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"Full" Classification of the Symmetries of 4HDM

- Extensions of abelian symmetry groups --- [JHEP 10 (2023) 070]
- And a look towards future work

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[1] Review & Motivation

- Partial list of 4HDM literature
- Many use discrete groups in model building
- **Q:** Which to choose? Guess work?
- **A:** Some math results needed to guide phenomenological search

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[2] Rephasing & Permutation transformation

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- The groups: abelian $\phi_k \mapsto \phi_k e^{\frac{2\pi i n_k}{N}}$

$$\mathbb{Z}_N \simeq \langle a \rangle$$

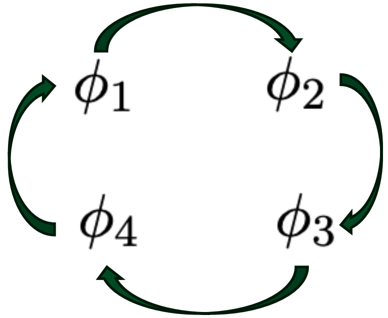
$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \ldots \simeq \langle a, a', \ldots \rangle$$

- Not all are symmetries of 4HDM
- Igor et al, [arXiv:1112.1660]
- Symmetry of 4HDM --- order < 8

$$\mathbb{Z}_8\,,\quad \mathbb{Z}_7\,,\quad \mathbb{Z}_6\,,\quad \mathbb{Z}_5\,,\quad \mathbb{Z}_4\,,\quad \mathbb{Z}_3\,,\quad \mathbb{Z}_2\,,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2\,,\ \mathbb{Z}_4 \times \mathbb{Z}_2\,,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\,,\ \mathbb{Z}_4 \times \mathbb{Z}_4$$

- https://github.com/JiazhenShao/4HDM-Toolbox.git ***--- my code,*** `4HDM toolbox`

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$$



$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\phi_1 \longleftrightarrow \phi_3$$
$$\phi_4 \longleftrightarrow \phi_2$$

- The groups: non-abelian, List of non-abelian symmetry of 4HDM ?

[3] Constructing non-abelian groups

- Start with rephasing symmetry group, e.g. $A = \langle a \rangle$
- Add permutation symmetry *--- we don't consider other symmetries in this work*
- What permutation? **don't want to guess.**
- Inspired by Igor et al, [arXiv:1206.7108], [arXiv:1210.6553], define $b \in \mathrm{Aut}(A)$.

permute + rephase + Permute back = Another rephasing

- $b^{-1} A\, b = A$, b as automorphism of A : $b \in \mathrm{Aut}(A)$. $b^{-1} a\, b = a'$. As **an equation**

[4] Effects of transformation on 4HDM potential

Renormalizable : $V = m_{ij}^2(\phi_i^\dagger \phi_j) + \Lambda_{ijkl}(\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l)$

Potential with abelian symmetry A : $V(A) = V_0 + V'(A)$.

Potential with additional permutation symmetry b : specify relation among coefficients

[5] Strategy of our work

Classify discrete symmetry groups of 4HDM **scalar sector**

- For A in $$\begin{array}{c} \mathbb{Z}_8\,, \quad \mathbb{Z}_7\,, \quad \mathbb{Z}_6\,, \quad \mathbb{Z}_5\,, \quad \mathbb{Z}_4\,, \quad \mathbb{Z}_3\,, \quad \mathbb{Z}_2\,, \\ \mathbb{Z}_2\times\mathbb{Z}_2\,, \ \mathbb{Z}_4\times\mathbb{Z}_2\,, \ \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\,, \ \mathbb{Z}_4\times\mathbb{Z}_4 \end{array}$$ do:
    - Find out $\mathrm{Aut}(A)$
    - Determine and solve equation $b^{-1}ab = a'$
    - Find out the group, see whether it's a symmetry of 4HDM potential

[6] Example --- extending Z_5 model

- Using `4HDM toolbox` https://github.com/JiazhenShao/4HDM-Toolbox.git
- **Unique** choice of Z_5 4HDM model, Invariant under $Z_5 \simeq \langle a \rangle$:

$$V(\mathbb{Z}_5) = V_0 + \lambda_1(\phi_2^\dagger\phi_1)(\phi_4^\dagger\phi_1) + \lambda_2(\phi_3^\dagger\phi_4)(\phi_2^\dagger\phi_4) + \lambda_3(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + \lambda_4(\phi_4^\dagger\phi_3)(\phi_1^\dagger\phi_3)$$
$$+ \lambda_5(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_4) + \lambda_6(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_2) + h.c.$$

$$V_0 = \sum_{i=1}^{4}\left[m_{ii}^2(\phi_i^\dagger\phi_i) + \Lambda_{ii}(\phi_i^\dagger\phi_i)^2\right] + \sum_{i<j}\left[\Lambda_{ij}(\phi_i^\dagger\phi_i)(\phi_j^\dagger\phi_j) + \tilde{\Lambda}_{ij}(\phi_i^\dagger\phi_j)(\phi_j^\dagger\phi_i)\right]$$

