi} C_F \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} \cdots \right)$$

soft & collinear nature.

Indeed, recall $\ln = \int \frac{dx}{x}$

Hence, the logs in

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 dy_1\, dy_2\, dy_3\, (y_1 y_2 y_3)^{-\epsilon}\, \delta(1-y_1-y_2-y_3)$$

$$\times \left\{ \frac{2}{y_2 y_3} + \frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon \right\}$$

$\Theta(\Phi_3)$

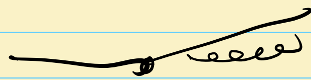
double log

single log.

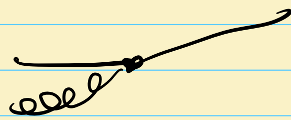
Comes from $y_2 \to 0$, $y_3 \to 0$.

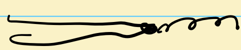
but $y_1 \to 0$ does not generate any logs

Pictorially:

$y_2 \to 0$  $+$ soft g $\Rightarrow \tau = y_2$

recall $\tau = \min(y_i)$

$y_3 \to 0$  $+$ soft g $\Rightarrow \tau = y_3$

$y_1 \to 0$  $+$ soft quark $\Rightarrow \tau = y_1$.

In this limit:

$$\sigma_{real} = \sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \frac{e^{\gamma_E \epsilon}}{\Gamma(1-\epsilon)} \cdot \left(\frac{4\pi\mu^2}{s}\right)^{-\epsilon}$$

$$\times \int_0^1 \cancel{dy_1}\, dy_2\, dy_3 \,(\cancel{y_1}^{\,1} y_2 y_3)^{-\epsilon}\, \cancel{\delta(1-y_1-y_2-y_3)}$$

$$\times \left\{ \frac{2}{y_2 y_3} + \cancel{\frac{-2+y_3-\epsilon y_3}{y_2} + \frac{-2+y_2-\epsilon y_2}{y_3} - 2\epsilon} \right\}$$

if keep only double log.

$\Theta(\Phi_3)$

$\hookrightarrow \left[\delta(\tau - y_2)\, \Theta(\tau < y_3) + y_2 \leftrightarrow y_3 + \text{log suppressed} \right]$

$\overset{\epsilon\to 0}{=} -\sigma_0 \cdot \frac{\alpha_s}{2\pi} C_F \; \frac{2\times 2}{\tau} \log\tau \quad + \text{ log suppressed}$

$\longrightarrow -\sigma_0 \frac{\alpha_s}{2\pi} C_F \; 4 \left(\frac{\log\tau}{\tau}\right)_+$ by "Unitarity"

cumulative $\longrightarrow -\sigma_0 \frac{\alpha_s}{2\pi} C_F \cdot 2 \ln^2 \delta$ & one power from soft
one power from coll.

soft-collinear limit

reproduce the leading log structure

To all orders, we have structures $\frac{1}{\epsilon^{2n}} \overset{n}{\cdots} \frac{1}{\epsilon}$

$\sigma \sim 1 \qquad\qquad L^{2n} \cdots L$

NLO $+ \; \alpha_s L^2 \quad + \alpha_s L \quad + \#$

NNLO $+ \; \alpha_s^2 L^4 \quad + \alpha_s^2 L^3 \quad + \alpha_s^2 L^2 \quad + \alpha_s^2 L \quad + \#$

$N^3LO \; + \; \alpha_s^3 L^6 \quad + \alpha_s^3 L^5 \quad + \alpha_s^3 L^4 \quad + \alpha_s^3 L^3 \quad + \alpha_s^3 L^2 \cdots$

$\vdots \qquad\qquad \vdots$ resummation

When $\alpha_s L^2 \sim 1$, we can not truncate the series at fixed α_s

since the $\alpha_s^n L^{2n}$ for $n > 1$ are equally important. We need to resum $\alpha_s L^2$ to all orders.

$\Delta \sim 1$

NLO $+ \alpha_s L^2 + \alpha_s L + \#$

NNLO $+ \alpha_s^2 L^4 + \alpha_s^2 L^3 + \alpha_s^2 L^2 + \alpha_s^2 L + \#$

$N^3LO + \alpha_s^3 L^6 + \alpha_s^3 L^5 + \alpha_s^3 L^4 + \alpha_s^3 L^3 + \alpha_s^3 L^2 \cdots$

$\vdots \qquad \vdots$

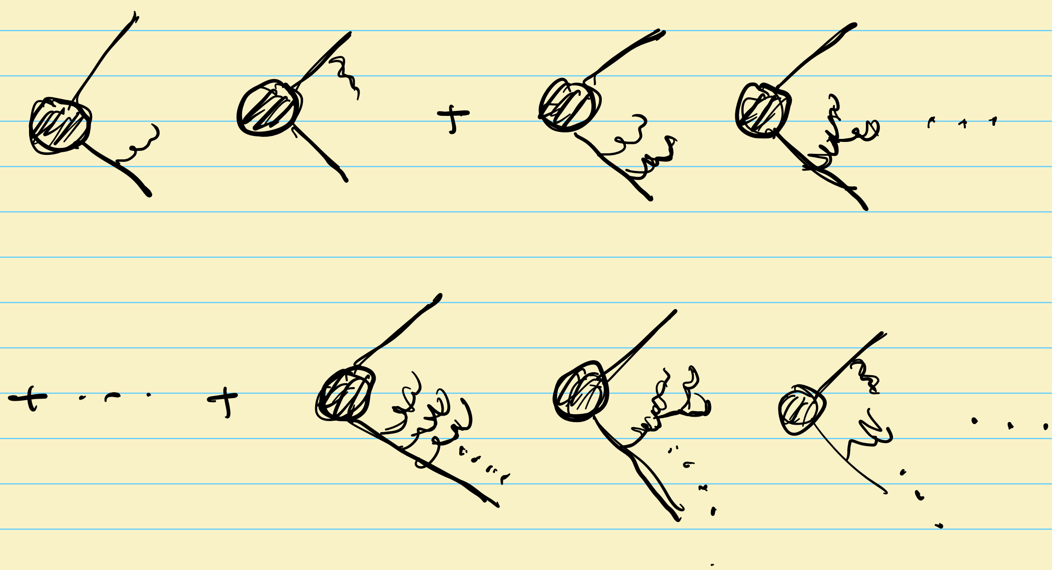
dominant logs, DL

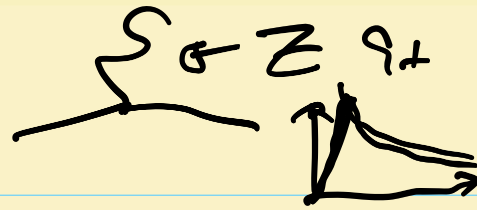
For certain observables, the logs can be resummed in to an exponetiation form

$$\exp[\underbrace{L g_1(\alpha_s L)}_{\text{Leading log (LL)}} + \underbrace{g_2(\alpha_s L)}_{\text{Next-to-leading log (NLL)}} + \underbrace{\alpha_s g_3(\alpha_s L)}_{\text{NNLL}} + \cdots]$$

Or equivalently.

