反问题方法

——非微扰计算的新方法

于福升

兰州大学

2022.11.26，　北大高能中心高能物理前沿中心讲座

Based on [Ao-Sheng Xiong, Ting Wei, FSY, arXiv:2211.xxxxx]

and [Ao-Sheng Xiong, Jian-Peng Wang, Ting Wei, FSY, unfinished]

感谢合作者五年来的支持与鼓励：李湘楠、徐繁荣、Hiroyuki Umeeda

感谢合作者三年来尤其最近在疫情下的辛苦工作：熊傲昇、汪建鹏、魏婷

感谢我的家人长期以来尤其最近两个月在疫情下的支持

感谢所有听众在疫情下的支持与批评指正

Outline

1. The foundation of the inverse problem approach

- The main idea of the inverse problem to non-perturbative QCD
- Dispersion relation and its inverse problem, ill-posedness and Tikhonov regularization, toy models
- Physical applications and discussions, perspectives

2. More details on the inverse problem approach

- Introduction and mathematical basis of the inverse problems
- $D^0 - \overline{D}^0$ mixing and its inverse problem.

Outline

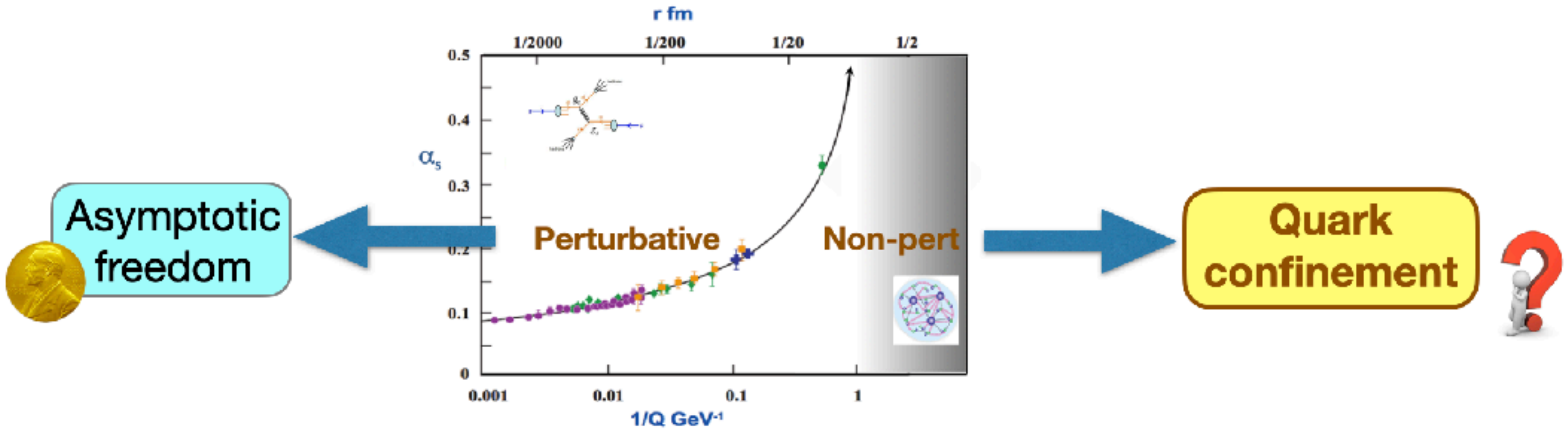
1. The foundation of the inverse problem approach

- The main idea of the inverse problem to non-perturbative QCD
- Dispersion relation and its inverse problem, ill-posedness and Tikhonov regularization, toy models
- Physical applications and discussions, perspectives

2. More details on the inverse problem approach

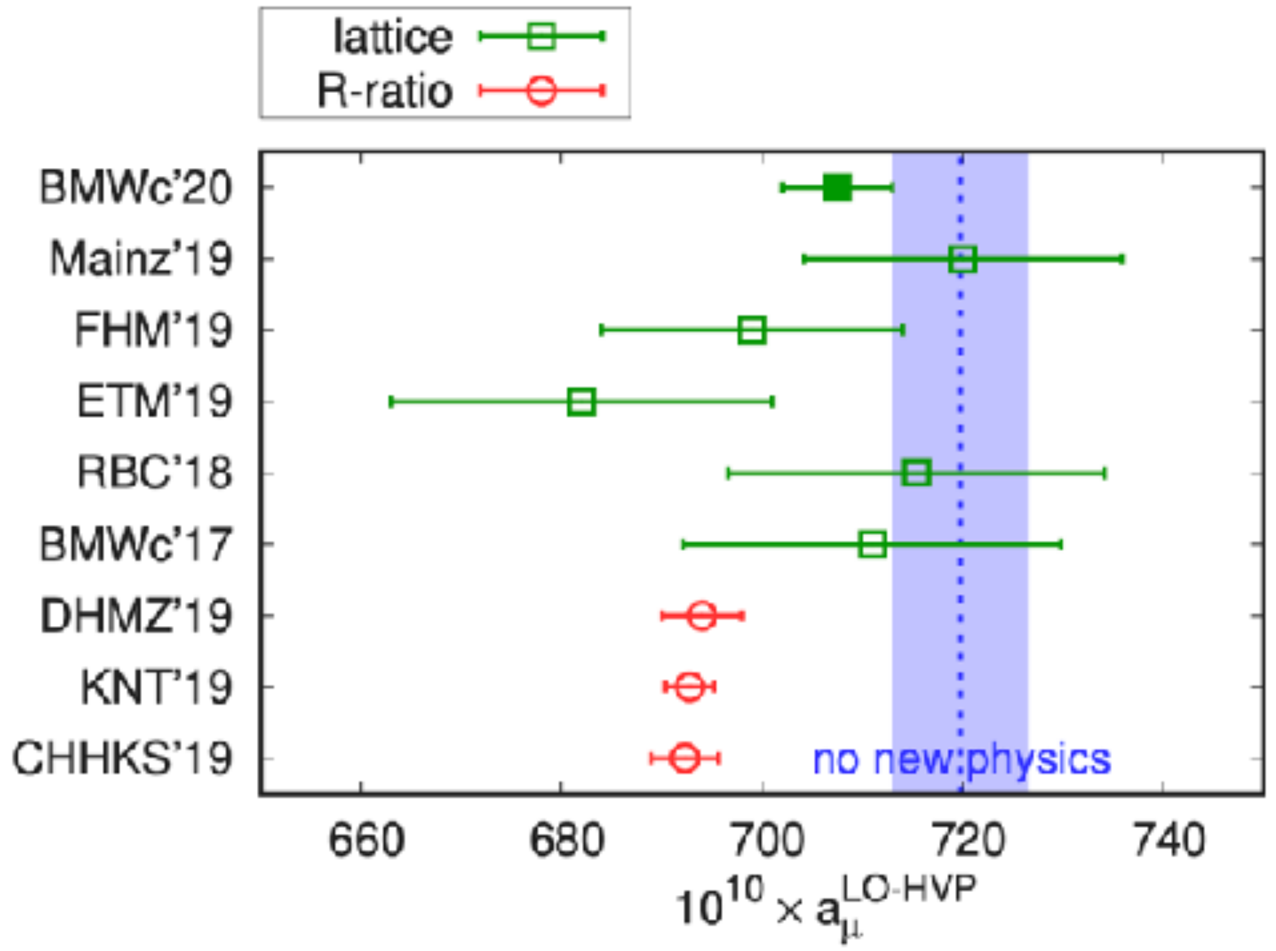
- Introduction and mathematical basis of the inverse problems
- $D^0 - \overline{D}^0$ mixing and its inverse problem.

Motivation: Problems of non-perturbation



Asymptotic freedom

Quark confinement



- **Particle physics: color confinement**
- **New physics**: muon g-2, $Br(B \to D^{(*)}\tau\nu)/Br(B \to D^{(*)}\ell\nu)$
- **Parton physics**: mass and spin of nucleon, PDF, GPD, TMD, LCDA
- **Hadron physics**: tetraquarks, pentaquark, glueballs
- **High energy nuclear physics**: QCD phase transition, critical point
- **Low energy nuclear physics**: nuclear force

Motivation: non-perturbative approaches

- Lattice QCD
- QCD sum rules
- Dyson-Schwinger Equation
- Chiral perturbation theory
- Other EFTs and phenomenological models

- Each of them has its advantages and shortcomings.
- It is always welcome to develop a new theoretical method for non-perturbation, to make complimentary predictions what are difficult by the above methods.

Criteria of a good theoretical approach

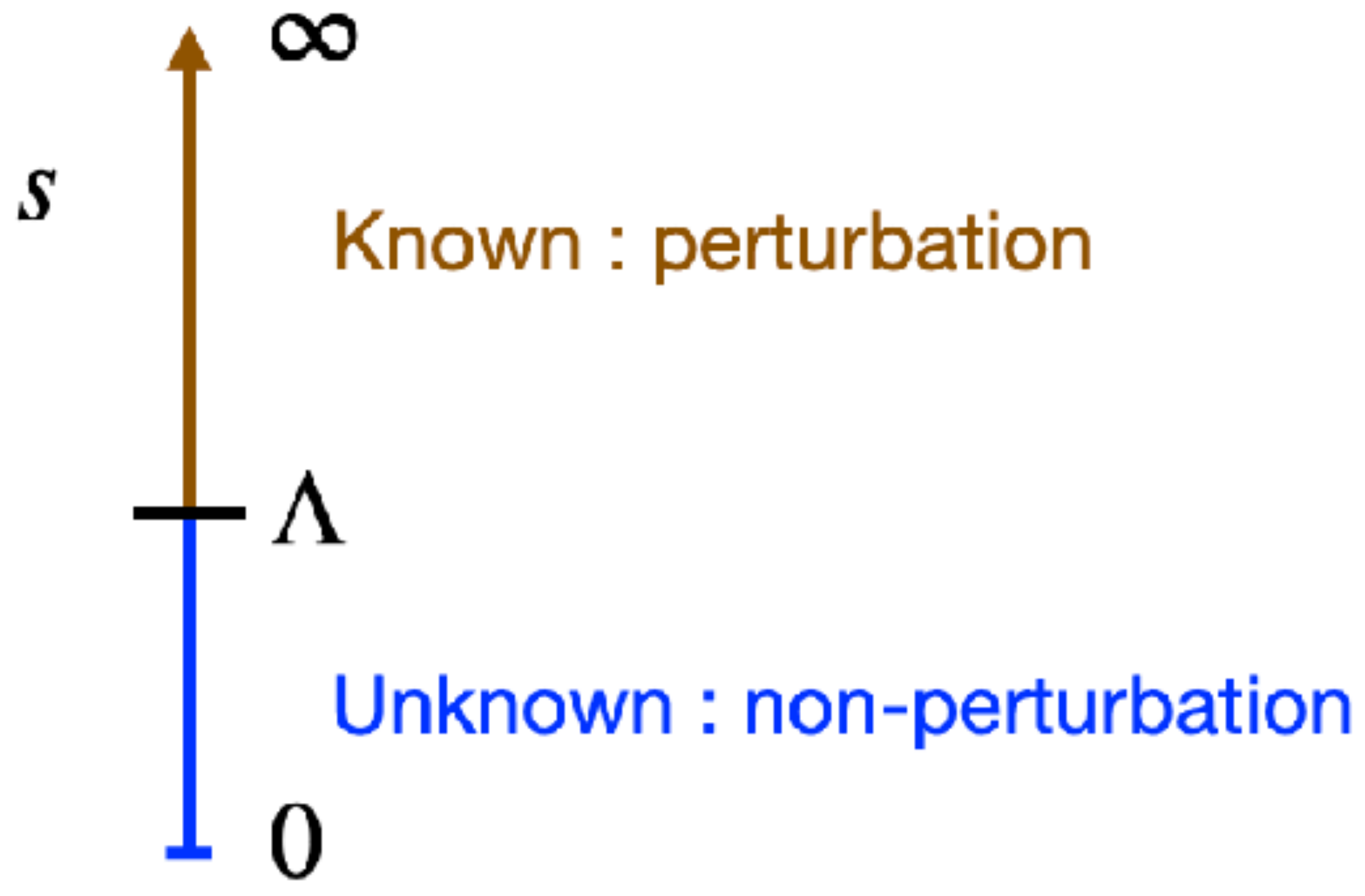
(1) Well defined in mathematics

(2) Realization in numerical calculations

(3) Can be systematically improved

(4) Simple at the beginning

The main idea of the inverse problem approach



∞

s

Known : perturbation

Λ

Unknown : non-perturbation

0

$$\Pi(q^2) = i\int d^4x e^{iq\cdot x}\,\langle O(x)O(0)\rangle$$

Dispersion Relation:

$$\mathcal{R}e[\Pi(s)] = \frac{1}{\pi}\mathcal{P}\int_0^{\infty}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

If $s > \Lambda$,

$$\mathcal{P}\int_0^{\Lambda}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_{\Lambda}^{\infty}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

The main idea of the inverse problem approach

If $s > \Lambda$,

$$\mathcal{P}\int_0^{\Lambda}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_{\Lambda}^{\infty}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

- With the dispersion relation of QFT, the non-perturbative quantities are obtained by solving the inverse problem with the perturbative calculations as inputs.
- Using the regularization method, the solutions are stable, and can be converged to the true value as the input errors approaching zero.
- The precision of the predictions can be systematically improved, without any artificially assumptions.

The main idea of the inverse problem approach

1. Dispersion relation and its inverse problem
2. Proof of ill-posedness
3. Regularization method
4. Test of some toy models
5. Physical discussions and perspectives

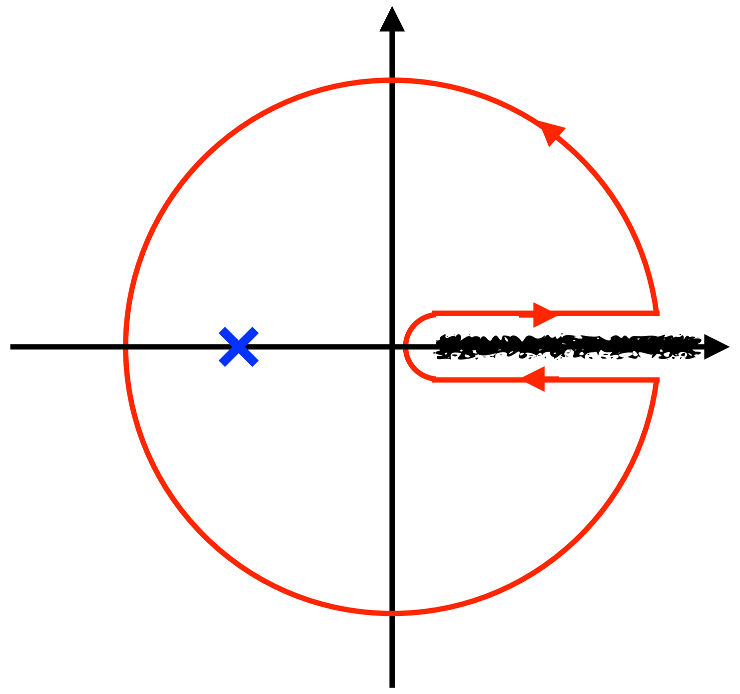
1. Dispersion relations and inverse problems

Dispersion relation:

- Based on Quantum Field Theory and correlation functions
- Analyticity of QFT, relation between a physical point and the curves, or relation between the real and imaginary parts

$$\Pi(q^2) = i\int d^4x e^{iq\cdot x} \langle \mathcal{O}(x)\mathcal{O}(0)\rangle$$

$$\mathrm{Re}[\Pi(s)] = \frac{1}{\pi}P\int_0^\infty \frac{\mathrm{Im}\,[\Pi\,(s')]}{s-s'}ds'$$



1. Dispersion relations and inverse problems

If $s > \Lambda$,

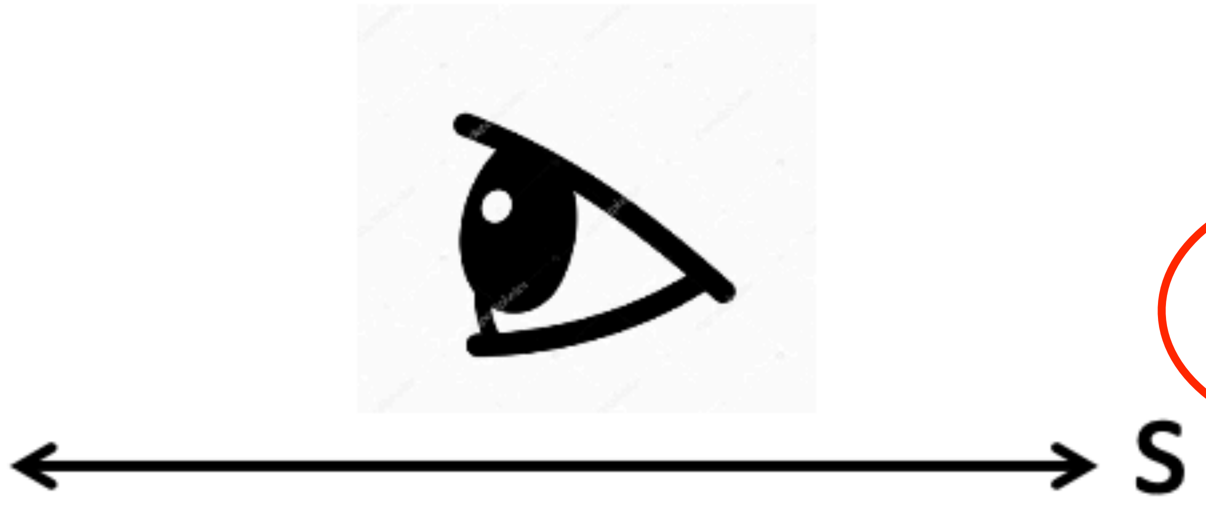
$$\mathcal{P}\int_0^\Lambda \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_\Lambda^\infty \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

"charge distribution" at low s

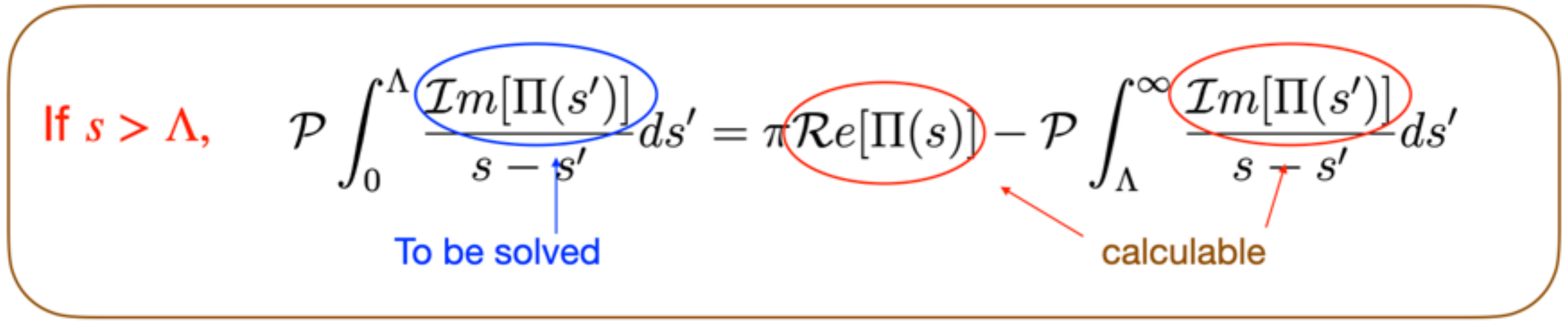
"potential" at high s



$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')e^{ik(\vec{x}-\vec{x}')}}{|\vec{x}-\vec{x}'|}dV'$$

2. Proof of the ill-posedness

- Dispersion relation: first-class Fredholm integration equation



If $s > \Lambda$, $\mathcal{P}\int_0^\Lambda \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_\Lambda^\infty \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$

To be solved

calculable

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

- Proof of the ill-posedness
  - Existence of solutions
  - Uniqueness of the solution
  - Instability of the solution

Most of inverse problems are ill-posed.

Solving such problems is non-trivial.

2. ill-posedness of the inverse problem

- Most of inverse problems are ill-posed

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5 \end{cases} \quad \longrightarrow \quad x_1 = 1, \;\; x_2 = 1$$

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5.001 \end{cases} \quad \longrightarrow \quad x_1 = -5, \;\; x_2 = 5$$

- A very small noise might cause a large change of solutions

2. ill-posedness of the inverse problem

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5 \end{cases} \quad \Rightarrow \quad x_1 = 1, \;\; x_2 = 1$$

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5.001 \end{cases} \quad \Rightarrow \quad x_1 = -5, \;\; x_2 = 5$$

- A very small noise might cause a large change of solutions

$$Ax = y, \qquad\qquad x = A^{-1}y$$

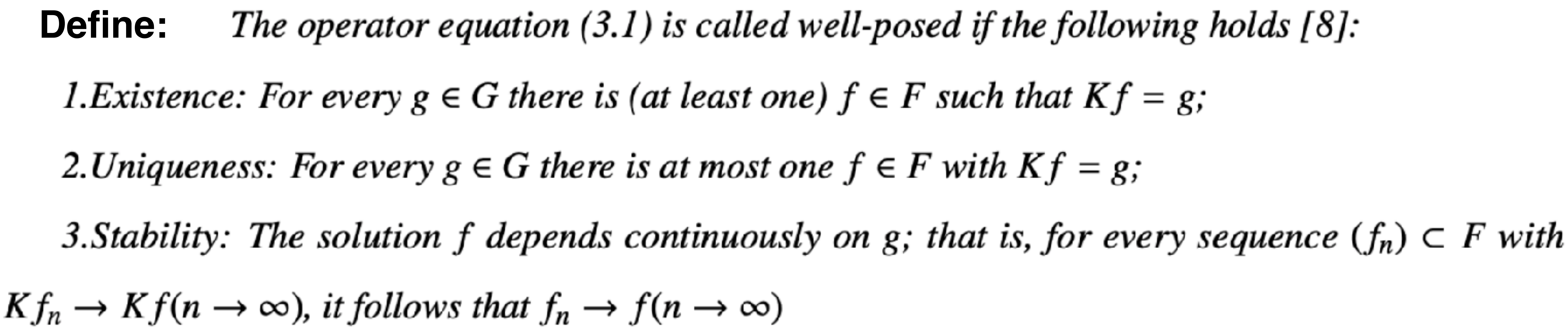
$$A = \begin{pmatrix} 2 & 3 \\ 1.9999 & 3.0001 \end{pmatrix}, \qquad |A| = 0.0005, \qquad A^{-1} = \frac{A^*}{|A|} = \begin{pmatrix} 6000.2 & -6000 \\ -3999.8 & 4000 \end{pmatrix}$$

A^{-1} enhances the errors

2. ill-posedness of the inverse problem

- The operator $K : X \to Y$, $Kx = y, \quad x \in X, \quad y \in Y$
- Inverse problem: solve x by known of K and y, $x = K^{-1}y$
- Definition of well-posedness:

Define:



The operator equation (3.1) is called well-posed if the following holds [8]:

1.Existence: For every $g \in G$ there is (at least one) $f \in F$ such that $Kf = g$;

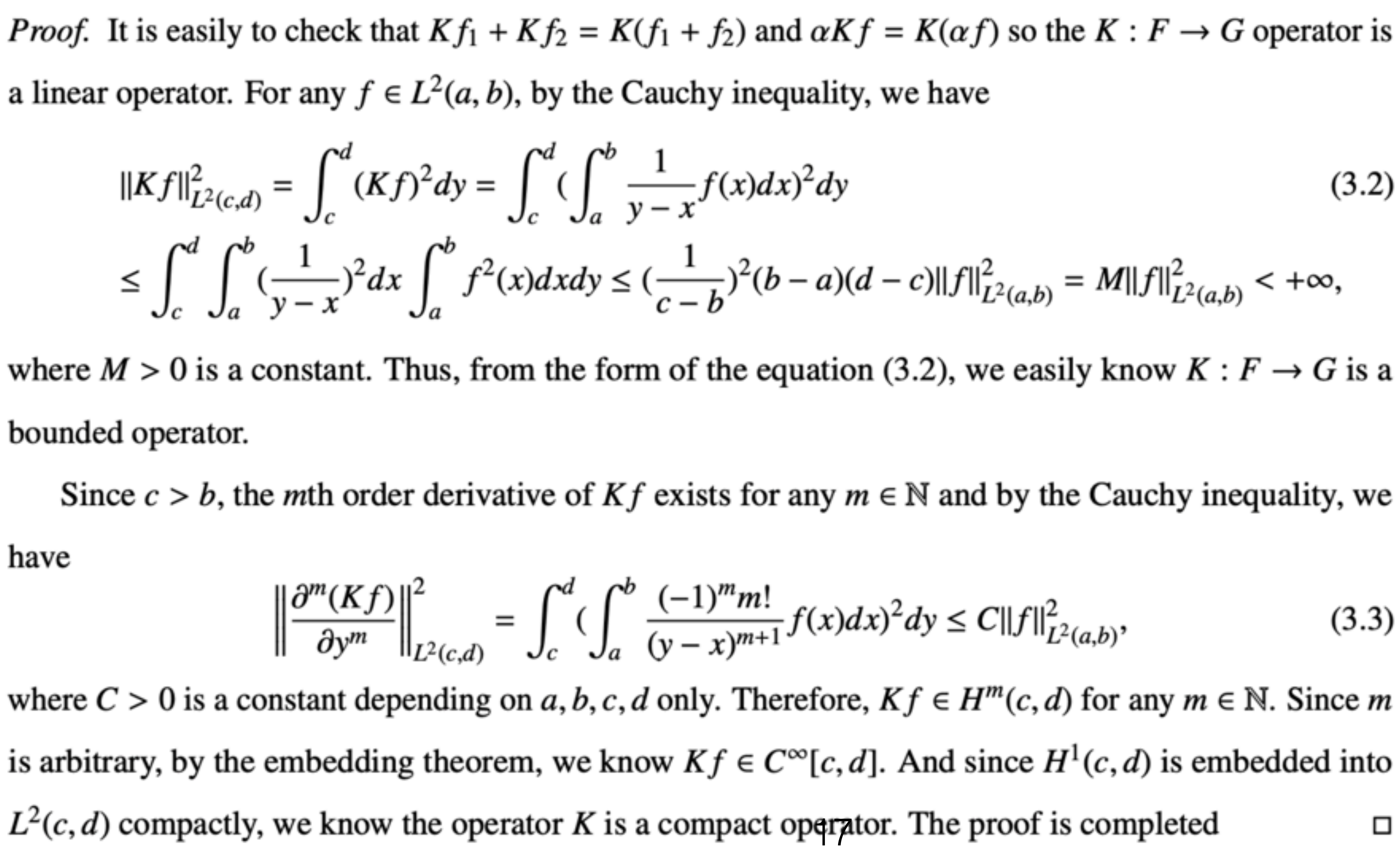
2.Uniqueness: For every $g \in G$ there is at most one $f \in F$ with $Kf = g$;

3.Stability: The solution f depends continuously on g; that is, for every sequence $(f_n) \subset F$ with $Kf_n \to Kf (n \to \infty)$, it follows that $f_n \to f (n \to \infty)$

- Ill-posedness: At least one of the above conditions is not satisfied
- If well-posed, K^{-1} must be a bounded or continuous operator, otherwise ill-posed.

2. ill-posedness of the inverse problem

- The operator $K : X \to Y$, $\qquad Kx = y, \quad x \in X, \quad y \in Y$
- The inverse problem of dispersion relation must be ill-posed.
- K is a linear bounded compact operator. It doesn't have a bounded inverse operator in the infinite dimensional space.



Proof. It is easily to check that $Kf_1 + Kf_2 = K(f_1 + f_2)$ and $\alpha Kf = K(\alpha f)$ so the $K : F \to G$ operator is a linear operator. For any $f \in L^2(a, b)$, by the Cauchy inequality, we have

$$\|Kf\|^2_{L^2(c,d)} = \int_c^d (Kf)^2 dy = \int_c^d (\int_a^b \frac{1}{y-x} f(x)dx)^2 dy \tag{3.2}$$
$$\leq \int_c^d \int_a^b (\frac{1}{y-x})^2 dx \int_a^b f^2(x)dxdy \leq (\frac{1}{c-b})^2 (b-a)(d-c)\|f\|^2_{L^2(a,b)} = M\|f\|^2_{L^2(a,b)} < +\infty,$$

where $M > 0$ is a constant. Thus, from the form of the equation (3.2), we easily know $K : F \to G$ is a bounded operator.

Since $c > b$, the mth order derivative of Kf exists for any $m \in \mathbb{N}$ and by the Cauchy inequality, we have

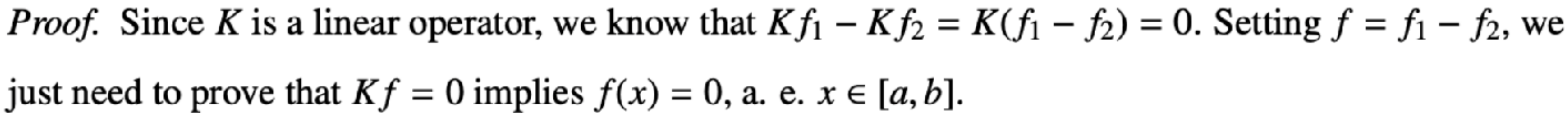
$$\left\|\frac{\partial^m (Kf)}{\partial y^m}\right\|^2_{L^2(c,d)} = \int_c^d (\int_a^b \frac{(-1)^m m!}{(y-x)^{m+1}} f(x)dx)^2 dy \leq C\|f\|^2_{L^2(a,b)}, \tag{3.3}$$

where $C > 0$ is a constant depending on a, b, c, d only. Therefore, $Kf \in H^m(c, d)$ for any $m \in \mathbb{N}$. Since m is arbitrary, by the embedding theorem, we know $Kf \in C^\infty[c, d]$. And since $H^1(c, d)$ is embedded into $L^2(c, d)$ compactly, we know the operator K is a compact operator. The proof is completed □

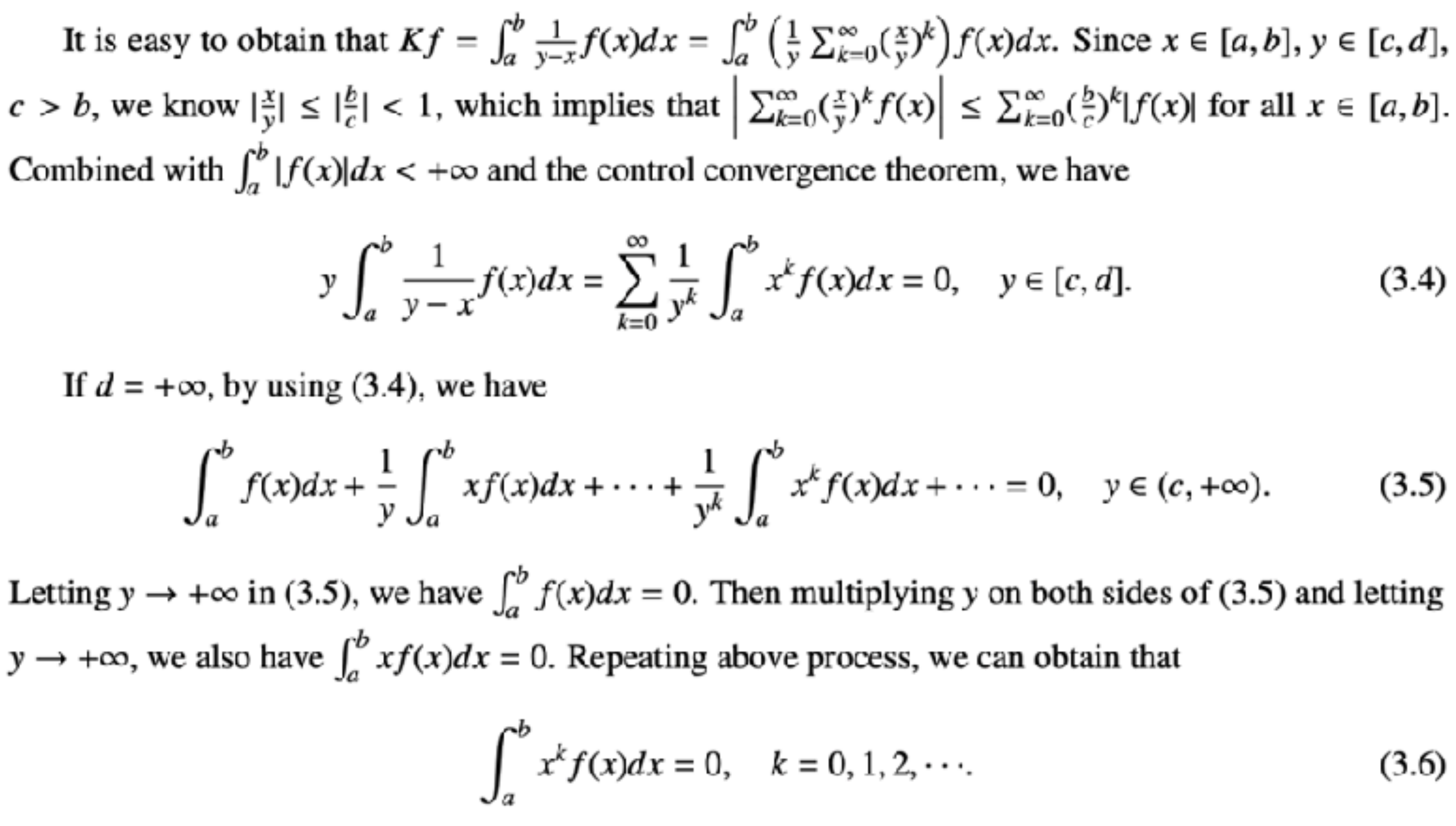
2. Proof of the ill-posedness

Proof of uniqueness:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$



Proof. Since K is a linear operator, we know that $Kf_1 - Kf_2 = K(f_1 - f_2) = 0$. Setting $f = f_1 - f_2$, we just need to prove that $Kf = 0$ implies $f(x) = 0$, a. e. $x \in [a,b]$.



