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Helicity Amplitude Method

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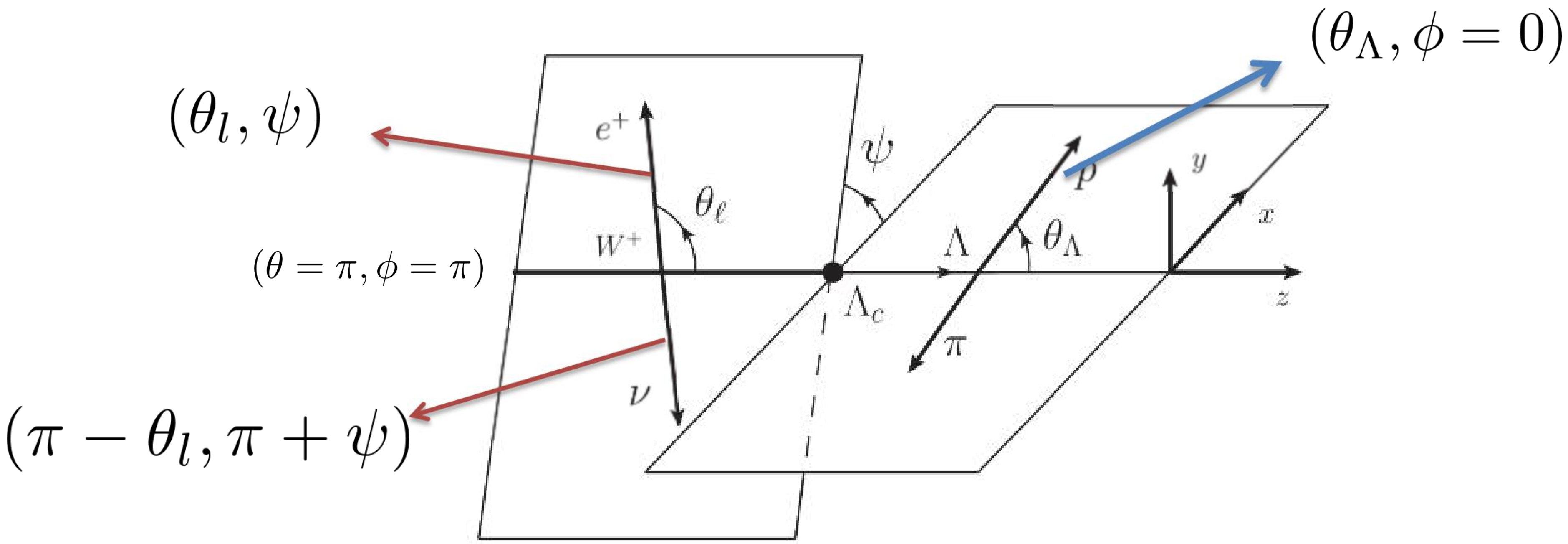
Helicity Amplitude



$$d\Gamma = \frac{1}{2m_{\Lambda_c}}(2\pi)^4\delta^4(P_I - P_f)\Pi_f \frac{d^3p_f}{(2\pi)^3 2E_f}\overline{|\mathcal{M}|^2}$$

Helicity Amplitude

$$g_{\mu\nu} = -\sum_{\lambda} \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) + \frac{q_\mu q_\nu}{q^2}$$

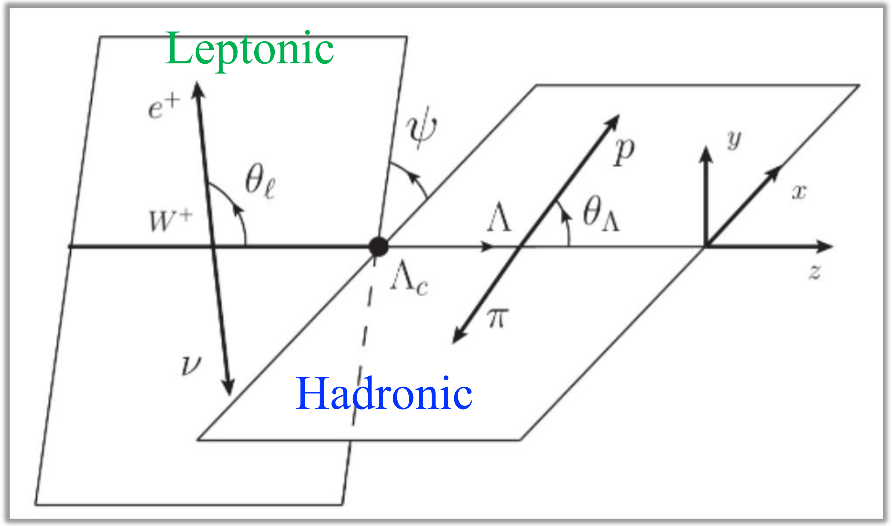
$$\mathcal{M} = \frac{G_F V_{cs}}{\sqrt{2}} \langle\Lambda|\bar{s}\gamma^\mu(1-\gamma^5)c|\Lambda_c\rangle \langle\bar{\ell}\nu_\ell|\bar{\nu}\gamma^\nu(1-\gamma^5)\ell|0\rangle \times g_{\mu\nu}$$