We sum up the most singular behavior of the soft & coll. radiations to all orders





There are different Approaches to Resummation

- CSS formalism

  Collins, Soper, Sterman Nucl. Phys. B 250 (1985) 199-224
  hep-ph/0409313 (1989)

this Lecture

- Coherent branching algorithm

  Catani et al. nucl. phys. B327 (1989) 323-352
  nucl. phys. B407 (1993) 3-42

- Soft collinear effective theory

  Bauer et al. hep-ph/0202088, hep-ph/0109045
  hep-ph/0107001, hep-ph/0011336
  Beneke et al. hep-ph/0206152

- CAESAR + PANSCALE

  Banfi et al. hep-ph/0112156

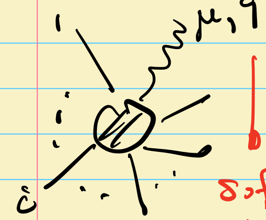
All Approaches Now have the ability to go beyond NLL.

Most Singular

Possibilities → soft & collinear

- Infrared behaviour in QCD

→ a lesson from QED gluon → photon



soft photon

$q \equiv |\vec{q}| \ll E_i$

Photon phase space:

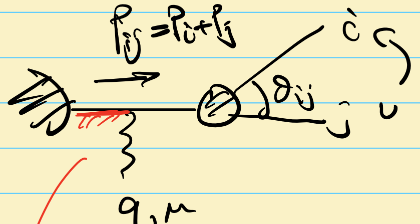
$$\frac{d^3q}{2q^0(2\pi)^3} = \frac{q^2\,dq\,d\Omega}{2q(2\pi)^3} = \boxed{\frac{1}{2(2\pi)^2}\,q\,dq\,d\cos\theta}$$

$$\left|\frac{dq}{q}\right|$$

we are then interested in term $\propto \frac{1}{q^2}$ in the matrix element square.

- possible single emission

1. photon from internal line

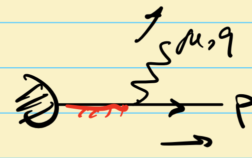


hard, $\theta_{ij} \sim 1$, E_i, E_j large

$$\propto \frac{1}{(P_{ij}+q)^2} = \frac{1}{2P_{ij}\cdot q + P_{ij}^2} \xrightarrow{q\to 0} \frac{1}{P_{ij}^2}$$

regular as $q \to 0$

2. photon from external line



$$\bar u(p_i)(-ie\gamma^\mu)\, i\, \frac{\not p_i + \cancel{\not q}}{2p_i\cdot q}\cdots$$

No $\frac{1}{q}$ enhancement

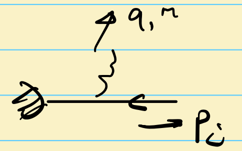
$$= \bar u(p_i)\, e\, \frac{\gamma^\mu \not p_i + \cancel{\not p_i \gamma^\mu}}{2p_i\cdot q}$$

0, since $\not p_i u(p_i) = 0$

eikonal current

$$= \bar u(p_i)\, e\, \frac{p_i^\mu}{p_i\cdot q}$$

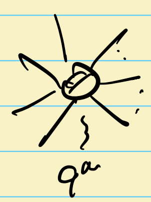
will give $\frac{1}{q}$ enhancement we need

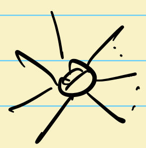


$$\bar u(p_i)(-ie)\gamma^\mu\, i\, \frac{-\not p_i}{2p_i\cdot q}$$

$$= \bar u(p_i)(-e)\frac{p_i^\mu}{p_i\cdot q}$$

absorb "−" sign into the electric charge.

$\Rightarrow$ 

$\longrightarrow \sum_i e_i \frac{p_i^\mu}{p_i\cdot q}\cdot$ 

- $q^\mu \mathcal{M}_\mu = \sum_i e_i \frac{q\cdot p_i}{p_i\cdot q} = \sum_i e_i = 0$

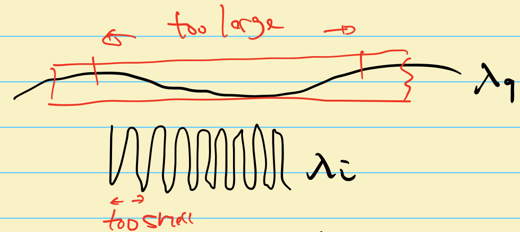
  charge conservation, gauge invariance.

- factorized soft current $J_i^\mu = e_i \frac{p_i^\mu}{p_i\cdot q}$

  independent of spin, Low theorem.

- a factorization of the long & short distance physics.

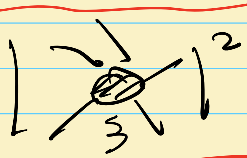
$$\frac{1}{q} \sim \lambda_q >>> \frac{1}{E_i} \sim \lambda_i$$



Can not resolve the internal structure

$\boxed{\text{Q: How about radiate a soft fermion? } \Rightarrow \text{ soft leads to logs?}}$

Now we square the Matrix element



$$\Big|\ \ \Big|^2 \rightarrow -\sum_{i\neq j} e_i e_j \frac{p_i\cdot p_j}{p_i\cdot q\; p_j\cdot q} \left|M_0(\{p_i\})\right|^2$$

For single emission

define $$dw_1(q) = -d\Phi_1 \sum_{ij} e_i e_j \frac{p_i\cdot p_j}{p_i\cdot q\; p_j\cdot q}$$

$$= -\sum_{ij} \frac{e_i e_j}{8\pi^2} \frac{p_i\cdot p_j}{p_i\cdot q\; p_j\cdot q} \cdot q\,dq\,d\cos\theta$$

$\frac{1}{2(2\pi)^2} q\,dq\,d\cos\theta$ Single photon emission prob.

Suppose we do a measurement v on the photon. that proposes the phase space restriction $\Theta_1(v)$ the prob. for the photon emission is

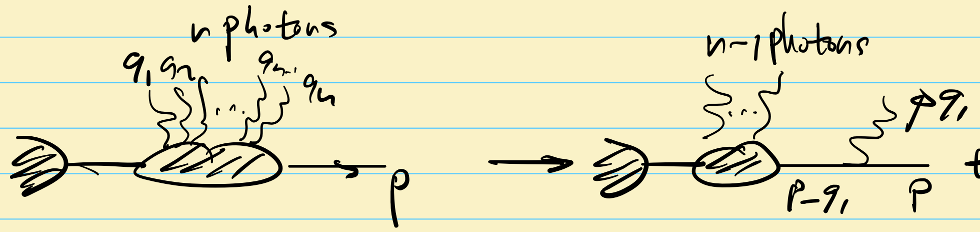
$$W_1(v)$$

$$= \int dv \Big\{ \underbrace{\frac{dw_1(q)}{dv} \Theta_1(v)}_{\text{real}} - \underbrace{\int dv' \frac{dw_1(q)}{dv'}}_{\text{virt.}} \delta(v) \Big\}$$

since $\int dv\,(\text{virt} + \text{real}) = 0$ + small corrections

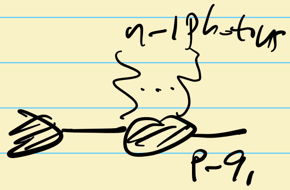
$$\Rightarrow \boxed{W_1(v) = \int dv \left[\Theta_1(v) - 1\right] \frac{dw_1}{dv}}$$