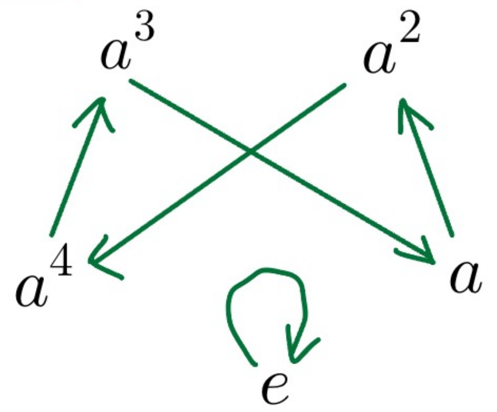
$$a = \eta\cdot\mathrm{diag}(\eta,\, \eta^2,\, \eta^3,\, 1) = \mathrm{diag}(\eta^2,\, \eta^{-2},\, \eta^{-1},\, \eta)\,, \quad \eta \equiv e^{2\pi i/5}\,, \quad \eta^5 = 1.$$

[7] Abelian group $A = Z_5$ ---- looking for equations

$$\mathrm{Aut}(\mathbb{Z}_5) = \mathbb{Z}_4$$

Option 1

$a \mapsto a^2$



$$\begin{cases} b^4 = \mathrm{id} \in \mathrm{Aut}(\mathbb{Z}_5) \\ b^{-1}ab = a^2 \end{cases}$$

Option 2

$a \mapsto a^4$

$$a^3 \longleftrightarrow a^2$$

$$a^4 \longleftrightarrow a$$

$$e$$

$$\begin{cases} b^2 = \mathrm{id} \in \mathrm{Aut}(\mathbb{Z}_5) \\ b^{-1}ab = a^4 \end{cases}$$

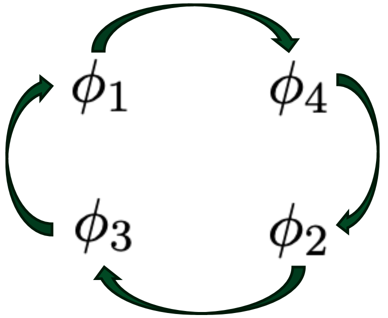
[8] Abelian group $A = Z_5$ ---- solving equations, option 1

$$b^{-1}ab = a^2 \Leftrightarrow ab = ba^2$$

$$a = \mathrm{diag}(\eta^2, \eta^{-2}, \eta^{-1}, \eta)\,, \quad \eta \equiv e^{2\pi i/5}\,, \quad \eta^5 = 1.$$

$$\Rightarrow \quad b = \begin{pmatrix} 0 & 0 & 0 & b_{14} \\ 0 & 0 & b_{23} & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & 0 & 0 \end{pmatrix} \xrightarrow{\text{Absorb in } \phi_i} b = i^{1/2} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad GA(1,5) \simeq \langle\, a,\, b \,|\, a^5 = b^4 = e\,,\, b^{-1}ab = a^2 \,\rangle$$

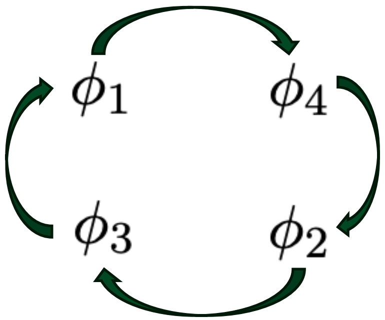


[9] Abelian group $A = Z_5$ ---- $GA(1,5)$ invariant potential

$$V'(\mathbb{Z}_5) = \lambda_1(\phi_2^\dagger\phi_1)(\phi_4^\dagger\phi_1) + \lambda_2(\phi_3^\dagger\phi_4)(\phi_2^\dagger\phi_4) + \lambda_3(\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + \lambda_4(\phi_4^\dagger\phi_3)(\phi_1^\dagger\phi_3)$$
$$+ \lambda_5(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_4) + \lambda_6(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_2) + h.c.$$

Invariance under

$$b = i^{1/2} \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



- 10 real free parameters
- λ, λ' can be complex

$$V'(GA(1,5)) = \lambda[(\phi_2^\dagger\phi_1)(\phi_4^\dagger\phi_1) + (\phi_3^\dagger\phi_4)(\phi_2^\dagger\phi_4) + (\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2) + (\phi_4^\dagger\phi_3)(\phi_1^\dagger\phi_3)]$$
$$+ \lambda'[(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_4) + (\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_2)] + h.c.$$