It is easy to obtain that $Kf = \int_a^b \frac{1}{y-x} f(x)dx = \int_a^b \left(\frac{1}{y}\sum_{k=0}^{\infty}(\frac{x}{y})^k\right) f(x)dx$. Since $x \in [a,b]$, $y \in [c,d]$, $c > b$, we know $|\frac{x}{y}| \le |\frac{b}{c}| < 1$, which implies that $\left|\sum_{k=0}^{\infty}(\frac{x}{y})^k f(x)\right| \le \sum_{k=0}^{\infty}(\frac{b}{c})^k |f(x)|$ for all $x \in [a,b]$. Combined with $\int_a^b |f(x)|dx < +\infty$ and the control convergence theorem, we have

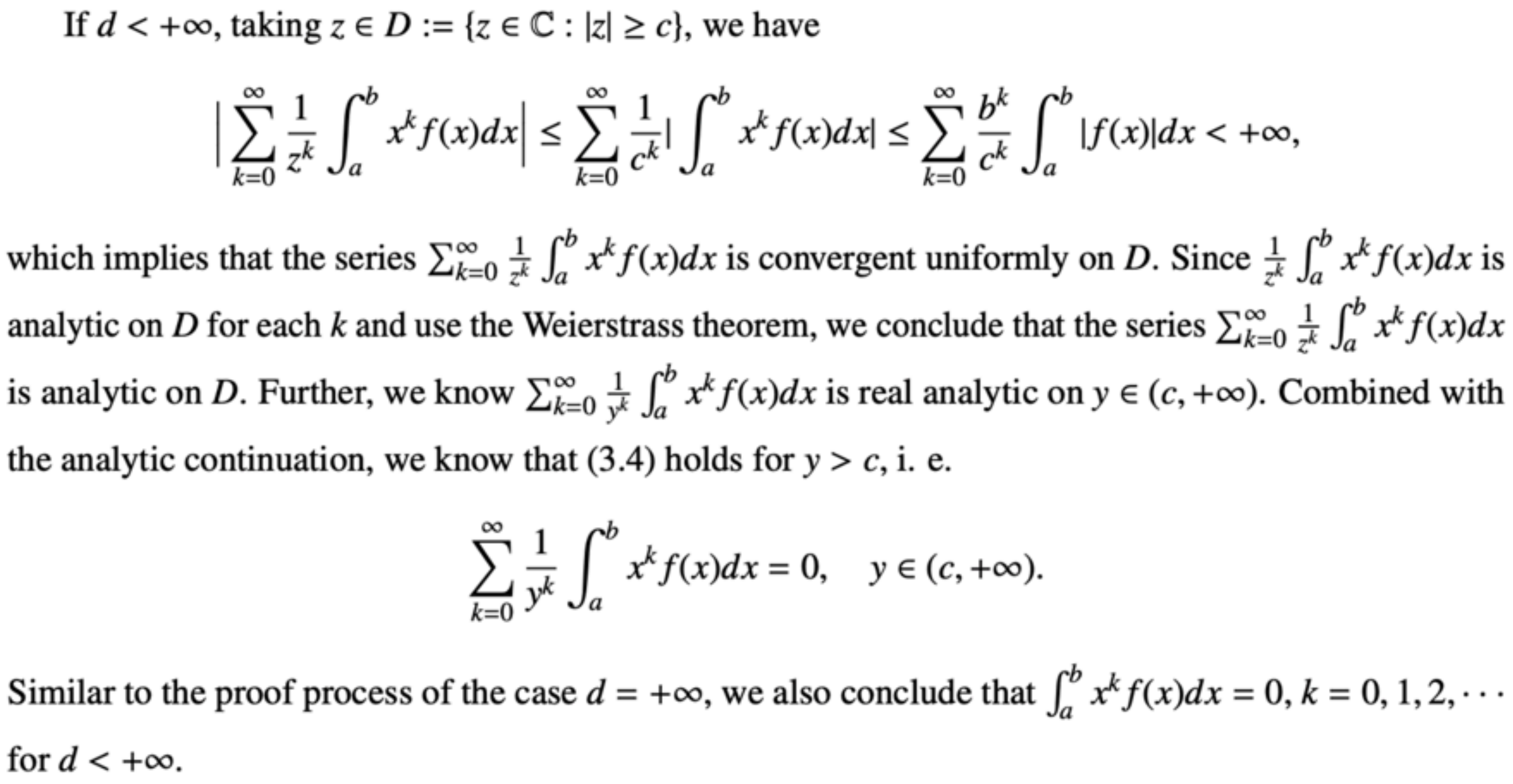
$$y\int_a^b \frac{1}{y-x} f(x)dx = \sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx = 0, \quad y \in [c,d]. \tag{3.4}$$

If $d = +\infty$, by using (3.4), we have

$$\int_a^b f(x)dx + \frac{1}{y}\int_a^b x f(x)dx + \cdots + \frac{1}{y^k}\int_a^b x^k f(x)dx + \cdots = 0, \quad y \in (c, +\infty). \tag{3.5}$$

Letting $y \to +\infty$ in (3.5), we have $\int_a^b f(x)dx = 0$. Then multiplying y on both sides of (3.5) and letting $y \to +\infty$, we also have $\int_a^b x f(x)dx = 0$. Repeating above process, we can obtain that

$$\int_a^b x^k f(x)dx = 0, \quad k = 0, 1, 2, \cdots. \tag{3.6}$$



If $d < +\infty$, taking $z \in D := \{z \in \mathbb{C} : |z| \ge c\}$, we have

$$\left|\sum_{k=0}^{\infty}\frac{1}{z^k}\int_a^b x^k f(x)dx\right| \le \sum_{k=0}^{\infty}\frac{1}{c^k}|\int_a^b x^k f(x)dx| \le \sum_{k=0}^{\infty}\frac{b^k}{c^k}\int_a^b |f(x)|dx < +\infty,$$

which implies that the series $\sum_{k=0}^{\infty}\frac{1}{z^k}\int_a^b x^k f(x)dx$ is convergent uniformly on D. Since $\frac{1}{z^k}\int_a^b x^k f(x)dx$ is analytic on D for each k and use the Weierstrass theorem, we conclude that the series $\sum_{k=0}^{\infty}\frac{1}{z^k}\int_a^b x^k f(x)dx$ is analytic on D. Further, we know $\sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx$ is real analytic on $y \in (c, +\infty)$. Combined with the analytic continuation, we know that (3.4) holds for $y > c$, i. e.

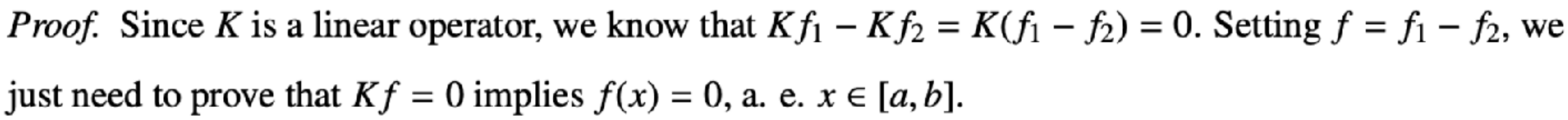
$$\sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx = 0, \quad y \in (c, +\infty).$$

Similar to the proof process of the case $d = +\infty$, we also conclude that $\int_a^b x^k f(x)dx = 0$, $k = 0, 1, 2, \cdots$ for $d < +\infty$.

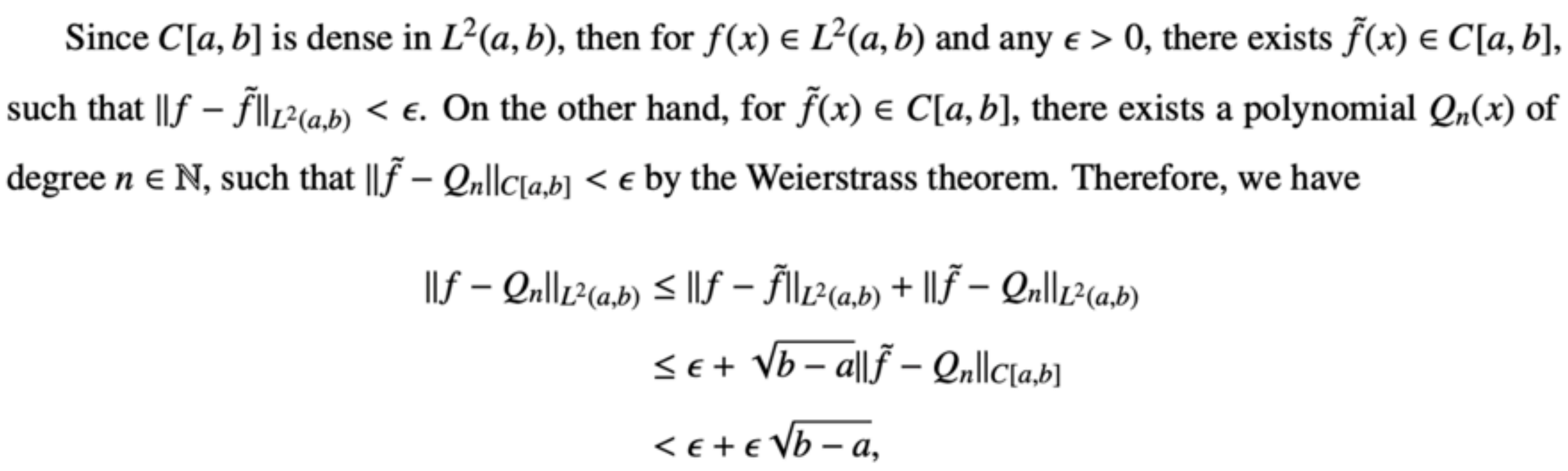
2. Proof of the ill-posedness

Proof of uniqueness:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

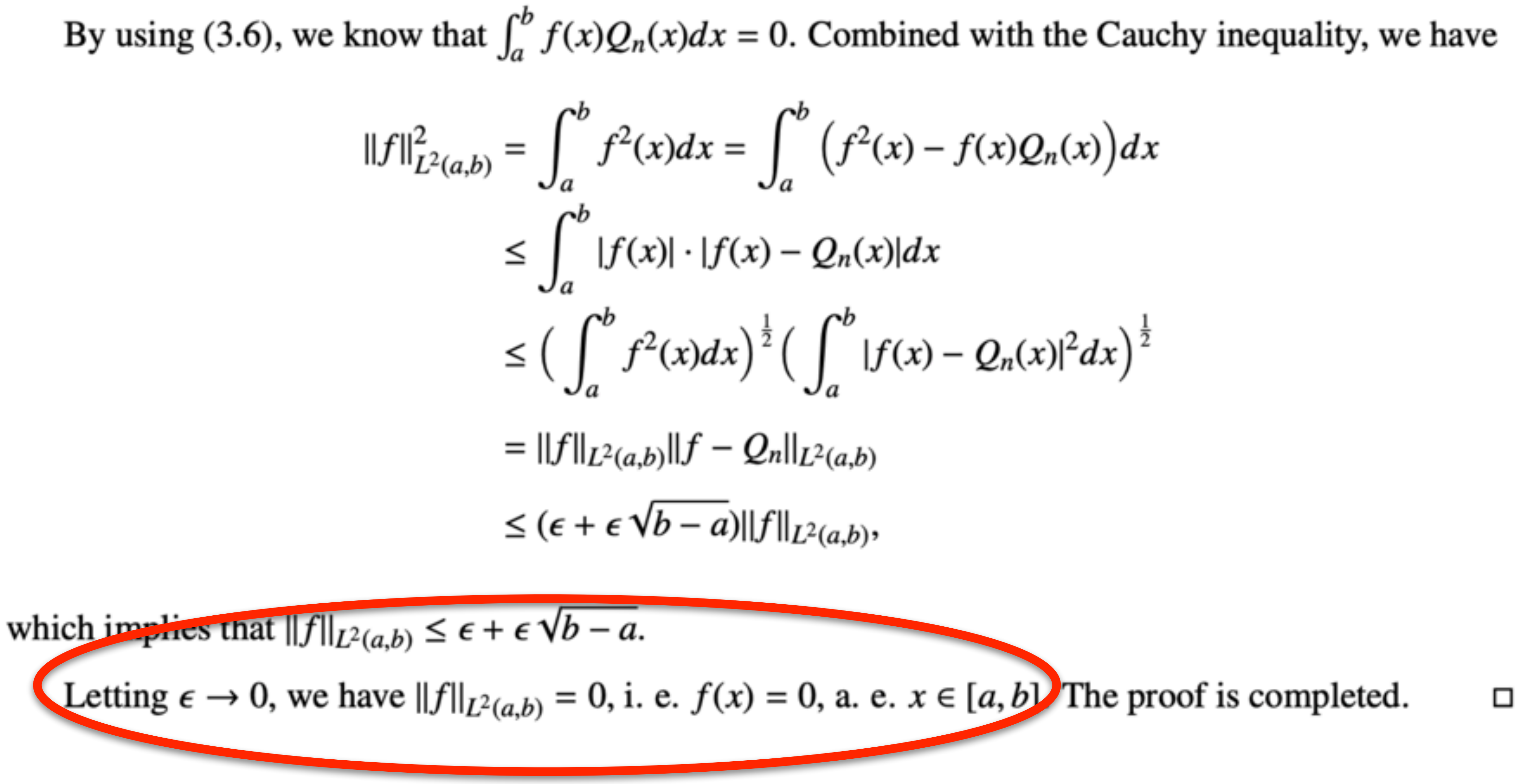


Proof. Since K is a linear operator, we know that $Kf_1 - Kf_2 = K(f_1 - f_2) = 0$. Setting $f = f_1 - f_2$, we just need to prove that $Kf = 0$ implies $f(x) = 0$, a. e. $x \in [a,b]$.



Since $C[a,b]$ is dense in $L^2(a,b)$, then for $f(x) \in L^2(a,b)$ and any $\epsilon > 0$, there exists $\tilde{f}(x) \in C[a,b]$, such that $\|f - \tilde{f}\|_{L^2(a,b)} < \epsilon$. On the other hand, for $\tilde{f}(x) \in C[a,b]$, there exists a polynomial $Q_n(x)$ of degree $n \in \mathbb{N}$, such that $\|\tilde{f} - Q_n\|_{C[a,b]} < \epsilon$ by the Weierstrass theorem. Therefore, we have

$$\begin{aligned}\|f - Q_n\|_{L^2(a,b)} &\le \|f - \tilde{f}\|_{L^2(a,b)} + \|\tilde{f} - Q_n\|_{L^2(a,b)}\\ &\le \epsilon + \sqrt{b-a}\|\tilde{f} - Q_n\|_{C[a,b]}\\ &< \epsilon + \epsilon\sqrt{b-a},\end{aligned}$$



By using (3.6), we know that $\int_a^b f(x)Q_n(x)dx = 0$. Combined with the Cauchy inequality, we have

$$\begin{aligned}\|f\|^2_{L^2(a,b)} &= \int_a^b f^2(x)dx = \int_a^b \left(f^2(x) - f(x)Q_n(x)\right)dx\\ &\le \int_a^b |f(x)|\cdot|f(x) - Q_n(x)|dx\\ &\le \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}\left(\int_a^b |f(x) - Q_n(x)|^2dx\right)^{\frac{1}{2}}\\ &= \|f\|_{L^2(a,b)}\|f - Q_n\|_{L^2(a,b)}\\ &\le (\epsilon + \epsilon\sqrt{b-a})\|f\|_{L^2(a,b)},\end{aligned}$$

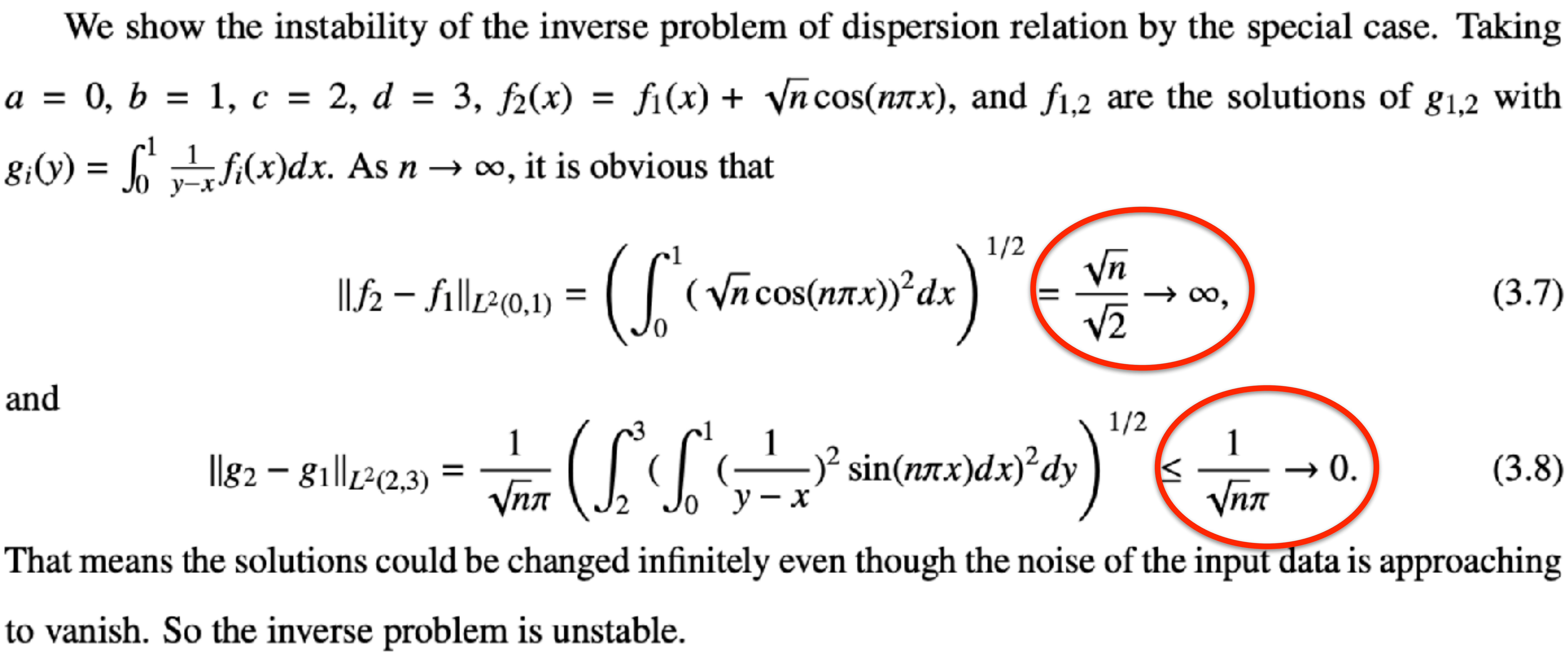
which implies that $\|f\|_{L^2(a,b)} \le \epsilon + \epsilon\sqrt{b-a}$.

Letting $\epsilon \to 0$, we have $\|f\|_{L^2(a,b)} = 0$, i. e. $f(x) = 0$, a. e. $x \in [a,b]$. The proof is completed. ☐

2. Proof of the ill-posedness

Proof of instability:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$



We show the instability of the inverse problem of dispersion relation by the special case. Taking $a = 0$, $b = 1$, $c = 2$, $d = 3$, $f_2(x) = f_1(x) + \sqrt{n}\cos(n\pi x)$, and $f_{1,2}$ are the solutions of $g_{1,2}$ with $g_i(y) = \int_0^1 \frac{1}{y-x} f_i(x)dx$. As $n \to \infty$, it is obvious that

$$\|f_2 - f_1\|_{L^2(0,1)} = \left(\int_0^1 (\sqrt{n}\cos(n\pi x))^2 dx\right)^{1/2} = \frac{\sqrt{n}}{\sqrt{2}} \to \infty, \tag{3.7}$$

and

$$\|g_2 - g_1\|_{L^2(2,3)} = \frac{1}{\sqrt{n\pi}}\left(\int_2^3 (\int_0^1 (\frac{1}{y-x})^2 \sin(n\pi x)dx)^2 dy\right)^{1/2} \leq \frac{1}{\sqrt{n\pi}} \to 0. \tag{3.8}$$

That means the solutions could be changed infinitely even though the noise of the input data is approaching to vanish. So the inverse problem is unstable.

2. Proof of the ill-posedness

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

1) Existence ✓

2) Uniqueness ✓

3) Stability ✗

The inverse problem of dispersion relation is ill-posed

How to find a good solution is the most important issue.

The main idea of the inverse problem approach

1. Dispersion relation and its inverse problem: Well defined
2. Proof of ill-posedness: Instability
3. Regularization method
4. Test of some toy models
5. Physical discussions and perspectives

3. Regularization method

Define: *A regularization strategy is a family of linear and bounded operators $R_\alpha : G \to F, \alpha > 0$, such that $\lim_{\alpha \to 0} R_\alpha K f = f$ for all $f \in F$, where the α is the regularization parameter [8].*

- Construct a bounded operator which is approximate to K^{-1},
- Ill-posed problem -> well-posed approximate problem, so that $f_\alpha^\delta = R_\alpha g^\delta$
- f_α^δ is the approximate solution related to both α and δ.
- An effective regularization strategy is to satisfy $f_\alpha^\delta \to f$, as $\|g^\delta - g\| \le \delta \to 0$

$$\begin{aligned}\left\|f_\alpha^\delta - f\right\|_F &\le \left\|R_\alpha g^\delta - R_\alpha g\right\|_F + \|R_\alpha g - f\|_F \\ &\le \|R_\alpha\| \left\|g^\delta - g\right\|_G + \|R_\alpha K f - f\|_F \\ &\le \underset{\substack{\downarrow \\ \infty}}{\delta \|R_\alpha\|} + \underset{\substack{\downarrow \\ 0}}{\|R_\alpha K f - f\|_F} \qquad \alpha \to 0\end{aligned}$$

$$Kf = g, f \in F, g \in G$$

- To keep a balance, α can be neither too large nor too small

3. Tikhonov Regularization

$$R_\alpha := (\alpha I + K^* K)^{-1} K^* : G \to F$$

$$\alpha f_\alpha^\delta + K^* K f_\alpha^\delta = K^* g^\delta$$

$$\boxed{f_\alpha^\delta = \underset{f \in L^2(a,b)}{\arg\min}\, J(f), \quad J(f) = \frac{1}{2}\|Kf - g^\delta\|^2_{L^2(c,d)} + \frac{\alpha}{2}\|f\|^2_{L^2(a,b)}}$$

$$\left\|f_\alpha^\delta - f\right\|_F \leq \delta\,\|R_\alpha\| + \|R_\alpha K f - f\|_F$$

$$\|f_\alpha^\delta - f\|_F \leq \frac{\delta}{2\sqrt{\alpha}} + \frac{\sqrt{\alpha}E}{2}$$

A priori condition: $f = K^* v,\ v \in G,\ \|v\|_G \leq E$

Take $\alpha = \delta/E$

$$\boxed{\|f_\alpha^\delta - f\|_F \leq \sqrt{\delta E} \to 0, \quad \delta \to 0}$$

- **The most important: the uncertainty converges to vanishing as $\delta \to 0$**

3. Selection rules of the Regularization parameter

A-priori methods are always difficult to use in practice.

A-posterior methods can be tried.

L-curve method: $$\alpha = \underset{f_\alpha^\delta \in L^2(a,b)}{\arg\min} \left(\left\| f_\alpha^\delta \right\|_F \left\| g^\delta - K f_\alpha^\delta \right\|_G \right)$$

Both of $||f_\alpha^\delta||$ and $||g^\delta - Kf_\alpha^\delta||$ should be minimized together,

considering $$f_\alpha^\delta = \underset{f \in L^2(a,b)}{\arg\min} J(f), \quad J(f) = \frac{1}{2}||Kf - g^\delta||_{L^2(c,d)}^2 + \frac{\alpha}{2}||f||_{L^2(a,b)}^2$$

The main idea of the inverse problem approach

1. Dispersion relation and its inverse problem: Well defined
2. Proof of ill-posedness: Instability
3. Regularization method: Tikhonov
4. Test of some toy models
5. Physical discussions and perspectives

4. Test of Toy Models

- Questions on the inverse problem approach:

(1) **Regularization:** How important are the regularization methods? Can the solutions be systematically improved by the regularization method and the method of selecting the regularization parameter?

(2) **Impact of input uncertainties:** What is the dependence of the errors of solutions on the uncertainties of inputs? Larger, smaller or similar?

(3) **Impact of α and Λ:** How sensitive are the solutions to the parameters α and Λ? Does it exist a plateau?

(4) **Impact of more conditions:** Can the solutions be improved if we known more conditions?

4. Test of Toy Models

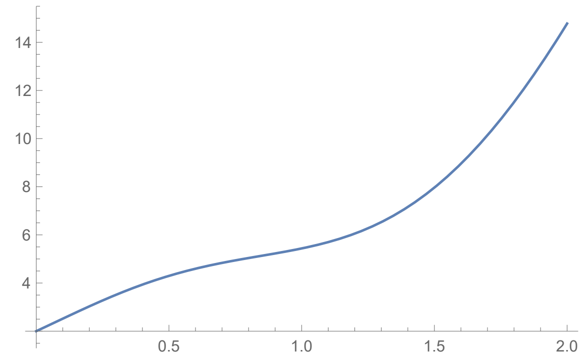
- Simple at the beginning: Tikhonov regularization + L-curve method for the regulator
  - They are simple in mathematics and in practice and thus are very helpful to develop the new approach in the future.
- Uncertainties are the most important issue. $b_i = \mu_i \pm \sigma_i$

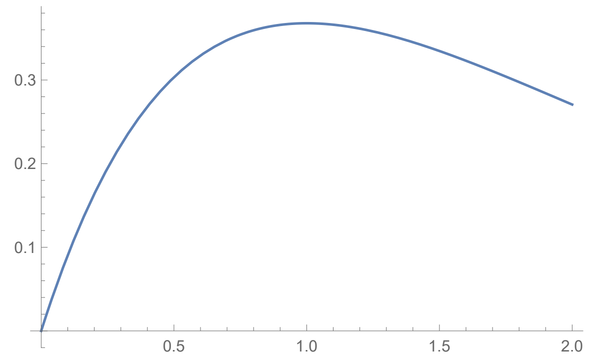
$$f(x) = a_1 f_1(x) + a_2 f_2(x) \qquad g^{\delta}(y) = b_1 g_1(y) + b_2 g_2(y) \qquad g_i(y) = \int_a^b \frac{f_i(x)}{y-x} dx.$$

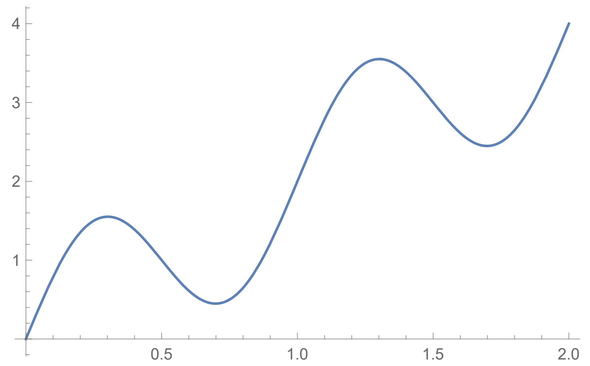
Model 1: a monotonic function as $f_1(x) = \sin(\pi x)$, $f_2(x) = e^x$;

Model 2: a simple non-monotonic function as $f_1(x) = xe^{-x}$, $f_2(x) = 0$;

Model 3: an oscillating function as $f_1(x) = \sin(2\pi x)$, $f_2(x) = x$.



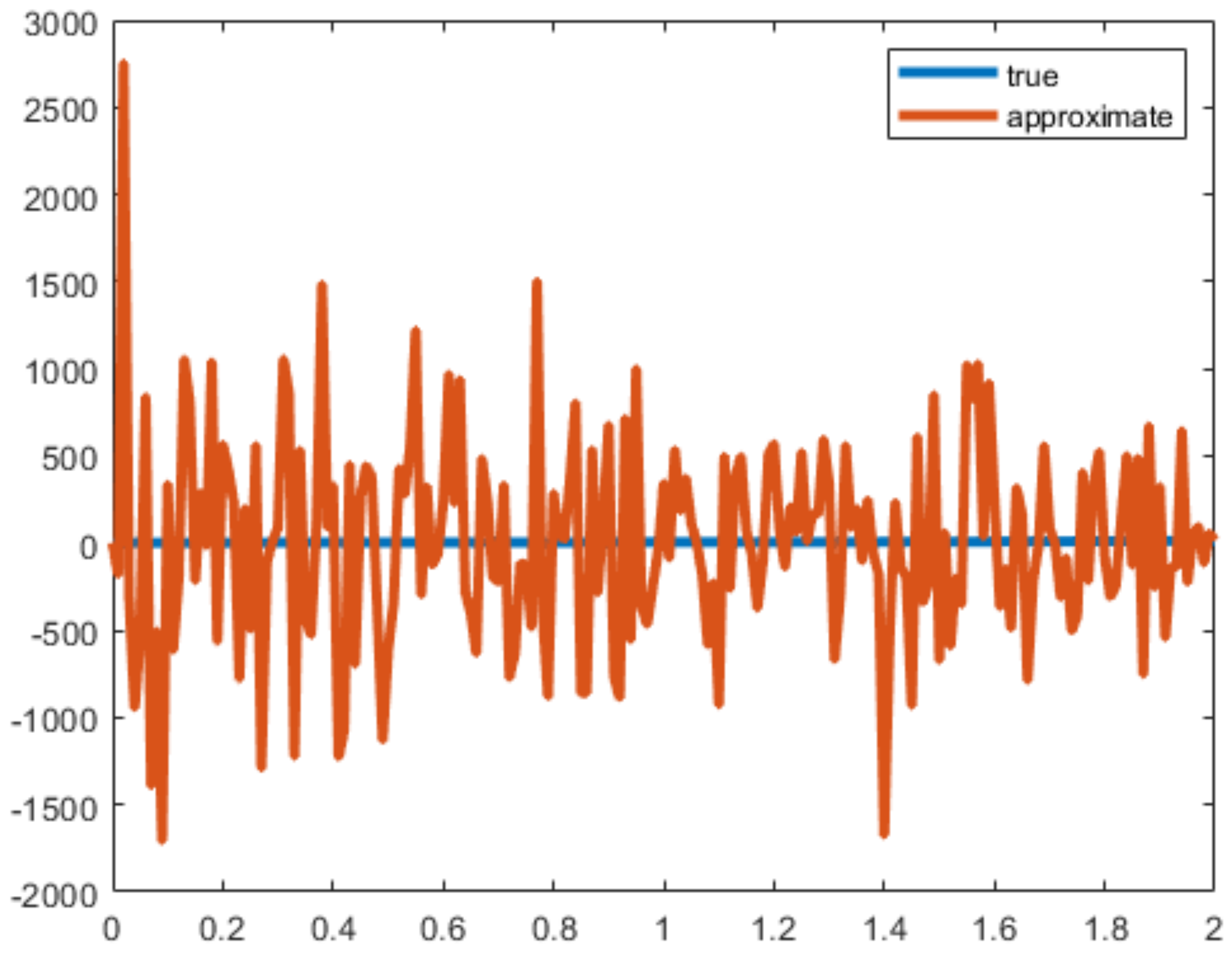




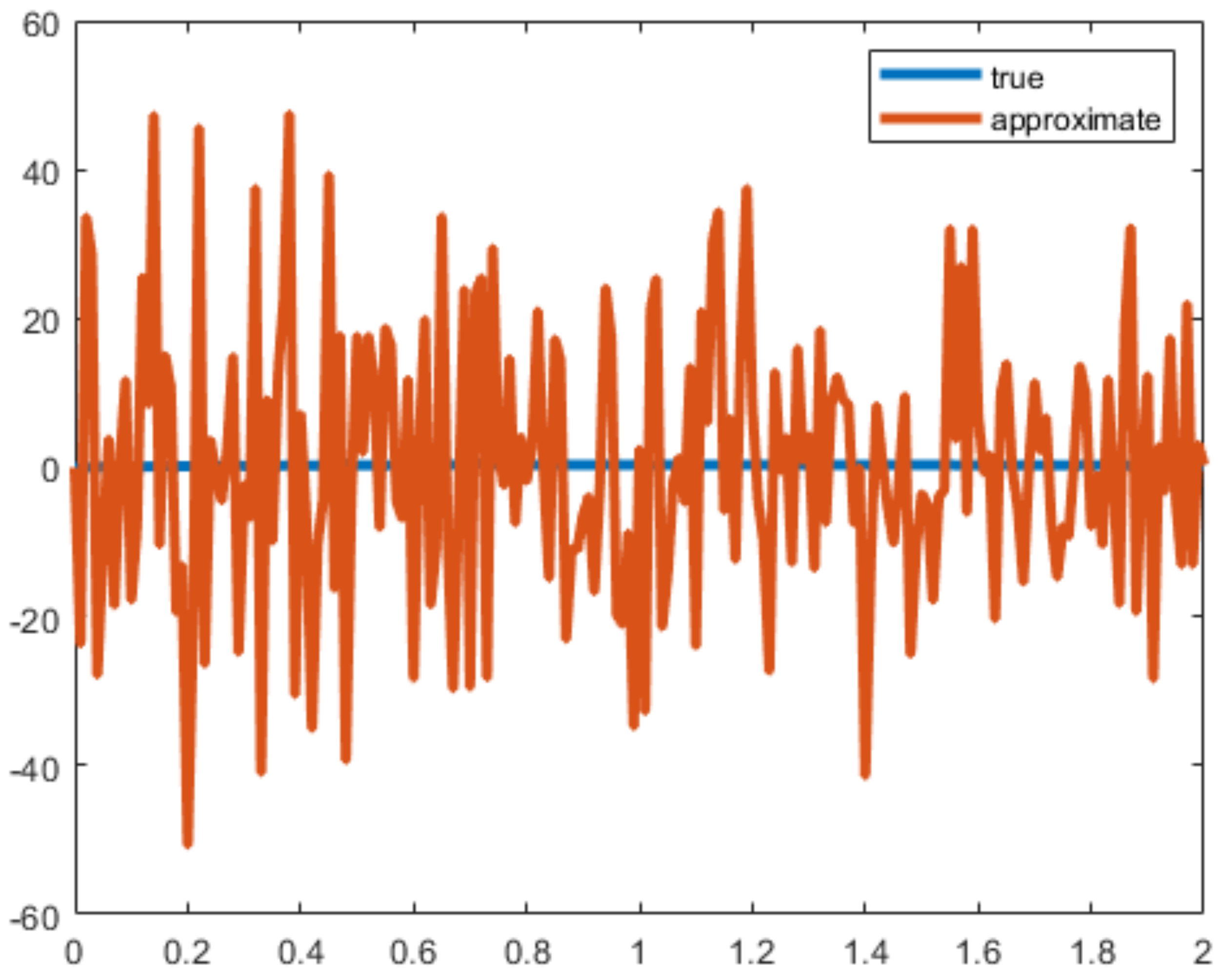
They are either helpful to clarify the properties of inverse problems or close to the real physical problem

