Relativistic Baryon Chiral Perturbation Theory

—an introduction—

相对论重子手征微扰论简介

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Part I:

Baryon Chiral Perturbation Theory

（概览）

QCD Lagrangian

✦ The general QCD Lagrangian

$$\mathscr{L}_{\rm QCD} = \bar{q}\left(i\gamma^\mu D_\mu - m\right)q - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu,a} - \frac{1}{2\xi}\left(\partial_\mu G^{\mu a}\right)^2 + \bar{C}^a\partial^\mu D^{ab}_\mu C^b$$

$\mathscr{L}_G$: Gluon

$\mathscr{L}_{GF}$: Gauge Fixing

$\mathscr{L}_{FP}$: Fadeev-Popov

- **Gluon fields**: $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g f^{abc} G^b_\mu G^c_\nu$
- **Covariant derivatives in color space**: $D_\mu = \partial_\mu - ig\frac{\lambda^a}{2}G^a_\mu\,,\quad D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc}G^c_\mu$
- **Quark fields and mass matrix:**

$$q = \begin{pmatrix} u \\ d \\ s \\ \vdots \end{pmatrix} \qquad m = \begin{pmatrix} m_u & & & \\ & m_d & & \\ & & m_s & \\ & & & \ddots \end{pmatrix}$$

- **Flavour symmetry:**

$SU(N_f)$

$N_f = 2$ or $N_f = 3$

$$\begin{pmatrix} m_u = (1.7\text{--}3.3)\,\text{MeV} \\ m_d = (4.1\text{--}5.8)\,\text{MeV} \\ m_s = (80\text{--}130)\,\text{MeV} \end{pmatrix} \ll 1\,\text{GeV} \leq \begin{pmatrix} m_c = 1.27^{+0.07}_{-0.09}\,\text{GeV} \\ m_b = 4.19^{+0.18}_{-0.06}\,\text{GeV} \\ m_t = (172.0 \pm 0.9 \pm 1.3)\,\text{GeV} \end{pmatrix}$$

Chiral symmetry ($m_q \to 0$) **Heavy quark symmetry ($M_Q \to \infty$)**

Global Chiral Symmetry

✦ QCD Lagrangian in the chiral limit $m_q = 0$

- **Left and right-handed quark fields** $q_{L/R} = \frac{1 \mp \gamma_5}{2} q \qquad \bar{q}_{L/R} = \bar{q}\frac{1 \pm \gamma_5}{2}$

$$\mathscr{L}^0_{\text{QCD}} = \bar{q}\left(i\gamma^\mu D_\mu\right) q + \mathscr{L}_G + \mathscr{L}_{GF} + \mathscr{L}_{FP}$$

$$= \left(\bar{q}_L + \bar{q}_R\right)\left(i\gamma^\mu D_\mu\right)\left(q_L + q_R\right) + \mathscr{L}_G + \mathscr{L}_{GF} + \mathscr{L}_{FP}$$

$$= \bar{q}_L\left(i\gamma^\mu D_\mu\right) q_L + \bar{q}_R\left(i\gamma^\mu D_\mu\right) q_R + \mathscr{L}_G + \mathscr{L}_{GF} + \mathscr{L}_{FP}$$

- **Global chiral symmetry**

$SU(N_f)_L$ **transformation** $\qquad q_L \to e^{-i\theta_L^a T_L^a} q_L \equiv L q_L \qquad L \in SU(N_f)_L$

$SU(N_f)_R$ **transformation** $\qquad q_R \to e^{-i\theta_R^a T_R^a} q_R \equiv R q_R \qquad R \in SU(N_f)_R$

$SU(N_f)_L \otimes SU(N_f)_R$

θ_L^a and θ_R^a are constants， corresponding to global transformations.

Local Chiral Symmetry

✦ QCD Lagrangian in the presence of external sources

$$\mathscr{L}_{\rm QCD} = \mathscr{L}^0_{\rm QCD} + \bar{q}\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right) q - \bar{q}\left(s - i\gamma_5 p\right) q$$

Vector fields $v_\mu = v_\mu^a T^a$

Scalar fields $s = s^a T^a$

Axial vector fields $a_\mu = a_\mu^a T^a$

Pseudo-scalar fields $p = p^a T^a$

- **L-R form of the source terms**

$$\bar{q}\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right) q = \left(\bar{q}_L + \bar{q}_R\right)\gamma^\mu\left(v_\mu + \gamma_5 a_\mu\right)\left(q_L + q_R\right)$$

$$= \bar{q}_L\gamma^\mu v_\mu q_L + \bar{q}_R\gamma^\mu v_\mu q_R - \bar{q}_L\gamma^\mu a_\mu q_L + \bar{q}_R\gamma^\mu v_\mu q_R$$

$$= \bar{q}_L\gamma^\mu\left(v_\mu - a_\mu\right) q_L + \bar{q}_R\left(v_\mu + a_\mu\right) q_R$$

$$\bar{q}\left(s - i\gamma_5 p\right) q = \left(\bar{q}_L + \bar{q}_R\right)\left(s - i\gamma_5 p\right)\left(q_L + q_R\right)$$

$$= \bar{q}_L s q_R + \bar{q}_R s q_L + \bar{q}_L\left(-i\gamma_5 p\right) q_R + \bar{q}_R\left(-i\gamma_5 p\right) q_L$$

$$= \bar{q}_L\left(s - ip\right) q_R + \bar{q}_R\left(s + ip\right) q_L$$

Local Chiral Symmetry

✦ Local transformation

- **Quark kinetic terms**

$$\bar{q}_L i\gamma^\mu\partial_\mu q_L \xrightarrow{L} \left[\bar{q}_L e^{i\theta_L^a(x)T_L^a}\right] i\gamma^\mu\partial_\mu \left[e^{i\theta_L^a(x)T_L^a} q_L\right] = \bar{q}_L i\gamma^\mu\partial_\mu q_L + \bar{q}_L L^\dagger \left[-\gamma^\mu\partial_\mu\theta_L^a(x)T_L^a\right] L q_L$$

$$\bar{q}_R i\gamma^\mu\partial_\mu q_R \xrightarrow{R} \bar{q}_R i\gamma^\mu\partial_\mu q_R + \bar{q}_R R^\dagger \left[-\gamma^\mu\partial_\mu\theta_R^a(x)T_R^a\right] R q_R$$

- **External sources terms**

$$L^\dagger\left(i\partial_\mu L\right)L + L^\dagger\left(v'_\mu - a'_\mu\right)L = v_\mu - a_\mu \qquad L^\dagger\left(s' - ip'\right)R = s - ip$$

$$R^\dagger\left(i\partial_\mu R\right)R + R^\dagger\left(v'_\mu + a'_\mu\right)R = v_\mu + a_\mu \qquad R^\dagger\left(s' + ip'\right)L = s + ip$$

⬇ **Cancel the new pieces from quark kinetic terms**

$$(v_\mu - a_\mu)' = L(v_\mu - a_\mu - i\partial_\mu L)L^\dagger = L(v_\mu - a_\mu + i\partial_\mu)L^\dagger$$

$$(v_\mu + a_\mu)' = R(v_\mu + a_\mu - i\partial_\mu R)R^\dagger = R(v_\mu + a_\mu + i\partial_\mu)R^\dagger$$

$$(s - ip)' = L(s - ip)R^\dagger$$

$$(s + ip)' = R(s + ip)L^\dagger$$

➡ **Local** $SU(N_f)_L \otimes SU(N_f)_R$

Now, $\mathscr{L}_{\rm QCD}$ is invariant under local $SU(N_f)_L \otimes SU(N_f)_R$ transformation.