[10] Option 2, extend by Z_2

$$b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\phi_1 \longleftrightarrow \phi_2$$

$$\phi_4 \longleftrightarrow \phi_3$$

- 18 real free parameters
- λ_i can be complex

$$D_5 \simeq \langle a,\, b \,|\, a^5 = b^2 = e\,,\, b^{-1}ab = a^4 \rangle$$

Subgroup of $GA(1,5)$

$$\begin{aligned} V'(D_5) = {} & \lambda_1[(\phi_2^\dagger\phi_1)(\phi_4^\dagger\phi_1) + (\phi_1^\dagger\phi_2)(\phi_3^\dagger\phi_2)] + \lambda_2[(\phi_3^\dagger\phi_4)(\phi_2^\dagger\phi_4) + (\phi_4^\dagger\phi_3)(\phi_1^\dagger\phi_3)] \\ & + \lambda_5(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_4) + \lambda_6(\phi_4^\dagger\phi_1)(\phi_3^\dagger\phi_2) + h.c. \end{aligned}$$

[11] Results --- 1

1. Some model can't be extended: Z_8
2. CP conservation: T_7
3. Many choices of Z_6, Z_4 models
4. Some model Z_4,has different extensions: D_4, Q_4 --- appendix B
5. Novel case: $GA(1,5)$

A	extension	G	$\|G\|$	irreps
$\mathbb{Z}_2$	—	—	—	—
$\mathbb{Z}_3$	$\mathbb{Z}_3 \rtimes \mathbb{Z}_2$	S_3	6	$1+1+2$
$\mathbb{Z}_4$ 3	$\mathbb{Z}_4 \rtimes \mathbb{Z}_2$	D_4	8	$1+1+2$ or $2+2$
	$\mathbb{Z}_4 \,.\, \mathbb{Z}_2$ 4	Q_4	8	$1+1+2$
$\mathbb{Z}_5$	$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$	$GA(1,5)$ 5	20	4
	$\mathbb{Z}_5 \rtimes \mathbb{Z}_2$	D_5	10	$2+2$
$\mathbb{Z}_6$ 3	$\mathbb{Z}_6 \rtimes \mathbb{Z}_2$	D_6	12	$1+1+2$ or $2+2$
$\mathbb{Z}_7$	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$	T_7 2	21	$1+3$
$\mathbb{Z}_8$	— 1	—	—	—

[12] The rest of the list --- more involved. Why?

$$\mathbb{Z}_2 \times \mathbb{Z}_2\,,\ \mathbb{Z}_4 \times \mathbb{Z}_2\,,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\,,\ \mathbb{Z}_4 \times \mathbb{Z}_4$$

- 1: $Aut(Z_n)$ is abelian but $Aut(Z_n \times Z_m \times \cdots)$ isn't
- 2: Choices of a, a' is not unique……
- 3: $Z_n \times \cdots$ generated by a, a' ..., ⇒ system of equations

e.g.
$Z_2 \times Z_2$
$Z_4 \times Z_4$

- 4: the automorphism group is huge:

A "Theorem" we proved

  - $|Aut(Z_2 \times Z_2 \times Z_2)| = |GL(3, F_2)| = 168$, 179 subgroups, $\boxed{\text{12 conjugacy classes}}$
  - $|Aut(Z_4 \times Z_4)| = |GL(2, Z_4)| = 96$, 234 subgroups, 62 conjugacy classes

[13] Results --- 2

A	extensions	G	$\lvert G\rvert$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$A \rtimes \mathbb{Z}_2$	D_4	8
	$A \rtimes \mathbb{Z}_3$	A_4	12
	$A \rtimes S_3$	S_4	24
$\mathbb{Z}_4 \times \mathbb{Z}_2$	$A \rtimes \mathbb{Z}_2$	SmallGroup(16,3)	16
	$A \rtimes \mathbb{Z}_2$ 1	$\mathbb{Z}_2 \times D_4$	16
	$A.\mathbb{Z}_2$	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	16
	$A \rtimes \mathbb{Z}_4$	Faithful $E_8 \rtimes \mathbb{Z}_4$	32
	$A \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	$\mathbf{2}^{\mathbf{5}}_{+}$	32

$(\mathbb{Z}_2)^3$ 2	$A \rtimes \mathbb{Z}_2$	$\mathbb{Z}_2 \times D_4$	16
	$A.\mathbb{Z}_2$	SmallGroup(16,3)	16
	$A \rtimes \mathbb{Z}_3$	$\mathbb{Z}_2 \times A_4$	24
	$A \rtimes \mathbb{Z}_4$	Faithful $E_8 \rtimes \mathbb{Z}_4$	32
	$A \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$	$\mathbf{2}^{\mathbf{5}}_{+}$	32
	$A \rtimes S_3$	$\mathbb{Z}_2 \times S_4$	48
$\mathbb{Z}_4 \times \mathbb{Z}_4$ 3	$A \rtimes \mathbb{Z}_2$	$\Sigma(32)$	32
	$A \rtimes \mathbb{Z}_4$	SmallGroup(64,34)	64
	$A \rtimes \mathbb{Z}_6$	SmallGroup(96,72)	96

- 1. split extensions may be different
- 2. trend: bigger A have richer extensions
- 3. trend: but too big G may lead to continuous symmetry

[14] Future...