4. Test: Importance of regularization

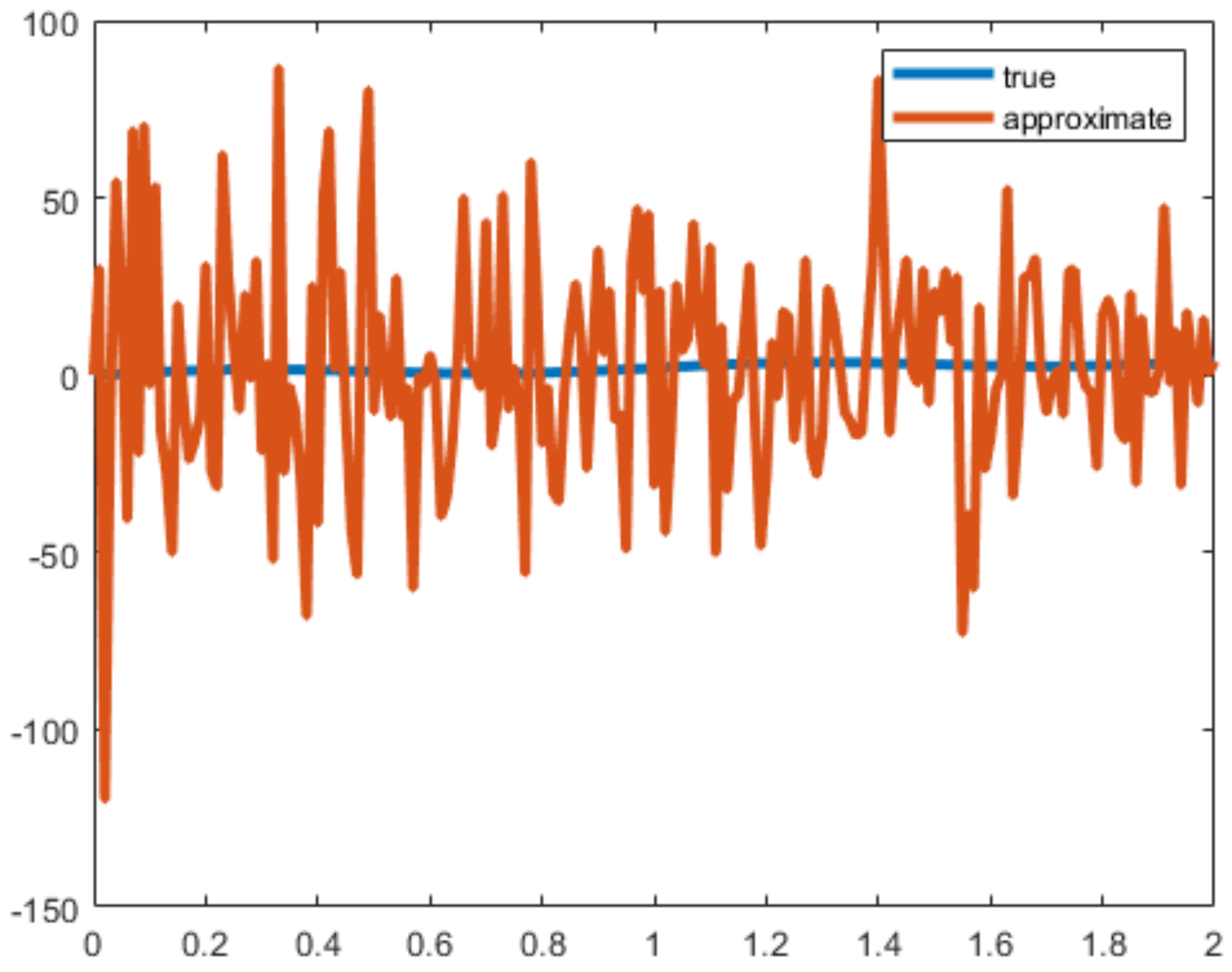
The solutions without any regularization:



model 1



model 2

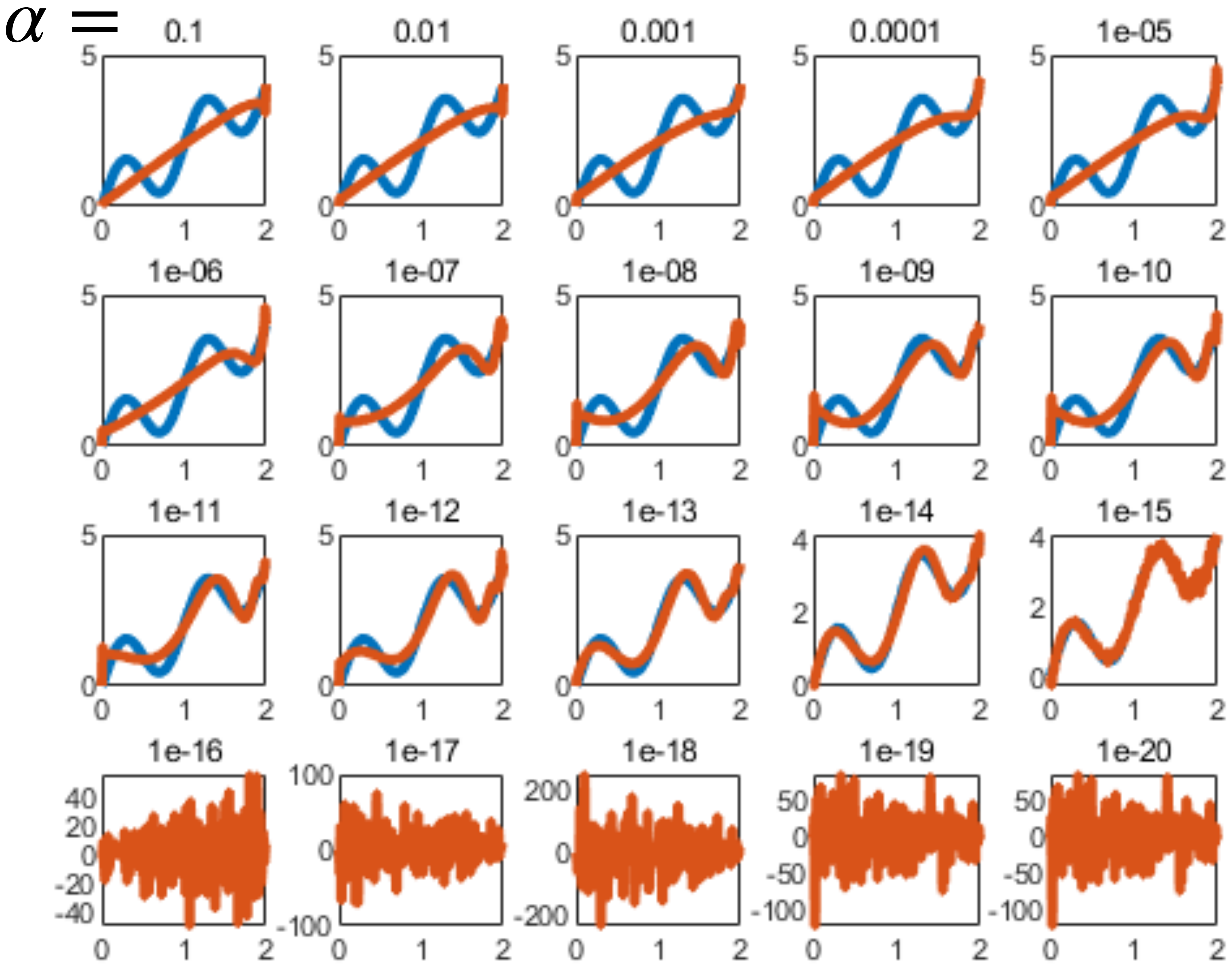


model 3

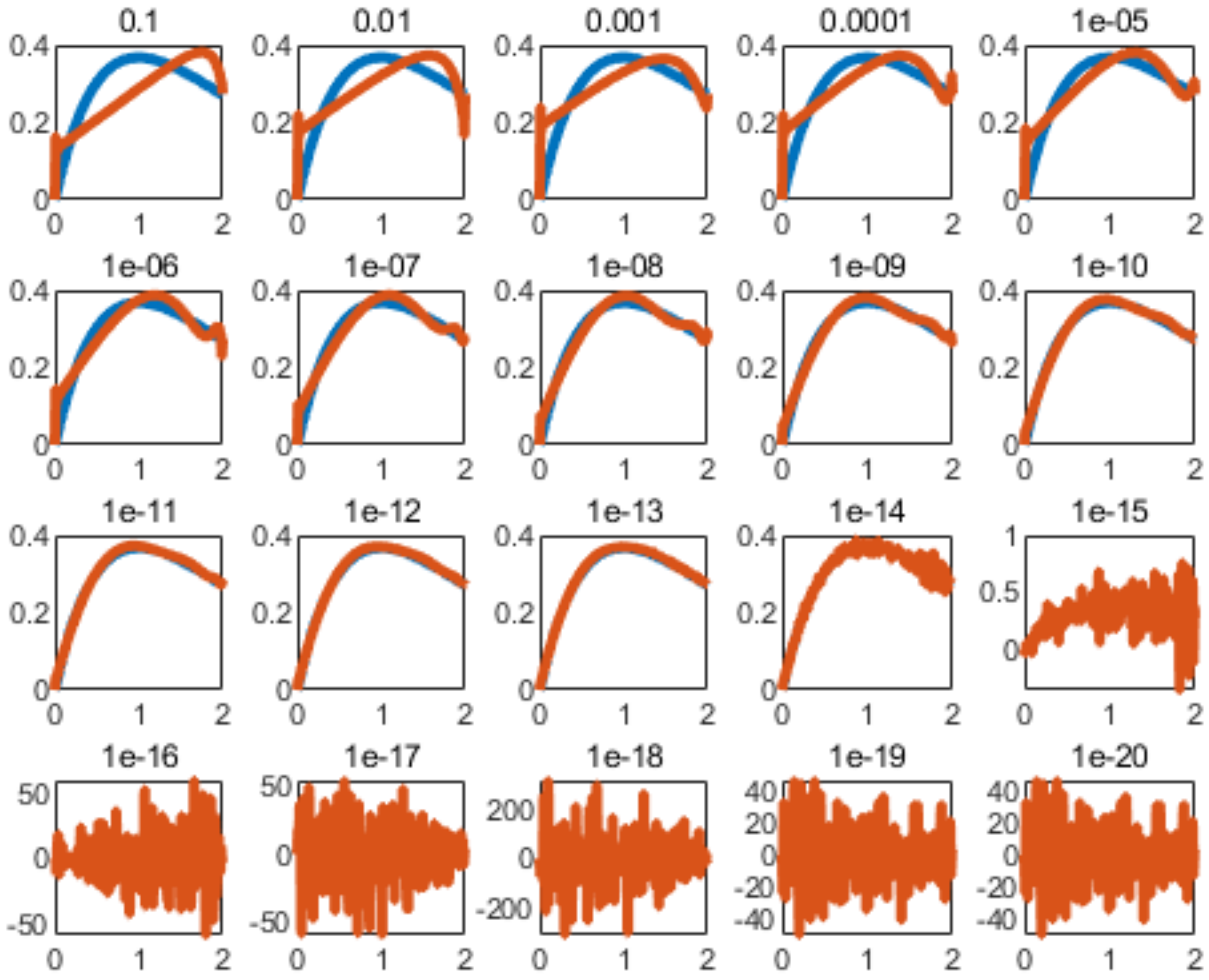
- It can be clearly seen that the solutions are unstable and far from the true values.
- The ill-posed inverse problems can not be solved without any regularization.

4. Test: Importance of regularization

The solutions with Tikhonov regularization:



model 3

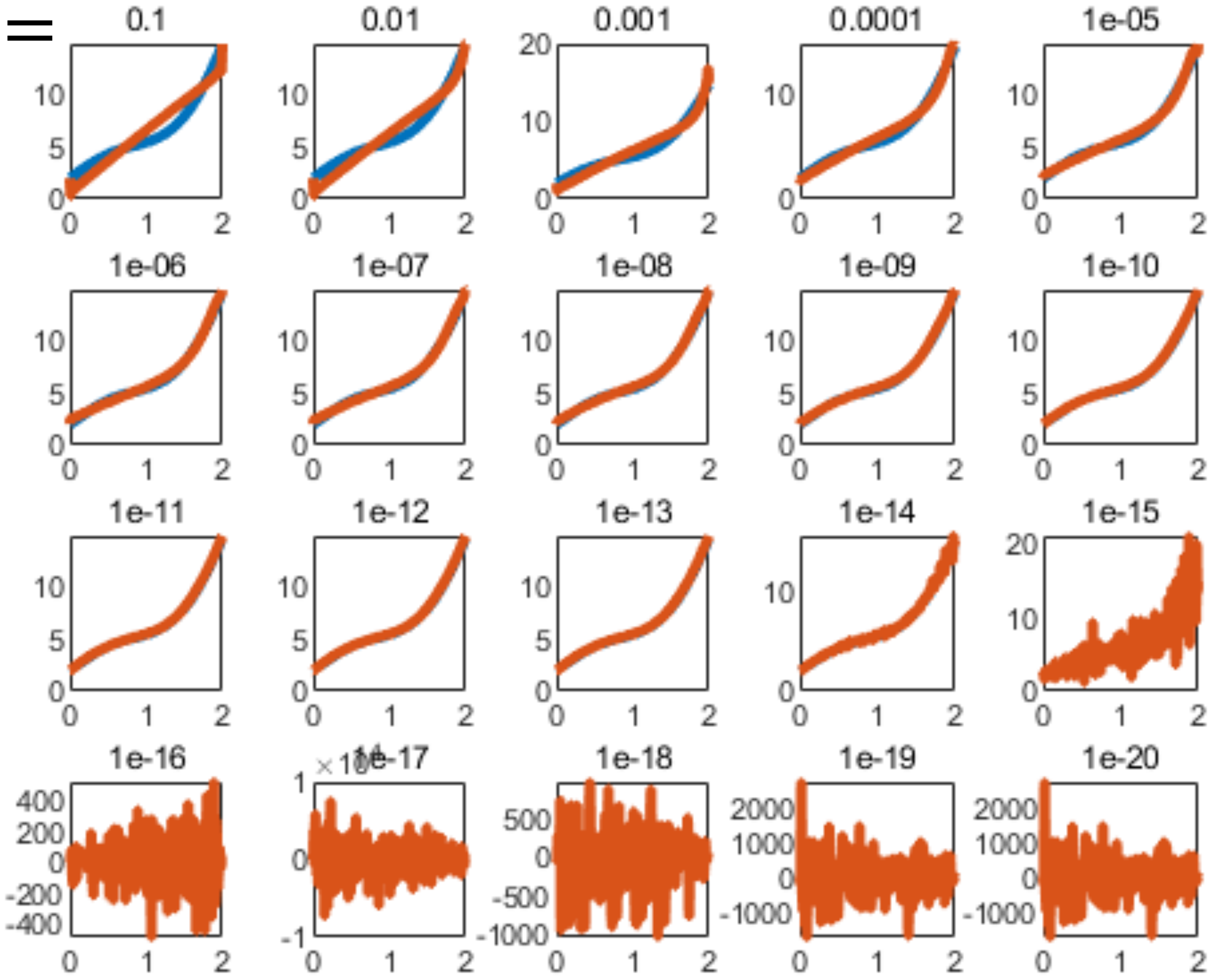


model 2

4. Test: Importance of regularization

The solutions with Tikhonov regularization:

$\alpha =$



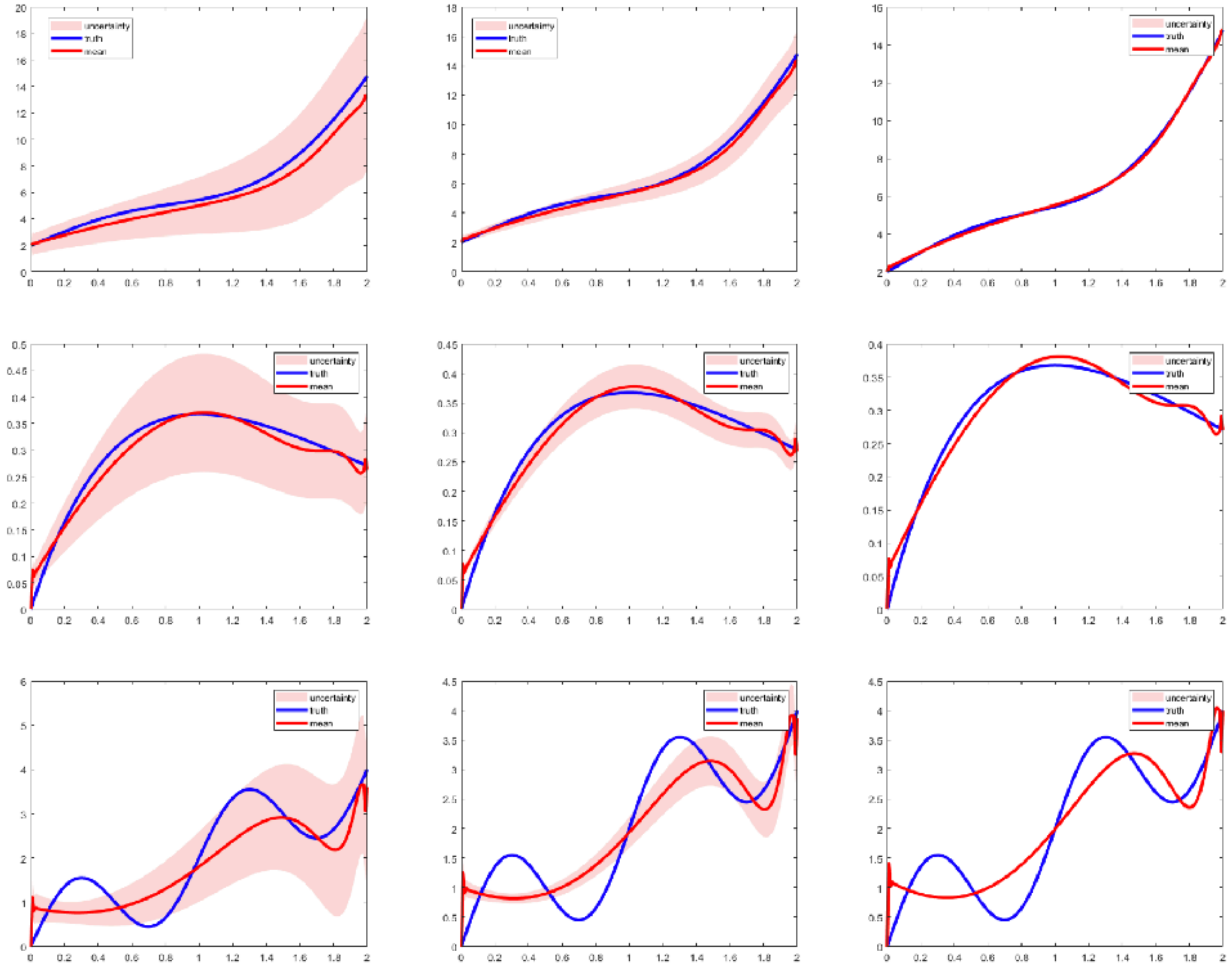
model 1

- It can be seen clearly that some values of regularization parameters can give good results.
- The ill-posed inverse problems can be solved by regularization.
- The regularization parameter can be neither too small (not enough for regularization), nor too large (dominate over the original problem)
- But α still works by ranging several orders of magnitude.
- The regularization methods are very important in solving the inverse problems.

4. Test: Impact of input uncertainties

- **The most important issue is to control the uncertainties!**

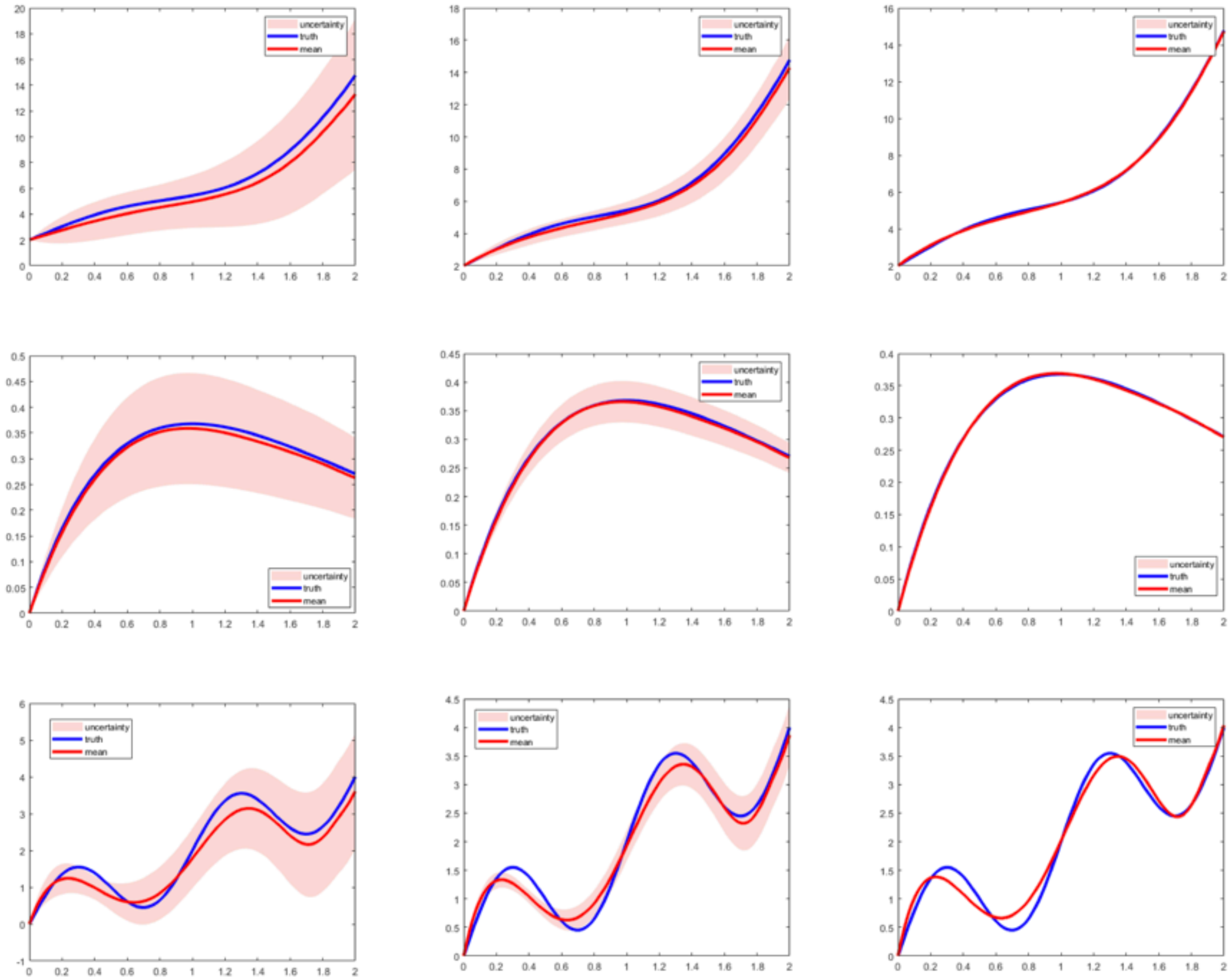
  - The uncertainties of the solutions are almost at the same level of the input errors.
  - The smaller the input errors are, the more precise the solutions are.
  - **The precision of the predictions can be systematically improved by lowering down the input errors.**



Input errors: 30% 10% 1%

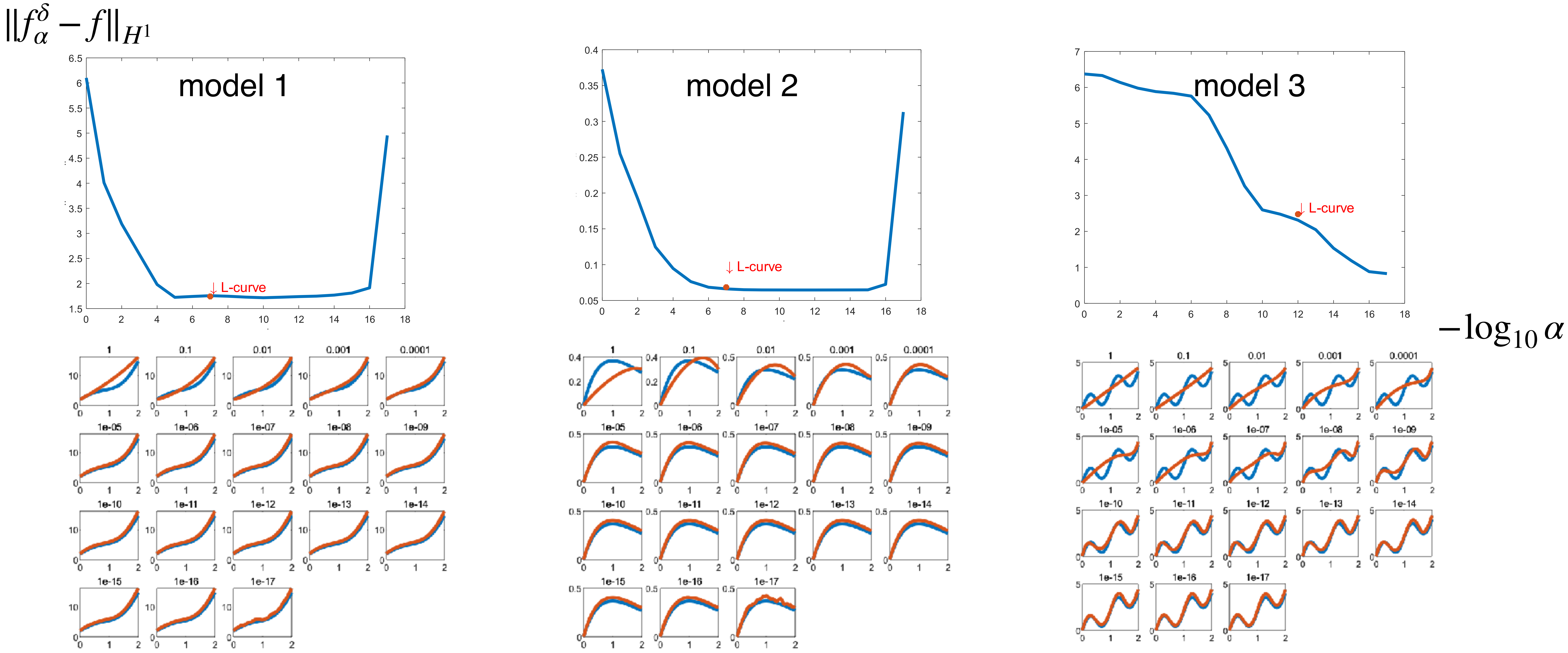
4. Test: Impact of improved regularization method

- **The regularization method can be modified according to the problem of physics**
  - The norm space of f(x) is changed from $L^2(a,b)$ to $H^1(a,b)$
  - The solutions are perfect for model 1 and 2. Model 3 is also significantly improved.
  - The uncertainties stemming from the regulator α is automatically included in the final results. Don't need to estimate the uncertainties from α.



Input errors: 30% 10% 1%

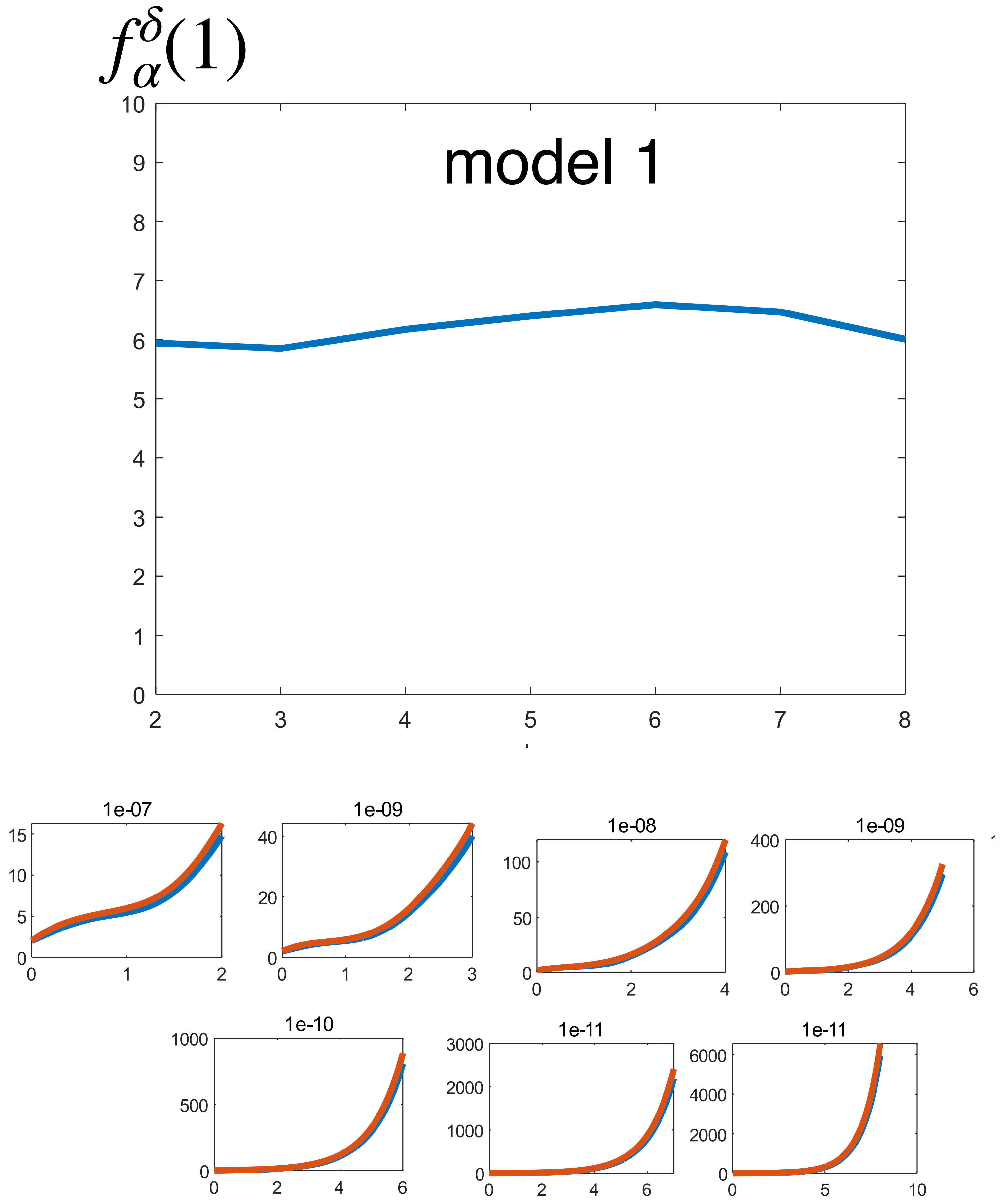
4. Test: Plateaus of the regularization parameter α

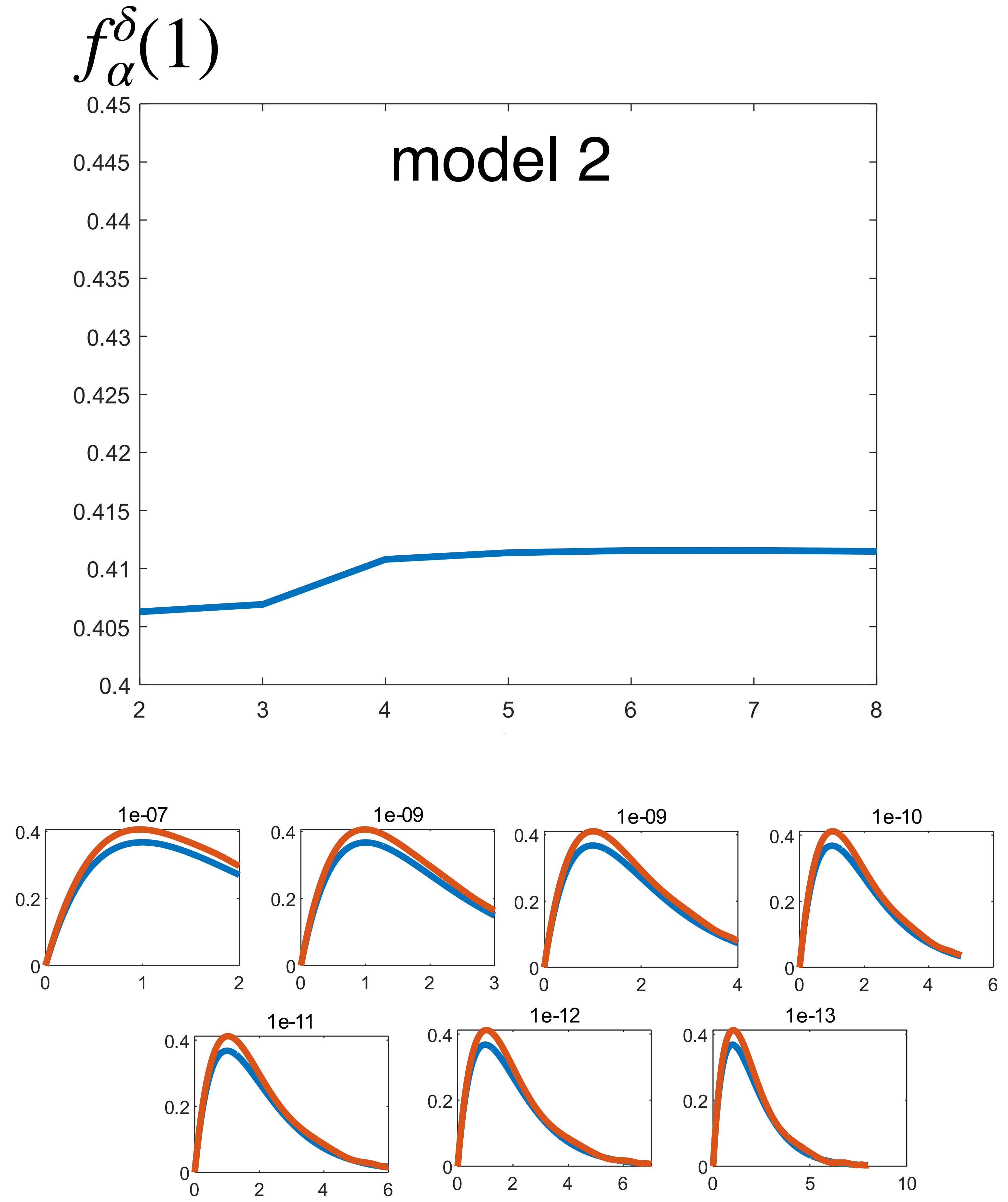


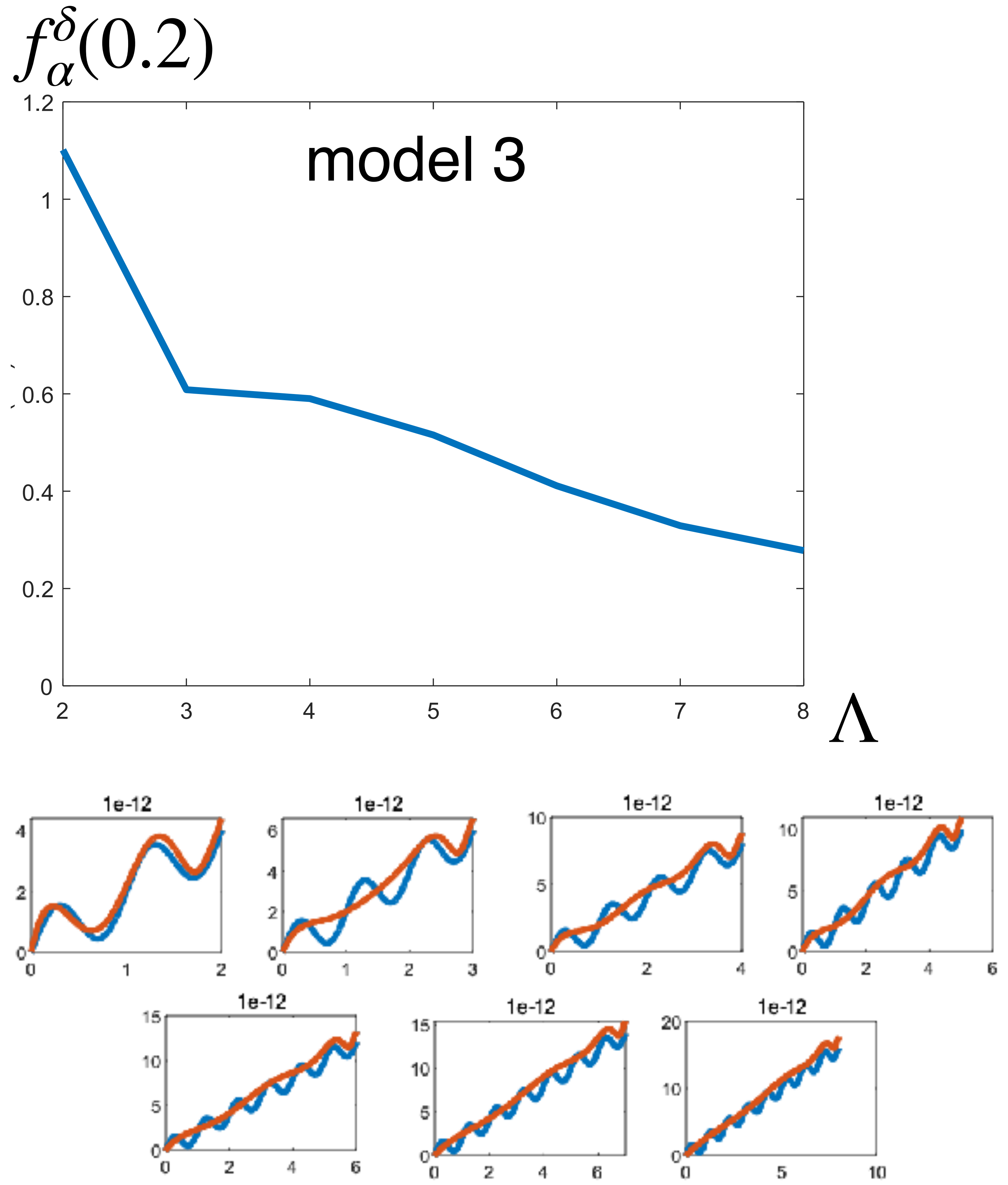
There exist plateaus. Solutions are insensitive to regularization parameter. L-curve method is suitable.