Multiple emissions :



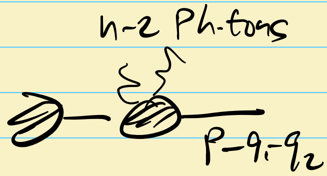
$+ \cdots$

$\longrightarrow$



$$e_i \frac{p^\mu}{p\cdot q_1} + \cdots$$

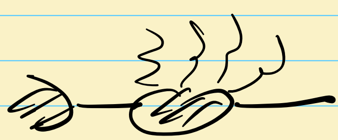
$\longrightarrow$



$$e_i \frac{p^\mu}{p\cdot q_1}\, e_i \frac{p^\nu}{p\cdot(q_1+q_2)} + \cdots$$

$\leftarrow$ from $(p+q_1+q_2)^2 \sim 2p\cdot(q_1+q_2)$

$\Rightarrow$



$$= \frac{1}{n!} \sum_{\substack{\text{All}\\ \text{permutations}}} e_i \frac{p^{\mu}}{p_i\cdot q_1}\, e_i \frac{p^{\mu}}{p_i\cdot(q_1+q_2)} \cdots e_i \frac{p^{\mu}}{p_i\cdot(q_1+\cdots+q_n)}$$

$$= \frac{1}{n!} \prod_{j=1}^{n} e_i \frac{p^\mu}{p_i\cdot q_j} \quad * \quad \longrightarrow \quad \boxed{dW_n = \frac{1}{n!}\prod^{n} dW_1}$$

"independent" emission

$\Longrightarrow$ $\sum_n$ 

$$\longrightarrow \sum_n \frac{1}{n!} \prod^n dw_i \, |M(\{p_i\})|^2$$

- We note that the emission probability also depends on the phase space restriction which is not necessarily independent

$$\Theta_n(v) = \Theta_n(v(\{q_i\})) \longrightarrow \prod_{i=1} \Theta_1(v)$$

- if the phase space restriction is also independent then

$$\sigma \longrightarrow \sigma_0 \exp[W_1(v)]$$

where

w_1 defined before including the "virtual".

$$W_1(v) = \int dv \left\{ \Theta_1(v) - 1 \right\} \frac{dw_1(q)}{dv}$$

You get the resummed form in QED for v.

* $\Theta_c(v) = 1 \Rightarrow$ inclusive $\Rightarrow W_1(v) = 0$

$\Rightarrow \sigma \to \sigma_0$, Unitarity.

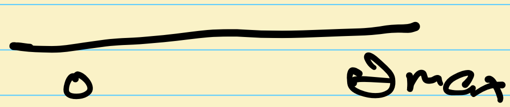
* $\Theta_c(v) = 0 \Rightarrow$ No emission

$\Rightarrow$

$$\sigma \to \sigma_0 \overbrace{\exp\Big[- \underbrace{\int dv \frac{dW_1(q)}{dv}}_{-\infty \text{ "virtual" only}}\Big]}^{\text{Sudakov Factor}} \to 0$$

$\Delta(\theta_{max}, 0)$

prob. of no emission below θ_{max}



* 2 photons.

$$\frac{p_i^\mu}{p_i \cdot q_1} \frac{p_i^\mu}{p_i \cdot (q_1+q_2)} + \frac{p_i^\mu}{p_i \cdot q_2} \frac{p_i^\mu}{p_i \cdot (q_1+q_2)}$$

$$= p_i^m p_i^n \frac{p_i\cdot q_2 + p_i\cdot q_1}{p_i\cdot q_1\, p_i\cdot q_2\, p_i\cdot(q_1+q_2)} = \frac{p_i^m}{p_i\cdot q_1}\frac{p_j^m}{p_j\cdot q_2} \checkmark$$

suppose $\displaystyle\sum_{\text{perm.}} \frac{p_i^n}{p_i\cdot q_1}\frac{p_i^n}{p_i\cdot(q_1+q_2)}\cdots\frac{p_j^m}{p_i\cdot(q_1+\cdots q_{n-1})}$

$$= \frac{p_j^m}{p_i\cdot q_1}\cdots\frac{p_i^n}{p_i\cdot q_{n-1}}$$

then $\displaystyle\sum_{\text{perm}} \frac{p_j^m}{p_j\cdot q_1}\cdots\frac{p_i^n}{p_i\cdot(q_1+\cdots q_{n-1})}\frac{p_j^n}{p_j\cdot(q_1+\cdots q_n)}$

$$= \sum_{\text{perm } j\neq n} \frac{p_i^n}{p_i\cdot q_1}\cdots\frac{p}{\cdots}\;\frac{p_i^m}{p_j\cdot(q_1\cdots q_n)}$$

$$+ \sum_{\text{perm } j\neq 1} \frac{p_i^m}{p_i\cdot q_2}\cdots\frac{p_i^n}{p_i(q_2+\cdots q_n)}\frac{p_j^n}{p_i\cdot(q_1+\cdots q_n)}$$

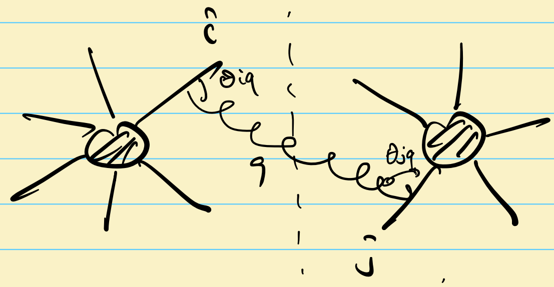
$$+ \cdots$$

$$= \left\{\prod_{j\neq 1}\frac{p_i^m}{p_i\cdot q_j} + \prod_{j\neq 2}\frac{p_i^n}{p_i\cdot q_j} + \cdots + \prod_{j\neq n}\frac{p_i^n}{p_i\cdot q_j}\right\}\frac{p_j^m}{p_i\cdot(q_1+\cdots+q_n)}$$

$$= \prod_j \left[\frac{p_i^n}{p_i\cdot q_j}(p_i\cdot q_1 + p_i\cdot q_2 + \cdots + p_i\cdot q_n)\right.\;\frac{p_i^m}{p_i\cdot(q_1+\cdots+q_n)}$$

$$= \prod_j \frac{p_i^m}{p_i\cdot q_j}$$

QED coherence



$$\propto -\sum_{i\neq j} e_i e_j \frac{P_i \cdot P_j}{P_i\cdot q \; P_j\cdot q}$$

suppose a θ_{qk} is the smallest among all angles.

$$= -\sum_{i<j} e_i e_j \frac{2P_i\cdot P_j}{P_i\cdot q\, P_j\cdot q} \qquad i=k \qquad q\parallel k \text{ almost}$$

$$= -e_k \sum_{j\neq k} e_j \frac{2P_k\cdot P_j}{E_k E_q(1-\cos\theta_{qk})} \frac{1}{E_j E_q(1-\cos\theta_{qj})} + \text{less singular}$$

$$= -\frac{e_k}{E_q^2} \frac{2}{1-\cos\theta_{qk}} \sum_{j\neq k} e_j \frac{P_k\cdot P_j}{E_j E_k(1-\cos\theta_{qj})} + \text{less singular}$$

$$\simeq -\frac{2e_k}{E_q^2} \frac{1}{1-\cos\theta_{qk}} \underbrace{\sum_{j\neq k} e_j \frac{E_k E_j(1-\cos\theta_{kj})}{E_k E_j(1-\cos\theta_{kj})}}_{-e_k \text{ since charge cons.}} + \text{less singular}$$

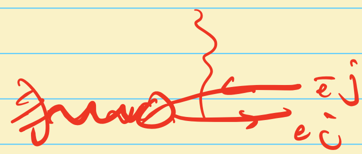
$$\simeq \frac{2e_k^2}{E_q^2} \frac{1}{1-\cos\theta_{qk}} \quad (*)$$

Here we used the coll. limit to derive the result but it can be derived as long as θ_{qk} is the smallest angle, no need for $\theta_{qk} \to 0$. See below.