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- Classification is complete using **this method: extension from abelian** $\boldsymbol{A}$
  - https://github.com/JiazhenShao/4HDM-Toolbox.git

Check all case by brute force using computer

- Proof: relies on the **solvability** of groups, works for 3HDM. (*think of numbers*)
- 4HDM discrete symmetry groups: solvability not proven
- discrete subgroups of $SU(4)$ is classified. [arXiv:hep-th/9905212v2]
- 4-D representation ⇒ full classification of discrete symmetry of 4HDM !

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Thanks for listening

- Me: 4th year undergraduate. Seeking graduate supervisors.
- Really welcome to discuss if we share interests

Jiazhen Shao

Igor Ivanov

[A1] The list of automorphism groups

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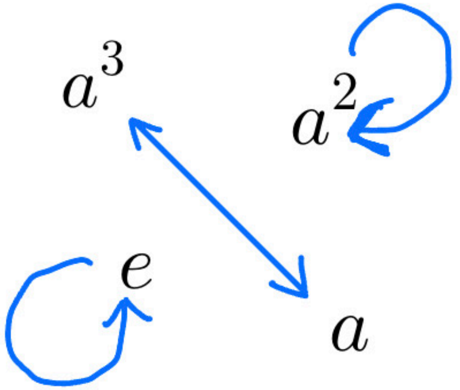
A	$\mathrm{Aut}(A)$
$\mathbb{Z}_2$	$\{e\}$
$\mathbb{Z}_3$	$\mathbb{Z}_2$
$\mathbb{Z}_4$	$\mathbb{Z}_2$
$\mathbb{Z}_5$	$\mathbb{Z}_4$
$\mathbb{Z}_6$	$\mathbb{Z}_2$
$\mathbb{Z}_7$	$\mathbb{Z}_6$
$\mathbb{Z}_8$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

A	$\mathrm{Aut}(A)$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	S_3
$\mathbb{Z}_2 \times \mathbb{Z}_4$	D_4
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$SL(3,2) \simeq PSL(2,7)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$GL(2,\mathbb{Z}/4\mathbb{Z})$

Table 1: The list of all finite abelian symmetry groups A of the 4HDM scalar sector and their automorphism groups $\mathrm{Aut}(A)$.

[B1] Why sending b_{ij} to 1? Additional equation $b^n = \cdots$

$a \mapsto a^3$



$\mathbb{Z}_4$ option 1: $a_1 = \sqrt{i} \cdot \mathrm{diag}(i, -1, -i, 1)$

$\mathbb{Z}_4$ option 2: $a_2 = \mathrm{diag}(i, -i, 1, 1)$

$\mathbb{Z}_4$ option 3: $a_3 = i^{3/4}\,\mathrm{diag}(i, i, -i, 1)$

$$ab = ba^3 \cdot i^r, \; a = \sqrt{i} \cdot \mathrm{diag}(i, \; -1, \; -i, \; 1)$$

$$b = \begin{pmatrix} 0 & 0 & b_{13} & 0 \\ 0 & b_{22} & 0 & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & b_{44} \end{pmatrix}, \quad b^2 = \begin{pmatrix} b_{13}b_{31} & 0 & 0 & 0 \\ 0 & b_{22}^2 & 0 & 0 \\ 0 & 0 & b_{13}b_{31} & 0 \\ 0 & 0 & 0 & b_{44}^2 \end{pmatrix} \propto \mathbf{1}_4 \text{ or } a^2$$

[B2] Two options in Z_4

Option 1:
Split extension

$$D_4 \simeq \langle a, b \,|\, a^4 = e, b^2 = e, b^{-1}ab = a^3 \rangle$$

- 13 real free parameters in $V'(D_4)$
- D_4 have a copy of $Z_2 = \langle b \rangle$

Option 2:
Non-split

$$Q_4 \simeq \langle a, b \,|\, a^4 = e, b^2 = a^2, b^{-1}ab = a^3 \rangle$$

- 9 real free parameters in $V'(Q_4)$ --- less but not too few
- Q_4 don't have a copy of $Z_2 = \langle b \rangle$

[B3] The D_4 and Q_4 model

$$
\begin{aligned}
V_1(\mathbb{Z}_4) = {} & \lambda_1(\phi_1^\dagger\phi_2)(\phi_1^\dagger\phi_4) + \lambda_2(\phi_2^\dagger\phi_1)(\phi_2^\dagger\phi_3) + \lambda_3(\phi_1^\dagger\phi_3)^2 \\
& + \lambda_4(\phi_4^\dagger\phi_1)(\phi_4^\dagger\phi_3) + \lambda_5(\phi_3^\dagger\phi_2)(\phi_3^\dagger\phi_4) + \lambda_6(\phi_2^\dagger\phi_4)^2 \\
& + \lambda_7(\phi_1^\dagger\phi_2)(\phi_4^\dagger\phi_3) + \lambda_8(\phi_1^\dagger\phi_3)(\phi_4^\dagger\phi_2) + \lambda_9(\phi_1^\dagger\phi_3)(\phi_2^\dagger\phi_4) + \lambda_{10}(\phi_1^\dagger\phi_4)(\phi_2^\dagger\phi_3) + h.c.
\end{aligned}
$$