The inverse problem approach works for the non-perturbative calculations.

4. Test: Plateaus of the separation scale Λ

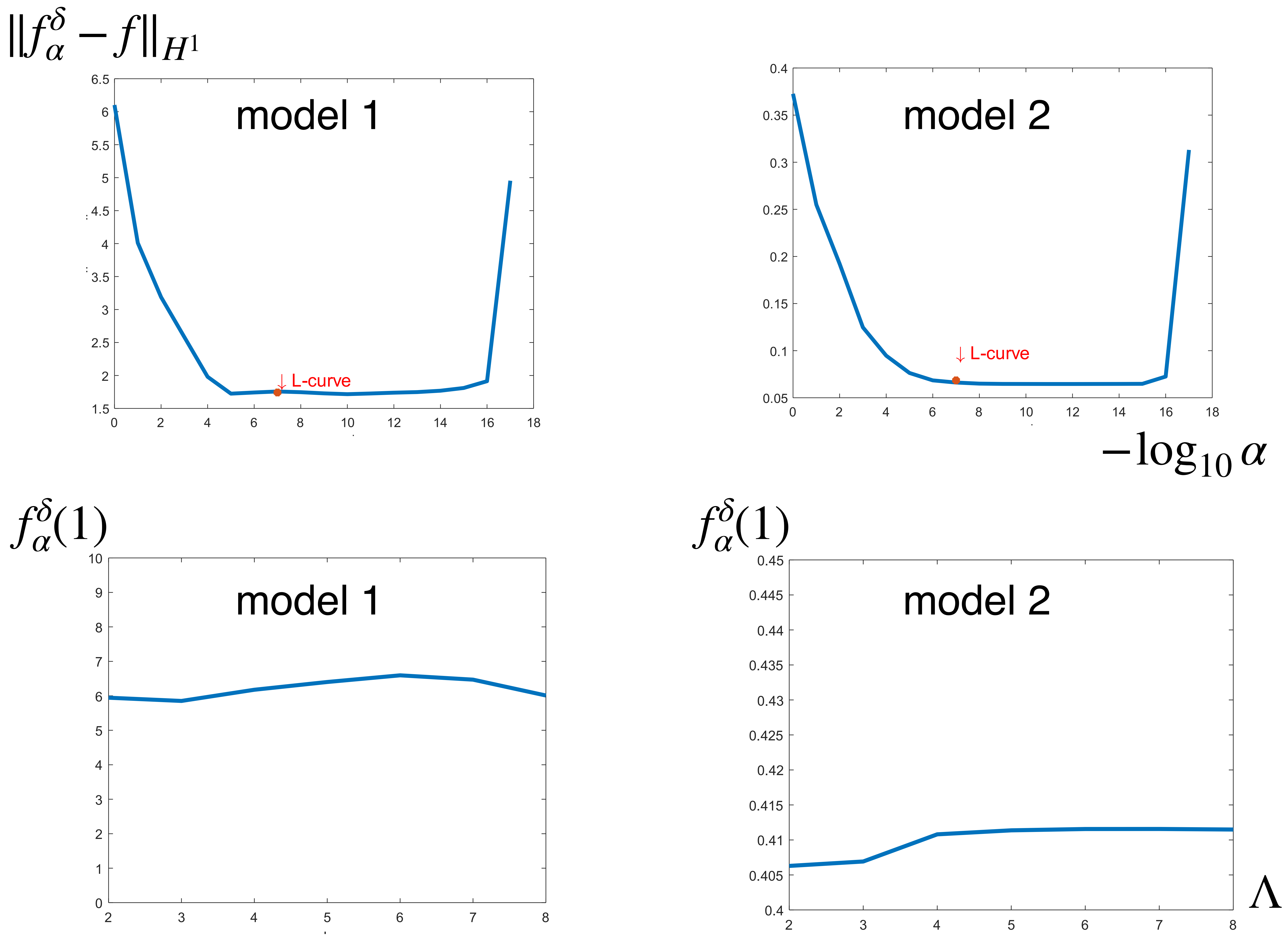






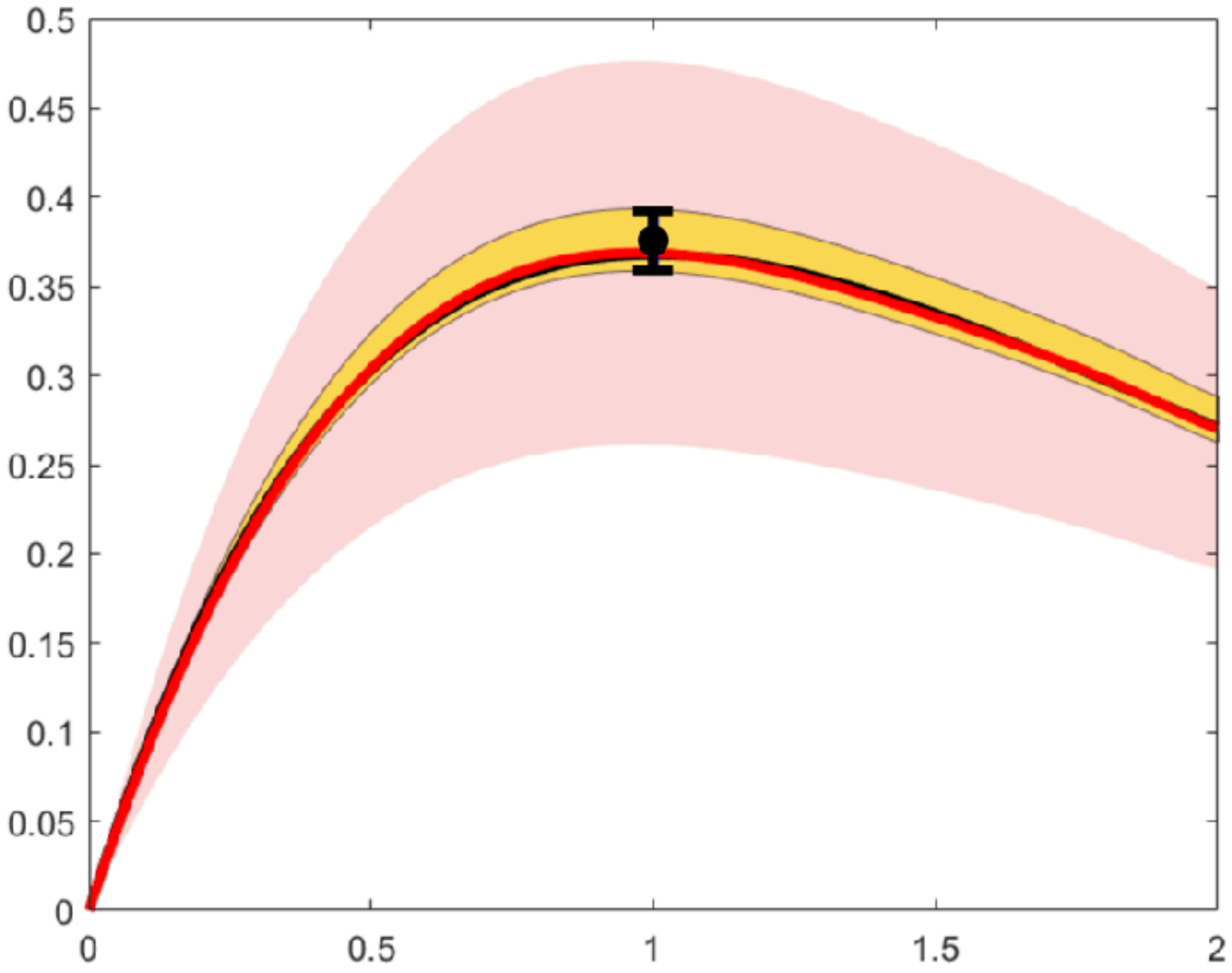
- There exist plateaus.
- Solutions are insensitive to the separation scale for monotonic and simple non-monotonic functions.
- The continuous condition at Λ might be even more helpful.

4. Test: Insensitivity to α and Λ



- Solutions are insensitive to the regularization parameter and the separation scale.
- The uncertainties of the inverse problem can be well controlled.

4. Test: Constrained data



- If there is an experimental data or lattice data with much smaller uncertainty than the original solutions, we can use it to constrain the solution to be more precise.
- Therefore, this method can combine with experiments and Lattice QCD to improve the precision of predictions

The main idea of the inverse problem approach

1. Dispersion relation and its inverse problem
2. Proof of ill-posedness
3. Regularization method
4. Test of some toy models
5. Physical discussions and perspectives

The precision can be systematically improved

Without any beyond-control assumptions, the precision can be systematically improved:

(1) Suitable regularization method and selection rule of the regulators

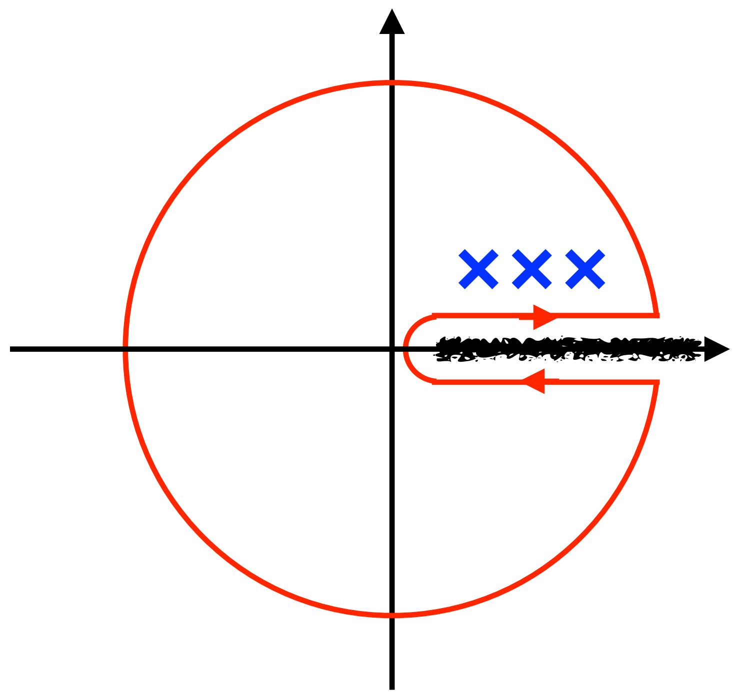
(2) Higher precision of input data

(3) Combination with higher precise data of experiments or Lattice QCD.

Criteria of a good theoretical approach

(1) Well defined in mathematics	Dispersion relation + proof of ill-posedness
(2) Realization in numerical calculations	Regularization methods
(3) Can be systematically improved	Converging to vanishing as $\delta \to 0$
(4) Simple at the beginning	Tikhonov regularization

5. Physical applications: neutral meson mixing

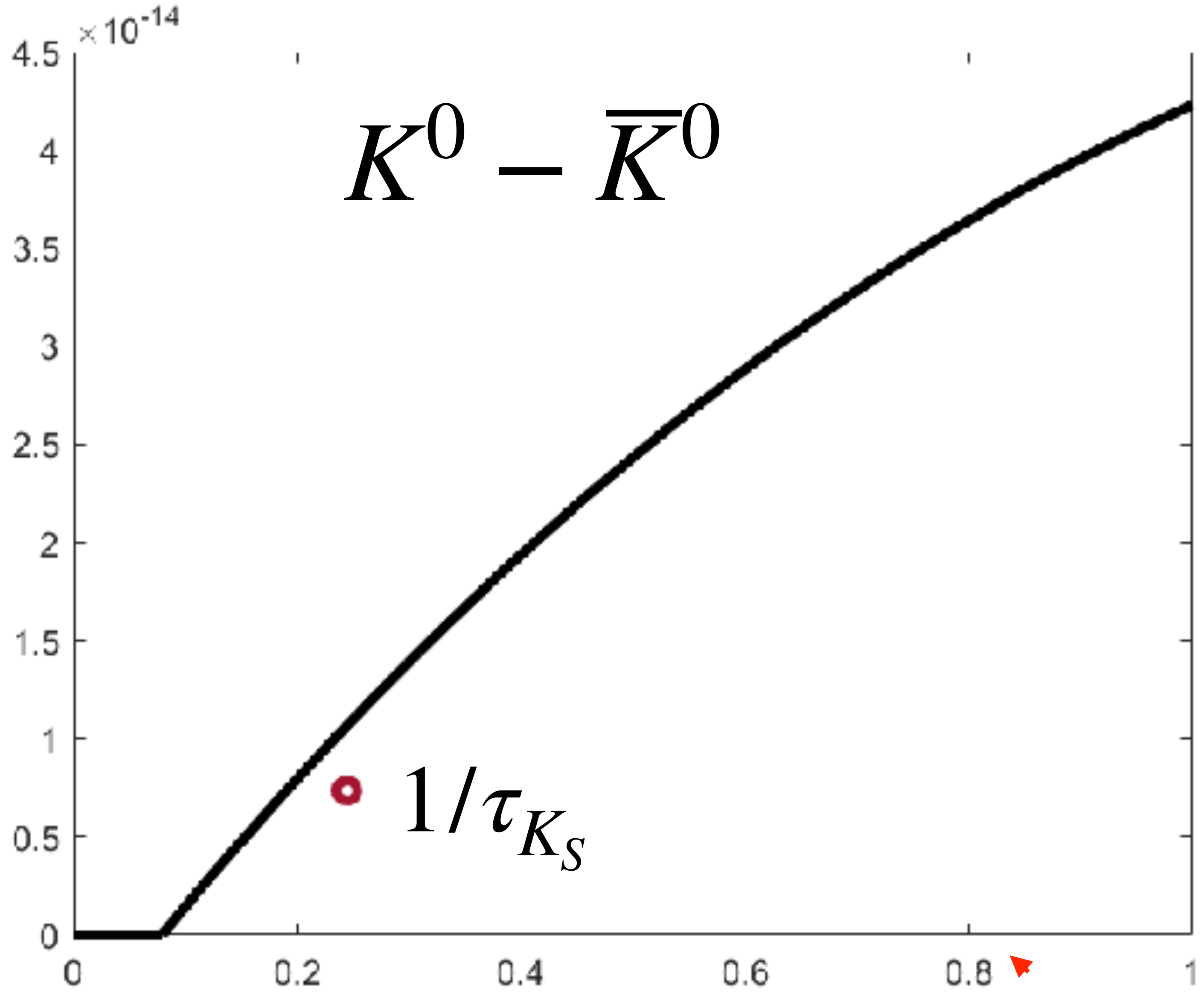


$$\frac{(s-s_1)(s_1-s_2)(s_2-s)}{2\pi}\int_{s_{th}}^{\Lambda}\frac{\Gamma_{12}(s')}{(s'-s)(s'-s_1)(s'-s_2)}ds'$$

$$=(s_1-s_2)M_{12}(s)+(s_2-s)M_{12}(s_1)+(s-s_1)M_{12}(s_2)$$

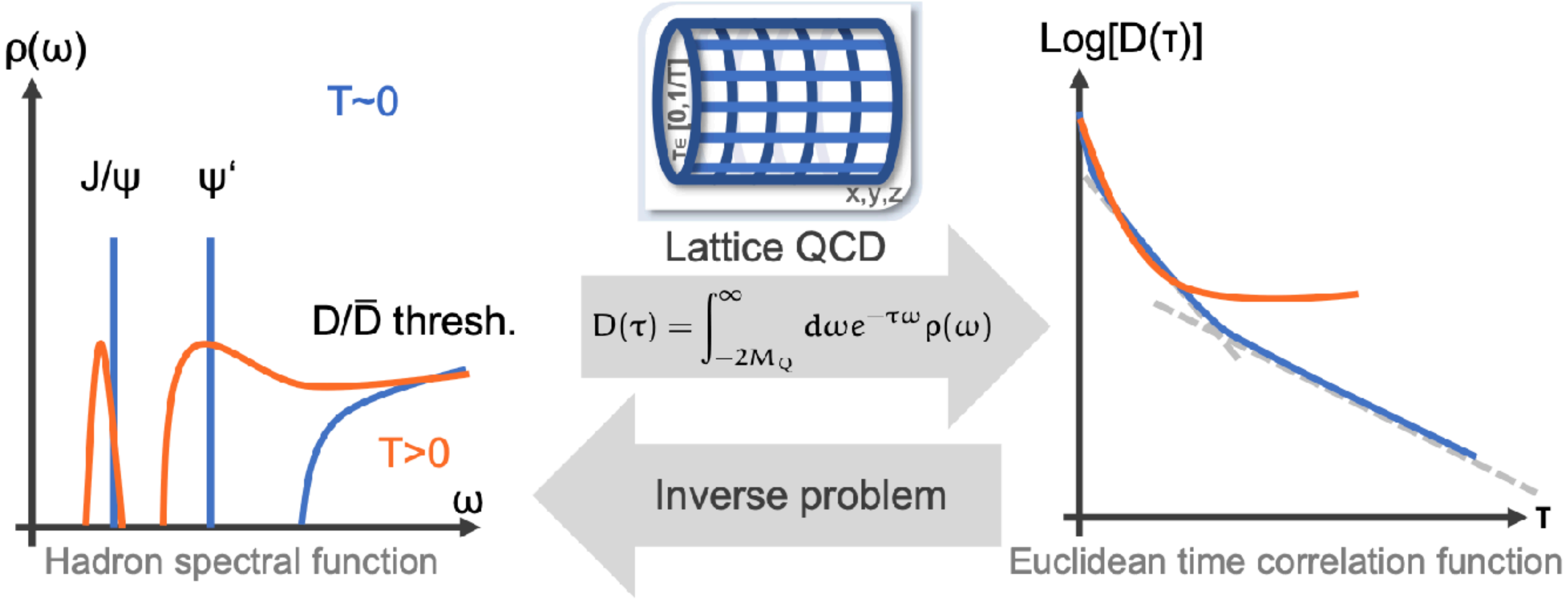
$$-\frac{(s-s_1)(s_1-s_2)(s_2-s)}{2\pi}\int_{\Lambda}^{\infty}\frac{\Gamma_{12}(s')}{(s'-s)(s'-s_1)(s'-s_2)}ds'$$

$$\Delta\Gamma_K = 2|\Gamma_{12}|$$



$$\Gamma^q_{21} = \frac{1}{2M_{B_q}}\text{Disc}\langle\overline{B}_q|i\int d^4x\, T\left(\mathcal{H}^{\Delta B=1}_{eff}(x)\mathcal{H}^{\Delta B=1}_{eff}(0)\right)|B_q\rangle$$

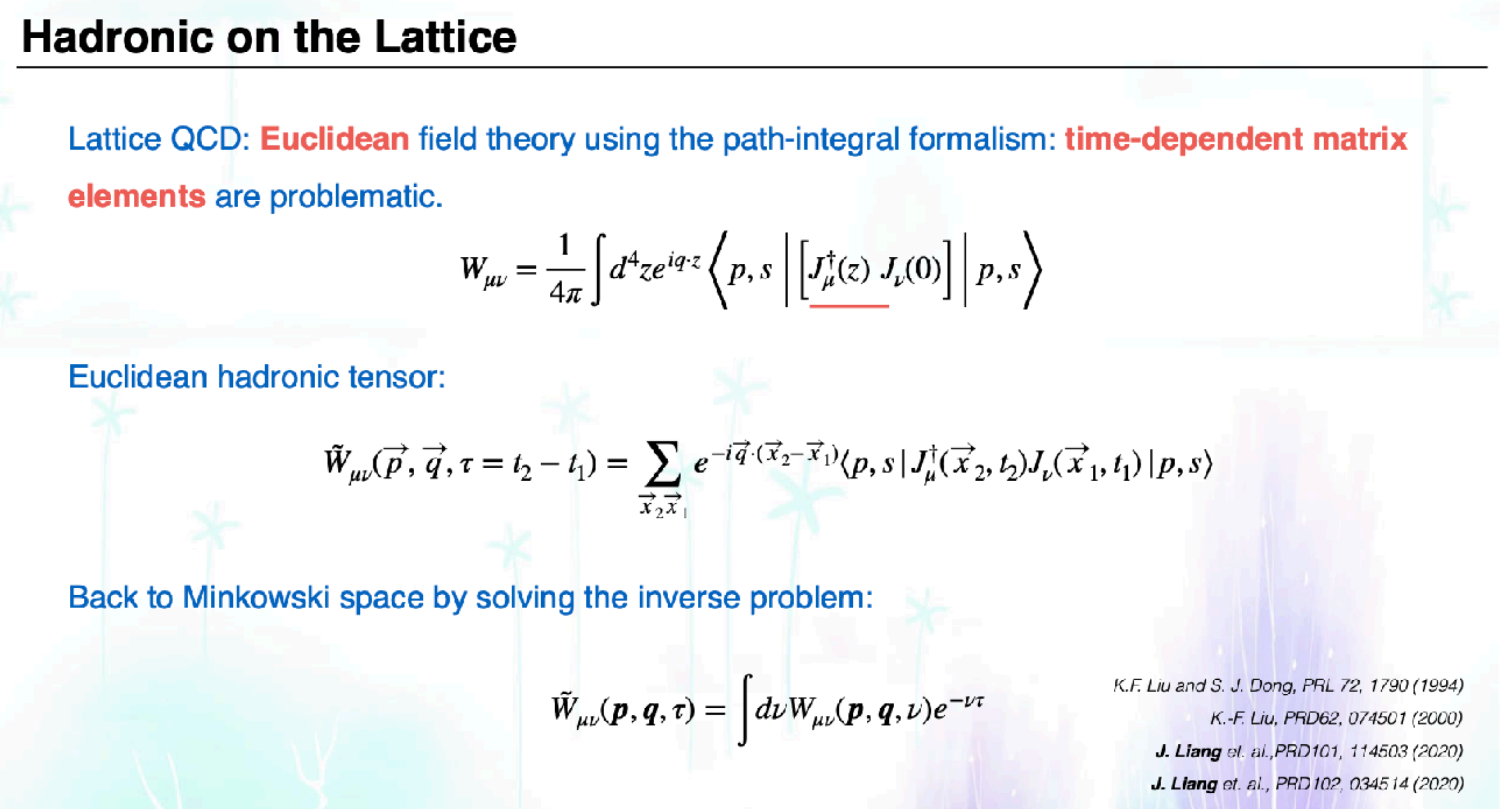
Inverse problem in Lattice QCD



Rothkopf, 2211.10680

Spectral function reconstruction from Euclidean lattices

Inverse problem in Lattice QCD



Hadronic on the Lattice

Lattice QCD: **Euclidean** field theory using the path-integral formalism: **time-dependent matrix elements** are problematic.

$$W_{\mu\nu} = \frac{1}{4\pi} \int d^4 z e^{iq\cdot z} \left\langle p, s \left| \left[\underline{J^\dagger_\mu(z)\ J_\nu(0)} \right] \right| p, s \right\rangle$$

Euclidean hadronic tensor:

$$\tilde{W}_{\mu\nu}(\vec{p}, \vec{q}, \tau = t_2 - t_1) = \sum_{\vec{x}_2 \vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2 - \vec{x}_1)} \langle p, s | J^\dagger_\mu(\vec{x}_2, t_2) J_\nu(\vec{x}_1, t_1) | p, s \rangle$$

Back to Minkowski space by solving the inverse problem:

$$\tilde{W}_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \tau) = \int d\nu W_{\mu\nu}(\boldsymbol{p}, \boldsymbol{q}, \nu) e^{-\nu\tau}$$

K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

K.-F. Liu, PRD62, 074501 (2000)

***J. Liang** et. al.,PRD101, 114503 (2020)*

***J. Liang** et. al., PRD102, 034514 (2020)*

Jian Liang's talk @ 2nd EicC CDR workshop

Maximum entropy method (MEM)

$$D(\tau) = \int_0^\infty K(\tau,w)A(w)\,dw$$

- **MEM is a method to circumvent these difficulties** by making a statistical inference of the most probable SPF (or sometimes called the image in the following) as well as its reliability on the basis of a limited number of noisy data.

- Its **basis is Bayes' Theorem**:

$$P[X|Y] = \frac{P[Y|X]P[X]}{P[Y]}$$

From Bayes' Theorem, we can get :

$$\boxed{P[A|DH]} = \frac{P[D|AH]P[A|H]}{P[D|H]}.$$

The most probable image is $A(w)$ that satisfies the condition: $\frac{\delta P[A|DH]}{\delta A} = 0.$

(1) Firstly, they make:

$$\begin{aligned} P[D|AH] &= \frac{1}{Z_L}\,e^{-L}\ , \\ L &= \frac{1}{2}\sum_{i,j}(D(\tau_i) - D_A(\tau_i))C_{ij}^{-1}(D(\tau_j) - D_A(\tau_j)), \end{aligned}$$

In the case where $P[A|H] = 0$, **maximizing $P[A|DH]$ is equivalent to standard $\chi^2 -$ fitting**. However, the $\chi^2 - fitting$ does not work.

hep-lat/0011040

5. Physical perspectives

(1) Provide the quantities at the whole non-perturbative region

(2) Advantage for the excited states. Either calculate directly, or combine with LQCD for ground states

(3) Advantage for non-local correlation functions: widths and lifetimes, inclusive processes, distribution amplitudes

(4) QCD sum rules with modification on the quark-hadron duality.

(5) Constrain some input parameters.

(6) Solving some inverse problems in Lattice QCD.

(7) More efforts on perturbative calculations to improve the input precision.

(8) And many others…

Outline

1. The foundation of the inverse problem approach
    - The main idea of the inverse problem to non-perturbative QCD
    - Dispersion relation and its inverse problem, ill-posedness and Tikhonov regularization, toy models
    - Physical applications and discussions, perspectives
2. More details on the inverse problem approach
    - Introduction and mathematical basis of the inverse problems
    - $D^0 - \overline{D}^0$ mixing and its inverse problem.

反问题是什么?

● 反问题：

$$\underset{\text{原因或输入}}{x?} \rightarrow \boxed{\underset{\text{过程或模型}}{K}} \rightarrow \underset{\text{结果或输出}}{y}$$

例：

		正问题	反问题
小学期间	x和K是整数	求$y = Kx$	$x = \frac{1}{K} * y$
中学期间	x是实数 K是映射或函数	求$y = K(x)$	隐函数定理
大学期间	x是向量 K是矩阵	求$y = K \ast x$	$x = K^{-1}y$
	$x = x(t)$是函数 K是积分运算	求$y(t) = \int \frac{x(s)}{t-s} ds$	?
泛函：算子方程	x是函数空间 K是算子	求$y = Kx$	$x = K^{-1}y$

反问题是什么?

● 反问题：

$x?$	$\rightarrow$	$\boxed{K}$	$\rightarrow$	y
原因或输入		过程或模型		结果或输出

例：

常规一维函数	若 $y = ax + b(a \neq 0)$，	已知y_1，求$x_1 = ?$	(存在且唯一)
	$y = sin(x)$，	已知y_2，求$x_2 = ?$	(存在但不唯一)
矩阵问题	$y = Kx$,	已知y_3，求$x_3 = ?$	依赖具体情况而定
	矩阵K非奇异		(存在且唯一)
	矩阵K奇异：需要额外验证其他条件		(存在但不唯一) (不存在)

反问题是什么?