From QCD to ChPT: the bridge

✦ Generating functional Z

- **Path integral representation in terms of quarks and gluons**

$$\left\langle O_{\rm out} | O_{\rm in} \right\rangle_{v,a,s,p} = \quad e^{iZ[v,a,s,p]} = \int \mathscr{D}G_\mu \mathscr{D}q \mathscr{D}\bar{q} e^{i\int {\rm d}^4x \mathscr{L}(q,\bar{q},G_\mu;v,a,s,p)}$$

Vacuum to vacuum transition amplitude

- **Green functions (examples)**

$\langle 0 | T[A_a^\mu(x) P_b(y)] | 0 \rangle$ — **pion decay**

$\langle 0 | T[P_a(x) J^\mu(y) P_b(z)] | 0 \rangle$ — **Pion electromagnetic form factor**

$\langle 0 | T[P_a(\omega) P_b(x) P_c(y) P_d(z)] | 0 \rangle$ — **Pion-pion scattering**

- **Path integral representation in terms of Goldstones**

$$\int \mathscr{D}G_\mu \mathscr{D}q \mathscr{D}\bar{q} e^{i\int {\rm d}^4x \mathscr{L}(q,\bar{q},G_\mu;v,a,s,p)} = e^{iZ[v,a,s,p]} = N \int \mathscr{D}U e^{i\int {\rm d}^4x \mathscr{L}_{\rm eff}(U;v,a,s,p)}$$

A theory of quarks and gluons — **A theory of effective d.o.f of hadrons**

The same green functions can be deduced at hadronic level

Effective dofs: the Goldstone Bosons

✦ Spontaneous Chiral Symmetry Breaking

- **Goldstone bosons**

$$\begin{array}{ccc} G = SU(N_f)_L \times SU(N_f)_R & \xrightarrow{\textbf{SCSB}} & SU(N_f)_V = H \\ \uparrow & & \uparrow \\ n_G & & n_H \end{array}$$

where n_G and n_H are the numbers of group elements.

Goldstone theorem

There are $n = n_G - n_H$ **Goldstone bosons** $\phi_1 , \cdots , \phi_n$ $\qquad \tilde{\Phi} = \left(\phi_1 , \cdots , \phi_n\right)$

- **Non-linear realization**

$$\begin{array}{ccc} \tilde{\Phi} & \xrightarrow{\quad g \quad} & \tilde{\Phi}' \\ \downarrow & & \downarrow \\ \tilde{g}H & \xrightarrow{g = (L , R) \in G} & g\tilde{g}H \end{array}$$

Quotient G/H

An isomorphic mapping between the quotient G/H **and the Goldstone boson fields.**

- **The introduction of** U

Let $\tilde{g} \equiv \left(\tilde{L} , \tilde{R}\right) \in G$, parameterize $\tilde{g}H$ by $SU(N_f)$ matrix $U = \tilde{R}\tilde{L}^\dagger$ $[U^\dagger U = I , \det U = 1]$

$$\tilde{g}H = \left(I , \tilde{R}\tilde{L}^\dagger\right) H , \quad g\tilde{g}H = \left(L , R\tilde{R}\tilde{L}^\dagger\right) H = \left(I , R\tilde{R}\tilde{L}L^\dagger\right) (L , L) H = \left(I , R\tilde{R}\tilde{L}^\dagger L^\dagger\right) H$$

➡ **Transformation law:** $U = \tilde{R}L^\dagger \xrightarrow{g} R\left(\tilde{R}^\dagger\tilde{L}^\dagger\right) L^\dagger = RUL^\dagger \qquad U = e^{i\Phi}$

Effective dofs: the Goldstone Bosons

✦ Non-linear realisation of the GBs

- **Transformation property** $[U^\dagger U = 1 \ \&\ \det U = 1]$

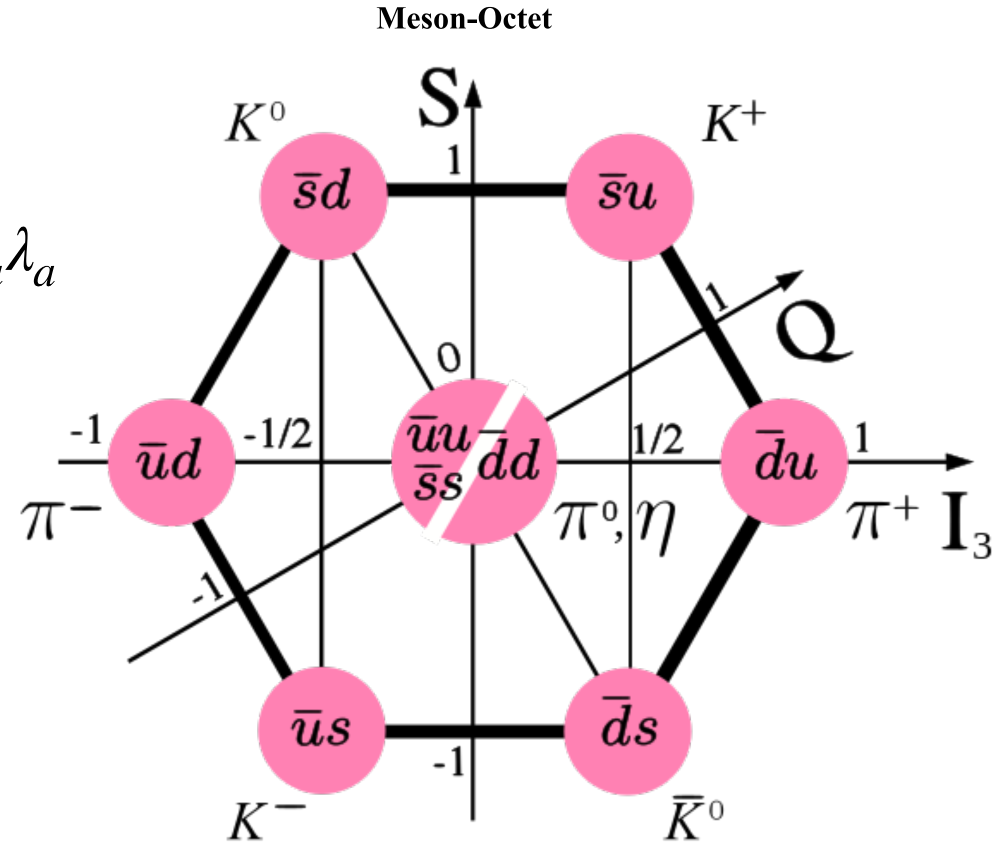
$$U \equiv e^{i\frac{\Phi}{\sqrt{2}F}} \qquad U \stackrel{SU(N_f)_L\otimes SU(N_f)_R}{\longrightarrow} RUL^\dagger$$

the parameter F has mass dimension

- **Goldstone bosons field are non-linearly realized.**

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{1}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{\sqrt{2}}{\sqrt{3}}\eta \end{pmatrix} = \frac{1}{\sqrt{2}}\phi_a\lambda_a$$

$$\Phi(x) = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\phi_i\tau_i$$



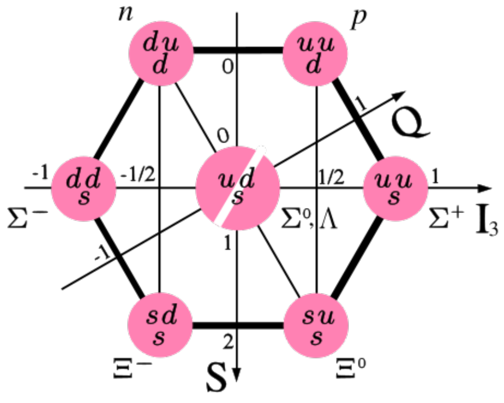
The Incorporation of Baryons

✦ Transformation properties of the Baryon fields

- **SU(3) case:** $B \overset{g}{\longmapsto} B' = K(L,R,U)BK(L,R,U)^{-1}$

$$B = \sum_{a=1}^{8} \frac{B_a \lambda_a}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix}$$

Baryon-Octet



- **SU(2) case:** $\Psi \overset{g}{\longmapsto} \Psi' = K(L,R,U)\Psi \qquad \Psi = \begin{pmatrix} p \\ n \end{pmatrix}$

