What is an Axion String?

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Axion cosmic strings have for some time been considered a potential source of enhancement of axion dark matter production, and have been the subject of extensive simulations in recent years. Also recently consideration of superconducting versions (e.g. Fukuda, Manohar, Murayama and Telem).

But axion strings are rather peculiar entities. We'll explore some aspects of these objects, and suggest that they are not likely to play a distinguished role in early universe cosmology.

Axions have long been considered a promising dark matter candidate. If the Peccei-Quinn transition occurs before inflation, they are produced by the so-called misalignment mechanism, with the result depending on an essentially random parameter, θ_0, the initial value of the angle θ within our present horizon. If the transition occurs after inflation, in a hot universe, then the misalignment mechanism still contributes, but the initial angle varies by amounts of order 2π on Hubble scales. In this case, the dark matter density from misalignment is, in principle, computable, since one averages over angles in different domains of the universe.

Today we will focus on the “Post Inflation" scenario.

The textbook (e.g. Kolb and Turner) estimate of the axion density assumes that the axion begins to oscillate around the minimum of its potential once $3H\left(T_{\rm osc}\right) = m_a\left(T_{\rm osc}\right)$. The number density of axions is of order $\rho_a\left(T_{\rm osc}\right)/m_a\left(T_{\rm osc}\right)$. So the axion energy density behaves, for small initial misalignment angle, as:

$$\rho_a = \rho_a(T_{\rm osc})\frac{m_a(T)}{m_a(T_{\rm osc})}\frac{R^3(T_{\rm osc})}{R^3(T)}\,, \qquad (1)$$

where R is the scale factor. In the case of PQ transition after inflation, one then averages over the initial misalignment angle.

Axions are the subject of current and planned future experiments. At the order one level there are several sources of uncertainty in the axion density as a function of its mass:

1. Uncertainty as to the topological susceptibility, χ, the second derivative of the free energy at $\theta = 0$: χ is known analytically only at very low and very high temperature. In the intermediate regime, one can estimate χ by interpolating between these results
2. Hubble scale variations of the axion field: These are expected to contribute to the energy density an amount at least of order $f_a^2 H^2$, which is comparable to the assumed zero momentum contribution.
3. Simply averaging the potential, proportional to $\sin^2\theta$, over θ, does not take into account the non-linearity of the axion equations. Here topological objects such as strings and domain walls, might enter. As with the previous item, these are likely to produce at least order one modifications of the axion density.

Might any of these be more than order one effects? There is a large literature exploring this question, including analytic studies and sophisticated simulations (see 2012.13065 [hep-ph] for extensive references). Cosmic strings have been a major focus.

I will argue today that each of these uncertainties translate into order one (but not larger) uncertainties in the axion dark matter density. It would be desirable to reduce them, but their effect on estimates of the axion mass as a function of the dark matter density would seem modest. Our principle focus today will be on the last item, and the possible role of cosmic strings (and at the final stages, domain walls).

In a post-inflationary scenario, a linearized treatment of the axion field is not reliable, as a/f_a varies by 2π or more on Hubble scales. Allowing such variation, one expects to encounter Hubble-scale closed loops for which

$$\oint \frac{a}{f_A} dl = 2\pi n \tag{2}$$

for some integer n. Shrinking the loop to smaller and smaller size, we must find regions in which the modulus of the underlying complex scalar field vanishes: cosmic strings.

We can consider a theory with a complex field, Φ, transforming under a spontaneously broken global $U(1)$ symmetry as $\Phi \to e^{i\alpha}\Phi$, spontaneously broken:

$$\Phi = (f_a + \sigma(x))\, e^{ia(x)/f_a}. \tag{3}$$

Such theories admit cosmic string solutions:

$$\Phi_{cl} = f(\rho)e^{in\phi}. \tag{4}$$

$f(\rho) \to 0$ as $\rho \to 0$.
$f \to f_a$ as $\rho \to \infty$. Away from the string core:

$$\vec{\nabla}\Phi = i\frac{n}{\rho} f_a e^{in\phi}\hat{\phi}. \tag{5}$$

The energy stored in the string configuration per unit length is logarithmically divergent The UV cutoff of the logarithmic divergence is provided by the size of the string core. The IR cutoff comes from physical considerations. E.g. for closed, roughly circular, strings, the IR cutoff would be provided by the string circumference.; for a long string-antistring, the string separation. In a cosmic string network with $O(1)$ string per Hubble volume, we would expect the cutoff to be of order H^{-1}, and the effective string tension to be of order:

$$T = 2\pi f_a^2 \log(f_a/H) \equiv 2\pi f_a^2 \xi \tag{6}$$

Prospects for Density Enhancement

The parameter $\xi = \log(f_a/H) \sim 70$ at the QCD phase transition, if $f_a \sim 10^{12}$ GeV. It is the possibility of enhancement of the axion dark matter density by powers of ξ which gives rise to the interest in cosmic axion strings (and domain walls) in the early universe.

Skepticism as to an enhancement by powers of $\log(f_a/H)$

- Due to the the infrared divergences, one must consider an effective action containing both string core collective coordinates and axions with momenta less than the inverse core size, where the cutoff length defining the core is chosen arbitrarily, but is short compared to the actual infrared cutoff on the system. It is precisely the *low momentum* axions which have the potential to evolve into the dark matter axions.
- We will study the matching of the expected axion distributions at this cutoff with those of the axion strings. We will see that the logarithmic enhancement of the axion energy density from high momentum axions is indeed what is expected from considerations of Hubble scale variations of the phase of the Peccei-Quinn field. Cosmic strings are just a piece of this set of variations.

- For non-relativistic motion of the string, one can attempt to derive an effective action for the string core collective coordinates and the axion (homage to Sakita and his frequent collaborator, J.L. Gervais). This is nominally straightorward for an isolated infinite straight string, where it yields the Nambu-Goto Kalb Ramond action which has been used in many analyses, and which exhibits the cutoff-dependent tension. (Kalb-Ramond action: describes the interaction of the axion with a critical string. Utilized by Vilenkin-Vachaspati, Dabholkar et al). But for physical situations where the closed string radius or string-antistring separation serves as an infrared cutoff, the situation is more complicated.
- The Axion is a compact field, $0 < \frac{a}{f_a} < 2\pi$. This suggests that there is a limit to how much of the low momentum axion there can be.

Two limiting cases and expectations for axion radiation

In simple situations, one can guess the expected axion spectrum:

- **Adiabatic Approximation**: For long, parallel string and antistring, one can think of the separation of the strings as a dynamical variable, $b(t)$, and obtain an action for b. One can demonstrate that, starting from rest and well separated, the system initially evolves in an adiabatic fashion. This means that the potential energy of the separated strings is largely converted to kinetic energy of b; little is converted into low momentum axions at this early stage.

- **Sudden Approximation**: As the strings approach each other, a “sudden" description becomes more appropriate, and the axion field of the system on scale $b(t)$ is converted into axions of wavelength b^{-1}, with energy $f_a^2 \ell$ (ℓ is the string length) per change of b by a factor of e (the base of natural logarithms). These cases suggest no accumulation of axions at small k.

Collective Coordinates and Low Momentum Axions

First, a Collective Coordinate Review

The utility of the collective coordinate method follows from elementary considerations. Consider, for example, a kink in a translationally invariant theory in two dimensions, described by a classical solution, $\Phi_{cl}(x)$

$$\Phi(x,t) = \Phi_{cl}(x - X(t)) + \delta\Phi. \tag{7}$$

$\delta\Phi$ describes the fluctuations about the classical solution.

There is a kinetic term for X of the form

$$\mathcal{L}(X, \delta\Phi) = \frac{1}{2} M \dot{X}^2 + \ldots \tag{8}$$

and higher order terms in $\dot{X}$ and $\delta\phi$. The higher order terms are such that for a system in uniform motion, the equations of motion are solved by uniform motion of X.