Now we integrate over the photon phase space to find

$$dw_1(q) = \frac{\alpha}{\pi} \underbrace{\sum_k e_k^2 \frac{dq}{q}}_{\text{from a single independent emitter}} \frac{d\theta_{qk}^2}{\theta_{qk}^2} \underbrace{\Theta(\theta_{max} - \theta_{qk})}_{\text{QED coherence due to destructive interference}}$$

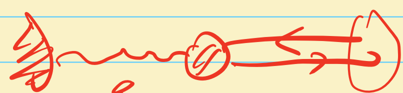
Here $\theta_{max} = \min\{\theta_{ij}\}$



$\theta_{ee} \ll \theta$

$p_j \parallel p_i$

$\Rightarrow$



see the overall charge 0

hence the wide angle emission is suppressed

e doesn't carry charge

$$(-e)\frac{p_j^\mu}{p_j\cdot q} + e\frac{p_i^\mu}{p_i\cdot q} = (-e+e)\frac{p^\mu}{p\cdot q} = 0$$

$$\Theta_n(v) = \prod_n \Theta_1(v)$$

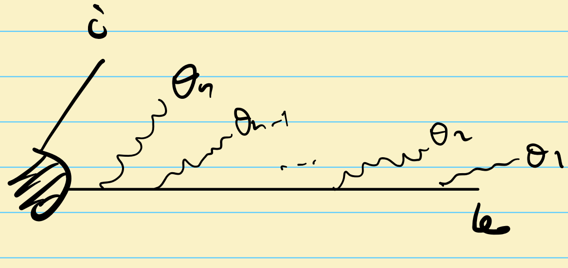
Therefore we find

$$\sigma = \sigma_0 \exp\left[\int dv \left\{\Theta_1(v)-1\right\} \frac{dw_1}{dv}\right]$$

$$\simeq \sigma_0 \prod_k \exp\left\{\underbrace{\frac{\alpha}{\pi} e_k^2 \int dv \int \frac{dq}{q}\int^{\theta_{max}} \frac{d\theta_{qk}^2}{\theta_{qk}^2}}_{\text{shower each leg}} \left[\Theta_1(v) - 1\right]\right\}$$

$$= \sigma_0 \prod_k \sum_{n=0}^{\infty} \frac{1}{n!}\prod_n \frac{\alpha^n}{\pi^n}\left(e_k^2\int\frac{dq}{q}\left[\Theta_1(v)-1\right]\right)^n \int^{\theta_{max}}\frac{d\theta_1^2}{\theta_1^2}\int^{\theta_{max}}\frac{d\theta_2^2}{\theta_2^2}\cdots\int^{\theta_{max}}\frac{d\theta_n^2}{\theta_n^2}$$

$$= \sigma_0 \prod_k \sum_{n=0}^{\infty} \prod_n \frac{\alpha^n}{\pi^n}\left(e_k^2\int\frac{dq}{q}\left[\Theta_1(v)-1\right]\right)^n \underbrace{\int^{\theta_{max}}\frac{d\theta_n^2}{\theta_n^2}\int^{\theta_{n-1}}\frac{d\theta_{n-1}^2}{\theta_{n-1}^2}\int^{\theta_{n-2}}\frac{d\theta_{n-2}^2}{\theta_{n-2}^2}\cdots\int^{\theta_2}\frac{d\theta_1^2}{\theta_1^2}}_{\text{angular order}}$$



angular ordering to account for the interference

Very useful for Monte Carlo event-generator.

$\circledast$

$$\frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} = S_{ij}^{(i)} + S_{ij}^{(j)}$$

$$S_{ij}^{(i)} = \frac{1}{2}\left\{\frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q} + \frac{1}{E^2}\frac{1}{1-\cos\theta_{iq}} - \frac{1}{E^2}\frac{1}{1-\cos\theta_{jq}}\right\}$$

Where

$E_q(1,\ \sin\theta_{iq}\cos\phi_{iq}\ \ \sin\theta_{iq}\sin\phi_{iq}.\ \ \cos\theta_{iq})$

$E_j(1,\ \sin\theta_{ij}\ \ ,\ \ 0\ \ ,\ \ \cos\theta_{ij})$

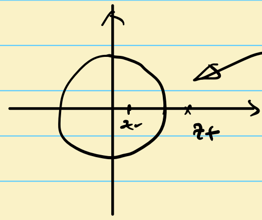
$$1-\cos\theta_{jq} = \underbrace{1-\cos\theta_{iq}\cos\theta_{ij}}_{c} - \underbrace{\sin\theta_{iq}\sin\theta_{ij}}_{s}\cos\phi_{iq}$$

Therefore

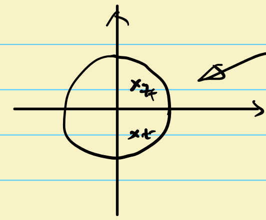
$$\int_0^{2\pi}\frac{d\phi_{iq}}{2\pi}\frac{1}{1-\cos\theta_{jq}} = \int_0^{2\pi}\frac{d\phi_{iq}}{2\pi}\frac{1}{c-s\cos\phi_{iq}} \qquad \text{let } z=e^{i\phi_{iq}}$$

$$= \frac{-i}{2\pi}\oint_{|z|=1}\frac{dz}{z}\frac{1}{c-s(z+1/z)} = \frac{i}{2\pi}\oint_{|z|=1}dz\,\frac{1}{sz^2-cz+s}$$

$$= \frac{i}{2\pi}\frac{1}{s}\oint_{|z|=1}dz\,\frac{1}{(z-z_+)(z-z_-)} \qquad \text{Where } z_\pm = \frac{1}{2}\frac{c}{s} \pm \frac{1}{2}\sqrt{\frac{c^2}{s^2}-1}$$



if $c/s > 1$



if $c/s < 1$

$$\Rightarrow \text{result} = -\frac{1}{s}\,\frac{1}{z_- - z_+}$$

$$= \frac{1}{\sqrt{c^2 - s^2}}$$

$$\Rightarrow \text{result} = 0$$

Therefore

$$\int \frac{d\phi_{iq}}{2\pi}\, S_{ij}^{(i)} = \frac{1}{2}\frac{1}{E^2}\int \frac{d\phi_{iq}}{2\pi}\left\{ \frac{1-\cos\theta_{ij}}{(1-\cos\theta_{iq})(1-\cos\theta_{jq})} + \frac{1}{1-\cos\theta_{iq}} - \frac{1}{1-\cos\theta_{jq}} \right\}$$

$$= \frac{1}{2}\frac{1}{E^2}\left[\left[\frac{1-\cos\theta_{ij}}{1-\cos\theta_{iq}} - 1 \right] \frac{1}{\sqrt{c^2 - s^2}} + \frac{1}{1-\cos\theta_{iq}} \right]$$

$$= \frac{1}{2E^2}\left\{ \frac{\cos\theta_{iq} - \cos\theta_{ij}}{1-\cos\theta_{iq}}\,\frac{1}{|\cos\theta_{iq} - \cos\theta_{ij}|} + \frac{1}{1-\cos\theta_{iq}} \right\}$$

$$= \begin{cases} \dfrac{1}{E^2}\,\dfrac{1}{1-\cos\theta_{iq}} & \theta_{iq} < \theta_{ij} \\ 0 & \theta_{iq} > \theta_{ij} \end{cases}$$