- **Definition of** $K(L,R,U)$

$$u^2(x) = U(x)$$

$$\sqrt{U(x)} = u(x) \overset{g}{\longmapsto} u'(x) = \sqrt{RUL^\dagger} \equiv RuK^{-1}(L,R,U) \qquad U \overset{g}{\longmapsto} RUL^\dagger$$

$$K(L,R,U) = u'^{-1}Ru = \sqrt{RUL^\dagger}^{-1} R\sqrt{U}$$

The Incorporation of Baryons

✦ Transformation properties of the Baryon fields

- **Proof: [$SU(2)$ case for example]**

  1. **Define a map**

$$\varphi(g): \quad \begin{pmatrix} U \\ \Psi \end{pmatrix} \xmapsto{g} \begin{pmatrix} U' \\ \Psi' \end{pmatrix} = \begin{pmatrix} RUL^\dagger \\ K(L,R,U)\Psi \end{pmatrix}$$

  2. **It is a homomorphism**

$$\varphi(g_1)\varphi(g_2)\begin{pmatrix} U \\ \Psi \end{pmatrix} = \varphi(g_1)\begin{pmatrix} R_2UL_2^\dagger \\ K(L_2,R_2,U_2)\Psi \end{pmatrix} = \begin{pmatrix} R_1R_2UL_2^\dagger L_1^\dagger \\ K(L_1,R_1,R_2UL_2^\dagger)K(L_2,R_2,U)\Psi \end{pmatrix}$$

$$\overset{\text{Exercise}}{=} \begin{pmatrix} (R_1R_2)U(L_1L_2)^\dagger \\ K(L_1L_2,R_1R_2,U)\Psi \end{pmatrix} = \varphi(g_1g_2)\begin{pmatrix} U \\ \Psi \end{pmatrix}$$

- **Similar for SU(3) case**

Baryon transition amplitudes

✦ Generating functional with baryon fields

- **The effective Lagrangian with Goldstones and baryons**

$$\mathscr{L}_{\rm eff} = \mathscr{L}_{\rm GB} + \mathscr{L}_B + \bar{\eta}^a B^a + \bar{B}^a \eta^a \qquad \mathscr{L}_B = \bar{B}^a D^{ab} B^b$$

- **Generating functional**

$$\langle O_{\rm Out} | O_{\rm in} \rangle_{v,a,s,p;\eta\bar{\eta}} = e^{i\mathscr{Z}[v,a,s,p;\eta\bar{\eta}]} = N' \int \mathscr{D}U \mathscr{D}B \mathscr{D}\bar{B} e^{i\int d^4x \mathscr{L}_{\rm eff}(B,\bar{B},U;v,a,s,p;\eta\bar{\eta})}$$

$$= N' \int \mathscr{D}U e^{i\int d^4x \mathscr{L}_{\rm GB}} \int \mathscr{D}B \mathscr{D}\bar{B} e^{i\int d^4x \left[\bar{B}^a D^{ab} B^b + \bar{\eta}^a B^a + \bar{B}^a \eta^a\right]}$$

$$= N' \int \mathscr{D}U e^{i\int d^4x \mathscr{L}_{\rm GB}} e^{-i\int d^4x \int d^4y \bar{\eta}^a(x) S_{ab}(x,y) \eta^b(y)} \det D$$

- **$S^{ab}(x, y \,|\, U; v, a, s, p)$ is the baryon propagator in the presence of the meson fields and external fields.**

$$D^{ac} S^{cb} = \delta^4(x-y)\delta^{ab}$$

- **Baryon propagator in the presence of the meson fields and external fields.**

$$S^{ab}(x, y \,|\, v, a, s, p) = \frac{\delta}{i\delta\eta^a(x)} \frac{\delta}{i\delta\eta^b(y)} \mathscr{Z} \,|_{\eta=\bar{\eta}=0} = \left\langle 0 \,|\, T\left[B^a(x) B^b(y)\right] |\, 0 \right\rangle$$

Baryon transition amplitudes

✦ Generating functional with baryon fields

- **Transfer to the momentum space**

$$\tilde{S}(p,p'|v,a,s,p) = \int \mathrm{d}^4x \int \mathrm{d}^4y e^{-ipx-ip'y} S(x,y|v,a,s,p)$$

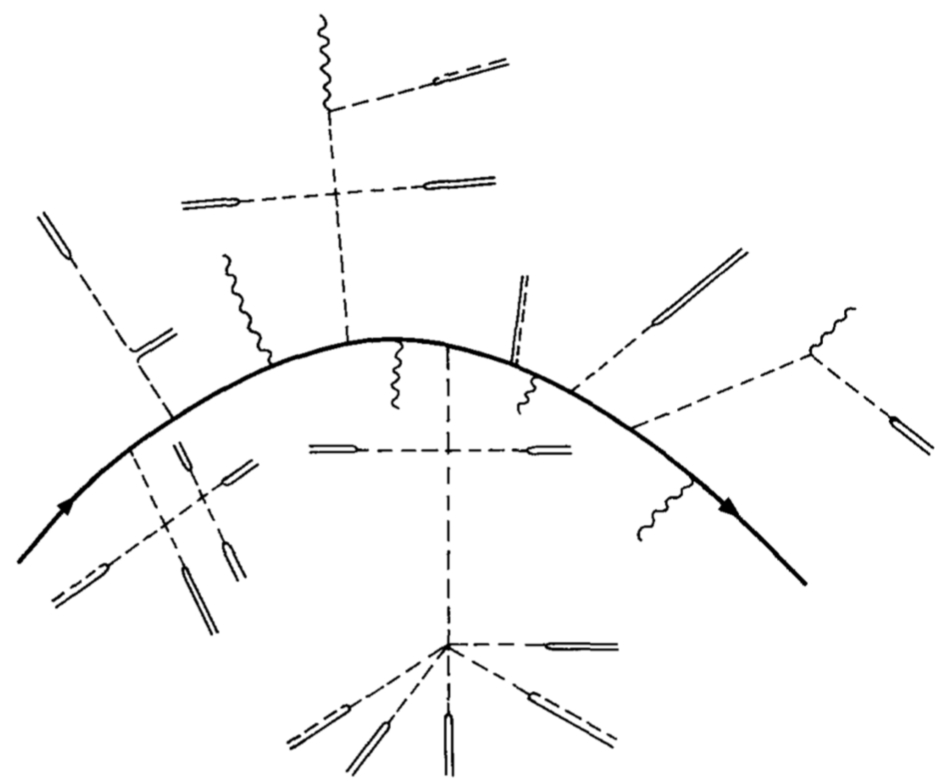
- **Baryon to baryon transition amplitude**

$$\mathscr{F}(p,p'|v,a,s,p) = \left\langle p'_{\text{out}} | p_{\text{in}} \right\rangle^{c}_{v,a,s,p} \propto \text{Residue of } \tilde{S}(p,p'|v,a,s,p)$$

  - **Examples : Baryon matrix elements**

$$\langle p'|\bar{q}(x)\gamma^\mu T^a q(x)|p\rangle = \frac{\delta}{i\delta v^a_\mu(x)}\mathscr{F}|_{v=0}$$

$$\left\langle p'|\bar{q}(x)\gamma^\mu\gamma^5 T^a q(x)|p\right\rangle = \frac{\delta}{i\delta a^a_\mu(x)}\mathscr{F}|_{a=0}$$



- **Our task is to construct $D^{ab}(U;v,a,s,p)$ order by order**

$$\mathscr{L}_B = \bar{B}^a D^{ab} B^b \qquad D^{ab} = \sum_i D^{ab,(i)}$$

Part II:

Construction of Chiral Effective Lagrangians

(算符构造基础)

Building Blocks

✦ Stuffs in hand

- **Meson fields:** $U \xmapsto{g} RUL^\dagger$
- **Baryon fields:** $B \xmapsto{g} KBK^\dagger \qquad (\Psi \xmapsto{g} K\Psi)$
- **A set of external fields:** $v_\mu\,, a_\mu\,, s\,, p$

✦ Recipe for BChPT

Building blocks transform under chiral symmetry group in the same way as baryon fields

$$A \xmapsto{g} KAK^\dagger \qquad A \in \{u_\mu, F^\pm_{\mu\nu}, \chi_\pm\} \qquad [D_\mu\,, X] = \partial_\mu X + [\Gamma_\mu\,, X]$$

- **The chiral connection**

$$\Gamma_\mu = \{u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger\}/2$$

- **The so-called chiral vielbein**

$$u_\mu \equiv i[u^\dagger(\partial_\mu - ir_\mu)u - u\left(\partial_\mu - il_\mu\right)u^\dagger]$$