The same sort of structure arises for other collective coordinates associated with symmetries of solutions, such as translational and dyonic excitations of monopoles. Infinite gauge strings in four dimensions have two collective coordinates, the transverse coordinates of the string solution, $\vec{X}_\perp$. Calling z the coordinate along the string, writing

$$\Phi(\vec{x}_\perp, z, t) = \Phi(\vec{x}_\perp - \vec{X}(z,t)) + \delta\Phi, \tag{9}$$

the action for these collective coordinates is:

$$\int d^2\sigma \mathcal{L}(X, \delta\Phi) = \int d^2\sigma \frac{1}{2} T \left(\frac{\partial \vec{X}}{\partial \sigma_\alpha} \right)^2 + \dots \tag{10}$$

Here $\sigma = (z, t)$. The first term is the non-relativistic limit of the usual Nambu-Gotto action.

These simple observations are the origin of the power of the collective coordinate method for such systems. It will not be the case for strings arising from the breaking of global symmetries in four dimensions (or vortices in $2 + 1$ dimensions). This is precisely because of the infrared divergences. The effects which cut off the divergence, such as finite circumference, will spoil the collective coordinate story, in the sense that the system as described by the collective coordinates alone does not obey the equations of motion. This is closely tied to the need to include other degrees of freedom in the effective action. The infrared divergence yields a dependence on the breaking scale of the symmetry. There will be a potential for the collective coordinates and coupling of the collective coordinates to $\delta\Phi$.

The Effective Action For the Collective Coordinates for a Long Global String

Consider a theory of a complex scalar, Φ, with action symmetric under $\Phi \to e^{i\alpha}\Phi$:

$$S = \int d^4x \left(|\partial_\mu \Phi|^2 + m^2|\Phi|^2 - \frac{\lambda}{2}|\Phi|^4 \right). \tag{11}$$

For this action,

$$\langle|\Phi|\rangle \equiv f_a \equiv v = \frac{m^2}{\lambda}. \tag{12}$$

String solutions: Taking the string to lie along the z axis, with coordinates $(x, y, z) \equiv (\vec{x}_\perp, z) = (\rho, \theta, z)$,

$$\Phi(t, z, \vec{x}_\perp) = \Phi_{cl}(\vec{x}_\perp) = f(\rho)e^{i\theta}. \tag{13}$$

The function $f(\rho)$ has the properties:

$$f(\rho) \to 0 \text{ as } \rho \to 0; \quad f(\rho) \to f_a(1 - \frac{1}{f_a\rho^2}) \text{ as } \rho \to \infty. \tag{14}$$

We first obtain the effective action for the collective coordinates for the string. Note, first that if we calculate the tension, thought of as the energy per unit length of the string, the result is infrared divergent:

$$T = \int d^2x_\perp \left(|\vec{\nabla}\Phi|^2 + V(\Phi) - V_0 \right) \approx 2\pi f_a^2 \int d\rho \frac{1}{\rho^2} = 2\pi \log(\Lambda/\mu). \quad (15)$$

Here Λ is an ultraviolet cutoff, of order the core size; we'll take $\Lambda = f_a$. μ is an infrared cutoff. If we have a closed string with radius R, $\mu = R^{-1}$; if we have a parallel string and antistring separated by a distance b, $\mu = b^{-1}$.

We can see more directly that T is the tension of the string by allowing slow variation in the transverse directions, introducing collective coordinates $X^i(z,t),\ i=1,2$. We also allow variation of the phase of Φ from $i\theta$, i.e. a spatially and time-dependent axion field:

$$\Phi(t,z,\vec{x}_\perp) = \Phi_{cl}(\vec{X}(t,z) - \vec{x}_\perp)e^{i\delta a(t,z,\vec{x}_\perp)}. \qquad (16)$$

Plugging into the action yields a sum of terms::

$$I_{transverse} = T \int dzdt \left(\left(\frac{\partial \vec{X}_\perp(t,z)}{\partial t} \right)^2 - \left(\frac{\partial \vec{X}_\perp(t,z)}{\partial z} \right)^2 \right) \quad (17)$$

with T given by our earlier expression. The infrared divergence appears again. The second term is:

$$I_{\delta a} = \int d^4x \epsilon_{ij} \frac{x^j}{|\vec{x}_\perp|^2} f_a^2 \left(\partial_t X^i \partial_t \delta a - \partial_z X^i \ \theta \partial_z \delta a \right). \quad (18)$$

This is the non-relativistic limit of the Nambu-Goto-Kalb-Ramond action, but with an infrared divergent tension.