- summarize for QED

- $W_1 = -4\pi\alpha \sum_{i\neq j} e_i e_j \int \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} (\Theta_1(v) - \underset{q}{1}) \, dv \, d\Phi_q$

virtual included

- if phase space factorize, then

$$d\sigma = d\sigma_0 \exp[W_1]$$

- Interference well approximated by time ordering

$$W_1 \to \frac{\alpha}{\pi} \sum_k e_k^2 \int \frac{dq}{q} \frac{d\theta^2}{\theta^2} \Theta(\theta_{max} - \theta_{qk}) (\Theta_1(v) - 1) dv$$

and

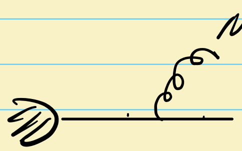
$$d\sigma = d\sigma_0 \exp[W_1]$$

$$= d\sigma \prod_k \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha}{\pi} e_k^2 \int \frac{dq}{q} \{\Theta_1(v) - 1\} dv\right]^n \underbrace{\int_0^{\theta_{max}} \frac{d\theta_n^2}{\theta_n^2} \int_0^{\theta_n} \frac{d\theta_{n-1}^2}{\theta_{n-1}^2} \cdots \frac{d\theta_1^2}{\theta_1^2}}_{\text{angular ordering.}}$$

→ soft emissions in QCD.

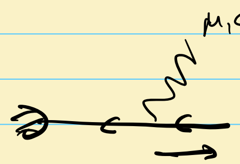
- single emission

Similar to the QED soft photon emission, we have the soft gluon radiated off a quark line.



$$= g_s t^a_{ij} \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_0^{j\ldots}$$

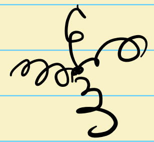
↳ with e_i replaced by the color charge, $g_s t^a_{ij}$ a matrix.

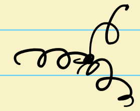


$$= g_s(-t^a_{ij}) \frac{p_i^\mu}{p_i \cdot q} \mathcal{M}_0^{j\ldots}$$

↳ From anti-quark

More than that, since now the gluon carries charge

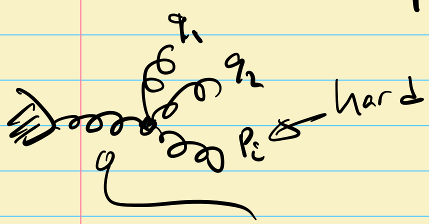




soft gluon can be emitted from hard gluons.

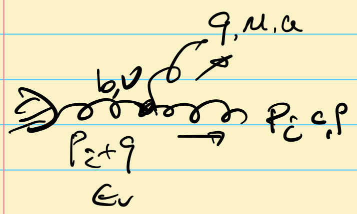
$$|\ | \propto \frac{1}{q^2}$$

q



$$\frac{g_s^2}{(p_i+q_1+q_2)^2} = \frac{g_s^2}{\underbrace{2p_i(q_1+q_2)}_{E_1+E_2}+\underbrace{2q_1\cdot q_2}_{\cancel{E_1E_2}\to 0}} \sim \frac{g_s^2}{\underbrace{2p_i\cdot(q_1+q_2)}_{\text{not singular}}}$$

Since from phase space we have $q_1 dq_1\ q_2 dq_2$, which returns suppressed results for either $q_1 \to 0$ or $q_2 \to 0$



$$\epsilon_\nu(p_i+q) \xrightarrow{q\ \text{soft}} \epsilon_\nu(p_i)$$

$$\propto \frac{1}{2p_i\cdot q} g_s f^{abc} \left\{ (\cancel{+}2p_i+q)^\rho g^{\mu\nu} - (2p_i+q)^\mu g^{\nu\rho} + (\cancel{p_i}-q)^\nu g^{\rho\mu} \right\} \epsilon_\rho \epsilon_\nu \mathcal{M}^b$$

gauge 0 — gauge 0 — 0 — 0 — 0 — non-singular

$$= -\frac{1}{2p_i\cdot q} g_s f^{abc}\, 2p_i^\mu\, \epsilon\cdot\epsilon\, \mathcal{M}_0^{b\ldots}$$

$$= g_s f^{abc} \frac{p_i}{p_i\cdot q} \mathcal{M}_0^{b\ldots}$$

c.f. gluon from quark: $g_s t^a_{ij} \frac{p_i}{p_i\cdot q}$

- same eikonal factor, spin-independent
- with the color charge $g_s f^{abc} = g_s f_c{}^a{}_b$

We can unify the soft approximation for both quark and gluon emitter by introducing the color charge $\vec{T}_i^a$ where i is the i-th parton

$$\vec{T}_i^a \to t^a \quad \text{if } i \text{ is a quark}$$

$$\to \bar{t}^a = -t^a \quad \text{if } i \text{ is an anti-quark}$$

$$\to f_{c\,b}^{\;a} \quad \text{if } i \text{ is a gluon.}$$

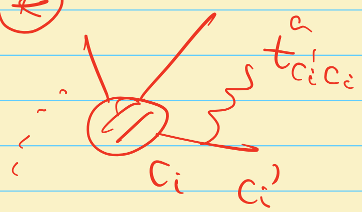
Color algebra:

We note that $\sum_i \vec{T}_i^a = 0$ color charge conservation

c.f. $\sum_i e_i = 0$ electric charge conservation

and if we define $\vec{T}_i^2 = \sum_a \vec{T}_i^a \vec{T}_i^a$, then

$$\vec{T}_i^2 = \begin{cases} C_F & \text{for quark \& anti-quark} \\ C_A & \text{for gluon.} \end{cases}$$