Hadronic Part:H

Leptonic Part:L

$$\begin{aligned}\mathcal{M}(\Lambda_c \to \Lambda e\nu) &= \frac{G_F}{\sqrt{2}} V_{cs} H^\mu L^\nu g_{\mu\nu} \\ &= \frac{G_F}{\sqrt{2}} V_{cs} H^\mu L^\nu \times \left(-\sum_{\lambda} \epsilon^*_\mu(\lambda)\epsilon_\nu(\lambda) + \frac{q_\mu q_\nu}{q^2}\right) \\ &= \frac{G_F}{\sqrt{2}} V_{cs} \left[-\sum_{\lambda} H\cdot\epsilon^*(\lambda) \times L\cdot\epsilon(\lambda) + H\cdot\epsilon^*(t) \times L\cdot\epsilon(t)\right]\end{aligned}$$

$$\lambda = 0, \pm 1$$



Leptonic Part

We introduce the notation,

$$L(\lambda_e, \lambda_\nu, \lambda_W = 0, \pm 1) = L^\mu \epsilon_\mu(\lambda) = \bar{u}_\nu \gamma^\mu (1-\gamma_5) v_e \epsilon_\mu(\lambda)$$

$$L(\lambda_e, \lambda_\nu, t) = \bar{u}_\nu \gamma^\mu (1-\gamma_5) v_e \frac{q_\mu}{\sqrt{q^2}}$$

The polarization vector:

$$\epsilon^\mu(+1) = \frac{1}{\sqrt{2}}(0, -\cos\theta\cos\phi + i\sin\phi, -\cos\theta\sin\phi - i\cos\phi, \sin\theta),$$

$$\epsilon^\mu(-1) = \frac{1}{\sqrt{2}}(0, \cos\theta\cos\phi + i\sin\phi, \cos\theta\sin\phi - i\cos\phi, -\sin\theta),$$

$$\epsilon^\mu(0) = \frac{1}{m}(|\vec{p}|, p^0\sin\theta\cos\phi, p^0\sin\theta\sin\phi, p^0\cos\theta).$$

Leptonic Part

After factoring a factor $f_\ell = i\sqrt{2(q^2 - m_\ell^2)}$, we have the results for the matrix elements:

$$L\left(\lambda_e = -\frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = -1\right) = e^{i\psi}\hat{m}_\ell \sin\theta_\ell,$$

$$L\left(\lambda_e = -\frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = 1\right) = -e^{-i\psi}\hat{m}_\ell \sin\theta_\ell,$$

$$L\left(\lambda_e = \frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = -1\right) = -e^{i\psi}(1 + \cos\theta_\ell),$$

$$L\left(\lambda_e = \frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = 1\right) = -e^{-i\psi}(1 - \cos\theta_\ell).$$

$$L\left(\lambda_e = -\frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = 0\right) = -\sqrt{2}\hat{m}_\ell \cos\theta_\ell,$$

$$L\left(\lambda_e = -\frac{1}{2}, \lambda_\nu = -\frac{1}{2}, t\right) = \sqrt{2}\hat{m}_\ell$$

$$L\left(\lambda_e = \frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = 0\right) = -\sqrt{2}\sin\theta_\ell,$$

Leptonic Part

neglect the electron mass:

$$L\left(\lambda_e = \frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = -1\right) = -e^{i\psi}(1 + \cos\theta_\ell), \quad L\left(\lambda_e = \frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = 0\right) = -\sqrt{2}\sin\theta_\ell,$$

$$L\left(\lambda_e = \frac{1}{2}, \lambda_\nu = -\frac{1}{2}, \lambda_W = 1\right) = -e^{-i\psi}(1 - \cos\theta_\ell).$$