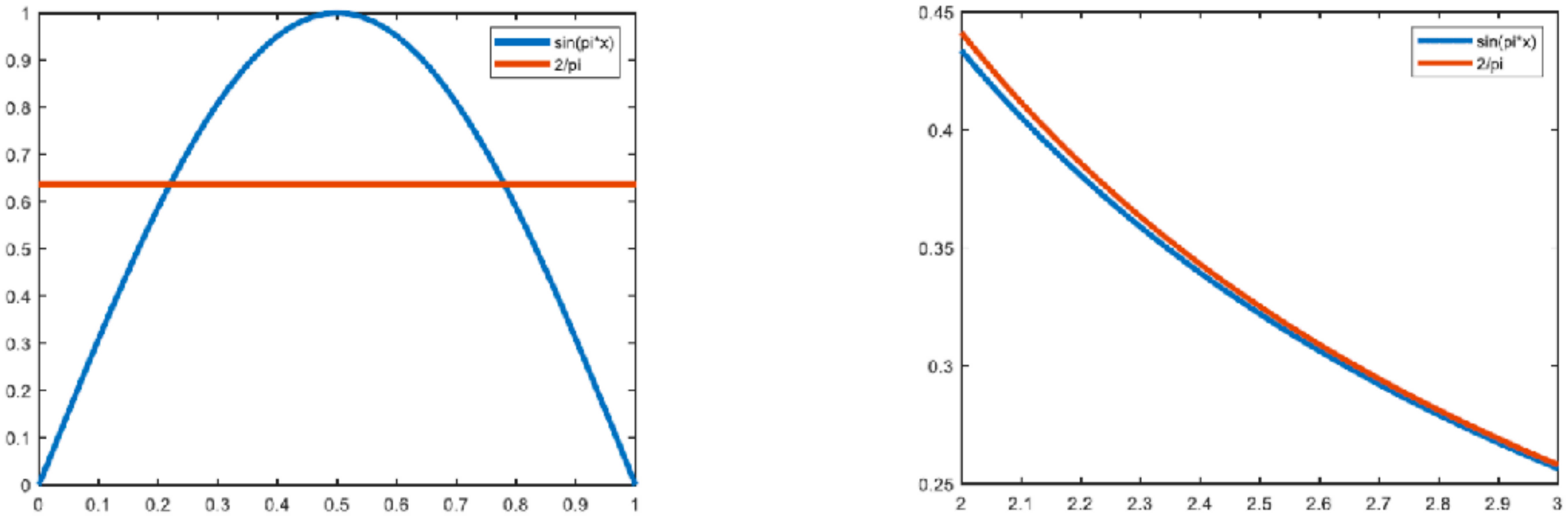
- **反问题:**

$$x? \rightarrow \boxed{K} \rightarrow y$$

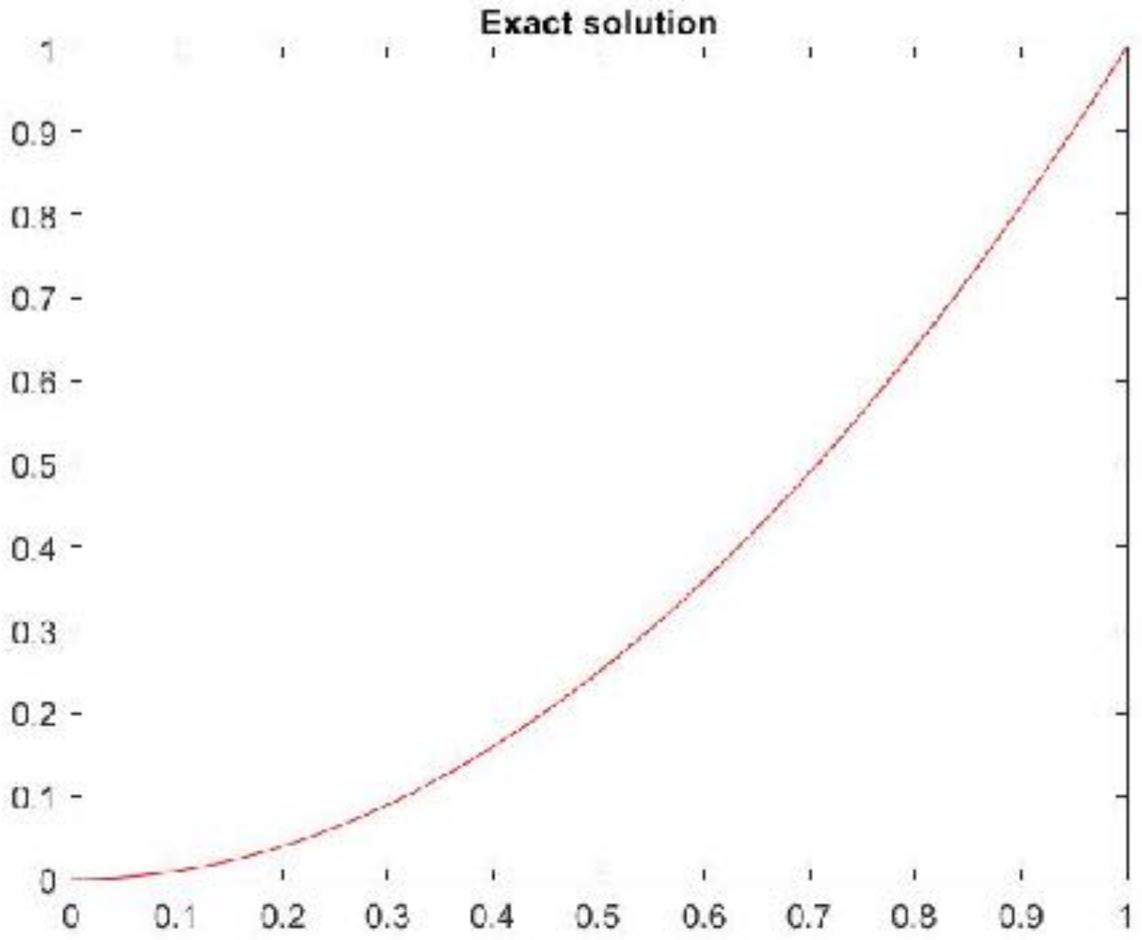
$x?$	→	K	→	y
原因或输入		过程或模型		结果或输出

例:

积分方程 $\int_0^1 \frac{f(x)}{y-x}dx = g(y), y \in [2,3]$ **知道**$g(y)$**，求**$f(x) =?$

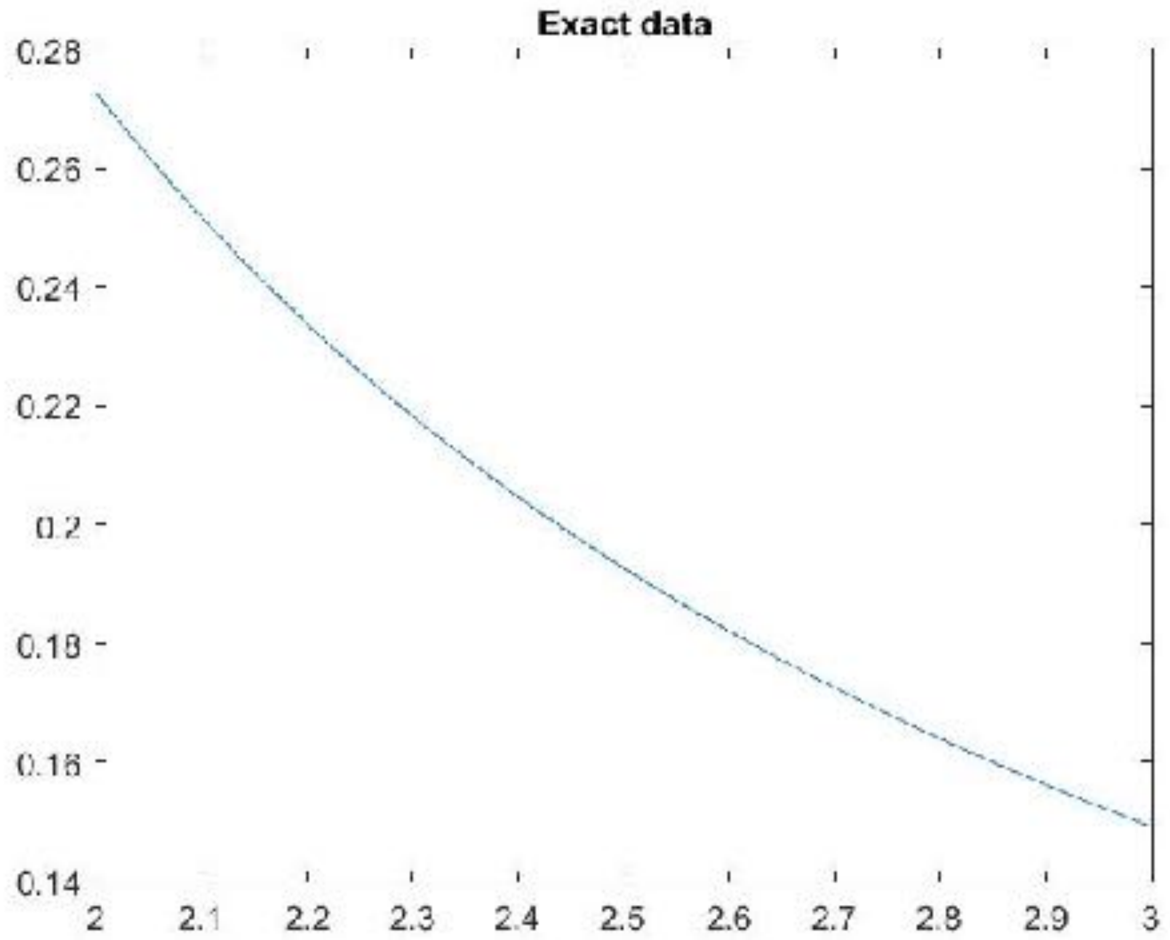


存在性未知;
唯一性未知;
但不稳定

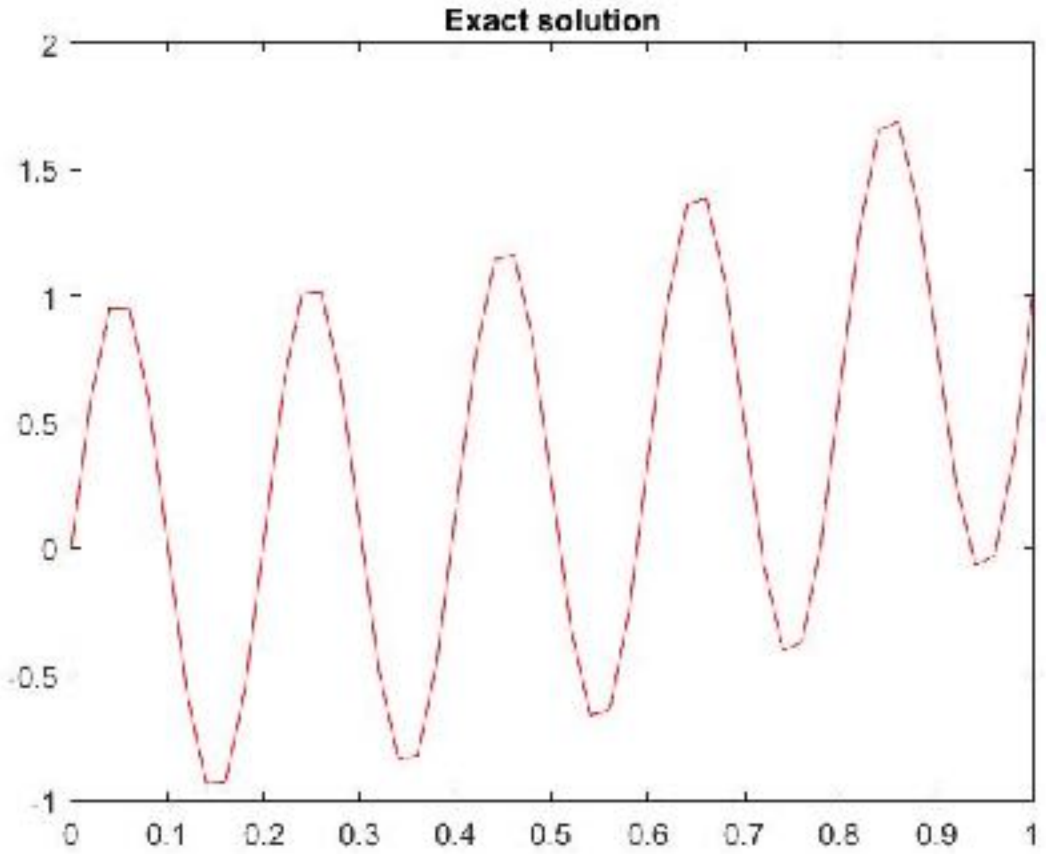


$x(t) = t^2$

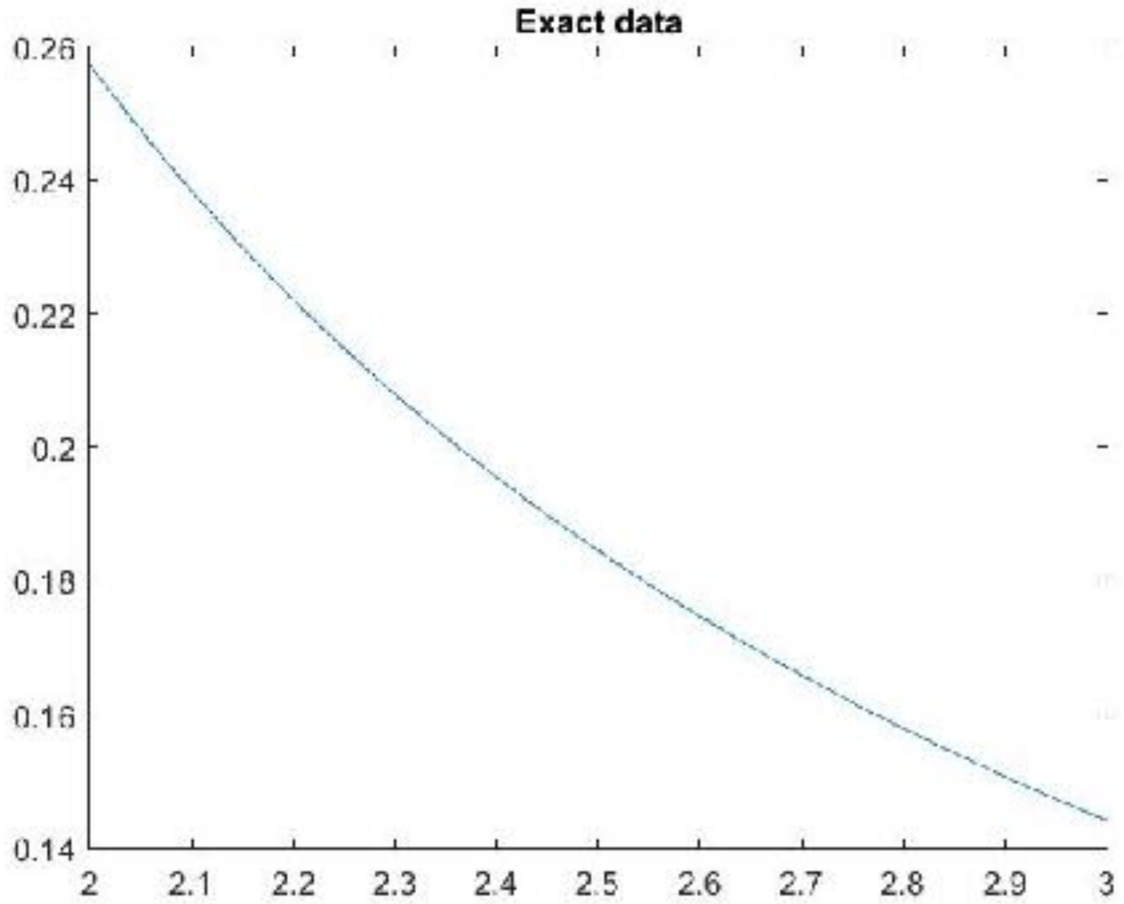
正问题



$y(s) = Kx$



$x(t) = t^2 + \sin(10\pi t)$



$y = Kx$

科学史上的著名反问题案例

1781 年, 天王星被确认为太阳系的第7 颗大行星. 40年后, 法国天文学家Bouvard 搜集了一个多世纪来的全部观测资料, 包括了1781 年之前的旧数据和之后的新数据, 试图用牛顿的天体力学原理来计算天王星的运动轨道. 他发现了一个奇怪的现象：用全部数据计算出的轨道与旧数据吻合得很好, 但是与新数据相比误差远超出精度允许的范围；如果仅以新数据为依据重新计算轨道, 得到的结果又无法和旧数据相匹配. Bouvard 的治学态度非常严谨, 他在论文中指出：“两套数据的不符究竟是因为旧的观测记录不可靠, 还是来自某个外部未知因素对这颗行星的干扰？我将这个谜留待将来去揭示.”

科学史上的著名反问题案例

首先, Bouvard 等天文学家核查了1750 年以后英国格林尼治天文台对各个行星所作的全部观测记录. 结果发现, 除天王星以外, 对于其它行星的观测记录与理论计算结果都符合得相当好. 似乎没有理由怀疑旧的天文观测唯独对天王星失准. 既然如此, 天文学家就需要对天王星的不规律运动作出科学的解释.

摆在天文学家面前的有两条路:

第一条路是质疑牛顿力学的普适性, 或许万有引力定律不适用于距离太阳遥远的天王星, 需要对之进行修正;

第二条路是寻找Bouvard所猜测的“未知因素”. 于是人们提出了“彗星撞击”、“未知卫星”和“未知行星”等多种可能.

科学史上的著名反问题案例

1841 年的暑期, 还是英国剑桥大学二年级学生的Adams 就定下计划, 不仅要确认天王星的轨道异常是否来自未知行星的引力作用, 还要尽可能地确认这颗新行星的轨道, 以便通过观测来发现之. 这不仅是一个新问题, 而且是一个反问题. 因为过去总是已知一颗行星的质量和轨道, 根据万有引力定律计算出它对另一颗行星产生的轨道摄动. 而现在则相反, Adams 要假定已知天王星轨道的摄动, 来计算出产生这一摄动的未知行星的质量和轨道. 由于未知因素很多, 实际计算起来是相当复杂和困难的.

Adams 于1845 年彻底解决了这个反问题. 他所运用的方法在当时是空前新颖的. 令人遗憾的是, 英国天文学家Airy 先入为主地认为天王星的轨道问题是引力定律不再适用的结果, 没有重视Adams 向他提交的新行星的轨道计算结果.

科学史上的著名反问题案例

几乎与此同时, 法国人Le Verrier 独立地解决了同样的反问题. 1846 年9月23 日, 柏林天文台的Galle 按照Le Verrier 提交的计算轨道着手观测, 当晚就在偏离预言位置不到1 度的地方发现了一颗新的八等星. 连续观测的数据都与Le Verrier 的预测结果吻合得很好, 证实这是一颗新行星. 这时英国天文台才想起了Adams 的工作, 悔之晚矣.

案子破了. 干扰天王星正常运行的那颗神秘天体正是太阳系的第8 颗大行星——海王星！不仅长期困扰天文界的天王星轨道异常问题在牛顿力学框架内得到了完满解释, 而且海王星的发现进一步验证了牛顿力学的正确性.

反问题发展简史

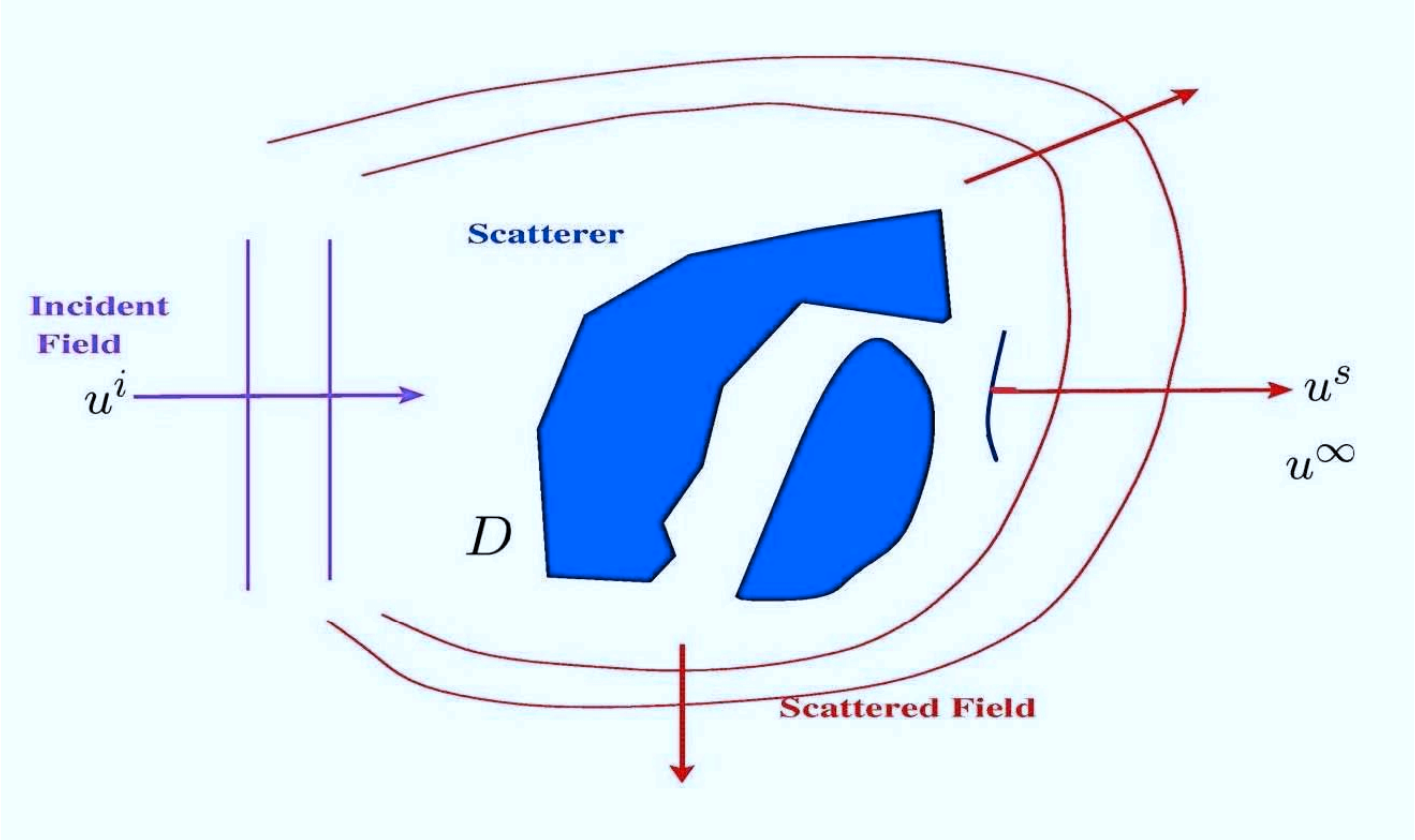
- 数学物理反问题最早出现在地球物理领域;
- 1846年, 法国人Le Verrier发现了海王星;
- 1880年, 美国学者J.A. Ewing等人发明了近代地震仪, 提出了地震记录的分析问题;
- 1907年, Herglog 提出了地震走时数据的反演;
- 1909年, A. Mohorovicic发现了莫霍面;
- 1912年, Beno Gutenbeg 发现了古登堡面;
- 1935年, Lehmann 发现了地球外核和内核的分界面;

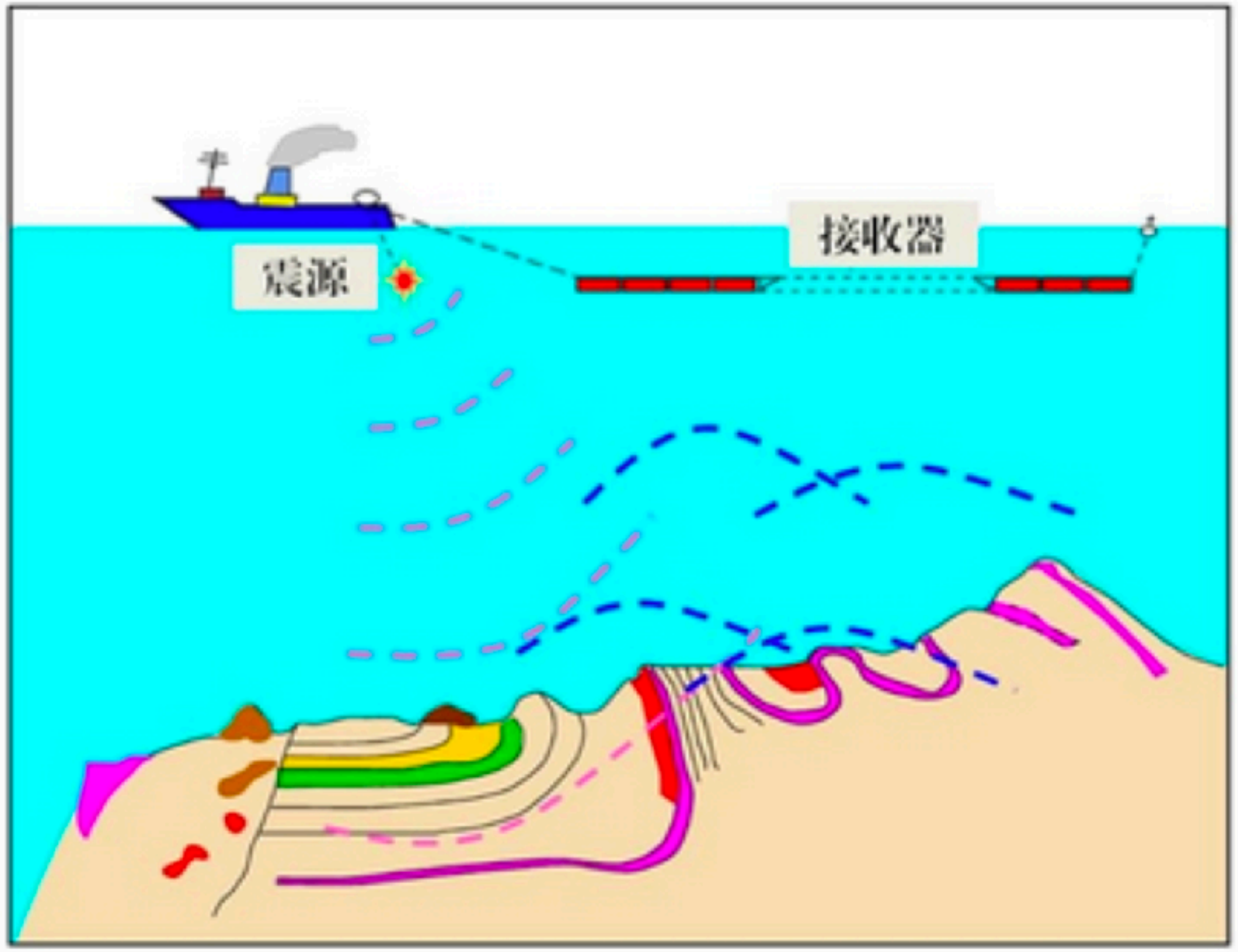
反问题发展简史

- 但在第一台数字计算机诞生之前, 反问题的发展非常缓慢, 反演方法只有选择法和量版法;
- 1967-1970年, 美国地球物理学家Backus 和应用数学家Gilbert 连续发表了三篇关于平均核法的文章, 奠定了反演理论的基础;
- Tikhonov (吉洪诺夫) 20世纪40年代, 提出了正则化方法(1977, Solutions of ill-posed problems, 美国. 中译本, 1979) ;
- 70年代初, 英国学者G. Honsfield 研制出了第一台医用 C T 机以及他和美国学者A.M.Cormack共同获得了1979年度生理性和医学诺贝尔奖, 大大推动了有关不可见物体层析成像的研究热潮, 也极大地推动了反问题数学理论、数值方法以及应用的发展;

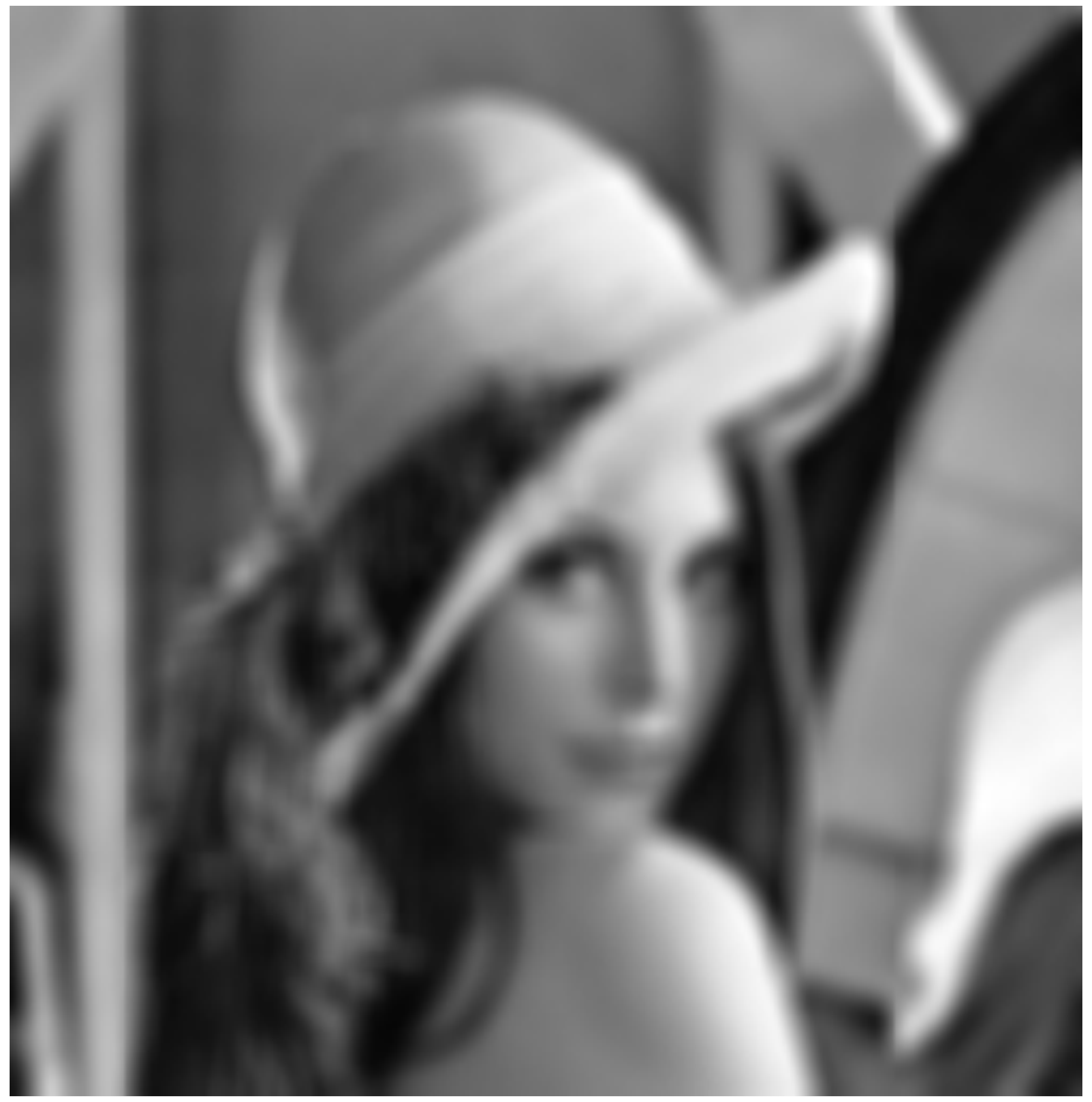
反问题发展简史

- 现代反问题开始于70年代末、80年代初，蓬勃发展至今，40多年的时间;
- 广泛应用于石油勘探、工程物探、无损探伤、航空航天、地下找水、光学、电子、控制等等领域，可以说无处不在(Everything is inverse problem!);
- 国际上四种反问题杂志：Inverse Problems, Inverse Problems and Imaging, Inverse Problems in Science and Engineering, Journal of inverse and ill-posed problems;
- 我国反问题的研究最早由计算数学家冯康倡导(1982)．他把反问题列为计算数学四大问题之一(正问题、反问题、逼近问题和代数问题)．


Scatterer
Incident
Field
u^i
D
u^s
u^∞
Scattered Field



海上人工地震数据的采集



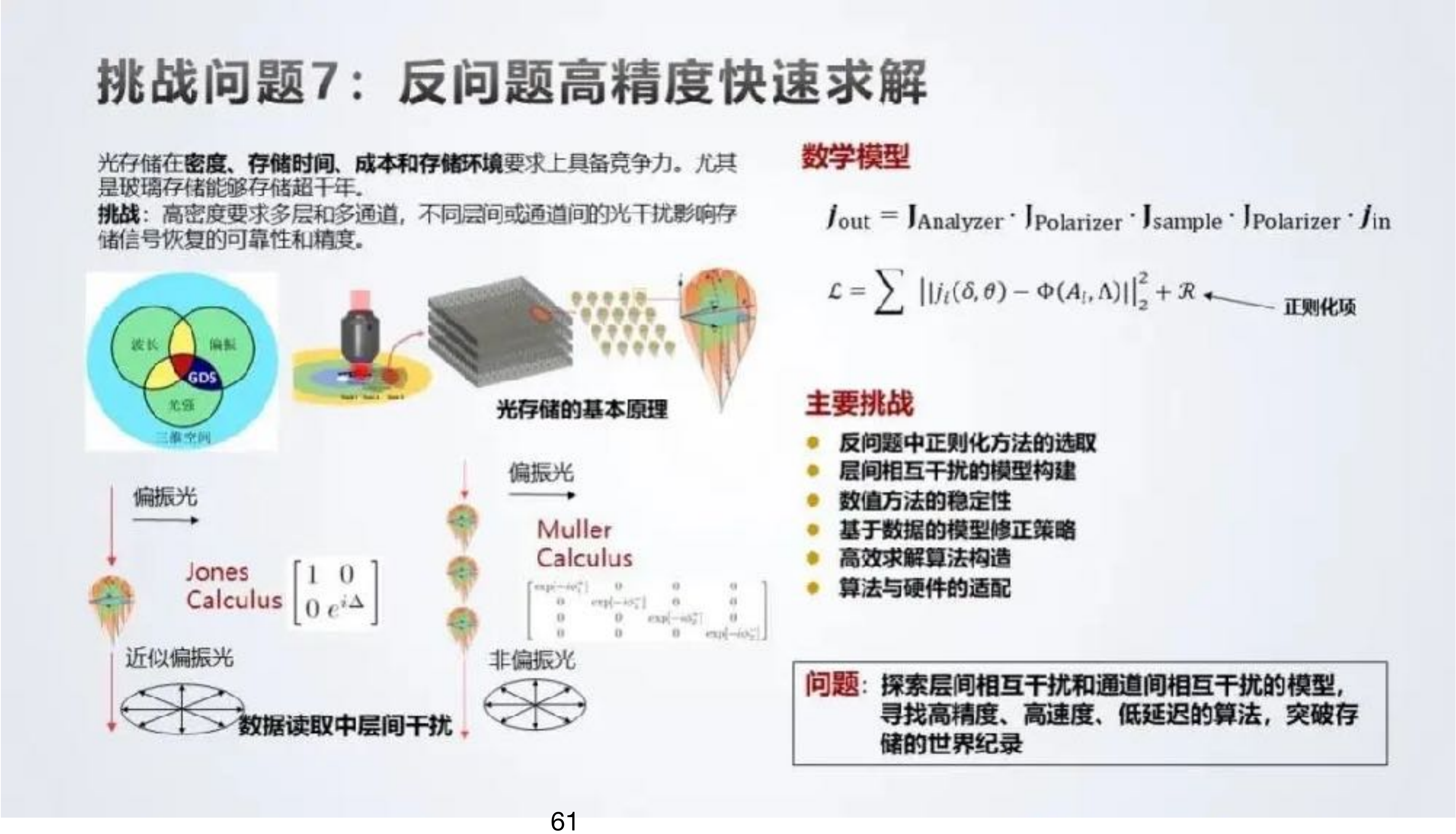
Blurred and noise

Recovered

后香农时代，数学决定未来发展的边界

徐文伟

华为董事、战略研究院院长



挑战问题7：反问题高精度快速求解

光存储在**密度、存储时间、成本和存储环境**要求上具备竞争力。尤其是玻璃存储能够存储超千年。

挑战：高密度要求多层和多通道，不同层间或通道间的光干扰影响存储信号恢复的可靠性和精度。

光存储的基本原理

数据读取中层间干扰

数学模型

$$J_{\text{out}} = \mathrm{J}_{\text{Analyzer}} \cdot \mathrm{J}_{\text{Polarizer}} \cdot \mathrm{J}_{\text{sample}} \cdot \mathrm{J}_{\text{Polarizer}} \cdot J_{\text{in}}$$

$$\mathcal{L} = \sum \left|\left|j_i(\delta,\theta) - \Phi(A_i,\Lambda)\right|\right|_2^2 + \mathcal{R}$$

正则化项

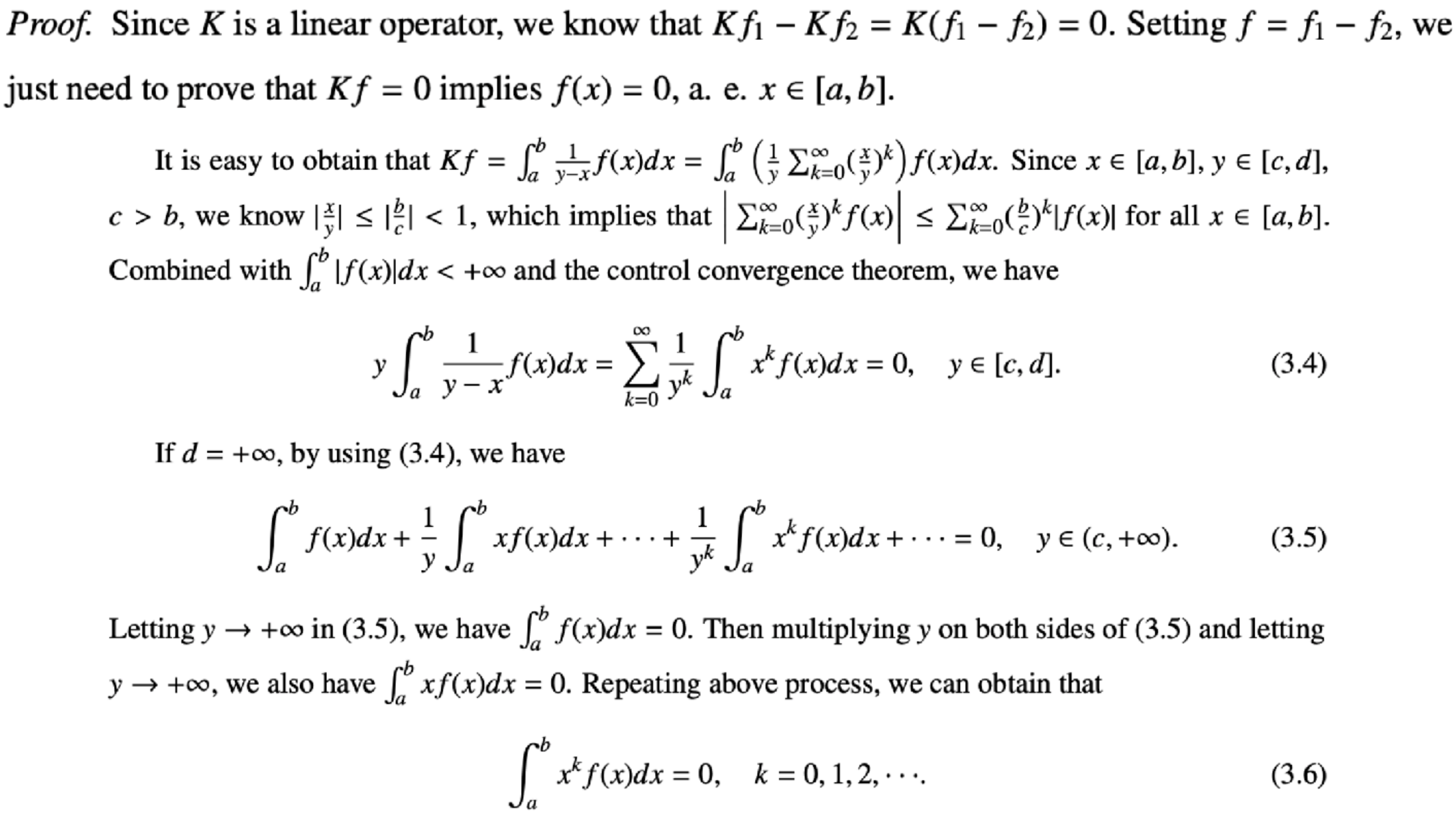
主要挑战

- 反问题中正则化方法的选取
- 层间相互干扰的模型构建
- 数值方法的稳定性
- 基于数据的模型修正策略
- 高效求解算法构造
- 算法与硬件的适配

问题：探索层间相互干扰和通道间相互干扰的模型，寻找高精度、高速度、低延迟的算法，突破存储的世界纪录

Proof of uniqueness:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$



Proof. Since K is a linear operator, we know that $Kf_1 - Kf_2 = K(f_1 - f_2) = 0$. Setting $f = f_1 - f_2$, we just need to prove that $Kf = 0$ implies $f(x) = 0$, a. e. $x \in [a,b]$.