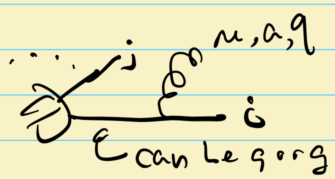
$$= g_s\, t^a_{c_i' c_i}\, \mathcal{M}^{\cdots c_i \cdots}$$

$$= g_s\, t^a_{c_i' c_i} \left| \cdots c_i \cdots \right\rangle$$

color operator acting on the color space, like a rotation

$$= g_s\, T^a_i \left| \cdots c_i \cdots \right\rangle$$

In this way



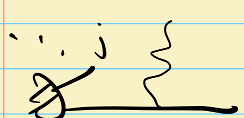
$$= g_s \sum_i \vec{T}_i^{\,a} \frac{p_i^\mu}{p_i \cdot q} \, |\mathcal{M}(\{p_i\})\rangle \qquad (*)$$

color charge operator acting on $|\mathcal{M}\rangle$

matrix element becomes a vector in the color space

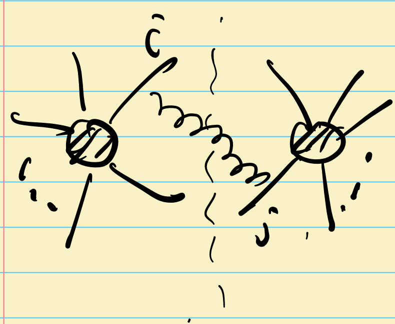
QCD is not truely factorized, there is color correlation remaining, since emitting gluon carries away color charge!!!

c.f. QED

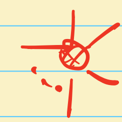


$$= e \sum_i e_i \frac{p_i^\mu}{p_i \cdot q} \, \mathcal{M}_0(\{p_i\})$$

Now square the matrix element. We find

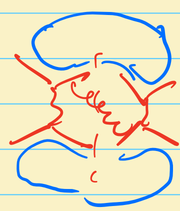
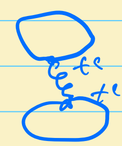
 $= -\sum_{i\neq j} g_s^2 \frac{p_i\cdot p_j}{p_i\cdot q\; p_j\cdot q}\langle \mathcal{M}_0|T_i\cdot T_j|\mathcal{M}_0\rangle$

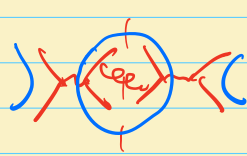
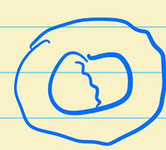
not entirely factorized

color correlation. depends on the color flow of the hard process. 

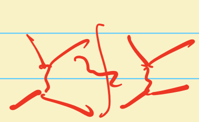
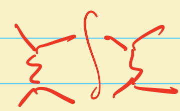
not proportional to $\langle \mathcal{M}_0|\mathcal{M}_0\rangle = |\mathcal{M}_0|^2$

QCD:

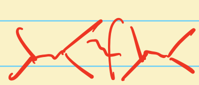
 $\xrightarrow{\text{Color Flow}}$  $= \mathrm{Tr}[t^a]\,\mathrm{Tr}[t^a] = 0$

 $\xrightarrow{\text{Color Flow}}$  $= \mathrm{Tr}[t^a t^a]\,\mathrm{Tr}[1]$

QED:

 $\sim e^2 \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q}$ 

exactly same!!

 $\sim e^2 \frac{p_i\cdot p_j}{p_i\cdot q\, p_j\cdot q}$ 

- multiple emissions

recall QED :

$$O_N(U) = \prod_N O_1(U)$$

$$d\sigma \sim d\sigma_0 \exp[W_1]$$

$$W_1(u) = -4\pi\alpha \sum_{i\neq j} e_i e_j \int \frac{p_i \cdot p_j}{p_i\cdot q\, p_j \cdot q} (O_1(u)-1)\, du\, d\Phi_q$$

direct exponentiation of 1-photo emission. (if the observable $\mathcal{V}$ is factorizable), since in QED,

$|M|^2 \sim |M_0|^2 e^2 \frac{p_i \cdot p_j}{p_i\cdot q\, p_j\cdot q}$ entirely factorizable.

How about QCD ?

Can we have

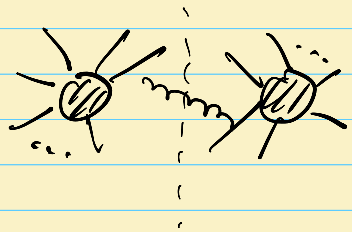
$$d\sigma \sim d\sigma_0 \exp[W_1]$$

$$W_1(u) = -4\pi\alpha \sum_{i\neq j} \vec{T}_i\cdot\vec{T}_j \int \frac{p_i \cdot p_j}{p_i\cdot q\, p_j \cdot q} (O_1(u)-1)\, du\, d\Phi_q \quad ?$$

No. Can not Factorize.!!

- color coherence

similar to QED. the Interference can be simplified in the Leading IR limit by angular ordering



$$= -g_s^2 \frac{p_i \cdot p_j}{p_i \cdot q \, p_j \cdot q} \langle M_0 | T_i \cdot T_j | M_0 \rangle$$

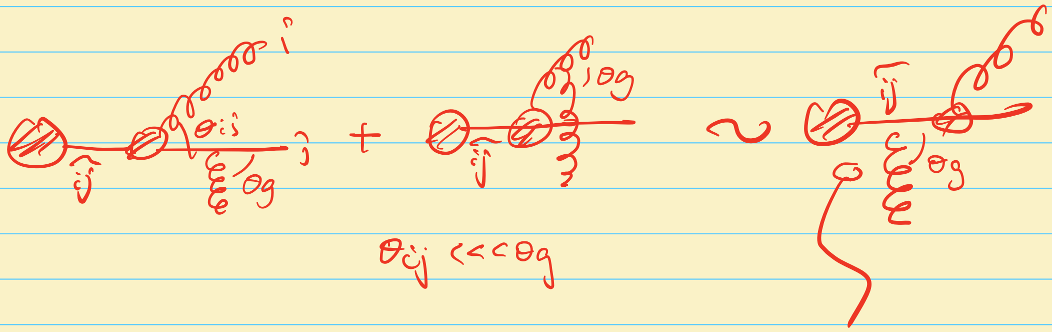
$$\simeq g_s \sum_i T_i^2 \frac{2}{E^2} \frac{1}{\theta_{iq}^2} \Theta(\theta_{max} - \theta_{iq}) |M_0|^2$$

$\sum_i T_i^2$ diagonal in the color space. c-#

does not "change" color

- QED-like Factorization
- angular ordering for Interference
- $T_i^2 = \begin{cases} C_F & \text{for } q \\ C_A & \text{for } g \end{cases}$

small angle radiation does not "change" the color charge.