Hadronic Part

The transition matrix elements can be parametrized as:

$$\Lambda \qquad\qquad\qquad \Lambda_C$$

$$\langle B_2(p_2,s_2)|V_\mu|B_1(p_1,s_2)\rangle = \bar{u}(p_2,s_2)\left[\gamma_\mu f_1(q^2) + i\sigma_{\mu\nu}\frac{q^\nu}{m_1}f_2(q^2) + \frac{q^\mu}{m_1}f_3(q^2)\right]u(p_1,s_2)\ ,$$

$$\langle B_2(p_2,s_2)|A_\mu|B_1(p_1,s_2)\rangle = \bar{u}(p_2,s_2)\left[\gamma_\mu g_1(q^2) + i\sigma_{\mu\nu}\frac{q^\nu}{m_1}g_2(q^2) + \frac{q^\mu}{m_1}g_3(q^2)\right]\gamma_5 u(p_1,s_2)$$

We introduce the notation:

$$H(\lambda_{\Lambda_c},\lambda_\Lambda,\lambda_W=0,\pm1) = H^\mu\epsilon_\mu(\lambda), \quad H(\lambda_{\Lambda_c},\lambda_\Lambda,t) = H^\mu\frac{q_\mu}{\sqrt{q^2}}.$$

The total helicity amplitudes :

$$H_{\lambda_\Lambda,\lambda_W} = \ H(\lambda_{\Lambda_c},\lambda_\Lambda,\lambda_W) = H_V(\lambda_{\Lambda_c},\lambda_\Lambda,\lambda_W) - H_A(\lambda_{\Lambda_c},\lambda_\Lambda,\lambda_W)$$

Hadronic Part

helicity amplitude:

$$H_V\left(\lambda_{\Lambda_c}=\frac{1}{2},\lambda_{\Lambda}=\frac{1}{2},0\right)=\frac{\sqrt{s_-}}{\sqrt{q^2}}(M_{\Lambda_c}+M_{\Lambda})f_+,$$
$$H_V\left(\lambda_{\Lambda_c}=-\frac{1}{2},\lambda_{\Lambda}=\frac{1}{2},1\right)=\sqrt{2s_-}f_\perp,$$
$$H_V\left(\lambda_{\Lambda_c}=\frac{1}{2},\lambda_{\Lambda}=\frac{1}{2},t\right)=\sqrt{s_+}(M_{\Lambda_c}-M_{\Lambda})f_0,$$
$$H_A\left(\lambda_{\Lambda_c}=\frac{1}{2},\lambda_{\Lambda}=\frac{1}{2},0\right)=\frac{\sqrt{s_+}}{\sqrt{q^2}}(M_{\Lambda_c}-M_{\Lambda})g_+ ,$$
$$H_A\left(\lambda_{\Lambda_c}=-\frac{1}{2},\lambda_{\Lambda}=\frac{1}{2},1\right)=\sqrt{2s_+}g_\perp,$$
$$H_A\left(\lambda_{\Lambda_c}=\frac{1}{2},\lambda_{\Lambda}=\frac{1}{2},t\right)=\sqrt{s_-}(M_{\Lambda_c}+M_{\Lambda})g_0.$$

Three-body semi-leptonic decays

$$d\Gamma = \frac{1}{2m_{\Lambda_c}}(2\pi)^4\delta^4(P_I - P_f)\Pi_f \frac{d^3p_f}{(2\pi)^3 2E_f}\overline{|\mathcal{M}|^2}$$

$$\frac{1}{(2\pi)^7}\frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}}{32m_{\Lambda_c}^2}(1 - m_\ell^2/q^2) \times d\cos\theta_\ell dq^2$$

The decay width formula:

$$d\Gamma = \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}}{512\pi^3 m_{\Lambda_c}^3}(1 - m_\ell^2/q^2) \times d\cos\theta_\ell dq^2 \times \overline{|\mathcal{M}(\Lambda_c \to \Lambda e\nu)|^2}$$

Sum over the polarization of the intermediate states, sum over the helicity of initial and final states:

$$\mathcal{M}(\Lambda_c \to \Lambda e\nu) = \frac{G_F}{\sqrt{2}}V_{cs}\left[-\sum_\lambda H\cdot\epsilon^*(\lambda)\times L\cdot\epsilon(\lambda) + H\cdot\epsilon^*(t)\times L\cdot\epsilon(t)\right]$$