- **Vector and axial vector sources**

$$F^\pm_{\mu\nu} \equiv u^\dagger F^R_{\mu\nu} u \pm u F^L_{\mu\nu} u^\dagger$$

- **Scalar and pseudo-scalar sources**

$$\chi_\pm \equiv u^\dagger \chi u^\dagger \pm u \chi^\dagger u$$

$$r_\mu = v_\mu + a_\mu$$

$$l_\mu = v_\mu - a_\mu$$

$$F^R_{\mu\nu} = \partial_\mu r_\nu - \partial_\nu r_\mu - i\left[r_\mu\,, r_\nu\right]$$

$$F^L_{\mu\nu} = \partial_\mu l_\nu - \partial_\nu l_\mu - i\left[l_\mu\,, l_\nu\right]$$

$$\chi = 2B\left(s + ip\right)$$

Building Blocks

✦ Lorentz transformation (also parity)

- **Bilinear**

$\bar{B}'B' = \bar{B}B$ **(Scalar)**

$\bar{B}'\gamma_5 B' = \det\Lambda\, \bar{B}\gamma_5 B$ **(Pseudo scalar)**

$\bar{B}'\gamma_\mu B' = \Lambda_\mu^{\ \nu}\bar{B}\gamma_\nu B$ **(Vector)**

$\bar{B}'\gamma_\mu\gamma_5 B' = \det\Lambda\Lambda_\mu^{\ \nu}\bar{B}\gamma_\nu\gamma_5 B$ **(Axial vector)**

- **Building blocks**

$\chi_+' = \chi_+$ **(Scalar)**

$\chi_-' = \det\Lambda\, \chi_-$ **(Pseudo Scalar)**

$D_\mu' = \Lambda_\mu^{\ \nu} D_\nu$ **(Vector)**

$u_\mu' = \det\Lambda\Lambda_\mu^{\ \nu} u_\nu$ **(Axial vector)**

$F_{\mu\nu}^{+'} = \Lambda_\mu^{\ \alpha}\Lambda_\nu^{\ \beta} F_{\alpha\beta}^+$ **(Tensor)**

$F_{\mu\nu}^{-'} = \det\Lambda\Lambda_\mu^{\ \alpha}\Lambda_\nu^{\ \beta} F_{\alpha\beta}^-$ **(Pseudo Tensor)**

Clifford algebra

$$\Gamma \in \{1\,,\gamma^5\,,\gamma^\mu\,,\gamma^5\gamma^\mu\,,\sigma^{\mu\nu}\}$$

Proof (Hints) :

0^- **particle**

$U = e^{i\Phi}$ $\quad v_\mu' = \Lambda_\mu^{\ \nu} v_\nu$

$\Phi' \stackrel{P}{=} -\Phi$ $\quad a_\mu' = \det\Lambda\Lambda_\mu^{\ \nu} a_\nu$

$U' \stackrel{P}{=} U^\dagger$ $\quad s' = s$

$u' \stackrel{P}{=} u^\dagger$ $\quad p' = \det\Lambda\, p$

$$\chi_- = u^\dagger(s+ip)u^\dagger - u(s+ip)^\dagger u$$

$\downarrow$

$$\chi_-' = u(s-ip)u - u^\dagger(s-ip)^\dagger u^\dagger = -\chi_-$$

Building Blocks

✦ Charge conjugation

- **Baryon field transform as** $B^C = C\bar{B}^T$

$$\left(\bar{B}\right)^C = \left(B^\dagger \gamma_0\right)^C = \left(B^C\right)^\dagger \gamma_0$$
$$= \left(C\bar{B}^T\right)^\dagger \gamma_0 = \left(\bar{B}C^T\right)^* \gamma_0$$
$$= B^T \gamma_0^* C^{T*} \gamma_0 = B^T \gamma_0 C^\dagger \gamma_0$$
$$= -\, B^T \gamma_0 C \gamma_0$$
$$= B^T C$$
$$= -\, B^T C^{-1}$$

- **Bilinear transform as**

$$\left(\bar{B}\Gamma B\right)^C = \bar{B}^C \Gamma B^C$$
$$= -\, B^T C^{-1} \Gamma C \bar{B}^T = -\, B^T C \Gamma C^{-1} \bar{B}^T$$
$$= -\, B^T (-1)^{C_\Gamma} \Gamma^T \bar{B}^T$$
$$= (-1)^{C_\Gamma} \left[-B^T \Gamma^T \bar{B}^T \right]$$
$$= (-1)^{C_\Gamma} \bar{B}\Gamma B$$

where we made use of $\left[B^T \Gamma^T \bar{B}^T\right] = -\left[\bar{B}\Gamma B\right]$

$$C\Gamma C^{-1} = (-1)^{C_\Gamma} \Gamma^T$$

Charge-conjugation matrix

$$C\gamma^\mu C^{-1} = -\, \gamma^{\mu T}$$
$$C\gamma^5 \gamma^\mu C^{-1} = \left(\gamma^5 \gamma^\mu\right)^T$$
$$C\gamma^5 C^{-1} = \gamma^{5T}$$
$$C\sigma^{\mu\nu} C^{-1} = -\left(\sigma^{\mu\nu}\right)^T$$

$$C = i\gamma^2\gamma^0 = -\, C^{-1} = -\, C^\dagger = -\, C^T = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Building Blocks

✦ Charge conjugation

- **Sources**

$$s^C = s^T \qquad p^C = p^T$$

$$v_\mu^C = -v_\mu^T \qquad a_\mu^C = a_\mu^T$$

How to prove? **QCD Lagrangian is invariant, e.g.**

$$[(\bar{q}\gamma^\mu v_\mu)q]^C = (\bar{q}\gamma^\mu v_\mu q)$$

$$\begin{aligned}\bar{q}^C\gamma^\mu v_\mu^C q^C &= -q^T C^{-1}\gamma^\mu C v_\mu^C \bar{q}^T \\ &= -q^T C\gamma^\mu C^{-1} v_\mu^C \bar{q}^T \\ &= q^T \gamma^{\mu T} v_\mu^C \bar{q}^T \\ &= -\left(\bar{q}\gamma^\mu v_\mu^{CT} q\right) = \bar{q}\gamma^\mu v_\mu q\end{aligned}$$

- **Meson fields**

$$\Phi^C = \Phi^T \quad \Longrightarrow \begin{cases} U^C = U^T \quad (U = e^{i\Phi}) \\ u^C = u^T \end{cases}$$

- **Building blocks**

$$u_\mu^C = u_\mu^T \qquad D_\mu^C = -D_\mu^T \qquad F_{\mu\nu}^{+C} = -F_{\mu\nu}^{+T} \qquad F_{\mu\nu}^{-C} = F_{\mu\nu}^{-T} \qquad \chi_\pm^C = \chi_\pm^T$$

✦ Hermiticity properties

- **Bilinear** $\left(\bar{B}\Gamma B\right)^\dagger = B^\dagger\Gamma^\dagger\gamma^0 B = \bar{B}\left(\gamma^0\Gamma^\dagger\gamma_0\right)B \xrightarrow{\gamma^0\Gamma\gamma^0 = (-1)^{h_\Gamma}\Gamma} \left(\bar{B}\Gamma B\right)^\dagger = (-1)^{h_\Gamma}\left(\bar{B}\Gamma B\right)$

- **Building blocks**

$$u_\mu^\dagger = -u_\mu \qquad D_\mu^\dagger = -D_\mu \qquad F_{\mu\nu}^{\pm\dagger} = F_{\mu\nu}^{\pm} \qquad \chi_-^\dagger = -\chi_- \qquad \chi_+^\dagger = \chi_+$$

Building Blocks

✦ Chiral power counting

- **Building blocks**

$$U = e^{i\Phi} \quad \sim \ O(1) \qquad\qquad u = \sqrt{U} \quad \sim \ O(1)$$

$$\partial_\mu U \quad \sim O(p)$$

Explanation: **Four momentum** $p = \left(E = \sqrt{M_\phi^2 + \vec{p}^{\,2}}, \vec{p}\right)$