D_4

$$
b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
$$

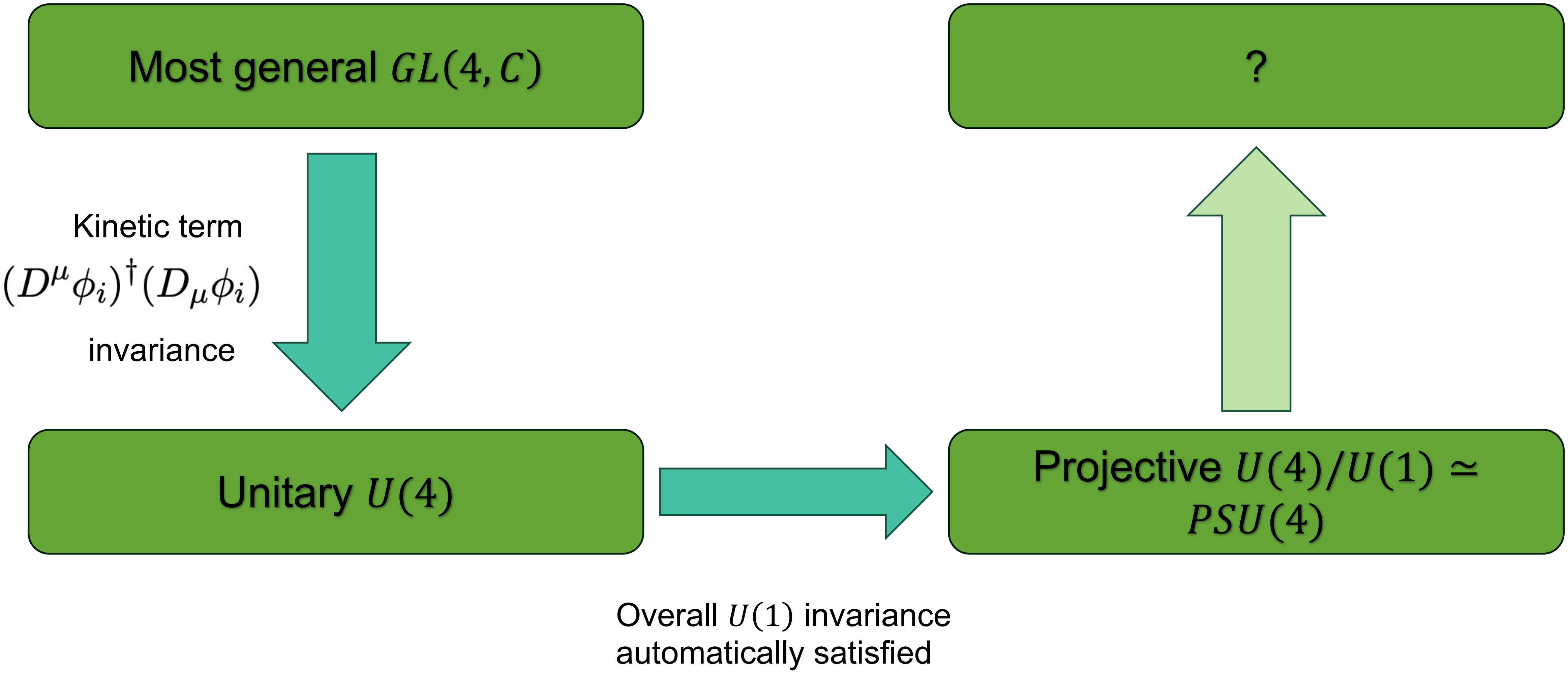
$$
\lambda_1 = \lambda_2\,, \quad \lambda_3 = \lambda_6\,, \quad \lambda_4 = \lambda_5\,, \quad \lambda_7, \lambda_8 \in \mathbb{R}\,.
$$

Q_4

$$
b'' = i^c \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -i\sigma & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}
$$

$$
\lambda_5 = \sigma\lambda_1\,, \quad \lambda_{10} = \sigma\lambda_7^*\,, \quad \lambda_9 = \sigma\lambda_8^*\,, \quad \lambda_3 \in \mathbb{R}
$$

[C1] Why tolerate i^r factor? more detailed reasoning



[C2] Formalism ---- what does a looks like

Projective $U(4)/U(1) \simeq PSU(4)$ → $SU(4)/Z_4$

$$U(4)/U(1) \simeq PSU(4) \simeq SU(4)/Z(SU(4)) \simeq SU(4)/\mathbb{Z}_4$$

- a is $PSU(4)$ transformation, i.e. a is $U(4)$ matrix ignoring overall $e^{i\alpha}$ difference
- $SU(4)$ easier to study, i.e. a is $SU(4)$ matrix ignoring overall $e^{i\alpha}$ difference
  - Where $e^{4i\alpha} = 1, \Rightarrow e^{i\alpha} = i^r, r = 0,1,2,3$