With a modest effort, we can show, formally, the collective coordinates and the axion perturbation of the solitonic string in field theory are described by the Nambu-Gotto-Kalb-Ramond (NGKR) action. We say formally both because of the infrared divergent tension and because of issues of causality: the axion field changes everywhere in response to transverse string displacements. These issues should be addressed by systems of closed strings, such as long, parallel strings and antistrings or circular strings, where string separations and/or radii will act as an infrared cutoff.

Systems of Closed Strings

Because of the infrared issues associated with the collective coordinate expression is not really sensible for an infinite string; it also implies an instantaneous change in the axion field throughout space.

For a closed system, e.g. a closed, roughly circular string, or alternatively for a long, parallel string and antistring, this potentially makes more sense, as the infrared divergences cancel. In this case, however, we have to consider what we mean by Φ_{cl}, since there are not static solutions of the equations of motion of this type. We might take this to be, for example, the minimal energy static (instantaneously) configuration with specified location of the string core.

We can attempt to introduce collective coordinates as before, and obtain an action for these coordinates. *But the question of what configuration our collective coordinates describe does not have a unique answer.* We have no reason to expect that as the system evolves it will remain in the corresponding lowest energy configuration.
Corrections to our would-be classical configuration, $\delta\Phi$ will be sourced by the motion of the collective coordinates, with the dominant modes produced being of order the frequency and scale of the collective coordinate motion.

We can illustrate with a long string-antistring pair, or a long, circular string. The analysis proceeds along the lines we have explored above. One finds a realization, at early times, of the adiabatic approximation we described, and at later times of a sudden approximation.

Conclusions

There has been a long running debate about whether cosmic strings play an important role in the determination of the axion density. Because these strings have a logarithmically enhanced tension, there is the possibility that they produce an enhanced contribution to the axion dark matter density. This would lead to a different relation, for example, between axion mass and dark matter density than otherwise. Lattice simulations provide some evidence both for and against this possibility. As we have stressed here, in order that these configurations be important, is it critical that they deposit, when they decay or annihilate, an order one fraction of their energy in Hubble-scale momentum. This looks to be highly unlikely.

ADDITIONAL SLIDES

The Kalb-Ramond Coupling for Critical Strings

Many analyses of the problem of axion radiation have begun with the *Kalb-Ramond* action, which describes an axion interacting with a critical string. In such a theory, the axion is dual to an antisymmetric tensor field:

$$\partial_\mu a = \epsilon_{\mu\nu\nu\sigma} H^{\nu\rho\sigma}. \tag{19}$$

Here H is a three form field strength which is the curl of a two form gauge field, $B_{\mu\nu}$.

The KR action is:

$$\int d^2\sigma \mathcal{L}_{KR} = \int d^2\sigma B_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \epsilon^{\alpha\beta}. \tag{20}$$

Appearance of the KR term for a Long, Slowly Moving String

Formally, for an infinitely long, straight string, the KR action *does* describe the coupling of the string collective coordinates to axion radiation for a slowly moving string. The a configuration defining the string changes as X^i, the string collective coordinates, change. If the collective coordinates change slowly enough, we might expect a to be, at each instant, nearly in its lowest energy (the $\nabla^2 a = 0$) configuration. The slight differences will correspond to radiation.