$M_\phi^2 \sim m_q \sim O(p^2)$ **three-momentum is small quantity**

$$\nabla_\mu U = \partial_\mu U - i\left(v_\mu + a_\mu\right)U + iU\left(v_\mu - a_\mu\right) \quad \sim O(p) \qquad \Longrightarrow v_\mu \sim a_\mu \sim O(p)$$

$$s + ip \quad \Longrightarrow \quad s \sim p \sim O(p^2)$$

$U, u \sim O(1)$ $\quad s, p \sim O(p^2)$ $\quad a_\mu, u_\mu \sim O(p)$

$u_\mu \sim O(p)$ $\quad F_{\mu\nu}^{\pm} \sim O(p^2)$ $\quad \chi_\pm \sim O(p^2)$ $\quad D_\mu \sim O(p)$

Building Blocks

✦ Chiral power counting

- **Positive-energy plane-wave solutions**

$$B(\vec{x},t) = e^{-ip\cdot x}\sqrt{E+M_B}\begin{pmatrix}\chi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+M_B}\chi\end{pmatrix}$$

⟶ **Large component**

⟶ **Small component**

$$B(\vec{x},t) \sim O(p^0)$$

$$\bar{B}(\vec{x},t) = B^\dagger\gamma^0 = e^{+ip\cdot x}\sqrt{E+M_B}\left(\chi^\dagger, \frac{\vec{\sigma}\cdot\vec{p}}{E+M_B}\chi^\dagger\right)\begin{pmatrix}1 & \\ & -1\end{pmatrix}$$

$$= e^{+ip\cdot x}\sqrt{E+M_B}\left(\chi^\dagger, -\frac{\vec{\sigma}\cdot\vec{p}}{E+M_B}\chi^\dagger\right) \sim O(p^0)$$

$$[D_\mu, B] \sim O(1) \qquad i\gamma^\mu[D_\mu, B] - M_0 B \sim O(p)$$

- **Bilinear** $\bar{B}\Gamma B$

$$\Gamma = \gamma_5 \qquad \sim O(p^1)$$

$$\gamma^5 = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$$

$$\Gamma = 1, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} \qquad \sim O(p^0)$$

$$\gamma^0 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \qquad \gamma^i = \begin{pmatrix}0 & \sigma_i \\ -\sigma_i & 0\end{pmatrix}$$

Construction of Chiral Effective Lagrangians

✦ Guiding rules

① Real (Hermiticity) ② Flavor neutral (Trace) ③ Scalar (Parity even)

④ Chiral transformation ⑤ Proper Lorentz transformations

⑥ Charge conjugation C ⑦ Parity P ⑧ Time reversal T

- **A general form**

$$\left\langle A_1 \bar{B} \Gamma A_2 B A_3 \right\rangle \longrightarrow \left\langle \bar{B} \Gamma A_2' B A_3' \right\rangle$$

$$A_i \in \left\{ F^{\pm}_{\mu\nu} , \chi_{\pm} , u_{\mu} , D_{\mu} \right\}$$

$$\Gamma \text{ (Clifford algebra)} \in \left\{ 1 , \gamma^5 , \gamma^{\mu} , \gamma^5 \gamma^{\mu} , \sigma^{\mu\nu} \right\}$$

Constraint from Flavor neutral, Scalar, Chiral and Proper Lorentz transformation

Lorentz indices contracted by $g_{\mu\nu}$, $\epsilon_{\mu\nu\rho\sigma}$

Chiral power counting $D = D_{\Gamma} + D_{A_2'} + D_{A_3'} \Longrightarrow O(p^D)$

Construction of Chiral Effective Lagrangians

✦ Guiding rules

① **Real (Hermiticity)** ② **Flavor neutral (Trace)** ③ **Scalar (Parity even)**

④ **Chiral transformation** ⑤ **Proper Lorentz transformations**

⑥ **Charge conjugation C** ⑦ **Parity P** ⑧ **Time reversal T**

- **More suitable form form by imposing charge and hermiticity transformation**

$$X = \langle \bar{B}\Gamma[A_1\,, [A_2\,\cdots\,, [A_n\,, B]_\pm\cdots]_\pm]_\pm\rangle$$

Hermiticity $X^\dagger = (-1)^{h_1+\cdots+h_n+h_\Gamma}\langle \bar{B}\Gamma[A_n\,, [A_2\,\cdots\,, [A_1\,, B]_\pm]_\pm\cdots]_\pm\rangle$

Charge conj. $X^c = (-1)^{c_1+\cdots+c_n+c_\Gamma}\langle \bar{B}\Gamma[A_n\,, [A_2\,\cdots\,, [A_1\,, B]_\pm]_\pm\cdots]_\pm\rangle$

$$\begin{cases} C\Gamma C^{-1} = (-1)^{c_\Gamma}\Gamma^T \\ \gamma_0\Gamma^\dagger\gamma_0 = (-1)^{h_\Gamma}\Gamma \end{cases} \qquad \begin{cases} A_k^C = (-1)^{c_k}A_k^T \\ A_k^\dagger = (-1)^{h_k}A_k \end{cases} \qquad X_{\text{allowed}} \Longrightarrow \begin{cases} \frac{1}{2}\left(X+X^\dagger\right) \\ \frac{1}{2}\left(X+X^c\right) \end{cases}$$

Construction of Chiral Effective Lagrangians

✦ Rewind of the differential operator D^{ab} in the generating functional

- **The effective Lagrangian**

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{GB}} + \mathscr{L}_B + \bar{\eta}^a B^a + \bar{B}^a \eta^a$$

$$\mathscr{L}_{\text{B}} \sim \langle \bar{B}\Gamma[A_1[A_2, \cdots, [A_n, B]_\pm \cdots]_\pm]_\pm \rangle$$

$$B = \frac{1}{\sqrt{2}} B_a \lambda^a$$

$$= \frac{1}{2} \bar{B}^a \langle \lambda^a \Gamma[A_1, [A_2, \cdots [A_n, \lambda^b]_\pm \cdots]_\pm]_\pm \rangle B^b$$

$$= \bar{B}^a D^{ab} B^b$$

Therefore, differential operator D^{ab} has the form of

$$D^{ab} = \frac{1}{2} \Gamma \langle \lambda^a [A_1, [A_2, \cdots [A_n, \lambda^b]_\pm \cdots]_\pm]_\pm \rangle$$

SU(3) Meson-Baryon Lagrangian

✦ Lowest order

- $O(p^0)$ **operators** $a\langle \bar{B} i\gamma^\mu [D_\mu , B]\rangle + b\langle \bar{B}B\rangle$

$$\Rightarrow \quad \langle \bar{B} i\gamma^\mu [D_\mu , B]\rangle - M_0\langle \bar{B}B\rangle \quad \sim \quad O(p^1)$$

- $O(p^1)$ **operators** $D\langle \bar{B} i\gamma^5\gamma^\mu \{u_\mu , B\}\rangle + F\langle \bar{B} i\gamma^5\gamma^\mu [u_\mu , B]\rangle$

- **The lowest-order chiral effective Lagrangian is**

$$\mathscr{L}^{(1)}_{\text{Baryon}} = \langle \bar{B} i\gamma^\mu [D_\mu , B]\rangle - M_0\langle \bar{B}B\rangle + D\langle \bar{B} i\gamma^5\gamma^\mu \{u_\mu , B\}\rangle + F\langle \bar{B} i\gamma^5\gamma^\mu [u_\mu , B]\rangle$$

- **D and F are unknown constants, called low-energy constants (LECs)**
- **Determined by fitting to semileptonic decay $B \to B' + e^- + \bar{\nu}_e$**

✦ Higher orders

- **Two and more traces**
- **Redundancy : minimal set (EOM, trace theorem, etc)**

	$O(p^2)$	$O(p^3)$	$O(p^4)$	
$SU(3)$:	16	84	540	[J.A. Oller, et al, JHEP 2006] [S.Z. Jiang , et al,PRD 2017]
$SU(2)$:	7	23	118	[N. Fetles, et al, Annals of physics 2000]

Electroweak interaction in BChPT

✦ Electromagnetic interaction ($\mathscr{A}_\mu$)