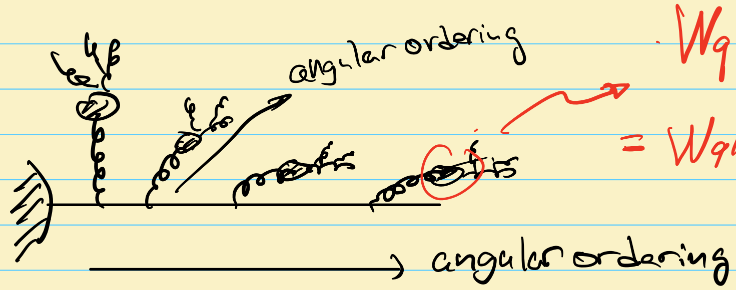
$\theta_{ij} \lll \theta_g$

$$T_i \frac{p^\mu}{p\cdot q} + T_j \frac{p^\mu}{p\cdot q} = \left\{\vec{T}_i + \vec{T}_j\right\} \frac{p^\mu}{p\cdot q} = \vec{T}_{\tilde{ij}} \frac{p^\mu}{p\cdot q}$$

as if radiated from the hard emitter $\tilde{ij}$

wide angle soft radiation only sees the overall color charge of the partons at smaller angles

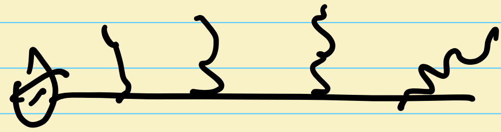
QCD muliple emission picture ⇒ coherent branching



$$W_q(V)\cdot M(p_1\cdots p_n, q)$$

$$= W_q(V)\,\frac{\alpha}{\pi}T_k^2\,\frac{dE_q}{E_q}\,\frac{d\theta_{qk}^2}{\theta_{qk}^2}\,\Theta_i(V)$$

QCD



$$W_k(V) = 1 + \frac{1}{n!}\prod_{n=0}^{\infty}\left\{\int\frac{\alpha_s}{\pi}T_k^2\frac{dE_q}{E_q}\int^{\theta_{max}}\frac{d\theta_{qk}^2}{\theta_{qk}^2}\left\{\Theta_i(V(q))\,W_q(V) - 1\right\}\right\}^n$$

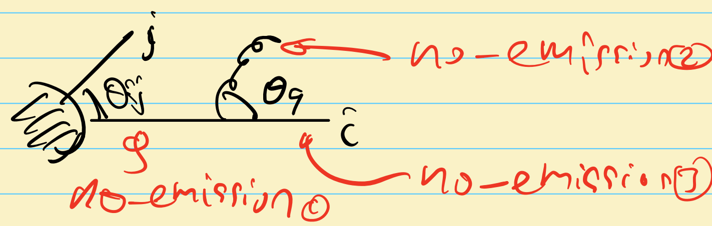
$$= \exp\left[\frac{\alpha_s}{\pi}T_k^2\int\frac{dE_q}{E_q}\int^{\theta_{max}}\frac{d\theta_{qk}^2}{\theta_{qk}^2}\left\{\Theta_i(V)\,W_q(V) - 1\right\}\right]$$

- the soft gluon as the new emitter.
- small angle $\theta_{max} \sim \theta_{qk}$ now

$$W(V) = \prod_k W_k(V), \qquad \sigma = \sigma_0 \cdot W(V)$$

- the fundament to parton shower

e.g. Prob. for one-emission



$$\underbrace{W_i(v=0,\theta_{ij},\theta_g)}_{\text{Sudakov ①}}\ \underbrace{\frac{\alpha_s}{\pi} T_i^2 \frac{dE_g}{E_g}}_{\text{1-emission prob.}}\ \frac{d\theta_g^2}{d\theta_g^2}\ \underbrace{W_g(v=0,\theta_g,0)}_{②}\ \underbrace{W_i(v=0,\theta_g,0)}_{③}$$

1-emission generated using Makov-chain

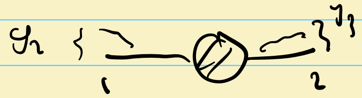
- can be improved to sub-leading logs

$$\frac{\alpha_s}{\pi} T_i^2 \frac{dE}{E}\frac{d\theta^2}{\theta^2} \to \frac{\alpha_s}{2\pi}\frac{d\theta^2}{\theta^2} P_{a\to ij}(z)\,dz$$

← splitting function

- applied to analytic Resummation

⊘ DL resumation for the thrust.



we have two branchings, and we consider $y_2 \to 0$. $y_3 \to 0$ can be included in the same way

We first consider the 1-gluon emission

Recall that for the thrust $\tau \ll 1$

$$\Theta_1(v) = \Theta(\delta-\tau)\,\delta(\tau-y_2)\,\Theta(y_3-y_2) + 2\leftrightarrow 3$$

$$+ \cancel{1\leftrightarrow 2} \to 0$$ $\rightsquigarrow$ does not lead to LL.

$$\simeq \Theta\Big(\delta - \underbrace{\frac{4E\cdot E_g}{4E^2}\frac{\theta^2}{2}}_{\tau}\Big)\,\Theta\Big(\frac{4EE_g}{4E^2}-\tau\Big)$$

$$= \Theta(\delta-\tau)\,\Theta\Big(\underbrace{\frac{E_g}{E}}_{z}-\tau\Big)$$

recall

$y_1 = \frac{s_{12}}{Q^2}$, $y_2 = \frac{s_{13}}{Q^2}$, $y_3 = \frac{s_{23}}{Q^2}$, $\tau = \min(y_1, y_2, y_3)$

Therefore we have for one-gluon case:

$$\frac{dE_g}{E_g}\frac{d\theta_g^2}{\theta_g^2}\,\Theta_1(v) = \int^1 \frac{d\tau}{\tau}\int_\tau^1 \frac{dz}{z}\,\Theta(\delta-\tau)$$

$$\mathcal{O}_n(v) \longrightarrow \prod_n A(v)$$

For n-emissions we have

$$\Theta_n(v) = \Theta\Big(\delta - \sum_i \tau_i\Big) \prod_i^n \delta(\tau_i - y_{2i})\,\Theta(y_{1i} - y_{2i})$$

$$= \underbrace{\Theta\Big(\delta - \sum_i \tau_i\Big)}_{\text{Does not factorize!!}} \underbrace{\prod_i^n (z_i - \tau)}_{\text{Factorized}}$$

Hence we have for n-emission

$$\frac{dE_i}{E_i}\frac{d\theta_i^2}{\theta_i^2}\,\Theta_n(v)$$

$$= \left\{ \prod_{i=1}^n \int^1 \frac{d\tau_i}{\tau_i} \int_{\tau_i}^1 \frac{dz_i}{z_i} \right\} \Theta\Big(\delta - \sum_i \tau_i\Big)$$

To factorize $\Theta(\delta - \sum_i \tau_i)$, we note that

$$\Theta\Big(\delta - \sum_i \tau_i\Big) = \frac{1}{2\pi i}\int \frac{d\nu}{\nu} e^{\nu\delta} e^{-\sum_i \tau_i \nu}$$

to find

$$\frac{dE_i}{E_i}\frac{d\theta_i^2}{\theta_i^2}\Theta_n(v)$$

$$= \frac{1}{2\pi i}\int\frac{d\nu}{\nu}e^{\nu\delta}\prod_{i=1}^{n}\int^1\frac{d\tau_i}{\tau_i}\int_{\tau_i}^1\frac{dz_i}{z_i}\underbrace{e^{-\tau_i\nu}}_{\Theta_1(v)}$$

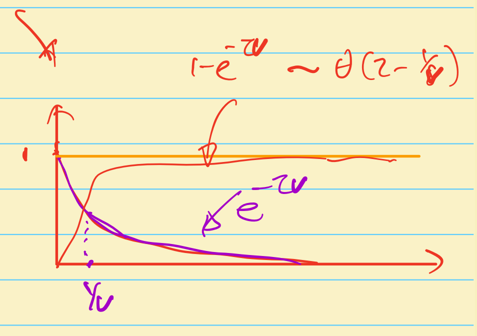
Now we plug it into $W(v)$

$$W(\delta) = \frac{1}{2\pi i}\int\frac{d\nu}{\nu}e^{\nu\delta}\exp\left[2\int^1\frac{d\tau}{\tau}\int_\tau^1\frac{dz}{z}\frac{\alpha_s}{\pi}C_F\left\{e^{-\tau\nu}-\underbrace{1}_{\text{virtual}}\right\}\right]$$

We further use the fact that

$$e^{-\tau\delta}-1 \simeq -\Theta(\tau - e^{-\gamma_E}\nu^{-1})$$

which holds up to NLL.