The Axion Field Surrounding a Straight String

We first express the axion field surrounding an infinitely long static string in terms of the antisymmetric tensor field. Consider an infinite static string along the z direction, i.e. $X^0 = t$, $X^z = z$. The KR action leads to the equation:

$$\nabla_\perp^2 B_{0z} = f_{KR}\delta(\vec{x}_\perp). \tag{21}$$

$$B_{0z} = G(\vec{x}_\perp) f_{KR} \tag{22}$$

where $G(\vec{x}_\perp)$ is the two dimensional Green's function,

$$G(\vec{x}_\perp) = \frac{1}{2\pi}\log(|\vec{x}_\perp|). \tag{23}$$

From this, we can construct

$$H_{0zi} = \partial_i B_{0z} = \epsilon_{0zij} \partial_j a \tag{24}$$

so

$$\partial_j a = \epsilon_{ij} \partial_i f_{KR} G = \epsilon_{ij} \partial_j f_{KR} \frac{x^i}{|\vec{x}_\perp|^2} \tag{25}$$

which is $f_{KR} \partial_j \theta$, the polar angle in cylindrical coordinates. So the static configuration is as expected. Note that this is the minimal energy configuration for fixed B (a), with the boundary condition of unit winding.

Axion Emission From the String

Now consider axion emission from a moving string. Consider the term in the action

$$\int d^4x \mathcal{L} \sim \int d^4x \left(H_{zij}^2 - H_{0ij}^2 \right). \tag{26}$$

Noting that

$$H_{zij}(t, z, \vec{x}_\perp) = \partial_0 a + \epsilon_{ij} \partial_i G^{(2)}(\vec{x}_\perp) \frac{\partial X^j}{\partial t} \tag{27}$$

yields for the action a sum of terms, $\mathcal{L} = \mathcal{L}_{X^i} + \mathcal{L}_{a-X^i}$

Here

$$\mathcal{L}_{X^i} = \int d^4x \mathcal{L} = \int dz\ dtx \left((\frac{dX^i}{dt})^2 - (\frac{dX^i}{dz})^2 \right) \int d^2x_\perp \left(\frac{1}{x_\perp^2} \right) \tag{28}$$

and

$$\mathcal{L}_{a-X^i} = \int dzdt\ d^2x_\perp \left(\partial_0 a \frac{dX^i}{dt} - \partial_z a \frac{dX^i}{dz} \right) \epsilon_{ij} \frac{x^j}{x_\perp^2}. \tag{29}$$

The first expression is identical to the action for the transverse fluctuations of the string found in the field theory; the second is the axion-string collective coordinate coupling found there.

We have argued that the compactness of the axion field bounds the density of Hubble-scale axions by $f_a^2 H^2$. By itself, this makes it hard to understand an enhancement of the axion density over the standard computation which yields an axion density enhanced by a power of ξ.

Axion strings, due to their infrared divergent tension, need to be carefully defined. In particular, we have focused on a low energy effective action consisting of axions with momenta below some cutoff, k_0, and string collective coordinates for strings defined by a cutoff in space of order k_0^{-1}. We have seen in a sharp sense that the string is a thickened object, with thickness of this order. Axion radiation is emitted from throughout the thickened string.

By studying the low momentum axion system in the conformal frame, we have seen that there is a natural matching of the low energy axion energy density to the string.

Considerations of two extremes, adiabatic and sudden motion of the strings, indicates that one does not expect an appreciable enhancement of low momentum axion production.

This leaves us, then, with several sources of order one uncertainty in the axion dark matter density. These include

1. Limited knowledge of the QCD free energy in the relevant region.
2. Imperfect knowledge of the low momentum axion distribution.

The first item requires improved lattice simulations of the topological susceptibility in finite temperature QCD; the latter of the universe evolution near the QCD phase transition.

Additional Slides

For a long string-antistring pair, separated by a distance b and moving slowly, we might take the corresponding field configuration, far from the string core, to be:

$$\Phi = f_a e^{i\frac{a(x,t)+\delta a}{f_a}} \quad (30)$$

where

$$a(x,t) = a_0(x_\perp - \frac{b(t)\hat{y}}{2}) - a_0(x_\perp + \frac{b(t)\hat{y}}{2}). \quad (31)$$

Here $a_0(\vec{x}_0)$ is the axion configuration around a single string. We can show that with initial conditions

$$b(0) = b_0; \;\; \dot{b}(0) = 0 \quad (32)$$

this is *almost* a solution of the equations of motion, for short times.