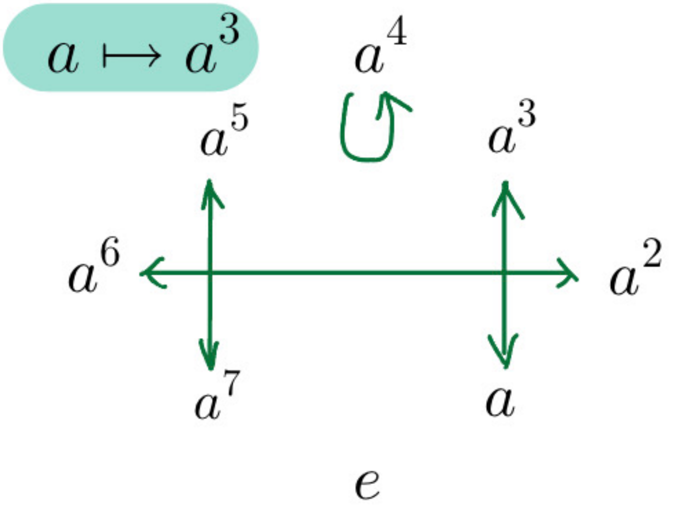
a is $PSU(4)$ transformation, technically represented by $SU(4)$ matrix with $i^r, r = 1,2,3,4$ difference ignored.

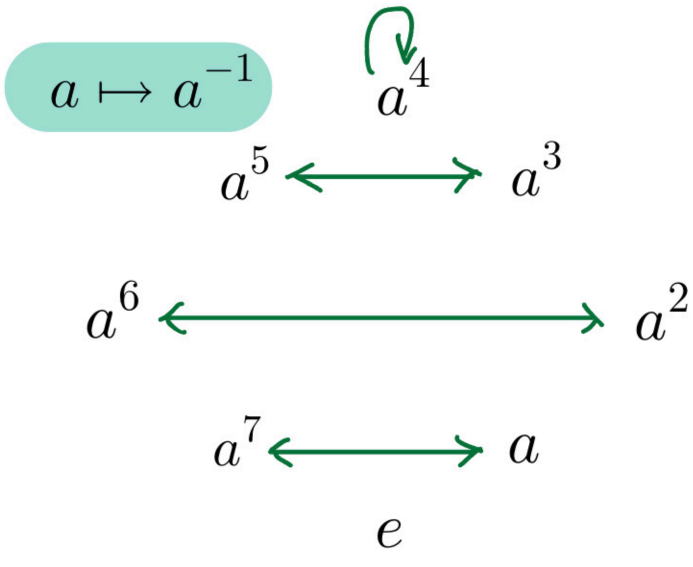
[C3] Solve equations --- e.g. $b^{-1} a\, b = a^3$

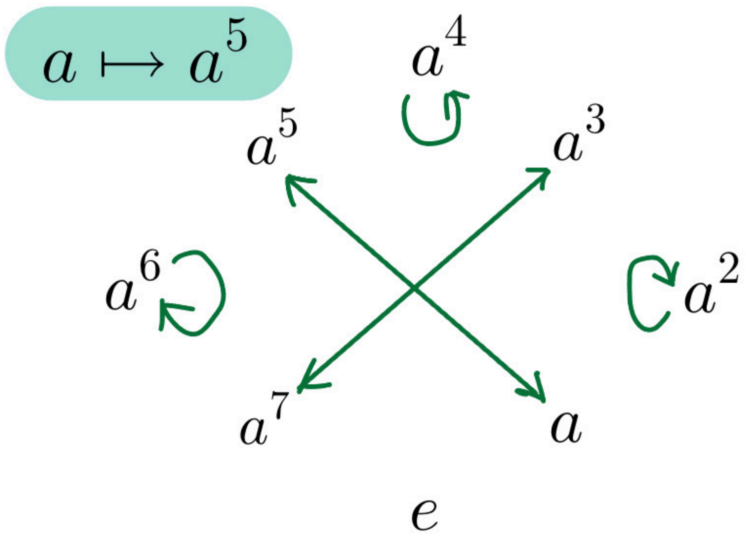
- How to solve $b^{-1} a\, b = a^3$? b as $U(4)$ matrix
- Multiply on the left by b
- $\Rightarrow ab = ba^3$
- $V[(\phi_i^\dagger \phi_j)]$ don't feel overall phase shift $\mathrm{U}(4)/\mathrm{U}(1) \simeq SU(4)/Z_4$
- $\Rightarrow ab = ba^3 \cdot i^r,\ r = 0,1,2,3$ b as $SU(4)$ matrix
- We can write the matrix explicitly

[D1] Abelian group $A = Z_8$ ---- looking for equations

- Automorphism group $\mathrm{Aut}(\mathbb{Z}_8) = \mathbb{Z}_2 \times \mathbb{Z}_2$
- Three b choices, $b^2 = e$