- $SU(2)$

$$r_\mu = l_\mu = -e\mathscr{A}_\mu \frac{\tau_3}{3} \qquad v_\mu^{(s)} = -\frac{e}{2}\mathscr{A}_\mu \qquad e>0 \quad \text{and} \quad \frac{e^2}{4\pi} \approx \frac{1}{137}$$

- $SU(3)$

$$r_\mu = l_\mu = -e\mathscr{A}_\mu Q \qquad Q = \mathrm{diag}\{\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\}$$

✦ Weak charged interaction ($W_\mu^\pm = (W_{1\mu} \mp iW_{2\mu})/\sqrt{2}$)

$$r_\mu = 0 \qquad l_\mu = -\frac{g}{\sqrt{2}}\left(W_\mu^\pm T_+ + h.c.\right)$$

- $SU(2)$ $\quad T_+ = \begin{pmatrix} 0 & v_{ud} \\ 0 & 0 \end{pmatrix}$
- $SU(3)$ $\quad T_+ = \begin{pmatrix} 0 & v_{ud} & v_{us} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Fermi constant $\quad G_F = \frac{\sqrt{2}g^2}{8M_W^2} = 1.16637(1) \times 10^{-5}\,\mathrm{GeV}^{-2}$

✦ Weak neutral interaction (Z_μ)

- $SU(2)$ $\quad r_\mu = e\tan(\theta_w) Z_\mu \frac{\tau_3}{2} \qquad l_\mu = -\frac{g}{\cos\theta_w} Z_\mu \frac{\tau_3}{2} + e\tan(\theta_w) Z_\mu \frac{\tau_3}{2} \qquad v_\mu^{(s)} = \frac{e\tan(\theta_w)}{2} Z_\mu$

SU(2) Meson-Baryon Lagrangian

✦ **The SU(2) Lagrangian can be constructed in the same way**

$$O(p^1) \qquad \mathscr{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\gamma^\mu D_\mu - m - \frac{g}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi \qquad \Psi = \begin{pmatrix} p \\ n \end{pmatrix}$$

$$O(p^2) \qquad \mathscr{L}_{\pi N}^{(2)} = \bar{\Psi}\left\{ c_1\langle\chi_+\rangle - \frac{c_2}{8m^2}[\langle u_\mu u_\nu\rangle D^{\mu\nu} + h.c.] + \frac{1}{2}c_3\langle u_\mu u^\mu\rangle + \frac{i}{4}c_4[u_\mu, u_\nu]\sigma^{\mu\nu} + c_5[\chi_+ - \frac{1}{3}\langle\chi_+\rangle] + \frac{c_6}{8m}F_{\mu\nu}^+\sigma^{\mu\nu} + \frac{c_7}{8m}\langle F_{\mu\nu}^+\rangle\sigma^{\mu\nu} \right\}\Psi$$

$$O(p^3) \qquad \mathscr{L}_{\pi N}^{(3)} = \sum_{i=1}^{23} d_i O_i^{(3)}$$

- **Remarks**

  - **# of LECs grows rapidly** — **No predictive power?**
  - **Operators with dimension larger than 4** — **Non-renomalizable?**

> **"A non-renormalizable theory is just as good as a renormalizable theory for computations, provided one is satisfied with a finite accuracy."—Aneesh V. Manohar**

Power counting rule is important!!!

Part III:

Phenomenological Applications at Tree Level

(树图水平现象学应用)

Meson-Baryon Lagrangian

✦ Expansion of the building blocks in terms of GB fields

- **The meson field u is defined by exponential function of matrix** $u = e^{\frac{i}{\sqrt{2}F}\Phi}$

$$u(\Phi) = 1 + \frac{i}{\sqrt{2}F}\Phi - \frac{1}{4F^2}\Phi^2 - \frac{i}{12\sqrt{2}F^3}\Phi^3 + \frac{1}{96F^4}\Phi^4 + \frac{i}{480\sqrt{2}F^5}\Phi^5 - \frac{1}{5760F^6}\Phi^6 + O(\Phi^7)$$

$$u^\dagger(\Phi) = 1 - \frac{i}{\sqrt{2}F}\Phi - \frac{1}{4F^2}\Phi^2 + \frac{i}{12\sqrt{2}F^3}\Phi^3 + \frac{1}{96F^4}\Phi^4 - \frac{i}{480\sqrt{2}F^5}\Phi^5 - \frac{1}{5760F^6}\Phi^6 + O(\Phi^7)$$

- **If $r_\mu = l_\mu$**

$$u_\mu = -\frac{\sqrt{2}}{F}\sum_{m=0}^{1} C_1^m(-\Phi)^m\left(\partial_\mu - il_\mu\right)\Phi^{1-m} + \frac{1}{6\sqrt{2}F^3}\sum_{m=0}^{3} C_3^m(-\Phi)^m\left(\partial_\mu - il_\mu\right)\Phi^{3-m} - \frac{1}{240\sqrt{2}F^5}\sum_{m=0}^{5} C_5^m(-\Phi)^m\left(\partial_\mu - il_\mu\right)\Phi^{5-m} + O(\Phi^7)$$

- **If $r_\mu = l_\mu = 0$**

$$\Gamma_\mu = \frac{1}{2}\left(u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right) \overset{n=2k-1}{=} \frac{1}{2}\sum_{k=1}^{\infty}\frac{-2}{(2k)(2k-1)!}\frac{i^{2k}}{2^kF^{2k}}[\underbrace{\Phi,\cdots[\Phi}_{2k-1\text{ times}},\partial_\mu\Phi]\cdots]$$

$$= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k(2k-1)!2^kF^{2k}}[\underbrace{\Phi,\cdots[\Phi}_{2k-1\text{ times}},\partial_\mu\Phi]\cdots]$$

Meson-Baryon Lagrangian

✦ Expansion of the building blocks in terms of GB fields

- **If** $p=0,\ \chi=\chi^\dagger=2B_0 s$

$$\chi_+ = 4B_0 s - \frac{B_0}{F^2}\sum_{m=0}^{2} C_2^m \Phi^m s \Phi^{2-m} + \frac{B_0}{24F^4}\sum_{m=0}^{4} C_4^m \Phi^m s \Phi^{4-m} - \frac{B_0}{1440F^6}\sum_{m=0}^{6} C_6^m \Phi^m s \Phi^{6-m} + O(\Phi^8)$$

$$\chi_- = -\frac{2\sqrt{2}B_0 i}{F}\sum_{m=0}^{1} C_1^m \Phi^m s \Phi^{1-m} + \frac{B_0 i}{3\sqrt{2}F^3}\sum_{m=0}^{3} C_3^m \Phi^m s \Phi^{3-m} - \frac{B_0 i}{120\sqrt{2}F^5}\sum_{m=0}^{5} C_5^m \Phi^m s \Phi^{5-m} + O(\Phi^7)$$

- **If** $r_\mu = l_\mu = -e\mathscr{A}_\mu Q$

$$F_+^{\mu\nu} = e(\partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu)\Big[-2Q + \frac{1}{2F^2}\sum_{m=0}^{2} C_2^m \Phi^m Q \Phi^{2-m} - \frac{1}{48F^2}\sum_{m=0}^{4} C_4^m \Phi^m Q \Phi^{4-m}$$
$$+\frac{1}{2880F^2}\sum_{m=0}^{6} C_6^m \Phi^m Q \Phi^{6-m} + O(\Phi^8)\Big]$$

$$F_-^{\mu\nu} = e(\partial^\mu \mathscr{A}^\nu - \partial^\nu \mathscr{A}^\mu)\Big[\frac{\sqrt{2}i}{F}\sum_{m=0}^{1} C_1^m (-\Phi)^m \partial_\mu \Phi^{1-m} - \frac{i}{6\sqrt{2}F^3}\sum_{m=0}^{3} C_3^m (-\Phi)^m \partial_\mu \Phi^{3-m}$$
$$+\frac{i}{240\sqrt{2}F^5}\sum_{m=0}^{5} C_5^m (-\Phi)^m \partial_\mu \Phi^{5-m} + O(\Phi^7)\Big]$$