Substituting Φ written in terms of the would-be collective coordinates into the field theory action, yields an action for b:

$$\int dtdzL \approx \int dtdzf_a^2(\dot{b}^2 \log(f_a b) - \log(b/b_0)). \tag{33}$$

as well as a coupling of the collective mode, b, to the axion, δa:

$$\int d^3x\, dt\, \left(\dot{b}\frac{\partial a}{\partial b}\partial_t \delta a\right) \tag{34}$$

So until $\dot{b} \sim 1$, most of the energy in the axion field is converted into kinetic energy of b, rather than radiation. $\dot{b} \sim 1$ once b is a few times f_a^{-1}. For a given separation, b, the typical wavelengths of δa are of order b^{-1}. So there is relatively little axion radiation at long wavelengths.

This is quite general. For circular motion, one instead studies a classical solution with a circular core of radius R. One can again start with a static configuration, minimizing the axion field energy as a function of R. The system becomes relativistic more quickly, but again the predominant radiation is at shorter wavelengths.

This analysis provides a realization of our notion of *adiabatic* and *sudden* behaviors. For a long parallel string and antistring, if the system starts at rest, the bulk of the energy in the axion field is converted into kinetic energy of b. Little radiation is produced. As b decreases and the velocity becomes relativistic, production of axions increases. Note, in particular, that one produces the mode C. This corresponds to the fact that, as the velocity increases, this motion is less and less an approximate solution of the equations of motion. All of this is consistent with the expectation that the energy in the axion field at a length scale $b(t)$ is converted into radiation of wavelength $b(t)$ except, possibly, at early stages of the motion, where long wavelength axions are suppressed if the velocity, $\dot{b}$, is small. More generally, b^{-1} sets the scale for typical momenta and wave numbers of the axions produced.

Additional Topics

- Expectations for the axion distribution $a(k, t)$ as a result of the Hubble expansion and matching to the string.
- Axion as a compact field. Enhancement of the low momentum part of the axion field by ξ implies a corresponding variation of the axion field on scales $\frac{1}{\xi H}$, which is hard to understand.

For the first item, it is straightforward to show that a typical distribution of $a(k)$ at the PQ phase transition evolves to a $1/k_\perp^4$ distribution, giving rise to a logarithmic distribution of energies with wave number. This matches onto the string solution at the cutoff scale which defines our action.

For the second item, one has to be a bit careful in defining the problem, as we are not in a finite box with discrete momenta. That said, it is difficult to understand how the axions could "pile up" at low momentum without yielding spatial variation rapid compared to H^{-1}.

More detail if time.

Naive Expectations of the Axion Number Density from Hubble Scale Variations

The scale factor satisfies, in a radiation dominated universe:

$$R(t) = \sqrt{\frac{t}{t_0}} R_0. \quad (35)$$

If we define the conformal time,

$$\tau = 2\sqrt{t t_0} R_0 = 2 t_0 R(\tau) \quad (36)$$

then the metric is:

$$ds^2 = R(\tau)^2 (d\tau^2 - d\vec{x}^2). \quad (37)$$

We can expand the axion field $a(x,\tau)$ field

$$a(x,\tau) = \int \frac{d^3k}{(2\pi)^3} \phi(k,\tau) e^{i\vec{k}\cdot\vec{x}}. \tag{38}$$

In the conformal frame, if we can neglect he potential:

$$\ddot{\phi}(k,\tau) + \frac{2}{\tau}\dot{\phi} + k^2\phi = 0. \tag{39}$$

So, for $k \ll \tau^{-1}$, $\phi(k,\tau)$ i.e. modes outside Hubble horizon, $\phi(k,\tau)$ is essentially constant; for $k \gg \tau^{-1}$,

$$\phi(k,\tau) = \phi(k,\tau_0)\frac{\tau_0}{\tau}\cos(k\tau). \tag{40}$$

To compute the energy density we need to make some assumption about the mean value of $\phi(k,\tau_0)$. Take:

$$\langle\phi(k,\tau_0)\phi^\dagger(k',\tau_0)\rangle \quad = \mathcal{J}(\vec{k},\vec{k}')\delta(\vec{k}-\vec{k}') \quad = \frac{f_a^2}{k^3}\delta(\vec{k}-\vec{k}') \tag{41}$$