Three-body semi-leptonic decays

$$
\begin{aligned}
d\Gamma =& \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}}{512\pi^3 m_{\Lambda_c}^3}(1 - m_\ell^2/q^2) \times d\cos\theta_\ell dq^2 \times \overline{|\mathcal{M}(\Lambda_c \to \Lambda e\nu)|^2} \\
=& \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}}{512\pi^3 m_{\Lambda_c}^3}(1 - \hat{m}_\ell^2) \times d\cos\theta_\ell dq^2 \times \frac{1}{2}\frac{G_F^2|V_{\rm CKM}|^2}{2} \times f_c^2 \\
& \times \Big\{ (|H_{1/2,0}|^2 + |H_{-1/2,0}|^2)(2\sin^2\theta_\ell + 2\hat{m}_\ell^2\cos^2\theta_\ell) + (|H_{1/2,t}|^2 + |H_{-1/2,0}|^2)2\hat{m}_\ell^2 \\
& - 2{\rm Re}[H_{1/2,t}H_{1/2,0}^* + H_{-1/2,t}H_{-1/2,0}^*]2\hat{m}_\ell^2\cos\theta_\ell \\
& + |H_{1/2,1}|^2(\hat{m}_\ell^2\sin^2\theta_\ell + (1-\cos\theta_\ell)^2) + |H_{-1/2,-1}|^2(\hat{m}_\ell^2\sin^2\theta_\ell + (1+\cos\theta_\ell)^2) \Big\}
\end{aligned}
$$

Integrating over the angle,

$$
\begin{aligned}
d\Gamma =& \frac{G_F^2|V_{\rm CKM}|^2 q^2\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}}{384\pi^3 m_{\Lambda_c}^3}(1 - \hat{m}_\ell^2)^2 \times dq^2 \\
& \times \Big\{ (|H_{1/2,0}|^2 + |H_{-1/2,0}|^2 + |H_{1/2,1}|^2 + |H_{-1/2,-1}|^2)(1 + \frac{1}{2}\hat{m}_\ell^2) + (|H_{1/2,t}|^2 + |H_{-1/2,t}|^2)\frac{3}{2}\hat{m}_\ell^2 \Big\}
\end{aligned}
$$

Four-body $\Lambda \to p\pi$

The $\Lambda \to p\pi$ transition is described by

$$\mathcal{A} = G_F m_\pi^2 \bar{u}_f (A - B\gamma_5) u_i$$

$$s = A, \text{ and } p = B \times p_p/(E_p + m_p)$$

The transition amplitude:

$$H_\Lambda(\lambda_\Lambda = 1/2, \lambda_p = 1/2) = f \cos\left(\frac{\theta_\Lambda}{2}\right)[s+p],$$

$$H_\Lambda(\lambda_\Lambda = 1/2, \lambda_p = -1/2) = -f \sin\left(\frac{\theta_\Lambda}{2}\right)[s-p],$$

$$H_\Lambda(\lambda_\Lambda = -1/2, \lambda_p = 1/2) = f \sin\left(\frac{\theta_\Lambda}{2}\right)[s+p],$$

$$H_\Lambda(\lambda_\Lambda = -1/2, \lambda_p = -1/2) = f \cos\left(\frac{\theta_\Lambda}{2}\right)[s-p].$$

The decay width:

$$\Gamma(\Lambda \to p\pi)$$

$$= \frac{1}{2m_\Lambda}(2\pi)^4 \frac{1}{(2\pi)^5} \frac{p_p d\cos\theta_\Lambda}{4m_\Lambda} \overline{|\mathcal{M}|^2}$$

$$= \frac{p_p f_\Lambda^2}{8\pi m_\Lambda^2}(|s|^2 + |p|^2)$$

Total Decay Width

Thus the differential decay width for $\Lambda_c \to \Lambda(\to p\pi)\ell^+\nu_\ell$ is given as:

$$d\Gamma = \frac{1}{2m_\Lambda}(2\pi)^4 \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}p_p(1-\hat{m}_\ell^2)}{(2\pi)^{10}\times 128 m_{\Lambda_c}^2 m_\Lambda} d\cos\theta_\ell d\cos\theta_\Lambda d\psi dq_\Lambda^2 dq^2 \times \overline{|\mathcal{M}|^2}$$

$$= \frac{1}{2m_\Lambda}(2\pi)^4 \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}p_p(1-\hat{m}_\ell^2)}{(2\pi)^{10}\times 128 m_{\Lambda_c}^2 m_\Lambda} d\cos\theta_\ell d\cos\theta_\Lambda d\psi dq_\Lambda^2 dq^2 \frac{1}{(q^2-m_\Lambda^2)^2+m_\Lambda^2\Gamma_\Lambda^2}\frac{G_F^2|V_{\mathrm{CKM}}|^2}{2}\times f_2^2 f_\Lambda^2$$