Meson-Baryon Lagrangian

✦ Feynman rule

- **LO Lagrangian**

$$\mathscr{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\gamma^\mu D_\mu - m\right)\Psi - \frac{g}{2}\bar{\Psi}\gamma^\mu\gamma_5 u_\mu \Psi \qquad D_\mu = \partial_\mu + \Gamma_\mu$$

$$= \bar{\Psi}\left(i\gamma^\mu \partial_\mu - m\right)\Psi + \bar{\Psi} i\gamma^\mu \Gamma_\mu \Psi - \frac{g}{2}\bar{\Psi}\gamma^\mu\gamma_5 u_\mu \Psi$$

- **The chiral connection and vielbein** $(r_\mu = l_\mu = 0)$

$$\Gamma_\mu = \frac{1}{2}\left\{u^\dagger\left(\partial_\mu - ir_\mu\right)u + u\left(\partial_\mu - il_\mu\right)u^\dagger\right\}$$

$$= \frac{1}{2}\left\{u^\dagger\partial_\mu u + u\partial_\mu u^\dagger\right\}$$

$$= -\frac{1}{4F^2}\left(\partial_\mu\Phi\Phi - \Phi\partial_\mu\Phi\right) + O(\Phi^4)$$

$$= -\frac{i}{4F^2}\left(\partial_\mu\vec{\phi}\times\vec{\phi}\right)\cdot\vec{\tau} + O(\vec{\phi}^4)$$

$$u_\mu = i\left\{u^\dagger\left(\partial_\mu - ir_\mu\right)u - u\left(\partial_\mu - il_\mu\right)u^\dagger\right\}$$

$$= i\left\{u^\dagger\partial_\mu u - u\partial_\mu u^\dagger\right\}$$

$$= -\frac{\sqrt{2}}{F}\partial_\mu\Phi + O(\Phi^3)$$

$$= -\frac{1}{F}\vec{\tau}\cdot\partial\vec{\phi} + O(\vec{\phi}^3)$$

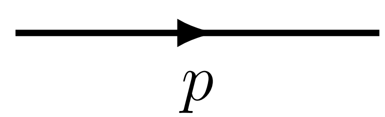
Meson-Baryon Lagrangian

✦ Feynman rule

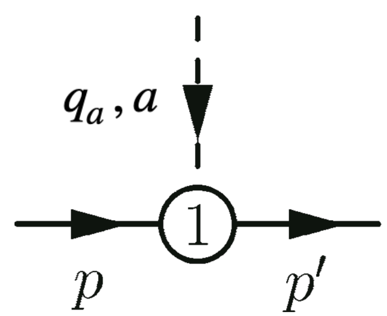
- **Rewrite Lagrangian**

$$\mathscr{L}^{(1)}_{\pi N} = \underbrace{\bar{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi}_{0\phi 2N} + \frac{1}{4F^2}\underbrace{\bar{\Psi}\left(\gamma^\mu\partial_\mu\vec{\phi}\times\vec{\phi}\right)\cdot\vec{\tau}\Psi}_{2\phi 2N} - \frac{g}{2F}\underbrace{\bar{\Psi}\gamma^\mu\gamma_5\vec{\tau}\cdot\partial_\mu\vec{\phi}\Psi}_{1\phi 2N} + \cdots$$

- **$O(p)$ Feynman rules**

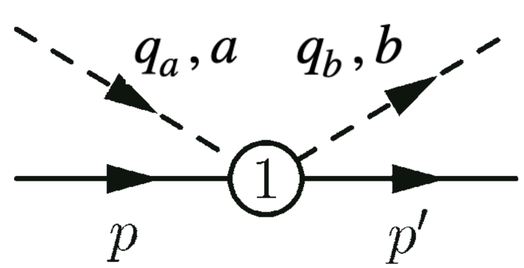


$$\frac{i}{\gamma^\mu p_\mu - m} \quad \sim \quad O(p^{-1})$$



$$-\frac{g}{2F}\gamma_\mu q_a^\mu \gamma_5 \tau_a \quad \sim \quad O(p^1)$$

- **overall i**
- **Incoming $\partial_\mu \to -iq_\mu$**



$$\frac{1}{4F^2}\epsilon_{abc}\tau_c\gamma_\mu\left(q_a^\mu + q_b^\mu\right) \quad \sim \quad O(p^1)$$

Goldberger-Treiman Relation

✦ Algebra calculation

- **Partially Conserved Axial Current (PCAC)**

$$\begin{cases} \partial_\mu V_a^\mu = i\bar{q}\left[\mathscr{M}, \frac{\tau_a}{2}\right] q & V_a^\mu = \bar{q}\gamma^\mu \frac{\tau_a}{2} q \\ \partial_\mu A_a^\mu = i\bar{q}\gamma_5 \left\{\frac{\tau_a}{2}, \mathscr{M}\right\} q & A_a^\mu = \bar{q}\gamma^\mu\gamma_5 \frac{\tau_a}{2} q \end{cases} \qquad a = 1,2,3$$

$SU(2)$ flavor symmetry is good

$$\mathscr{M} = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} = \hat{m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \partial_\mu V_a^\mu = 0 \\ \partial_\mu A^\mu = i\bar{q}\gamma_5\tau_a q = \hat{m}P(x) \end{cases}$$

$P(x)$ denotes pseudo scalar current

$$\Longrightarrow \left\langle N(p')\,|\,\partial_\mu A_a^\mu(x)\,|\,N(p)\right\rangle = \hat{m}\left\langle N(p')\,|\,P_a(x)\,|\,N(p)\right\rangle$$

Baryon to baryon transition amplitude

$$\Longrightarrow \partial_\mu \left\langle N(p')\,|\,e^{i\hat{p}\cdot x}A_a^\mu(0)e^{-i\hat{p}\cdot x}\,|\,N(p)\right\rangle = \hat{m}\left\langle N(p')\,|\,e^{i\hat{p}\cdot x}P_a(0)e^{-i\hat{p}\cdot x}\,|\,N(p)\right\rangle$$

$$\Longrightarrow \partial_\mu \left[e^{i(p'-p)\cdot x}\left\langle N(p')\,|\,A_a^\mu(0)\,|\,N(p)\right\rangle\right] = \hat{m}e^{i(p'-p)\cdot x}\left\langle N(p')\,|\,P_a(0)\,|\,N(p)\right\rangle$$

$$\overset{q^\mu=(p'-p)^\mu}{\Longrightarrow} iq_\mu \left\langle N(p')\,|\,A_a^\mu(0)\,|\,N(p)\right\rangle = \hat{m}\left\langle N(p')\,|\,P_a(0)\,|\,N(p)\right\rangle$$

Goldberger-Treiman Relation

✦ Algebra calculation

- **Form factors**

$$\left\langle N(p') \,|\, A_a^\mu(0) \,|\, N(p) \right\rangle = \bar{u}(p') \left[\gamma^\mu G_A(t) + \frac{(p'-p)^\mu}{2m_N} G_P(t) \right] \gamma_5 \frac{\tau_a}{2} u(p)$$

$$\hat{m} \left\langle N(p') \,|\, P_a(0) \,|\, N(p) \right\rangle = \frac{M_\pi^2 F_\pi}{M_\pi^2 - t} G_{\pi N}(t) i \bar{u}(p') \gamma_5 \tau_a u(p) \qquad (t = q^2 = (p'-p)^2)$$

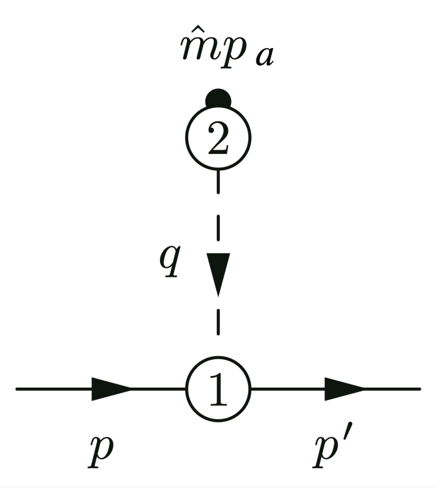
- **General relation**

$$2m_N G_A(t) + \frac{t}{2m_N} G_P(t) = 2 \frac{M_\pi^2 F_\pi}{M_\pi^2 - t} G_{\pi N}(t)$$