It is easy to obtain that $Kf = \int_a^b \frac{1}{y-x} f(x)dx = \int_a^b \left(\frac{1}{y}\sum_{k=0}^{\infty}(\frac{x}{y})^k\right) f(x)dx$. Since $x \in [a,b]$, $y \in [c,d]$, $c > b$, we know $|\frac{x}{y}| \leq |\frac{b}{c}| < 1$, which implies that $\left|\sum_{k=0}^{\infty}(\frac{x}{y})^k f(x)\right| \leq \sum_{k=0}^{\infty}(\frac{b}{c})^k |f(x)|$ for all $x \in [a,b]$. Combined with $\int_a^b |f(x)|dx < +\infty$ and the control convergence theorem, we have

$$y\int_a^b \frac{1}{y-x} f(x)dx = \sum_{k=0}^{\infty} \frac{1}{y^k} \int_a^b x^k f(x)dx = 0, \quad y \in [c,d]. \tag{3.4}$$

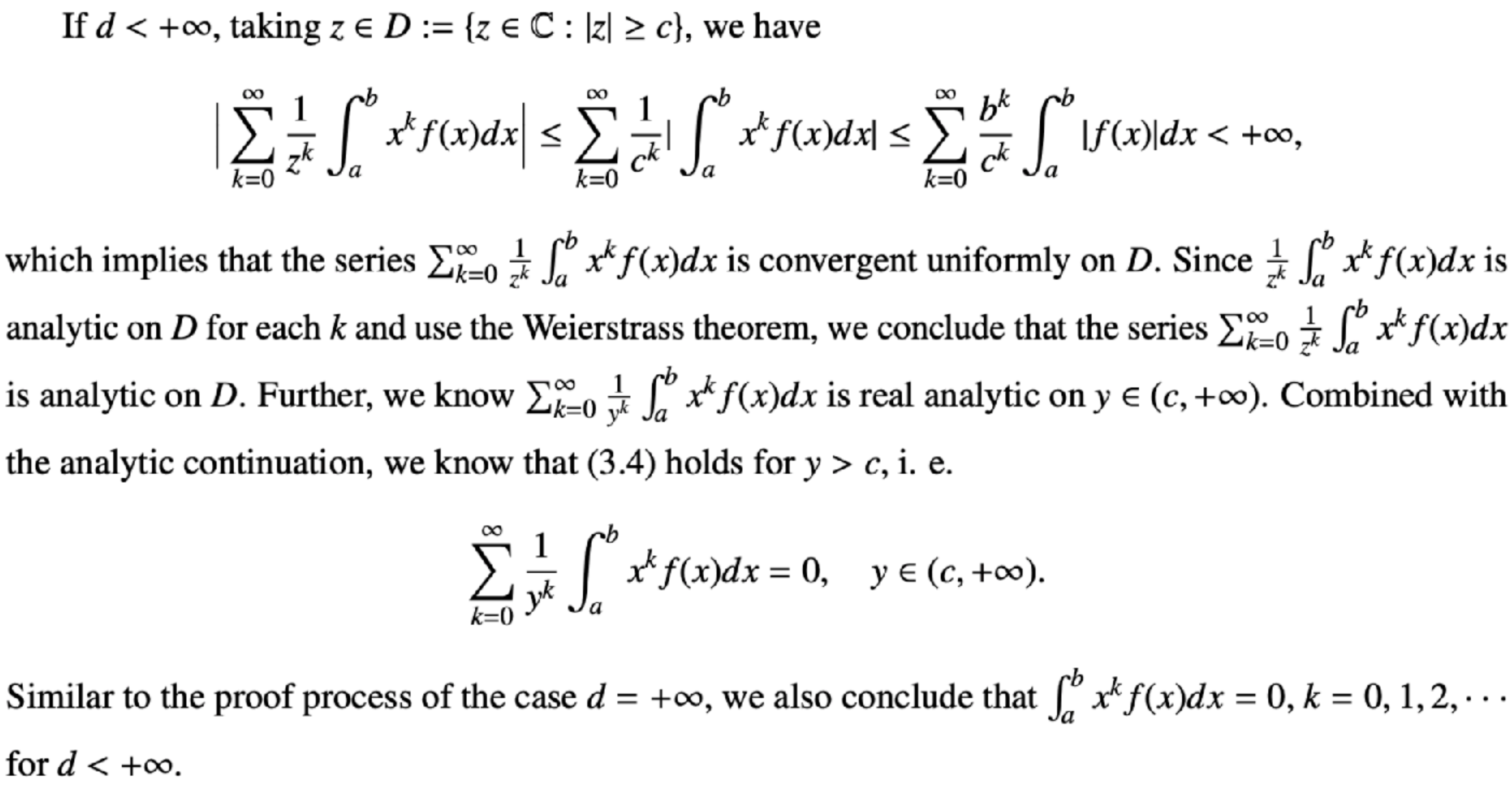
If $d = +\infty$, by using (3.4), we have

$$\int_a^b f(x)dx + \frac{1}{y}\int_a^b xf(x)dx + \cdots + \frac{1}{y^k}\int_a^b x^k f(x)dx + \cdots = 0, \quad y \in (c, +\infty). \tag{3.5}$$

Letting $y \to +\infty$ in (3.5), we have $\int_a^b f(x)dx = 0$. Then multiplying y on both sides of (3.5) and letting $y \to +\infty$, we also have $\int_a^b xf(x)dx = 0$. Repeating above process, we can obtain that

$$\int_a^b x^k f(x)dx = 0, \quad k = 0, 1, 2, \cdots. \tag{3.6}$$

Proof of uniqueness:



If $d < +\infty$, taking $z \in D := \{z \in \mathbb{C} : |z| \geq c\}$, we have

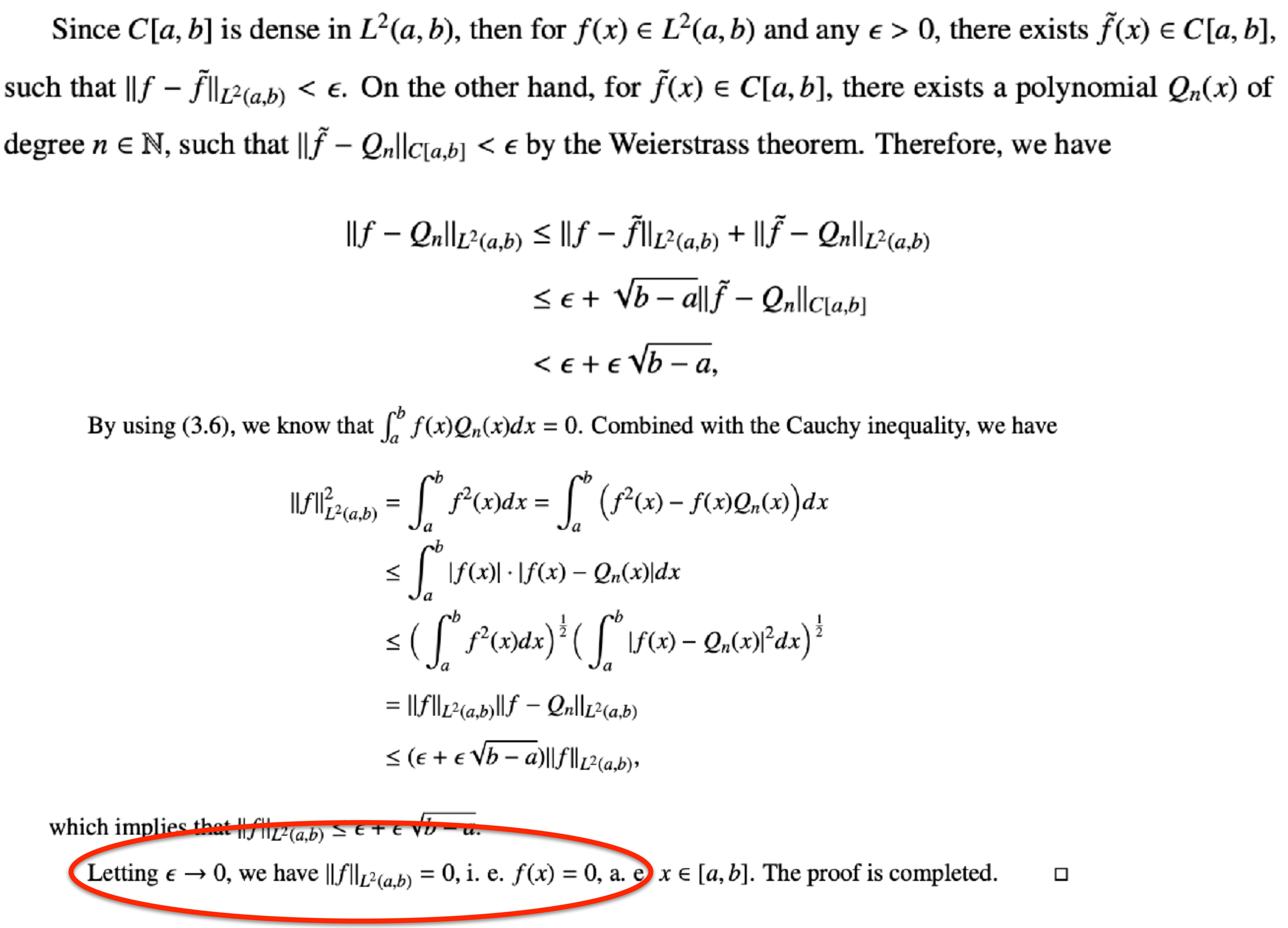
$$\left|\sum_{k=0}^{\infty} \frac{1}{z^k} \int_a^b x^k f(x)dx\right| \leq \sum_{k=0}^{\infty} \frac{1}{c^k} \left|\int_a^b x^k f(x)dx\right| \leq \sum_{k=0}^{\infty} \frac{b^k}{c^k} \int_a^b |f(x)|dx < +\infty,$$

which implies that the series $\sum_{k=0}^{\infty} \frac{1}{z^k} \int_a^b x^k f(x)dx$ is convergent uniformly on D. Since $\frac{1}{z^k} \int_a^b x^k f(x)dx$ is analytic on D for each k and use the Weierstrass theorem, we conclude that the series $\sum_{k=0}^{\infty} \frac{1}{z^k} \int_a^b x^k f(x)dx$ is analytic on D. Further, we know $\sum_{k=0}^{\infty} \frac{1}{y^k} \int_a^b x^k f(x)dx$ is real analytic on $y \in (c, +\infty)$. Combined with the analytic continuation, we know that (3.4) holds for $y > c$, i. e.

$$\sum_{k=0}^{\infty} \frac{1}{y^k} \int_a^b x^k f(x)dx = 0, \quad y \in (c, +\infty).$$

Similar to the proof process of the case $d = +\infty$, we also conclude that $\int_a^b x^k f(x)dx = 0$, $k = 0, 1, 2, \cdots$ for $d < +\infty$.

Proof of uniqueness:



Since $C[a,b]$ is dense in $L^2(a,b)$, then for $f(x) \in L^2(a,b)$ and any $\epsilon > 0$, there exists $\tilde{f}(x) \in C[a,b]$, such that $\|f - \tilde{f}\|_{L^2(a,b)} < \epsilon$. On the other hand, for $\tilde{f}(x) \in C[a,b]$, there exists a polynomial $Q_n(x)$ of degree $n \in \mathbb{N}$, such that $\|\tilde{f} - Q_n\|_{C[a,b]} < \epsilon$ by the Weierstrass theorem. Therefore, we have

$$\begin{aligned}\|f - Q_n\|_{L^2(a,b)} &\le \|f - \tilde{f}\|_{L^2(a,b)} + \|\tilde{f} - Q_n\|_{L^2(a,b)}\\ &\le \epsilon + \sqrt{b-a}\|\tilde{f} - Q_n\|_{C[a,b]}\\ &< \epsilon + \epsilon\sqrt{b-a},\end{aligned}$$

By using (3.6), we know that $\int_a^b f(x)Q_n(x)dx = 0$. Combined with the Cauchy inequality, we have

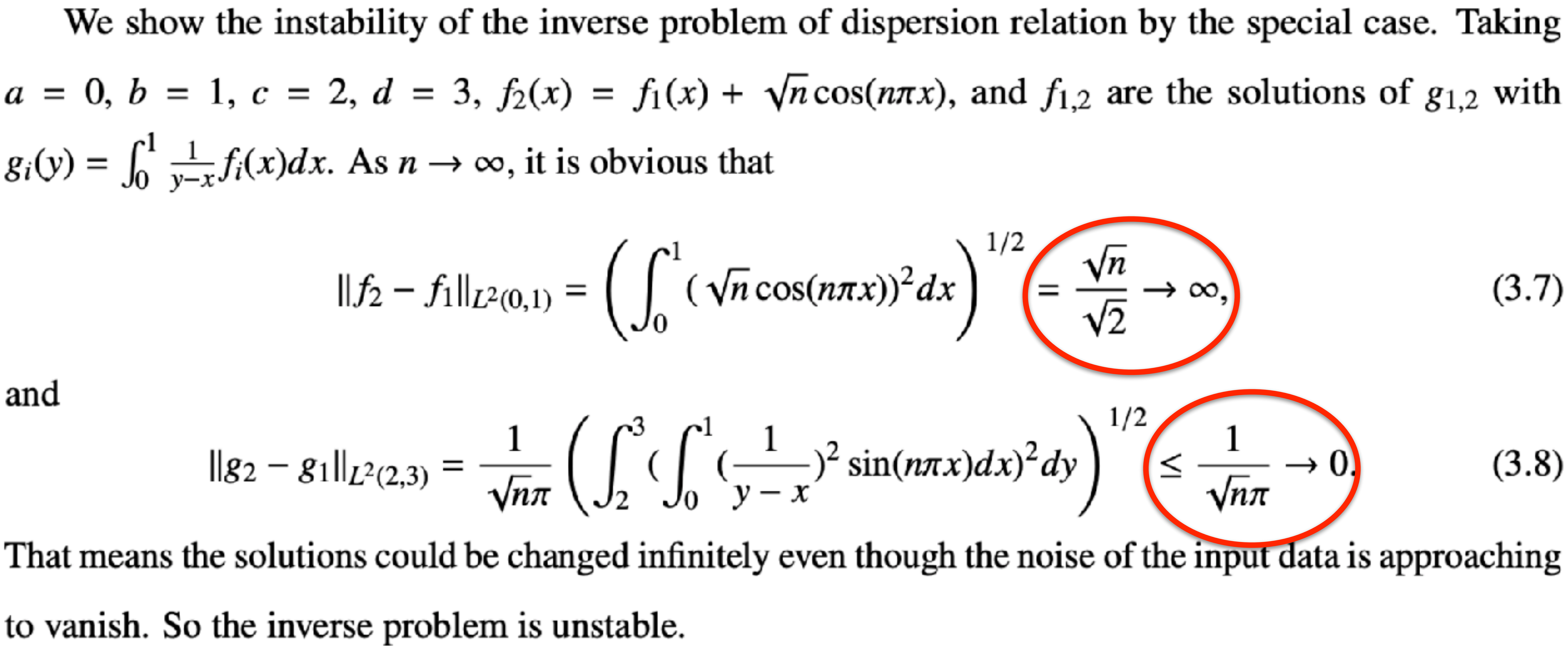
$$\begin{aligned}\|f\|^2_{L^2(a,b)} &= \int_a^b f^2(x)dx = \int_a^b \left(f^2(x) - f(x)Q_n(x)\right)dx\\ &\le \int_a^b |f(x)|\cdot|f(x) - Q_n(x)|dx\\ &\le \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}\left(\int_a^b |f(x) - Q_n(x)|^2dx\right)^{\frac{1}{2}}\\ &= \|f\|_{L^2(a,b)}\|f - Q_n\|_{L^2(a,b)}\\ &\le (\epsilon + \epsilon\sqrt{b-a})\|f\|_{L^2(a,b)},\end{aligned}$$

which implies that $\|f\|_{L^2(a,b)} \le \epsilon + \epsilon\sqrt{b-a}$.

Letting $\epsilon \to 0$, we have $\|f\|_{L^2(a,b)} = 0$, i. e. $f(x) = 0$, a. e $x \in [a,b]$. The proof is completed. □

Proof of instability:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c, d], \quad c > b, \quad a > 0$$



We show the instability of the inverse problem of dispersion relation by the special case. Taking $a = 0$, $b = 1$, $c = 2$, $d = 3$, $f_2(x) = f_1(x) + \sqrt{n}\cos(n\pi x)$, and $f_{1,2}$ are the solutions of $g_{1,2}$ with $g_i(y) = \int_0^1 \frac{1}{y-x} f_i(x)dx$. As $n \to \infty$, it is obvious that

$$\|f_2 - f_1\|_{L^2(0,1)} = \left(\int_0^1 (\sqrt{n}\cos(n\pi x))^2 dx\right)^{1/2} = \frac{\sqrt{n}}{\sqrt{2}} \to \infty, \tag{3.7}$$

and

$$\|g_2 - g_1\|_{L^2(2,3)} = \frac{1}{\sqrt{n}\pi}\left(\int_2^3 (\int_0^1 (\frac{1}{y-x})^2 \sin(n\pi x)dx)^2 dy\right)^{1/2} \le \frac{1}{\sqrt{n}\pi} \to 0, \tag{3.8}$$

That means the solutions could be changed infinitely even though the noise of the input data is approaching to vanish. So the inverse problem is unstable.

3. Numerical Method of Tikhonov Regularization

$$\boxed{f_\alpha^\delta = \underset{f\in L^2(a,b)}{\arg\min}\, J(f) = \underset{f\in L^2(a,b)}{\arg\min}\left(\frac{1}{2}\|Kf - g^\delta\|_{L^2(c,d)}^2 + \frac{\alpha}{2}\|f\|_{L^2(a,b)}^2\right)}$$

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h}, x \in [x_{i-1}, x_i], \\ -\frac{x-x_{i+1}}{h}, x \in [x_i, x_{i+1}], \\ 0, otherwise, \end{cases}$$

$$\varphi_0(x) = \begin{cases} -\frac{x-x_1}{h}, x \in [x_0, x_1], \\ 0, otherwise, \end{cases}$$

$$\varphi_n(x) = \begin{cases} \frac{x-x_{n-1}}{h}, x \in [x_{n-1}, x_n], \\ 0, otherwise. \end{cases}$$

$$X_n = \mathrm{span}\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$$

$$f_{\alpha,n}^\delta(x) = \sum_{i=0}^{n} c_i\varphi_i(x)$$

$$f_{\alpha,n}^\delta(x) = \sum_{i=0}^{n} c_i\varphi_i(x)$$

$$J(f_{\alpha,n}^\delta) = \frac{1}{2}\left\|\sum_{i=0}^{n} c_iK\varphi_i - g^\delta\right\|_{L^2(c,d)}^2 + \frac{\alpha}{2}\left\|\sum_{i=0}^{n} c_i\varphi_i\right\|_{L^2(a,b)}^2$$

$$= \frac{1}{2}\sum_{i,j=0}^{n} c_ic_j(K\varphi_i, K\varphi_j)_{L^2(c,d)} - \sum_{i=0}^{n} c_i(K\varphi_i, g^\delta)_{L^2(c,d)} + \frac{1}{2}(g^\delta, g^\delta)_{L^2(c,d)} + \frac{\alpha}{2}\sum_{i,j=0}^{n} c_ic_j(\varphi_i, \varphi_j)_{L^2(a,b)}$$

$$A_{ij} = (K\varphi_i, K\varphi_j)_{L^2(c,d)} \qquad B_{ij} = (\varphi_i, \varphi_j)_{L^2(a,b)} \qquad C = (c_0, c_1, \cdots, c_n)^T$$

$$(A + \alpha B)C = D \qquad D_i = (K\varphi_i, g^\delta)_{L^2(c,d)}$$

Theorem 4.3. *If the noise* δ *and the regularization parameter* α *are fixed, we have* $\|f_{\alpha,n}^\delta - f_\alpha^\delta\|_{L^2(a,b)} \to 0$, *as* $n \to \infty$.

$D^0 - \overline{D}^0$ Mixing

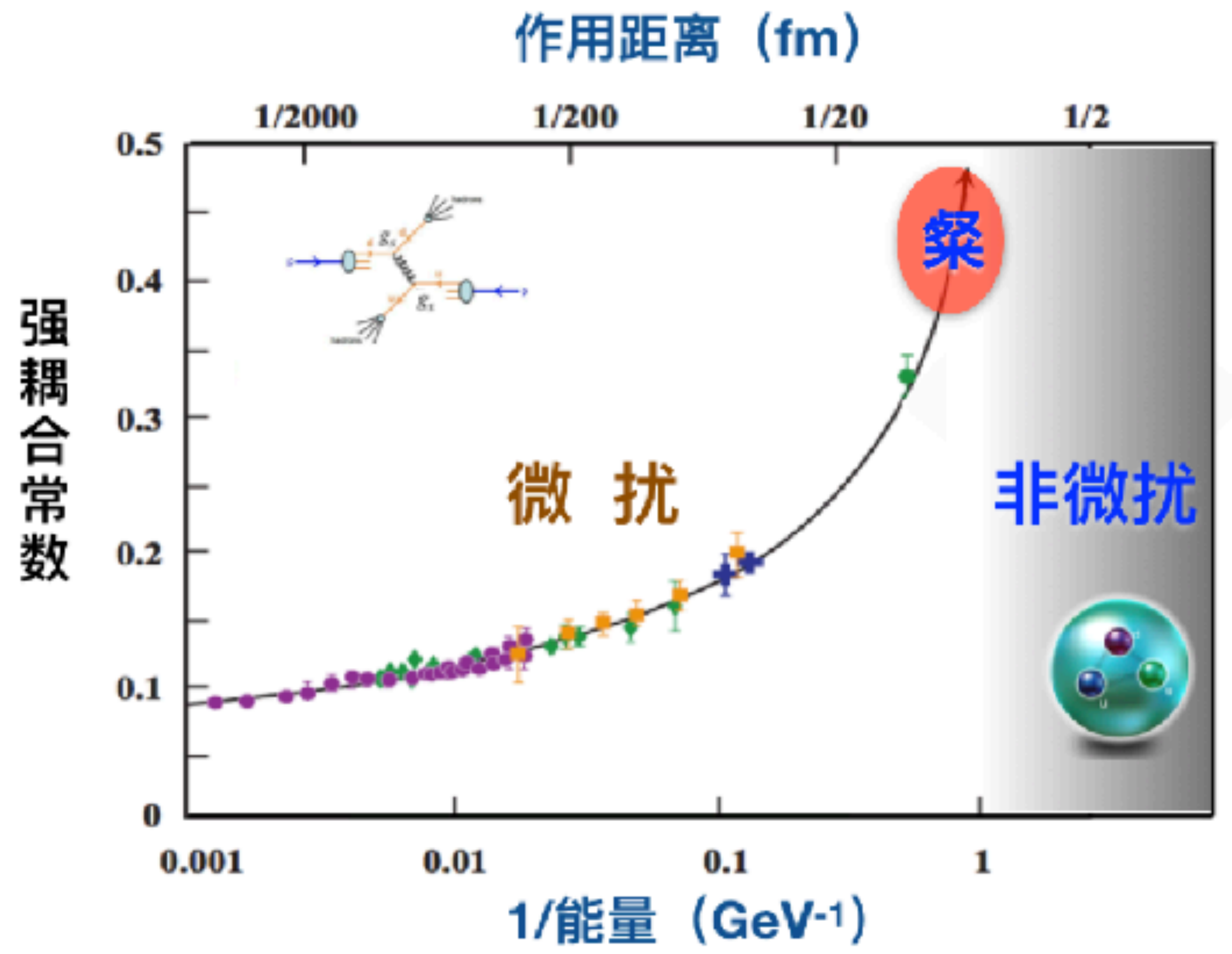
- The time evolution

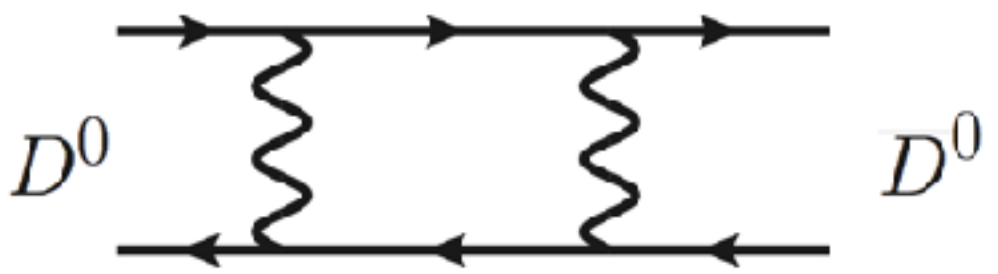
$$i\frac{\partial}{\partial t}\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\mathbf{\Gamma}\right)\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix}$$

- Mixing parameters: Mass and Width differences

$$\boxed{x \equiv \frac{\Delta m}{\Gamma} = \frac{m_1 - m_2}{\Gamma}} \qquad \boxed{y \equiv \frac{\Delta\Gamma}{2\Gamma} = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}}$$

- Useful to search for new physics,
- but less understood in the Standard Model

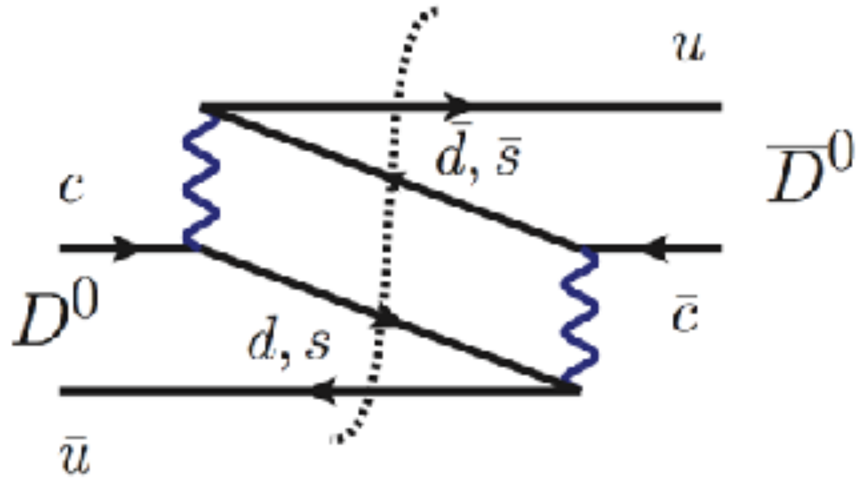


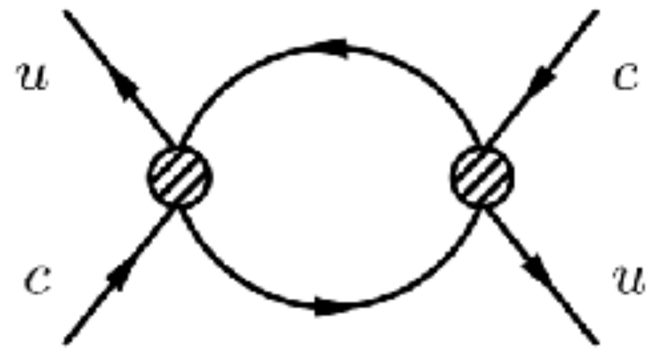


Falk, et al, ’02; Cheng, Chiang, ’10

$y_{\mathrm{PP+PV}} = (3.6 \pm 2.6) \times 10^{-3}$

- **Before 2017, exclusive approach is hopeful**

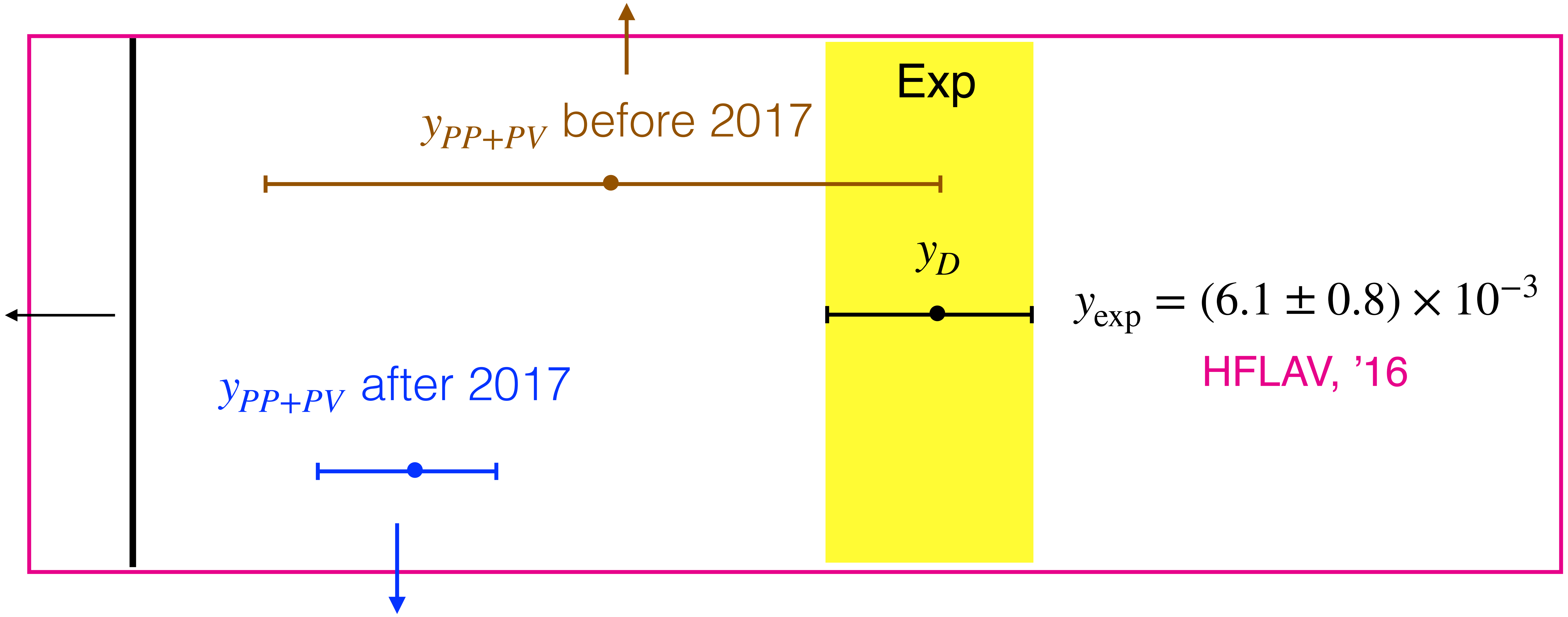




Inclusive approach doesn’t work

$y_{\mathrm{incl}} \sim 10^{-7}$

Lenz, *et al*, ’12



$y_{\mathrm{exp}} = (6.1 \pm 0.8) \times 10^{-3}$

HFLAV, ’16

- **After 2017, exclusive approach is dying**

$y_{\mathrm{PP+PV}} = (2.1 \pm 0.7) \times 10^{-3}$

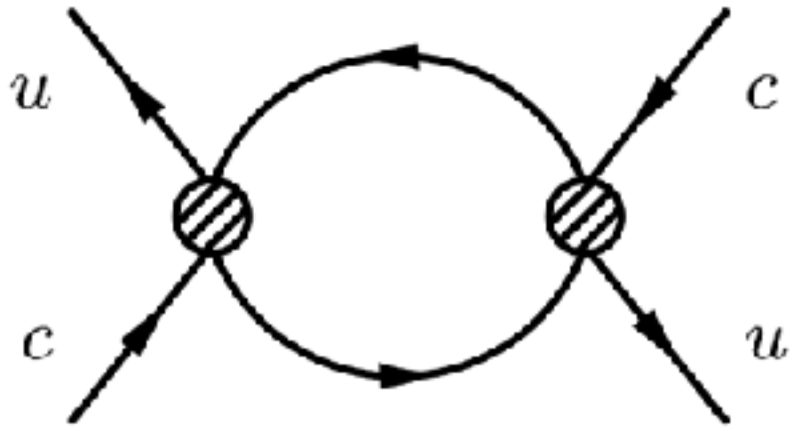
Jiang, **FSY**, Qin, Li, Lü, ’17

No theoretical methods work for D0 mixing

No theoretical predictions for indirect CP violation

Inclusive Approach

Theory / Exp. comparison (for inclusive)



quark level

Short-distance

D meson

Hagelin 1981, Cheng 1982
Buras, Slominski and Steger 1984
NLO QCD Golowich and Petrov 2005