$$\times\frac{1}{2}\left\{\sum_{\lambda_{\Lambda_c},\lambda_e,\lambda_\nu}\left|\sum_{\lambda_\Lambda,\lambda_W} H(\lambda_{\Lambda_c},\lambda_\Lambda,\lambda_W)\times L_\ell(\lambda_e,\lambda_\nu,\lambda_W)\times H_\Lambda(\lambda_\Lambda,\lambda_p)\right|^2\right\}$$

$$\Big\{|H_{1/2,t}H_\Lambda(\lambda_\Lambda=1/2,\lambda_p=-1/2)L_\ell(\lambda_e=-1/2,\lambda_\nu=-1/2,\lambda_W=t)$$
$$-H_{1/2,0}H_\Lambda(\lambda_\Lambda=1/2,\lambda_p=-1/2)L_\ell(\lambda_e=-1/2,\lambda_\nu=-1/2,\lambda_W=0)$$
$$-H_{-1/2,-1}H_\Lambda(\lambda_\Lambda=-1/2,\lambda_p=-1/2)L_\ell(\lambda_e=-1/2,\lambda_\nu=-1/2,\lambda_W=-1)|^2$$
$$+\text{another 7 terms}\Big\},$$

Total Decay Width

In the narrow-width limit, the integration over the $p\pi$ invariant mass will be conducted as :

$$\int dq_\Lambda^2 \frac{m_\Lambda \Gamma_\Lambda}{\pi} \frac{1}{(q_\Lambda^2 - m_\Lambda^2)^2 + m_\Lambda^2 \Gamma_\Lambda^2} = 1$$

$$\int dq_\Lambda^2 \frac{m_\Lambda \Gamma(\Lambda \to p\pi)}{\pi} \frac{1}{(q_\Lambda^2 - m_\Lambda^2)^2 + m_\Lambda^2 \Gamma_\Lambda^2} = \mathcal{B}(\Lambda \to p\pi)$$

$$\Gamma(\Lambda \to p\pi) = \frac{p_p f_\Lambda^2}{8\pi m_\Lambda^2}(|s|^2 + |p|^2)$$

$$d\Gamma = \frac{1}{2m_{\Lambda_c}}(2\pi)^4 \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}p_p}{(2\pi)^{10} \times 128 m_{\Lambda_c}^2 m_\Lambda} d\cos\theta_\ell d\cos\theta_\Lambda d\psi dq_\Lambda^2 dq^2 \frac{1}{(q^2 - m_\Lambda^2)^2 + m_\Lambda^2 \Gamma_\Lambda^2} \frac{G_F^2 |V_{\mathrm{CKM}}|^2}{2} \times f_2^2 \frac{8\pi m_\Lambda^2 \Gamma(\Lambda \to p\pi)}{p_p(|s|^2 + |p|^2)}$$

$$= \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)} q^2 G_F^2 |V_{\mathrm{CKM}}|^2}{(2\pi)^4 \times 256 m_{\Lambda_c}^3} d\cos\theta_\ell d\cos\theta_\Lambda d\psi dq^2 \frac{\mathcal{B}(\Lambda \to p\pi)}{(|s|^2 + |p|^2)} \times \sum |\mathcal{M}|^2$$

Total Decay Width

$$\frac{d\Gamma}{d\cos\theta_\ell d\cos\theta_\Lambda d\psi dq^2} = \frac{\sqrt{\lambda(m_{\Lambda_c}, m_\Lambda, q)}(1-\hat{m}_\ell^2)^2 q^2 G_F^2 |V_{\rm CKM}|^2}{(2\pi)^4 \times 256 m_{\Lambda_c}^3}\mathcal{B}(\Lambda \to p\pi)$$

$$\times\Big\{|H_{-1/2,-1}|^2(1-\alpha\cos\theta_\Lambda)(1+\cos\theta_\ell)^2 + |H_{1/2,1}|^2(1+\alpha\cos\theta_\Lambda)(1-\cos\theta_\ell)^2$$

$$+2|H_{1/2,0}|^2(1+\alpha\cos\theta_\Lambda)\sin^2\theta_\ell + 2|H_{-1/2,0}|^2(1-\alpha\cos\theta_\Lambda)\sin^2\theta_\ell$$

$$+2\sqrt{2}\alpha\cos\psi\sin\theta_\ell\sin\theta_\Lambda(1+\cos\theta_\ell){\rm Re}[H_{1/2,0}H^*_{-1/2,-1}]$$