- Just to check : $a \mapsto a^3 \mapsto a^9 = a\,;\;\; a^2 \mapsto a^6 \mapsto a^{18} = a^2 \ldots$
- $b^{-1} a\, b = a^n\;,\;\; n = 3, -1, 5$

[D2] Abelian group $A = Z_8$ ---- No solutions

$$b = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \quad a = \eta^{1/4} \cdot \begin{pmatrix} \eta & 0 & 0 & 0 \\ 0 & \eta^2 & 0 & 0 \\ 0 & 0 & \eta^4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\eta^{1/4} \begin{pmatrix} \eta\, b_{11} & \eta\, b_{12} & \eta\, b_{13} & \eta\, b_{14} \\ \eta^2 b_{21} & \eta^2 b_{22} & \eta^2 b_{23} & \eta^2 b_{24} \\ \eta^4 b_{31} & \eta^4 b_{32} & \eta^4 b_{33} & \eta^4 b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix} = \eta^{3/4} \eta^{2r} \begin{pmatrix} \eta^3 b_{11} & \eta^6 b_{12} & \eta^4 b_{13} & b_{14} \\ \eta^3 b_{21} & \eta^6 b_{22} & \eta^4 b_{23} & b_{24} \\ \eta^3 b_{31} & \eta^6 b_{32} & \eta^4 b_{33} & b_{34} \\ \eta^3 b_{41} & \eta^6 b_{42} & \eta^4 b_{43} & b_{44} \end{pmatrix}$$

Compare term by term ---- no $b \in PSU(4)$ solutions

⇒ same goes for $n = -1, 5$. sometimes have non-zero b_{ij}, but not enough to make b invertable

[E1] E.g. Abelian group $A = Z_2 \times Z_2$ --- looking for equations

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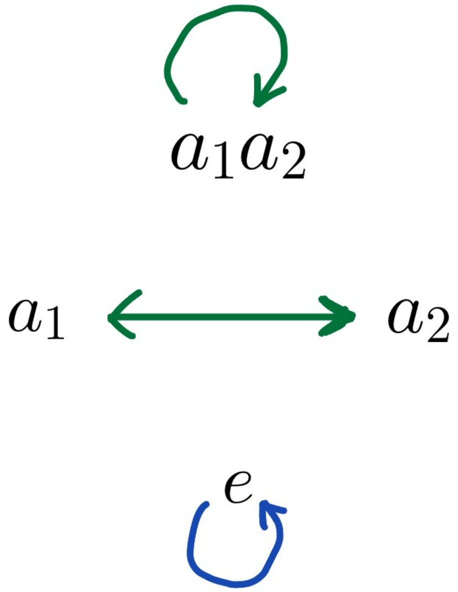
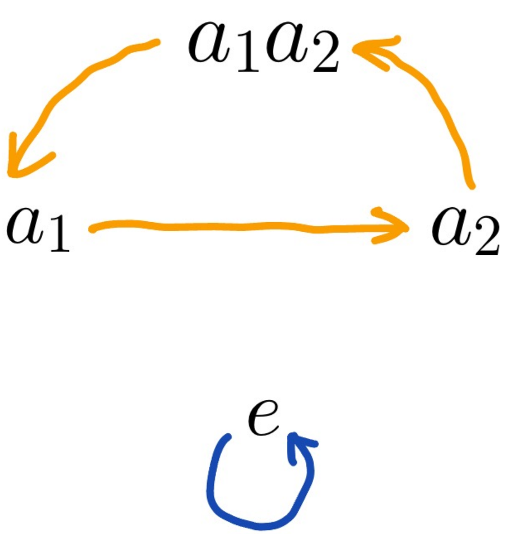
$$Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq S_3\,, \quad \mathbb{Z}_2, \mathbb{Z}_3 \subsetneq S_3$$

1



$$\begin{cases} b^{-1}a_1 b = i^k \cdot a_2 \\ b^{-1}a_2 b = i^l \cdot a_1 \\ b^2 = \dots \end{cases}$$

2



$$\begin{cases} c^{-1}a_1 c = i^k \cdot a_2 \\ c^{-1}a_2 c = i^l \cdot a_1 a_2 \\ c^3 = \dots \end{cases}$$

3

- Have b & c at the same time
- Full S_3 structure:
- $b^{-1}cb = c^{-1}$, $b^{-1}cbc = id$
- So $b^{-1}cbc = \cdots$

[E2] Abelian group $A = Z_2 \times Z_2$ ---- models

$$Aut(\mathbb{Z}_2 \times \mathbb{Z}_2) \simeq S_3 \,, \quad \mathbb{Z}_2, \mathbb{Z}_3 \subsetneq S_3$$

1

$$\begin{cases} b^{-1} a_1 b = i^k \cdot a_2 \\ b^{-1} a_2 b = i^l \cdot a_1 \\ b^2 = \ldots \end{cases}$$