Then

$$\langle\partial_i\phi\partial_j\phi g^{ij}\rangle_{PQ} \quad = \frac{1}{R^2(t)}\int d^3k\,\frac{k^2}{(2\pi)^3}\frac{f_a^2}{k^3}\theta(1/\tau_{PQ}-k) \tag{42}$$

$$= \frac{f_a^2 t_0^2}{\tau^4} \propto f_a^2 H_{PQ}^2 \tag{43}$$

This is exactly as we expect.

Now consider the contribution to the energy density from larger momenta, $k > \tau$. These modes start to oscillate for $k = \tau(k)$, and damp as $\tau(k)^2/\tau^2 = \frac{1}{k^2}/\tau^2$. So the corresponding contribution is:

$$\rho = \left(\frac{1}{R(t)}\right)^2 \int d^3k \frac{k}{(2\pi)^3} \frac{f_a^2}{k^2} \frac{1}{\tau^2 k^2} \theta(k - 1/\tau) \quad (44)$$

$$= \frac{f_a^2}{\tau^2 R(t)^2} \log(\Lambda\tau) = f_a^2 H^2 \log(\Lambda/H) \quad (45)$$

for some ultraviolet cutoff Λ.

If the bulk of this energy were converted into low momentum axions, it is conceivable that these could be the dominant source of axion dark matter.

If we write the correlator of two axion fields, in momentum space, as

$$\langle a(\vec{k})a(-\vec{k})\rangle = \frac{f_a^2 H^{q+1}}{k^{4+q}} \equiv \Delta(\vec{k}),\ |k| \geq H. \tag{46}$$

Then the axion energy density is:

$$\rho = \int \frac{dk}{2\pi^2}\frac{d\rho(k)}{dk} \equiv \int dk \frac{f_a^2 H^2 H^{q-1}}{k^q}. \tag{47}$$

The corresponding number density is:

$$n_{QCD} = \int \frac{dk}{2\pi^2} \frac{d\rho(k)}{dk} \equiv \int dk \frac{C}{k^{q+1}} \qquad (48)$$

Then if $q > 1$, there is the potential for a significant contribution to the density at low momentum

Our studies of adiabatic and sudden approximations suggest that $q < 1$ at low momenta.

Axion as a Compact Field

The fact that the axion is a compact field suggests that there is a limit to the density of low momentum axions. With Φ the PQ field, outside of the string core we have

$$\Phi(x) = f_a e^{i \frac{a(x)}{f_a}} \equiv f_a e^{i\theta} \tag{49}$$

If there is of order one string per Hubble volume, then, almost everywhere, Φ takes this form.

In other words, if the phase varies on scales H^{-1},

$$\langle \partial_i \Phi \partial_j \Phi \rangle \sim f_a^2 H^2 \tag{50}$$

and one expects that for k a momentum typical of the distribution. $k \sim H$.

In a recent paper of Gorghetto, Hardy, and Nicolaescu this observation is criticized with the comment:
“Recently it has been claimed that this cannot be true because the compactness of the axion feld bounds the energy that can be stored in low momentum modes. In fact, the periodicity of the axion only affects the zero-mode: all the other modes can be populated by arbitrarily large amplitudes."

So it is perhaps worth elaborating on our argument.