SM
$$\begin{cases} x \simeq 6 \times 10^{-7} \\ y \simeq 6 \times 10^{-7} \end{cases}$$

Suppressed by GIM

HFLAV at Moriond2019

Exp.
$$\begin{cases} x = (3.9^{+1.1}_{-1.2}) \times 10^{-3} \\ y = (6.51^{+0.63}_{-0.69}) \times 10^{-3} \end{cases}$$

B_s meson

Artuso, Borissov and Lenz, 2016

SM
$$\begin{cases} \Delta M_s = (18.3 \pm 2.7)\mathrm{ps}^{-1} \\ \Delta\Gamma_s = (0.088 \pm 0.020)\mathrm{ps}^{-1} \end{cases}$$

HFLAV

Exp.
$$\begin{cases} \Delta M_s = (17.757 \pm 0.021)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_s = (0.082 \pm 0.006)\ \mathrm{ps}^{-1} \end{cases}$$

B_d meson

Artuso, Borissov and Lenz, 2016

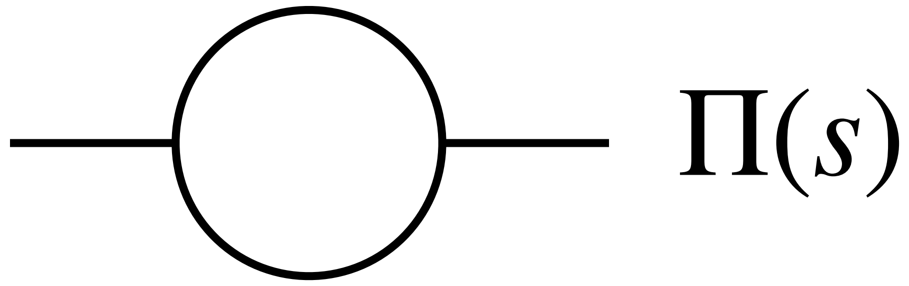
SM
$$\begin{cases} \Delta M_d = (0.528 \pm 0.078)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_d = (2.61 \pm 0.59) \cdot 10^{-3}\ \mathrm{ps}^{-1} \end{cases}$$

HFLAV

Exp.
$$\begin{cases} \Delta M_d = (0.5055 \pm 0.0020)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_d = 0.66(1 \pm 10) \cdot 10^{-3}\ \mathrm{ps}^{-1} \end{cases}$$

- For B_s, B_d mesons, the data are reproduced within 1σ.
- For D meson, the order of magnitude is not reproduced within leading-power.

Dispersion Relation

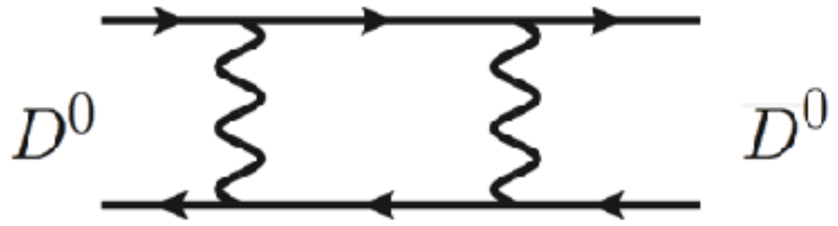


Dispersion Relation:

$$\mathcal{R}e[\Pi(s)] = \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

$D^0 - \overline{D}^0$ mixing:

$$M_{12} =$$

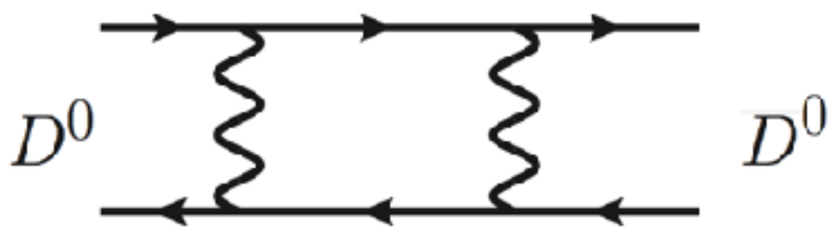


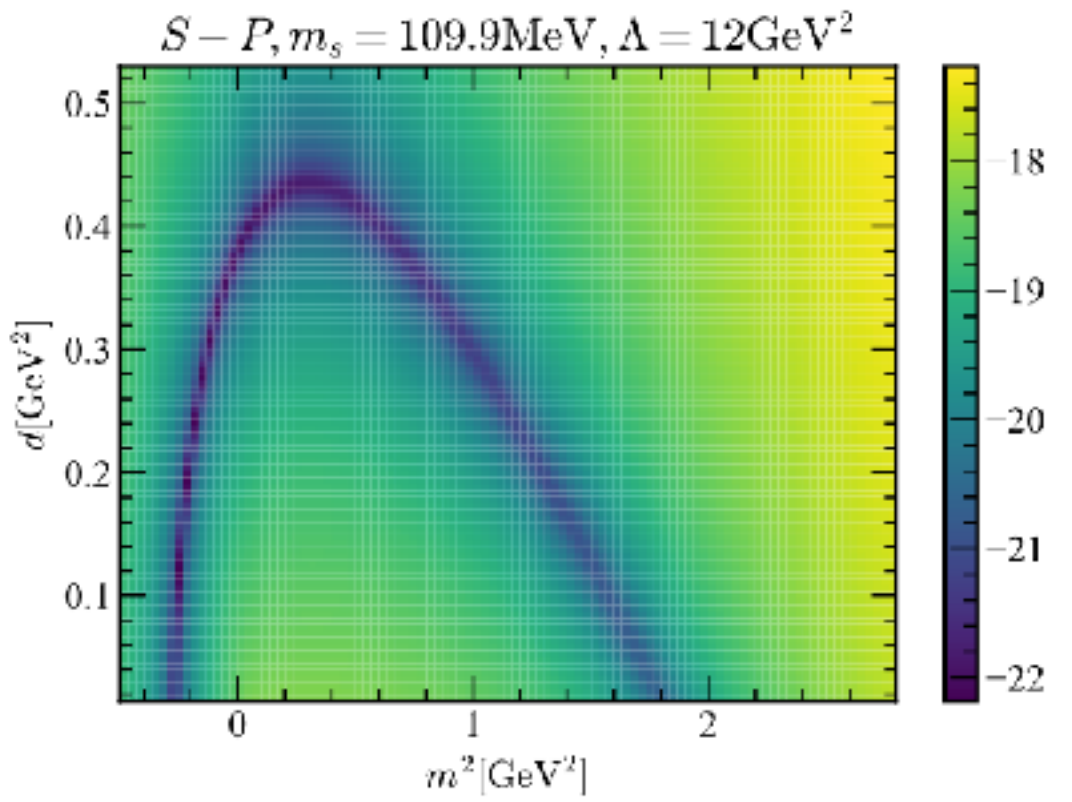
$$Re[M_{12}(s)] = \frac{1}{\pi}\int_0^\infty \frac{Im[M_{12}(s')]}{s-s'}ds'$$

$$x(s) = \frac{1}{\pi}\int_0^\infty \frac{y(s')}{s-s'}ds'$$

Inverse Problem

$D^0 - \overline{D}^0$ mixing



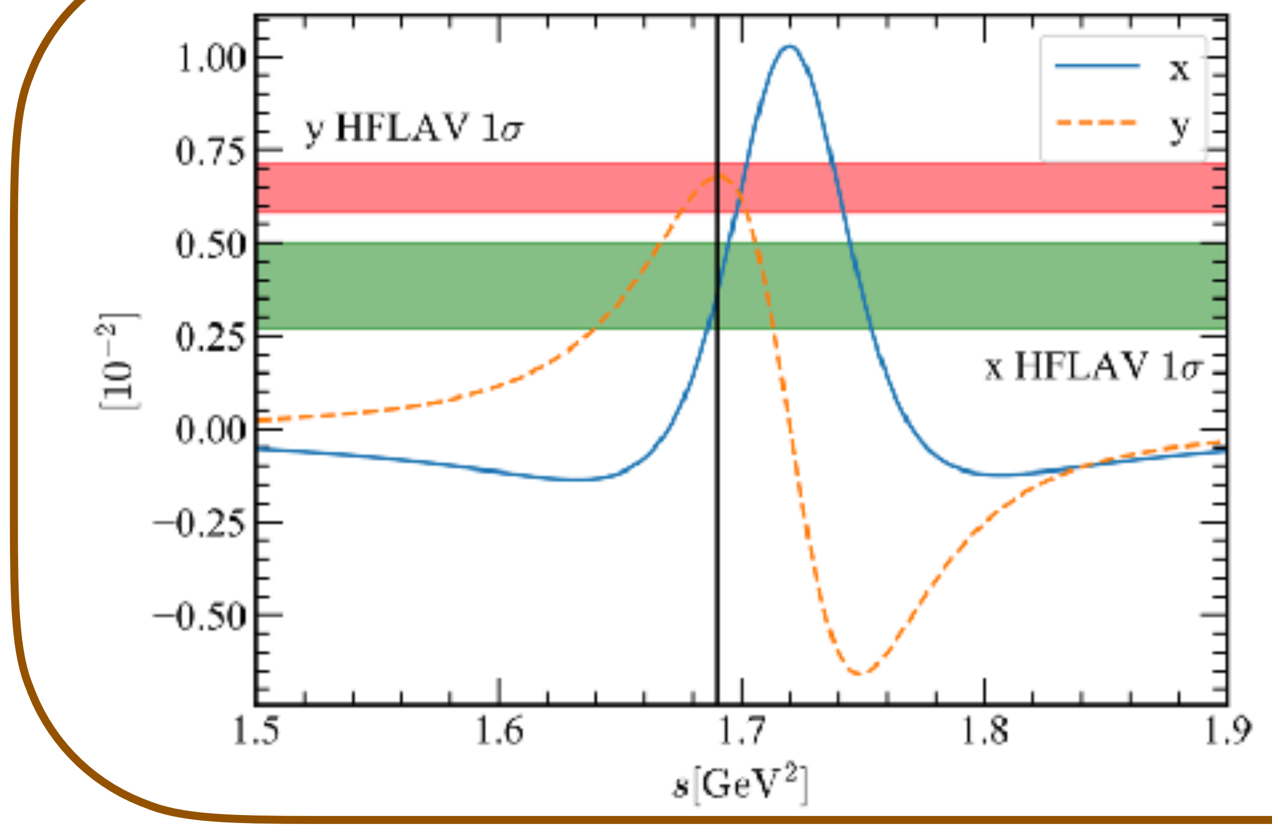


unstable solutions

$$\int_0^\Lambda ds' \frac{y(s')}{s-s'} = \pi x(s) - \int_\Lambda^\infty ds' \frac{y(s')}{s-s'} \equiv \omega(s)$$

parametrization:

$$y(s) = \frac{Ns[b_0 + b_1(s-m^2) + b_2(s-m^2)^2]}{[(s-m^2)^2 + d^2]^2}$$



Additional conditions:
data of x and y as inputs

Predict indirect CPV

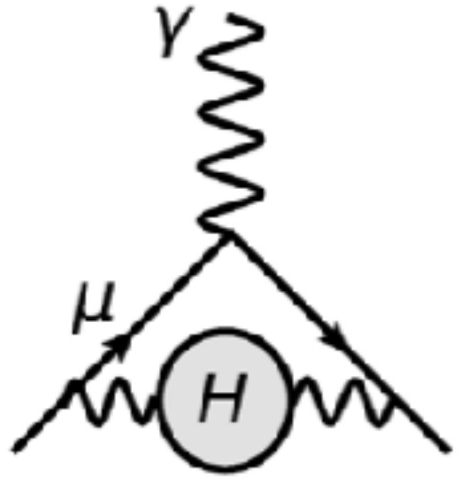
$$q/p = 1.0002e^{i0.006^\circ}$$

consistent with data

$$q/p = (0.969^{+0.050}_{-0.045})e^{i(-3.9^{+4.5}_{-4.6})^\circ}$$

Li, Umeeda, Xu, **FSY**, PLB(2020)

Applications of the Inverse Problem: muon g-2



- Muon g-2: 4.2σ deviation from the SM

Muon g-2, PRL(2021)

- Dominate uncertainty of the SM prediction: hadronic vacuum polarization (HVP)

Aoyama, et al, Phys.Rept(2020)

- Inverse Problem:

$$\int_{\lambda_r}^{\Lambda_r} ds' \frac{\mathrm{Im}\Pi_r(s')}{s'(s'+s)} - \pi \frac{\Pi_r(0)}{s} = -\pi \frac{\Pi_r(-s)}{s} - \int_{\Lambda_r}^{\infty} ds' \frac{\mathrm{Im}\Pi_r(s')}{s'(s'+s)}$$

$$r = \rho,\ \omega,\ \phi$$

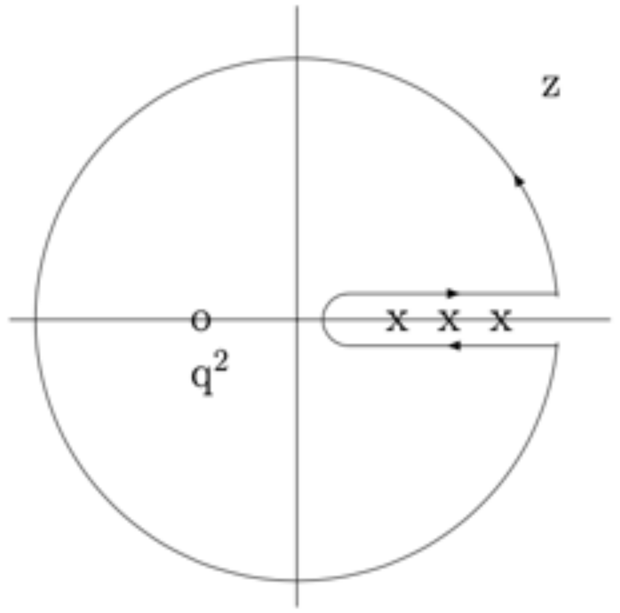
- Result: Inverse problem: $a_\mu^{\mathrm{HVP}} = (641^{+65}_{-63}) \times 10^{-10}$ H.n.Li, Umeeda, '20

Data driven: $a_\mu^{\mathrm{HVP}} = (693.9 \pm 4.0) \times 10^{-10}$ Davier, Hoecker, Malaescu, Zhang, '20

Lattice QCD: $a_\mu^{\mathrm{HVP}} = (654 \pm 32^{+21}_{-23}) \times 10^{-10}$ Della Morte et al, '17

Non-perturbative properties can be revealed from asymptotic QCD by solving an inverse problem.

Applications of the Inverse Problem: QCD sum rules



- Conventional QCD sum rules $\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle$

Dispersion relation: $\Pi(q^2) = \frac{1}{2\pi i}\oint ds \frac{\Pi(s)}{s-q^2} = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds \frac{\text{Im }\Pi(s)}{s-q^2-i\epsilon}$

$$\text{Im}\Pi(q^2) = \pi f_V^2 \delta(q^2 - m_V^2) + \pi\rho^h(q^2)\theta(q^2 - s_h)$$

Quark-hadron duality: $\rho^h(s) = \frac{1}{\pi}\text{Im}\Pi^{\text{pert}}(s)\theta(s - s_0)$

$$\int_{s_h}^{\infty} ds \frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_{s_0}^{\infty} ds \frac{\text{Im}\Pi^{\text{pert}}(s)}{s-q^2}$$

Uncertainty sources: quark-hadron duality and Borel transformation

Applications of the Inverse Problem: QCD sum rules

- **Inverse-Problem QCD sum rules**

$$\frac{1}{2\pi i}\oint ds \frac{\Pi(s)}{s-q^2} = \frac{1}{\pi}\int_{s_i}^{\Lambda} ds \frac{\mathrm{Im}\Pi(s)}{s-q^2} + \frac{1}{\pi}\int_{\Lambda}^{R} ds \frac{\mathrm{Im}\Pi^{\mathrm{pert}}(s)}{s-q^2} + \frac{1}{2\pi i}\int_C ds \frac{\Pi^{\mathrm{pert}}(s)}{s-q^2}$$

Involving excited states and parameterization:

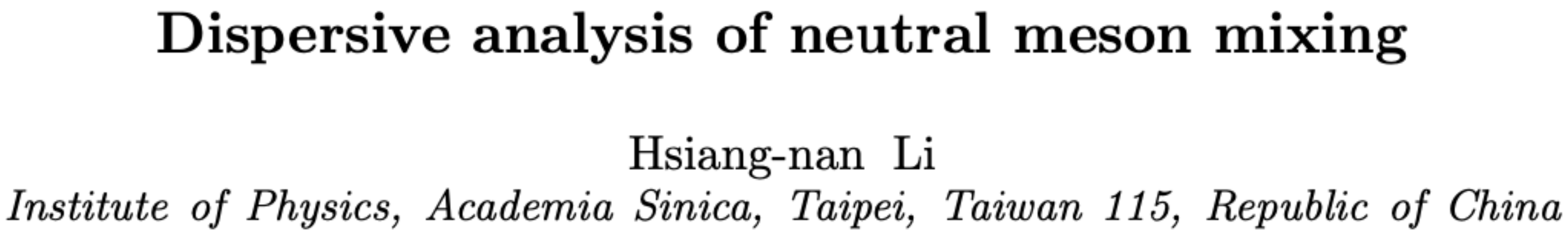
$$\begin{aligned}\mathrm{Im}\Pi(q^2) \;=\;& \pi f_\rho^2\delta(q^2-m_\rho^2) + \pi f_{\rho(1450)}^2\delta(q^2-m_{\rho(1450)}^2) + \pi f_{\rho(1700)}^2\delta(q^2-m_{\rho(1700)}^2)\\ &+\pi f_V^2\delta(q^2-m_V^2) + \pi\rho^h(q^2),\end{aligned}$$

$$\rho^h(y) = b_0P_0(2y-1) + b_1P_1(2y-1) + b_2P_2(2y-1) + b_3P_3(2y-1) + \cdots$$

$$m_{\rho(770)}(m_{\rho(1450)}, m_{\rho(1700)}, m_{\rho(1900)}) \approx 0.78\ (1.46,\ 1.70,\ 1.90)\ \mathrm{GeV}$$

$$f_{\rho(770)}(f_{\rho(1450)}, f_{\rho(1700)}, f_{\rho(1900)}) \approx 0.22\ \ (0.19,\ 0.14,\ 0.14)\ \ \mathrm{GeV}$$

H.n.Li, Umeeda, ’20



Dispersive analysis of neutral meson mixing

Hsiang-nan Li
Institute of Physics, Academia Sinica, Taipei, Taiwan 115, Republic of China

- Revisited by the inverse matrix method
- SU(3) breaking effects: physical thresholds of $D \to \pi\pi, K\pi, KK$
- The solutions are stable
- B mixing and kaon mixing are also studied in the same formalism

Inverse Problem: inverse matrix method

- The inverse matrix method was proposed by Hsiang-nan Li, on the studies of glueballs in arXiv:2109.04956 and on pion LCDA in arXiv:2205.06746
- A unique and stable solution can be attained before an ill posed nature appears. (Discretized regularization)

Inverse Problem: $$\int_0^\infty dv \frac{\Delta A_{ij}(v)}{u-v} = \int_0^\infty dv \frac{A_{ij}^{\text{box}}(v\Lambda + m_{IJ})e^{-v^2}}{u-v} + \int_{r_{ij}-r_{IJ}}^{0} dv \frac{A_{ij}^{\text{box}}(v\Lambda + m_{IJ})}{u-v}$$

To be solved

calculable

Notice that the range of $v = [0, +\infty)$

Inverse Problem: inverse matrix method

- Inverse Problem: $\int_0^\infty dv \frac{\Delta A_{ij}(v)}{u-v} = \int_0^\infty dv \frac{A_{ij}^{\mathrm{box}}(v\Lambda + m_{IJ})e^{-v^2}}{u-v} + \int_{r_{ij}-r_{IJ}}^{0} dv \frac{A_{ij}^{\mathrm{box}}(v\Lambda + m_{IJ})}{u-v}$
- Expansion by generalized Laguerre polynomials:

$$\Delta A_{ij}(v) = \sum_{n=1}^{N} a_n^{(ij)} v^\alpha e^{-v} L_{n-1}^{(\alpha)}(v),$$

Unknown coefficients: $a^{(ij)} = (a_1^{(ij)}, a_2^{(ij)}, \cdots, a_N^{(ij)}),$

$\alpha = 3/2$ by physical condition of $\Delta A_{ij}(v) \sim v^{3/2}$ at $v \to 0$

orthogonality: $\int_0^\infty v^\alpha e^{-v} L_m^{(\alpha)}(v) L_n^{(\alpha)}(v) dv = \frac{\Gamma(m+\alpha+1)}{m!}\delta_{mn}$ independent $a_n^{(ij)}$

- Inverse Problem: $\int_0^\infty dv \frac{\Delta A_{ij}(v)}{u-v} = \int_0^\infty dv \frac{A_{ij}^{\text{box}}(v\Lambda + m_{IJ})e^{-v^2}}{u-v} + \int_{r_{ij}-r_{IJ}}^{0} dv \frac{A_{ij}^{\text{box}}(v\Lambda + m_{IJ})}{u-v}$

- Expansion by generalized Laguerre polynomials: $\Delta A_{ij}(v) = \sum_{n=1}^{N} a_n^{(ij)} v^\alpha e^{-v} L_{n-1}^{(\alpha)}(v),$

- Taylor expansion of the integral: $\frac{1}{u-v} = \sum_{m=1}^{N} \frac{v^{m-1}}{u^m}$ for a sufficiently large $|u|$

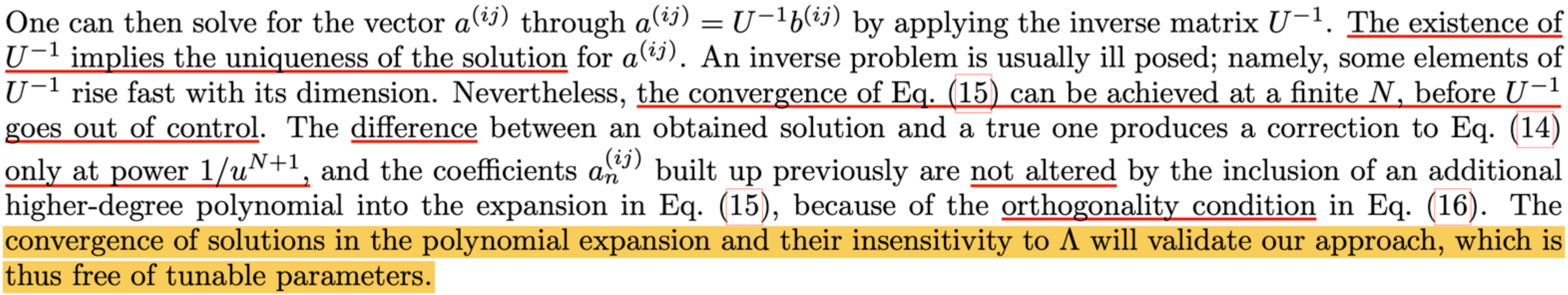
$$U_{mn} = \int_0^\infty dv v^{m-1+\alpha} e^{-v} L_{n-1}^{(\alpha)}(v) \qquad \boxed{U a^{(ij)} = b^{(ij)}}$$

$$b_m^{(ij)} = \int_0^\infty dv v^{m-1} A_{ij}^{\text{box}}(v\Lambda + m_{IJ})e^{-v^2} + \int_{r_{ij}-r_{IJ}}^{0} dv v^{m-1} A_{ij}^{\text{box}}(v\Lambda + m_{IJ})$$

- **Inverse Problem: inverse matrix method** $\boxed{a^{(ij)} = U^{-1} b^{(ij)}}$ $N \times N$: regularization

Inverse Problem: inverse matrix method

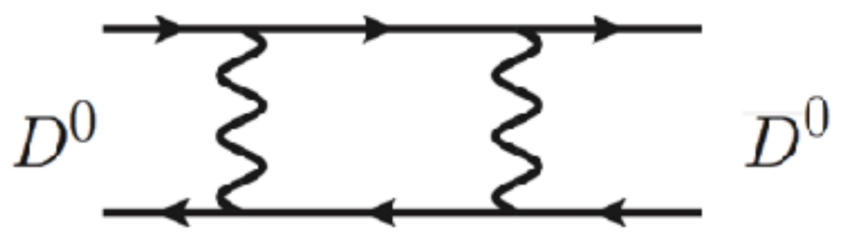
$$\boxed{a^{(ij)} = U^{-1} b^{(ij)}}$$



One can then solve for the vector $a^{(ij)}$ through $a^{(ij)} = U^{-1} b^{(ij)}$ by applying the inverse matrix U^{-1}. The existence of U^{-1} implies the uniqueness of the solution for $a^{(ij)}$. An inverse problem is usually ill posed; namely, some elements of U^{-1} rise fast with its dimension. Nevertheless, the convergence of Eq. (15) can be achieved at a finite N, before U^{-1} goes out of control. The difference between an obtained solution and a true one produces a correction to Eq. (14) only at power $1/u^{N+1}$, and the coefficients $a_n^{(ij)}$ built up previously are not altered by the inclusion of an additional higher-degree polynomial into the expansion in Eq. (15), because of the orthogonality condition in Eq. (16). The convergence of solutions in the polynomial expansion and their insensitivity to Λ will validate our approach, which is thus free of tunable parameters.

$D^0 - \overline{D}^0$ mixing

- Dispersion relation: $$M_{12}(s) = \frac{1}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \, \frac{\Gamma_{12}(s')}{s - s'}$$



$$|D_{1,2}\rangle = p|D^0\rangle \pm q|\bar{D}^0\rangle \qquad \frac{q}{p} = \sqrt{\frac{2M_{12}^* - i\Gamma_{12}^*}{2M_{12} - i\Gamma_{12}}}$$

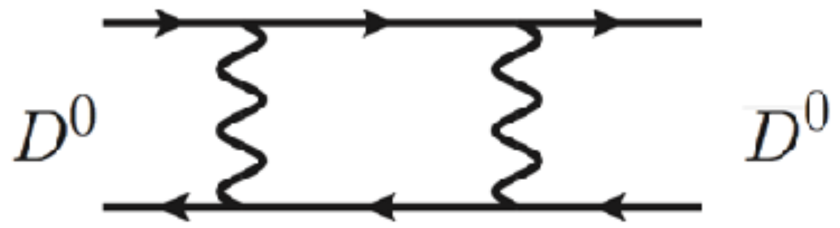
$$x \equiv \frac{m_2 - m_1}{\Gamma} = \frac{1}{\Gamma}\mathrm{Re}\left[\frac{q}{p}(2M_{12} - i\Gamma_{12})\right] \qquad y \equiv \frac{\Gamma_2 - \Gamma_1}{2\Gamma} = -\frac{1}{\Gamma}\mathrm{Im}\left[\frac{q}{p}(2M_{12} - i\Gamma_{12})\right]$$

In the CP-conserving case: $$x = \frac{2M_{12}}{\Gamma}, \quad y = \frac{\Gamma_{12}}{\Gamma}$$

- Absorptive piece: $$\Gamma_{12}(s) = \sum_{i,j} \lambda_i \lambda_j \Gamma_{ij}(s) \qquad i, j = d, s, b, \text{ and } \lambda_k \equiv V_{ck} V_{uk}^*, \; k = d, s, b$$

$$\Gamma_{12}(m_D^2) = \lambda_s^2[\Gamma_{dd}(m_D^2) - 2\Gamma_{ds}(m_D^2) + \Gamma_{ss}(m_D^2)] + 2\lambda_s\lambda_b[\Gamma_{dd}(m_D^2) - \Gamma_{ds}(m_D^2)] + \lambda_b^2\Gamma_{dd}(m_D^2)$$

$D^0 - \overline{D}^0$ mixing



- Dispersion relation: $$M_{12}(s) = \frac{1}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s-s'}$$
- Absorptive piece: $$\Gamma_{12}(s) = \sum_{i,j} \lambda_i \lambda_j \Gamma_{ij}(s) \qquad i,j = d,s,b, \text{ and } \lambda_k \equiv V_{ck}V_{uk}^*,\ k = d,s,b$$

At large s $$\Gamma_{ij}^{\rm box}(s) = \frac{G_F^2 f_D^2 m_W^3 B_D}{12\pi^2} A_{ij}^{\rm box}(s)$$

$$x_i = m_i^2/m_W^2 \qquad x_D = s/m_W^2$$

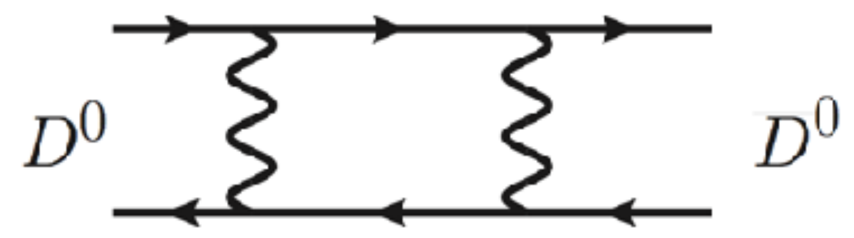
$$\begin{aligned} A_{ij}^{\rm box}(s) \ &= \ \frac{\pi}{2x_D^{3/2}} \frac{\sqrt{x_D^2 - 2x_D(x_i+x_j) + (x_i-x_j)^2}}{(1-x_i)(1-x_j)} \\ &\times \left\{ \left(1 + \frac{x_i x_j}{4}\right) [3x_D^2 - x_D(x_i+x_j) - 2(x_i-x_j)^2] + 2x_D(x_i+x_j)(x_i+x_j-x_D) \right\} \end{aligned}$$

$$M_{12}(s) = \sum_{i,j} \lambda_i \lambda_j M_{ij}(s)$$

- In principle, the dispersion relation, as a result of QCD dynamics which has nothing to do with the CKM factors, holds for each pair of the components Mij(s) and Γij(s)
- Inverse problem for each components of ij. $A_{ij}(s)$ is monotonic, easily for solutions.