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2 \simeq D_4$$

2

$$\begin{cases} c^{-1} a_1 c = i^k \cdot a_2 \\ c^{-1} a_2 c = i^l \cdot a_1 a_2 \\ c^3 = \ldots \end{cases}$$

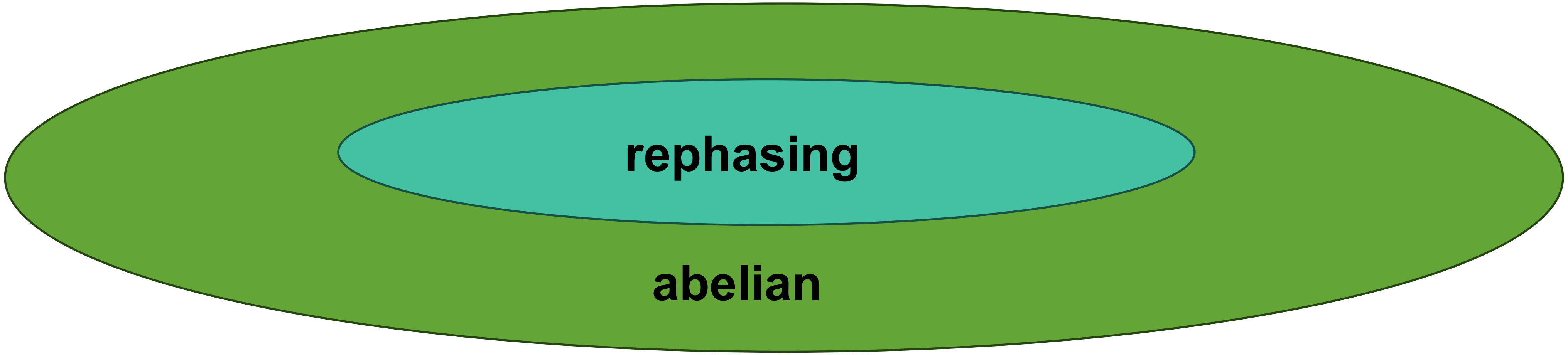
$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_3 \simeq A_4$$

3

$$\begin{cases} b^{-1} a_1 b = i^k \cdot a_2 \\ b^{-1} a_2 b = i^l \cdot a_1 \\ b^2 = \ldots \\ c^{-1} a_1 c = i^k \cdot a_2 \\ c^{-1} a_2 c = i^l \cdot a_1 a_2 \\ c^3 = \ldots \\ b^{-1} c b c = \ldots \end{cases}$$

$$(\mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes S_3 \simeq S_4$$

[F1] Abelian group $A = Z_4 \times Z_4$



$$\mathbb{Z}_4 \times \mathbb{Z}_4 \qquad a_1 = \sqrt{i} \cdot \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a_2 = \sqrt{i} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Commute in $PSU(4)$: $a_1 a_2 = -i a_2 a_1$
- System of equations have solutions like this: $\frac{i^{1/2}}{2} \cdot \begin{pmatrix} i\sqrt{i} & i\sqrt{i} & i\sqrt{i} & i\sqrt{i} \\ -i & 1 & i & -1 \\ -i\sqrt{i} & i\sqrt{i} & -i\sqrt{i} & i\sqrt{i} \\ -i & -1 & i & 1 \end{pmatrix}$

[G1] Example of the fact $b^{-1}ab = a'$

$$b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} e^{i\theta_1} & 0 & 0 & 0 \\ 0 & e^{i\theta_2} & 0 & 0 \\ 0 & 0 & e^{i\theta_3} & 0 \\ 0 & 0 & 0 & e^{i\theta_4} \end{pmatrix}, \quad a' = \begin{pmatrix} e^{i\theta_2} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & e^{i\theta_4} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} \xrightarrow{b} \begin{pmatrix} \phi_2 \\ \phi_1 \\ \phi_4 \\ \phi_3 \end{pmatrix} \xrightarrow{a} \begin{pmatrix} \phi_2\, e^{i\theta_1} \\ \phi_1\, e^{i\theta_2} \\ \phi_4\, e^{i\theta_3} \\ \phi_3\, e^{i\theta_4} \end{pmatrix} \xrightarrow{b^{-1}} \begin{pmatrix} \phi_1\, e^{i\theta_2} \\ \phi_2\, e^{i\theta_1} \\ \phi_3\, e^{i\theta_4} \\ \phi_4\, e^{i\theta_3} \end{pmatrix}$$

[H1] Why not other transformations

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- To **construct non-abelian groups**: something "fearful"
- In group theory, construct a larger group from smaller ones: **group extension**
- E.g. $G = H \times K, hk = kh \Leftrightarrow k^{-1}hk = h$
- If $k^{-1}hk = h' \in H$, is well defined
- Then $k^{-1}Hk = H$
- Then $k \in Aut(H), K \subseteq Aut(H)$
- So, to construct larger groups starting from abelian group A: start from $Aut(A)$, select its subgroup K, define $k^{-1}hk = h'$ --- an equation