Suppose

$$\langle a(k)a(-k)\rangle = f_a^2 H^2 (q-1)\,\xi \frac{H^{q-1}}{k^{4+q}} \equiv \Delta(k)\;\; q \neq 1 \tag{51}$$

Here the normalization is fixed by the requirement that

$$\rho = \int_H^{f_a} \frac{d^3k}{(2\pi)^3} k^2 \Delta(k) = f_a^2 H^2 \log(f_a^2/H^2) \equiv f_a^2 H^2 \xi. \tag{52}$$

Assuming a Gaussian distribution for the axion field:

$$\int [da(k)] e^{-\int d^3k a(k)\Delta(k)a(k) + \int d^3k J(k) a(-k)}. \tag{53}$$

we can compute:

$$\mathcal{M}(x) = \langle \Phi(x)\Phi(0) \rangle = f_a^2 \langle e^{\frac{ia(x)}{f_a}} e^{\frac{ia(0)}{f_a}} \rangle.$$

by taking $J(x') = i\delta(x - x') - i\delta(x')$. With

$$\Delta(x) = \int \frac{d^3k}{(2\pi)^3} \Delta(k) e^{i\vec{k}\cdot\vec{x}} \sim \xi(q-1) H^{1+q} f_a^2 |x|^{q+1} \; q \neq 1 \tag{54}$$

$$\langle \Phi(x)\Phi(0)\rangle = f_a^2 e^{(q-1)\xi H^{q+1}|x|^{q+1}} \quad q \neq 1. \tag{55}$$

For $q > 1$, Φ changes by e^{ξ}, when $|x|$ changes by amounts of order H^{-1}. Derivatives of Φ are enhanced by ξ. This would be consistent with a number of strings per unit volume scaling like a power of ξ, but, as I will explain shortly, this doesn't translate into enhanced low momentum axions. In other words, this sort of high occupation number for low momentum axions corresponds to very rapid variation in Φ, which seems unlikely.

An enhanced Density of Strings does not necessarily lead to a larger dark matter axion density

If there are of order $\mathcal{N}$ strings per hubble volume, then the effective ir cutoff is larger, and correspondingly the lowest string axion momenta are larger.

Some discussion of the mode C

Substituting Φ written in terms of the would-be collective coordinates into the field theory action, yields an action for b:

$$L \approx f_a^2 \ell(\dot{b}^2 \log(f_a b) + \log(f_a b)). \tag{56}$$

as well as a coupling of the collective mode, b, to the axion, δa:

$$\int d^3x \, dt \, \left(\dot{b} \frac{\partial a}{\partial b} \partial_t \delta a - \frac{\partial a}{\partial x^i} \frac{\partial \delta a}{\partial x^i} \right) \tag{57}$$

We can integrate by parts in the first term with respect to time. Provided that $\dot{b}$ is small, the source for δa is then proportional to $\ddot{b} = \frac{1}{b\log(f_a b)}$. The *energy* stored in δa is suppressed, at early times, by $\frac{1/}{\log(f_a b)}$ relative to the kinetic energy in b. But we can make a stronger statement. The system can be described as approximately adiabatic. Note that the source for the mode

$$\delta a = C(t)\frac{\partial a}{\partial b} \tag{58}$$

vanishes at early times as a consequence of the equation of motion for b.

To see this, note that, the kinetic term, after integration by parts, neglecting $\dot{b}$, yields:

$$\ddot{b}C\int d^2x_\perp\left(\frac{\partial a}{\partial b}\right)^2 \tag{59}$$

which is just C times the second derivative term in the equation of motion for b, while from the integration over the $\vec{\nabla}_\perp^2$ term one has:

$$C\frac{\partial}{\partial b}\int d^2x_\perp(\vec{\nabla}a)^2 = C\frac{\partial V(b)}{\partial b}. \tag{60}$$

So, provided $\dot{b} \approx 0$, the field configuration described by the collective coordinates is *almost* a solution of the equations of motion. But as $\dot{b}$ grows, C is significantly sourced. In particular, the $\dot{b}^2$ terms dominate once $\dot{b} > \log(f_a b)^{-1/2}$. At this point the adiabatic approximation is breaking down.
This is quite general. For circular motion, one instead studies a classical solution with a circular core of radius R. One can again start with a static configuration, minimizing the axion field energy as a function of R.