If we ignore the τ & z dependence in α_s which is sub-leading, we can have

$$W(\delta) = \frac{1}{2\pi i}\int \frac{d\nu}{\nu}\, e^{\nu\delta}\, e^{-\frac{\alpha_s}{\pi} C_F (\gamma_E + \log\nu)^2}$$

now we let $u = \nu\delta$ to have

$$W(\delta) = \frac{1}{2\pi i}\int du\, e^{u}\, e^{-\log u}\, e^{-\frac{\alpha_s}{\pi} C_F (\log u + \log e^{\gamma_E}\delta^{-1})^2}$$

$$= e^{-\frac{\alpha_s}{\pi} C_F \log^2 e^{\gamma_E}\delta^{-1}}\, \underbrace{\frac{1}{2\pi i}\int du\, e^{u}\, e^{-\log u}}_{\mathcal{O}(1)}\, e^{-\frac{\alpha_s}{\pi} C_F (\log^2 u + 2\ldots)}$$

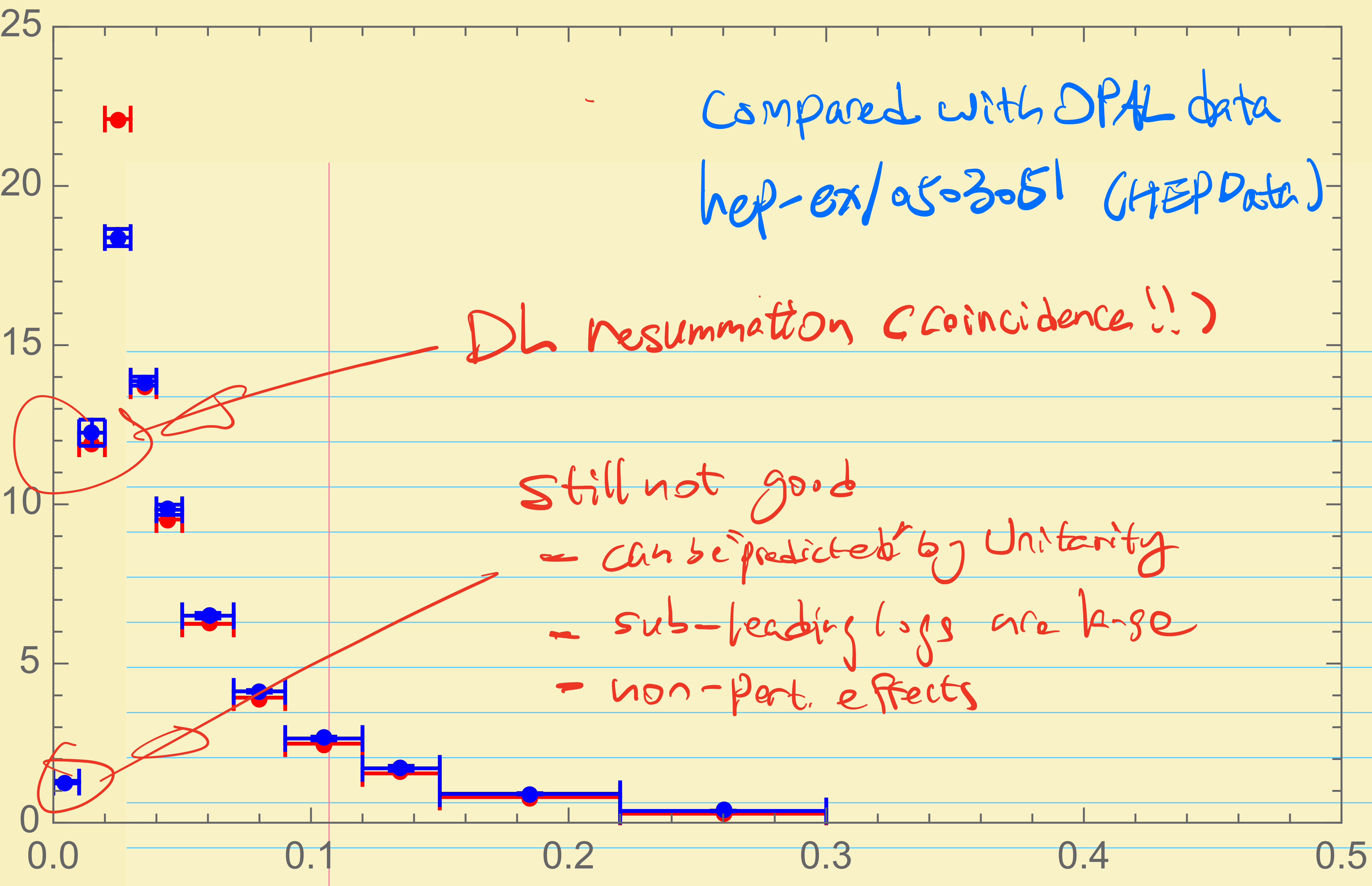
sub-leading in logs

$$\Rightarrow \frac{\sigma(\delta)}{\sigma_{tot}} = e^{-\frac{\alpha_s}{\pi} C_F \log^2\delta} \quad + \text{ sub-leading terms} \ldots$$

$$= 1 - \frac{\alpha_s}{\pi} C_F \log^2\delta + \underbrace{\left(\frac{\alpha_s}{\pi}\right)^2 C_F^2 \log^4\delta + \cdots}$$

reproduce the full NLO

predict higher orders


25
20
15
10
5
0
0.0
0.1
0.2
0.3
0.4
0.5
Compared with OPAL data
hep-ex/0503051 (HEPData)
DL resummation (coincidence!!)
Still not good
- can be "predicted" by Unitarity
- sub-leading logs are large
- non-pert. effects

Comment on the result:

- Can be improved to NLL by using the coherent branching to derive a RGE of the "Jet" Function

- DL = exp[NLO double log]

$$DL = \exp\left[\frac{2\alpha_s}{\pi}C_F\int_0^1 \frac{d\theta^2}{\theta^2}\int_\tau^1 \frac{dz}{z}\left\{\Theta(\delta-\tau)-1\right\}\right]$$

$$= 1 + \prod_{n\geq 1}^{\infty}\left[\frac{2\alpha_s}{\pi}C_F\int_0^1 \frac{d\theta^2}{\theta^2}\int_\tau^1 \frac{dz}{z}\left\{\Theta(\delta-\tau)-1\right\}\right]^n$$

with $\tau = z\theta^2/4$

Therefore at DL, we can replace

$$\Theta(\delta - \sum_i \tau_i) \longrightarrow \prod_{i=1}^{n}\Theta(\delta-\tau_i)$$

The emissions are all independent!

This holds in general for many observables