$$+2\sqrt{2}\alpha\cos\psi\sin\theta_\ell\sin\theta_\Lambda(1-\cos\theta_\ell){\rm Re}[H_{-1/2,0}H^*_{1/2,1}]$$

$$+\hat{m}_\ell^2\Big[(1+\alpha\cos\theta_\Lambda)2[|H_{1/2,t}|^2 + |H_{1/2,0}|^2\cos^2\theta_\ell + 2{\rm Re}[H_{1/2,0}H^*_{1/2,t}]\cos\theta_\ell]$$

$$+(1-\alpha\cos\theta_\Lambda)|H_{-1/2,-1}|^2\sin\theta_\ell^2$$

$$-2\alpha{\rm Re}[H_{1/2,0}H^*_{-1/2,-1}]\cos\psi\sqrt{2}\sin\theta_\ell\cos\theta_\ell\sin\theta_\Lambda$$

$$-2\alpha{\rm Re}[H_{1/2,t}H^*_{-1/2,-1}]\cos\psi\sqrt{2}\sin\theta_\ell\sin\theta_\Lambda$$

$$+(1-\alpha\cos\theta_\Lambda)2[|H_{1/2,t}|^2 + |H_{1/2,0}|^2\cos^2\theta_\ell + 2{\rm Re}[H_{1/2,0}H^*_{1/2,t}]\cos\theta_\ell]$$

$$+(1+\alpha\cos\theta_\Lambda)|H_{1/2,1}|^2\sin\theta_\ell^2$$

$$+2\alpha{\rm Re}[H_{1/2,0}H^*_{1/2,1}]\cos\psi\sqrt{2}\sin\theta_\ell\cos\theta_\ell\sin\theta_\Lambda$$

$$+2\alpha{\rm Re}[H_{1/2,t}H^*_{1/2,1}]\cos\psi\sqrt{2}\sin\theta_\ell\sin\theta_\Lambda\Big]\Big\}.$$

$$A_{\lambda_A,\lambda_B,\lambda_C} = H_{\lambda_B,\lambda_C} D^{J_A *}_{\lambda_A,\lambda_B-\lambda_C}(\phi,\theta,0)$$

$$H_{\lambda_B,\lambda_C} = \sum_{ls} g_{ls} \sqrt{\frac{2l+1}{2J_A+1}} \langle l,0;s,\delta | J_A,\delta\rangle \langle J_B,\lambda_B\ J_C,\lambda_C | s\delta\rangle$$

$$\frac{d\Gamma}{d\cos\theta d\phi d\cos\theta_2 d\phi_2} \propto \sum_{\lambda_{\Lambda_c}} \sum_{\lambda_\Lambda,\lambda_l,\lambda_{\nu_l}} |A^{\Lambda_c\to\Lambda+W} A^{W\to l+\nu_l}|^2$$

$$= \sum_{\lambda} |\sum_{\lambda_W} H_{\lambda_W,\lambda_\Lambda} D^{\frac{1}{2}*}_{\lambda_{\Lambda_c},\lambda_W-\lambda_\Lambda}(\phi,\theta,0) H_{\lambda_l,\lambda_{\nu_l}} D^{1*}_{\lambda_W,\lambda_l-\lambda_{\nu_l}}(\phi_2,\theta_2,0)|^2$$

for decay $W \to l + \nu_l, H_{\lambda_l,\lambda_{\nu_l}}$

$$H_{\frac{-1}{2},\frac{-1}{2}} = \frac{\sqrt{6}g_{0,1}}{6} - \frac{\sqrt{2}g_{1,0}}{2} - \frac{\sqrt{3}g_{2,1}}{3}$$

$$H_{\frac{1}{2},\frac{-1}{2}} = \frac{\sqrt{3}g_{0,1}}{3} + \frac{\sqrt{2}g_{1,1}}{2} + \frac{\sqrt{6}g_{2,1}}{6}$$

$$H_{\frac{-1}{2},\frac{1}{2}} = \frac{\sqrt{3}g_{0,1}}{3} - \frac{\sqrt{2}g_{1,1}}{2} + \frac{\sqrt{6}g_{2,1}}{6}$$

$$H_{\frac{1}{2},\frac{1}{2}} = \frac{\sqrt{6}g_{0,1}}{6} + \frac{\sqrt{2}g_{1,0}}{2} - \frac{\sqrt{3}g_{2,1}}{3}$$

Thanks!