$D^0 - \overline{D}^0$ mixing

- Dispersion relation: $$M_{12}(s) = \frac{1}{2\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\Gamma_{12}(s')}{s-s'}$$



$$\begin{aligned} A_{ij}^{\rm box}(s) &= \frac{\pi}{2x_D^{3/2}}\frac{\sqrt{x_D^2-2x_D(x_i+x_j)+(x_i-x_j)^2}}{(1-x_i)(1-x_j)} \\ &\times\left\{\left(1+\frac{x_ix_j}{4}\right)\left[3x_D^2-x_D(x_i+x_j)-2(x_i-x_j)^2\right]+2x_D(x_i+x_j)(x_i+x_j-x_D)\right\} \end{aligned}$$

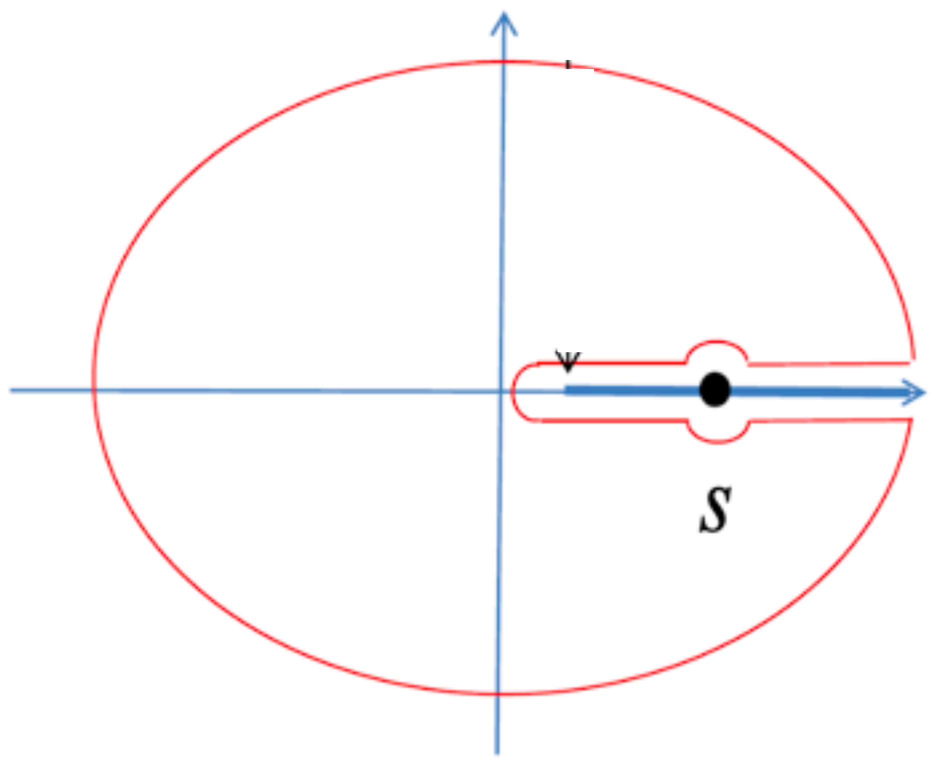
$\Gamma_{ij}(s)$ grows like $s^{3/2}$, so the integration is divergent.

Γ_{12} converges due to SU(3) cancellation.

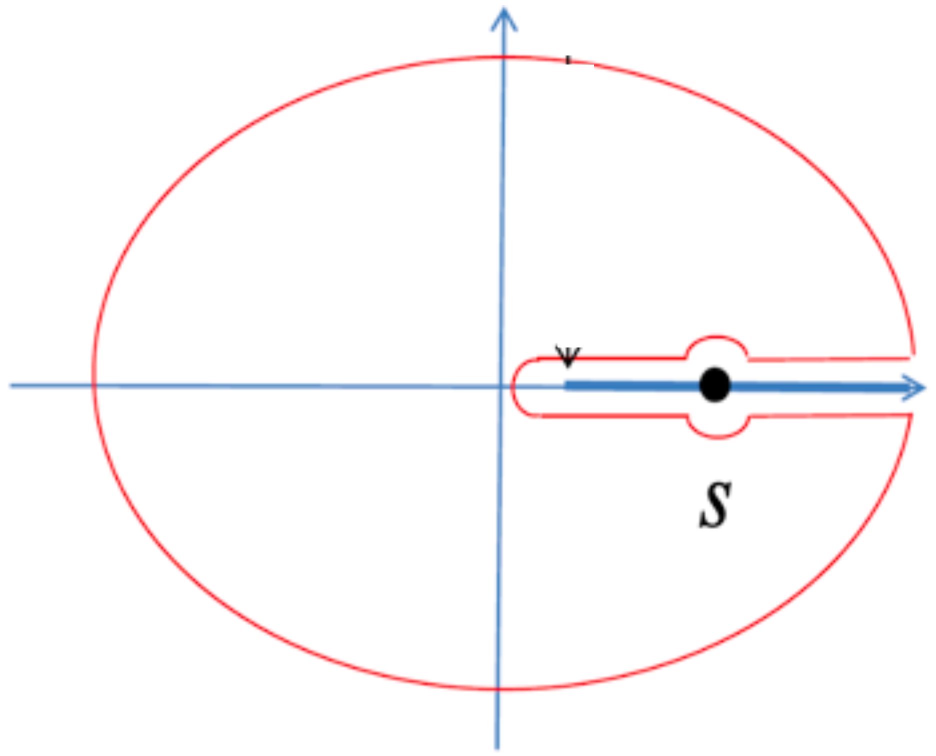
- Reformulate the dispersion relation: $$\Pi_{ij}(s) = M_{ij}(s) - i\Gamma_{ij}(s)/2$$

$$\frac{1}{2\pi i}\oint ds' \frac{\Pi_{ij}(s')}{s-s'} = 0$$

$$M_{ij}(s) = \frac{1}{2\pi}\int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s-s'} + \frac{1}{2\pi i}\int_{C_R} ds' \frac{\Pi_{ij}^{\rm box}(s')}{s-s'},$$



$D^0 - \overline{D}^0$ mixing



- Reformulate the dispersion relation: $\Pi_{ij}(s) = M_{ij}(s) - i\Gamma_{ij}(s)/2$

$$\frac{1}{2\pi i}\oint ds' \frac{\Pi_{ij}(s')}{s-s'} = 0$$

Physical threshold: $$M_{ij}(s) = \frac{1}{2\pi}\int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s-s'} + \frac{1}{2\pi i}\int_{C_R} ds' \frac{\Pi_{ij}^{\rm box}(s')}{s-s'},$$

$$m_{\pi\pi} = 4m_\pi^2, \;\; m_{\pi K} = (m_\pi + m_K)^2, \; m_{KK} = 4m_K^2,$$

Quark-level threshold: $$M_{ij}^{\rm box}(s) = \frac{1}{2\pi}\int_{m_{ij}}^{R} ds' \frac{\Gamma_{ij}^{\rm box}(s')}{s-s'} + \frac{1}{2\pi i}\int_{C_R} ds' \frac{\Pi_{ij}^{\rm box}(s')}{s-s'},$$

$$m_{dd} = 4m_d^2, \; m_{ds} = (m_d + m_s)^2, \; m_{ss} = 4m_s^2,$$

At large s, $M_{ij}(s) = M_{ij}^{\rm box}(s)$, as heavy meson mixings.

$$\int_{m_{IJ}}^{R} ds' \frac{\Gamma_{ij}(s')}{s-s'} = \int_{m_{ij}}^{R} ds' \frac{\Gamma_{ij}^{\rm box}(s')}{s-s'},$$

$D^0 - \overline{D}^0$ mixing

- Reformulate the dispersion relation:

$$\int_{m_{IJ}}^{R} ds' \frac{\boxed{\Gamma_{ij}(s')}}{s - s'} = \int_{m_{ij}}^{R} ds' \frac{\Gamma_{ij}^{\mathrm{box}}(s')}{s - s'},$$

To be solved

- Introduce a subtracted unknown function:

$$\Delta\Gamma_{ij}(s, \Lambda) = \Gamma_{ij}(s) - \Gamma_{ij}^{\mathrm{box}}(s)\{1 - \exp[-(s - m_{IJ})^2/\Lambda^2]\}$$

The scale Λ characterizes the order of s, at which $\Gamma_{ij}(s)$ transits to the perturbative expression $\Gamma_{ij}^{\mathrm{box}}(s)$

Alternative formula, like $1 - \exp[-(s - m_{IJ})^3/\Lambda^3]$, only vary the solution by few percent

The subtraction term can be regarded as an ultraviolet regulator.

- Inverse problem:

$$\int_{m_{IJ}}^{\infty} ds' \frac{\Delta\Gamma_{ij}(s', \Lambda)}{s - s'} = \int_{m_{IJ}}^{\infty} ds' \frac{\Gamma_{ij}^{\mathrm{box}}(s') \exp[-(s' - m_{IJ})^2/\Lambda^2]}{s - s'} + \int_{m_{ij}}^{m_{IJ}} ds' \frac{\Gamma_{ij}^{\mathrm{box}}(s')}{s - s'}.$$

Numerical results

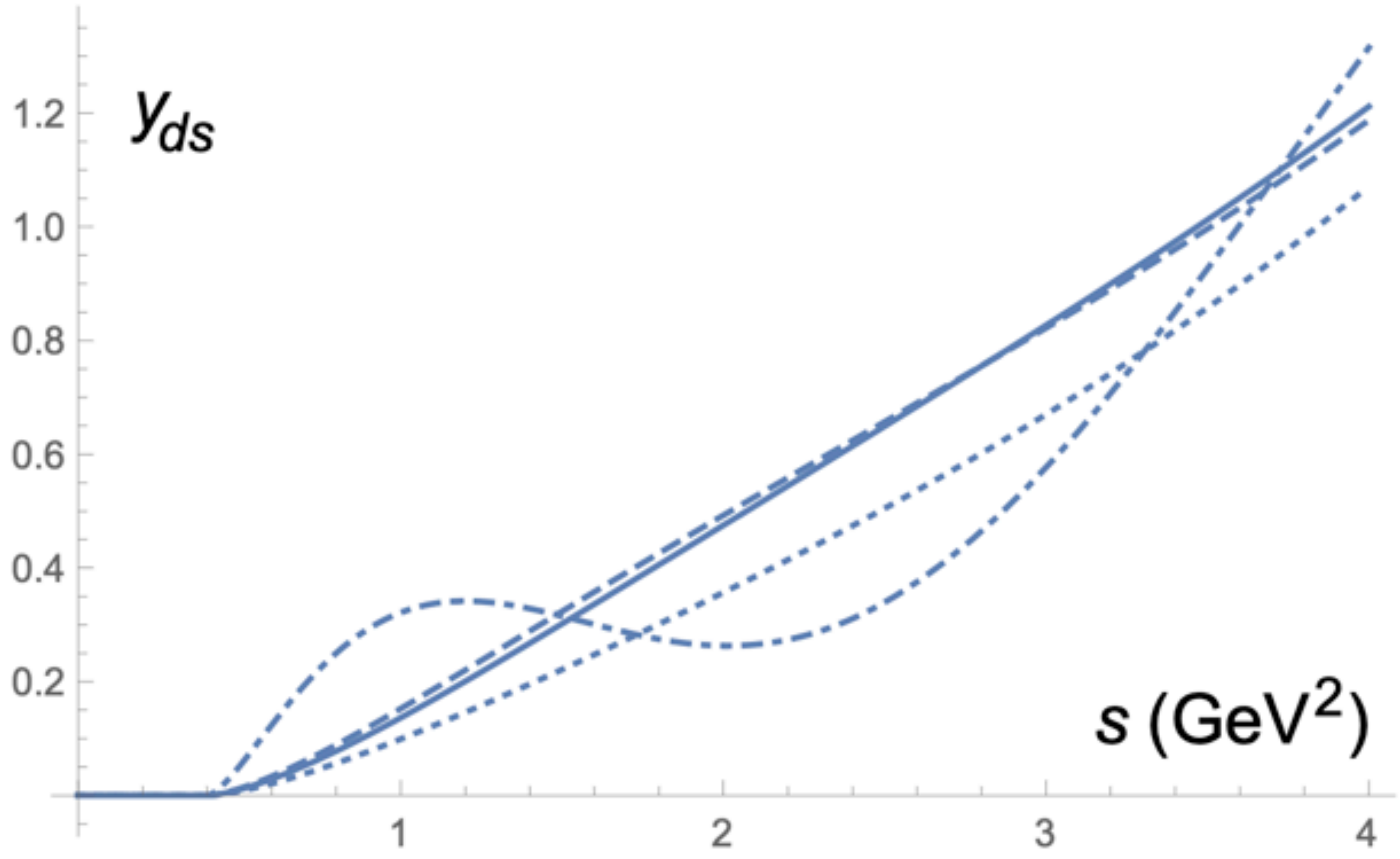


FIG. 1: Dependencies of $y_{ds}(s) \equiv \Gamma_{ds}(s)/\Gamma$ on s for $N = 3$ (dotted line), $N = 8$ (dashed line), $N = 13$ (solid line) and $N = 23$ (dot-dashed line) with $\Lambda = 5$ GeV^2.

$$\begin{aligned} & 10^5 \times (a_1^{(ds)}, a_2^{(ds)}, a_3^{(ds)}, \cdots, a_{12}^{(ds)}, a_{13}^{(ds)}, a_{14}^{(ds)}, \cdots, a_{22}^{(ds)}, a_{23}^{(ds)}) \\ = & \ (4.04, 2.47, 1.45, \cdots, -2.08 \times 10^{-2}, -4.59 \times 10^{-3}, 9.25 \times 10^{-3}, \cdots, 7.49 \times 10^{-2}, 1.04), \end{aligned}$$

Numerical results

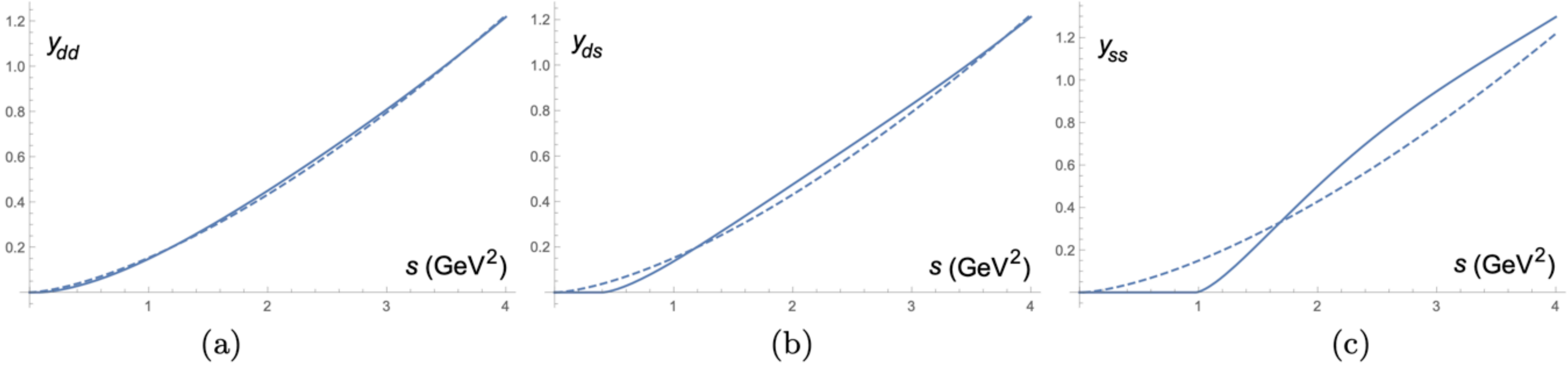
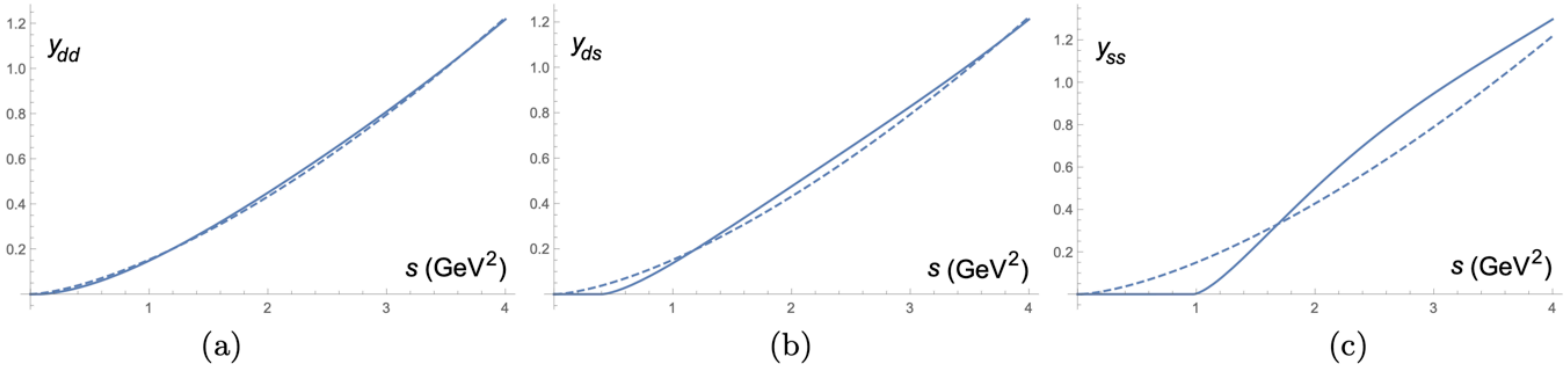


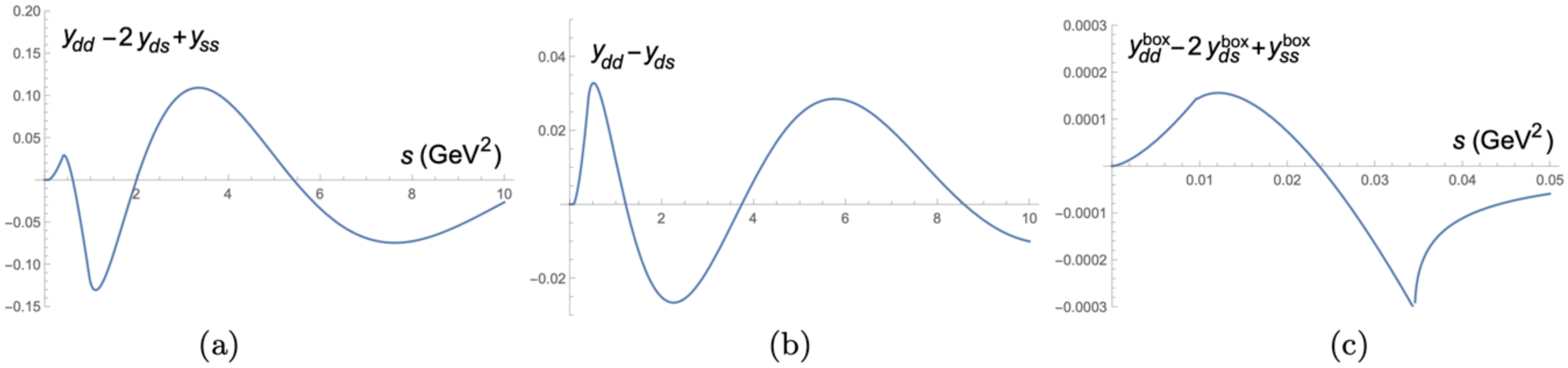
FIG. 2: Comparison of the solutions $y_{ij}(s) \equiv \Gamma_{ij}(s)/\Gamma$ (solid lines) with the inputs $y_{ij}^{\text{box}}(s) \equiv \Gamma_{ij}^{\text{box}}(s)/\Gamma$ (dashed lines) for (a) $ij = dd$, (b) $ij = ds$ and (c) $ij = ss$ at $\Lambda = 5\ \text{GeV}^2$.

Numerical results



(a) (b) (c)

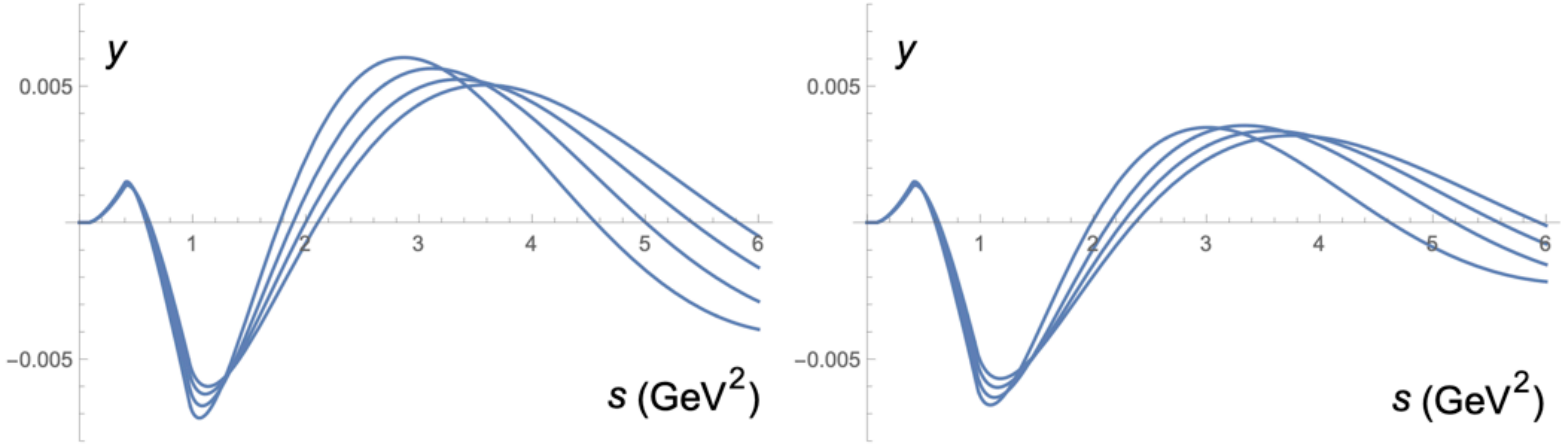
FIG. 2: Comparison of the solutions $y_{ij}(s) \equiv \Gamma_{ij}(s)/\Gamma$ (solid lines) with the inputs $y_{ij}^{\rm box}(s) \equiv \Gamma_{ij}^{\rm box}(s)/\Gamma$ (dashed lines) for (a) $ij = dd$, (b) $ij = ds$ and (c) $ij = ss$ at $\Lambda = 5$ GeV^2.



(a) (b) (c)

FIG. 3: Dependencies of (a) $y_{dd} - 2y_{ds} + y_{ss}$, (b) $y_{dd} - y_{ds}$ and (c) $y_{dd}^{\rm box} - 2y_{ds}^{\rm box} + y_{ss}^{\rm box}$ on s for $\Lambda = 5$ GeV^2.

Numerical results



(a) (b)

FIG. 4: Solutions of $y(s)$ for $\Lambda = 4.0$ GeV^2, 4.5 GeV^2, 5.0 GeV^2 and 5.5 GeV^2, corresponding to the curves with the peaks from left to right, in the cases (a) with and (b) without the second term in Eq. (20).

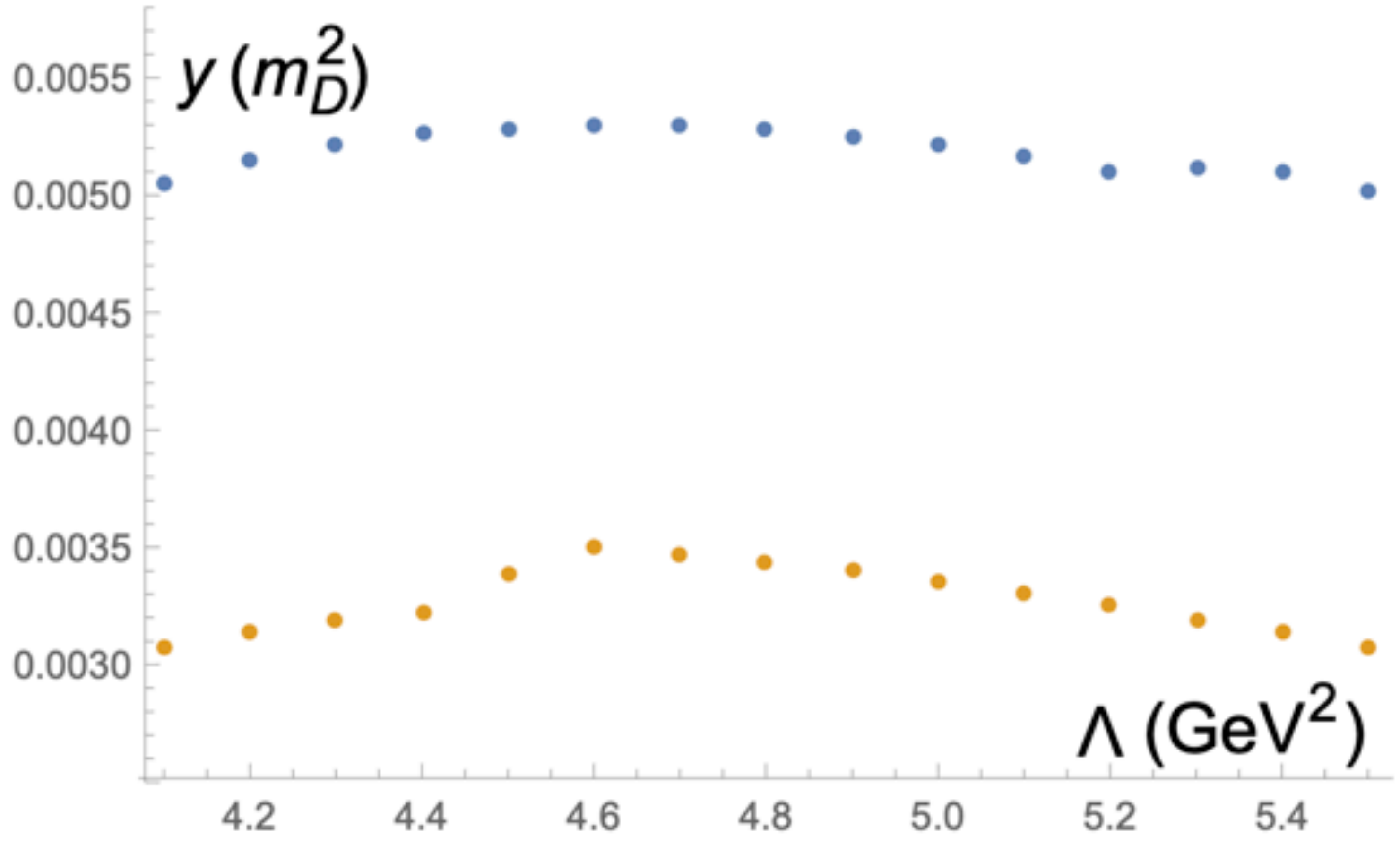


FIG. 5: Dependencies of $y(m_D^2)$ on Λ in the cases with (upper curve) and without (lower curve) the second term in Eq. (20).

Numerical results

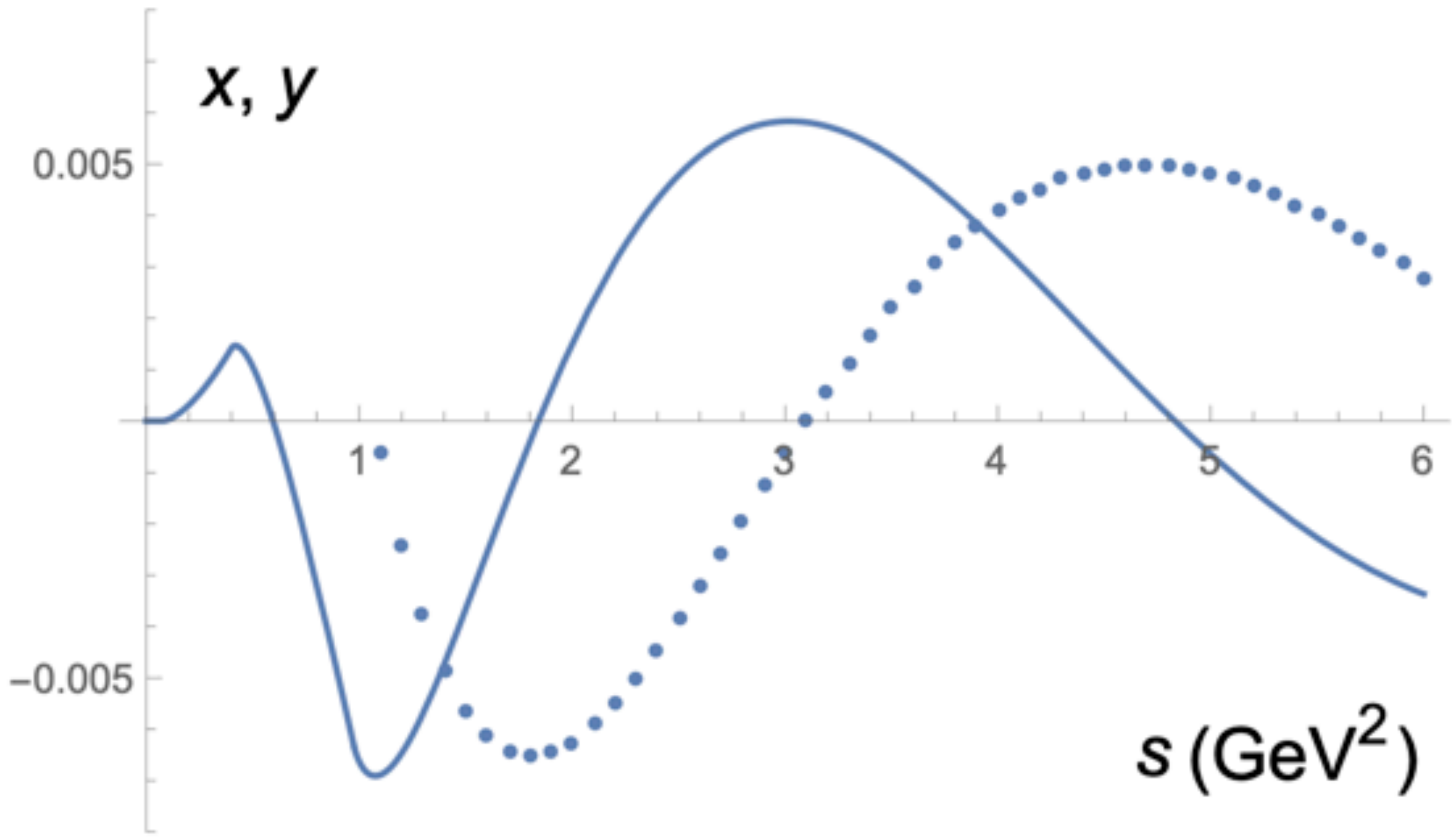
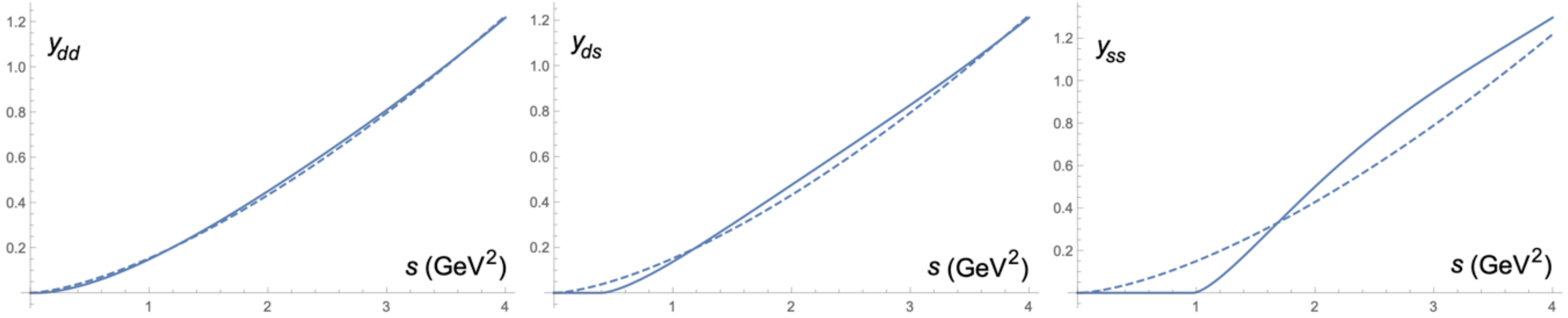


FIG. 7: Behaviors of $x(s)$ (dotted line) and $y(s)$ (solid line) for $\Lambda = 4.3$ GeV^2.

$$x(m_D^2) = (0.21^{+0.04}_{-0.07})\%, \quad y(m_D^2) = (0.52 \pm 0.03)\%.$$

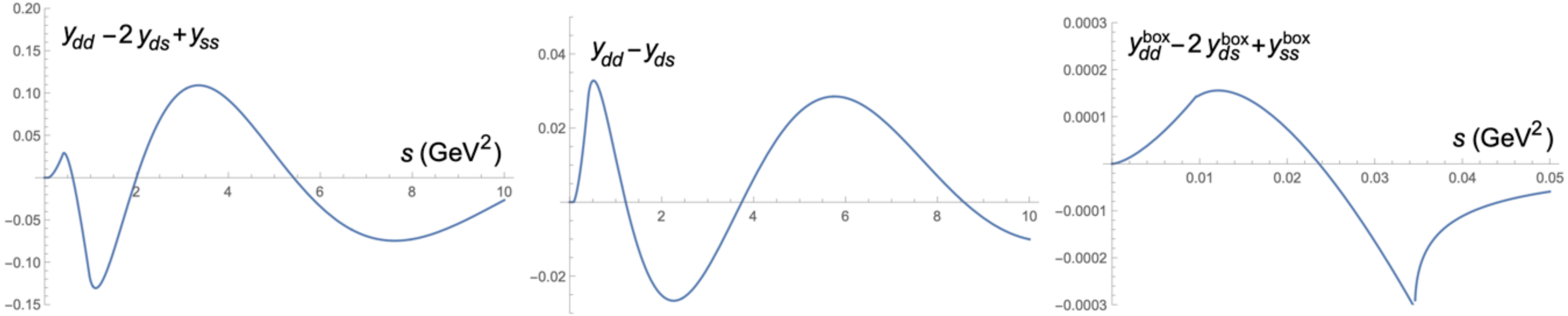
$$x = (0.44^{+0.13}_{-0.15})\%, \quad y = (0.63 \pm 0.07)\%,$$

Numerical results



(a) (b) (c)

FIG. 2: Comparison of the solutions $y_{ij}(s) \equiv \Gamma_{ij}(s)/\Gamma$ (solid lines) with the inputs $y_{ij}^{\text{box}}(s) \equiv \Gamma_{ij}^{\text{box}}(s)/\Gamma$ (dashed lines) for (a) $ij = dd$, (b) $ij = ds$ and (c) $ij = ss$ at $\Lambda = 5\ \text{GeV}^2$.



(a) (b) (c)

FIG. 3: Dependencies of (a) $y_{dd} - 2y_{ds} + y_{ss}$, (b) $y_{dd} - y_{ds}$ and (c) $y_{dd}^{\text{box}} - 2y_{ds}^{\text{box}} + y_{ss}^{\text{box}}$ on s for $\Lambda = 5\ \text{GeV}^2$.