- **GT relation** $t = 0$

$$G_A(t=0) = g_A \qquad G_{\pi N}(t=0) = g_{\pi NN} \qquad m_N g_A = F_\pi g_{\pi NN}$$

✦ ChPT calculation



$$\hat{m} 2 B F_\pi \frac{i}{t - M_\pi^2} \bar{u}(p') \left\{ -\frac{1}{2} \frac{g_A}{F} \gamma^\mu q_\mu \gamma_5 \tau_a \right\} u(p)$$

$$= M_\pi^2 F_\pi \frac{m g_A}{F} \frac{1}{M_\pi^2 - t} \bar{u}(p') \gamma_5 i \tau_a u(p)$$

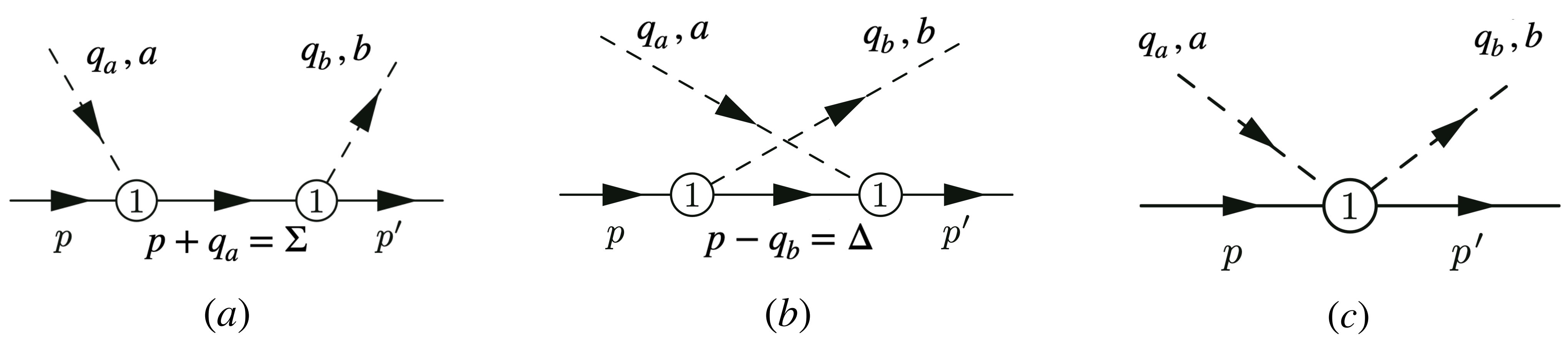
The lowest-order prediction

$$G_{\pi N}(t) = \frac{m}{F} g_A \qquad g_{\pi NN} = \frac{m_N}{F_\pi} g_A$$

Pion-Nucleon Scattering

✦ Leading-order calculation

- **Tree diagrams**



- **Tree amplitudes**

$$i\mathcal{M}_a=\bar{u}(p')\left\{\left[-\frac{g}{2F}\left(-\not{q}_b\right)\gamma_5\tau_b\right]\frac{i}{\not{p}+\not{q}_a-m}\left[-\frac{g}{2F}\left(\not{q}_a\right)\gamma_5\tau_a\right]\right\}u(p)$$

$$i\mathcal{M}_b=\bar{u}(p')\left\{\left[-\frac{g}{2F}\left(\not{q}_a\right)\gamma_5\tau_a\right]\frac{i}{\not{p}-\not{q}_b-m}\left[-\frac{g}{2F}\left(-\not{q}_b\right)\gamma_5\tau_b\right]\right\}u(p)$$

$$i\mathcal{M}_c=\bar{u}(p')\left\{\frac{1}{4F^2}\epsilon_{abc}\tau_c\left(\not{q}_a+\not{q}_b\right)\right\}u(p)$$

Pion-Nucleon Scattering

✦ Leading-order calculation

- **Mandelstam variables**

$$s = (p + q_a)^2 \qquad t = (p - p')^2 \qquad u = (p - q_b)^2$$

- **Independent scalar kinematical variables**

$$\nu = \frac{s-u}{4m_N} \qquad \nu_B = -\frac{q_a \cdot q_b}{2m_N} = \frac{t - 2M_\pi^2}{4m_N}$$

- **The simplified amplitude**

$$T^{ab}_{(a)} = \frac{g_A}{4F_\pi^2}\bar{u}(p')\left\{2m_N + \frac{1}{2}\left(\not{q}_a + \not{q}_b\right)\left(-1 - \frac{2m_N}{\nu - \nu_B}\right)\right\}\tau_b\tau_a u(p)$$

$$T^{ab}_{(b)} = \frac{g_A}{4F_\pi^2}\bar{u}(p')\left\{2m_N + \frac{1}{2}\left(\not{q}_a + \not{q}_b\right)\left(1 - \frac{2m_N}{\nu + \nu_B}\right)\right\}\tau_a\tau_b u(p)$$

$$T^{ab}_{(c)} = \frac{1}{4F_\pi^2}\bar{u}(p')\left\{\left(\not{q}_a + \not{q}_b\right)\left(-i\epsilon_{abc}\tau_c\right)\right\}u(p)$$

Pion-Nucleon Scattering

✦ Leading-order calculation

- **Amplitude analysis**

1. **Isospin decomposition** $\left(\tau_a\tau_b = \delta_{ab} + i\epsilon_{abc}\tau_c = \frac{1}{2}\left[\tau_a\,,\tau_b\right] + \frac{1}{2}\left\{\tau_a\,,\tau_b\right\}\right)$

$$T^{ab} = \frac{1}{2}\left\{\tau^b\,,\tau^a\right\}T^+ + \frac{1}{2}\left[\tau^b\,,\tau^a\right]T^- = \delta^{ab}T^+ - i\epsilon^{abc}\tau^c T^-$$

2. **Lorentz decomposition**

$$T^{\pm} = \bar{u}(p')\left[A^{\pm}(\nu\,,\nu_B) + \frac{1}{2}\left(\not{q}_a + \not{q}_b\right)B^{\pm}(\nu\,,\nu_B)\right]u(p)$$

3. **Relation between** $\left\{T^+\,,T^-\right\}$ **and** $\left\{T^{1/2}\,,T^{3/2}\right\}$

$$\begin{cases} T^{\frac{1}{2}} = T^+ + 2T^- \\ T^{\frac{3}{2}} = T^+ - T^- \end{cases}$$

Ten physical processes :

e.g. $T_{\pi^+p\to\pi^+p} = T^{\frac{3}{2}}$

…

4. **A, B functions**

$$A^+ = A^+_{(a)} + A^+_{(b)} + A^+_{(c)} = \frac{g_A^2 m_N}{F_\pi^2} \qquad B^+ = 0$$

$$A^- = -\frac{g_A^2}{F_\pi^2}\frac{m_N\nu}{\nu^2-\nu_B^2} \qquad B^- = \frac{1-g_A^2}{2F_\pi^2} - \frac{g_A^2}{F_\pi^2}\frac{m_N\nu}{\nu^2-\nu_B^2}$$

Pion-Nucleon Scattering

✦ Leading-order calculation

- *S*-wave scattering lengths

$$a_{0+}^{\pm} \equiv \frac{1}{8\pi(m_N+M_\pi)} T^{\pm}|_{\rm thr} = \frac{1}{4\pi(1+\mu)}\left[A^{\pm}+M_\pi B^{\pm}\right]_{\rm thr}$$

- **ChPT results of $O(p)$**

$$a_{0+}^{-} = \frac{M_\pi}{8\pi(1+\mu)F_\pi^2}\left(1+\frac{g_A^2\mu^2}{4}\frac{1}{1-\frac{\mu^2}{4}}\right) = \frac{M_\pi}{8\pi(1+\mu)F_\pi^2}\left[1+O(p^2)\right]$$

$$a_{0+}^{+} = -\frac{g_A^2 M_\pi}{16\pi(1+\mu)F_\pi^2}\frac{\mu}{1-\frac{\mu^2}{4}} = O(p^2)$$

- **Weinberg-Tomozawa relation**

Taking the linear combination $a^{\frac{1}{2}} = a_{0+}^{+} + 2a_{0+}^{-}$ **and** $a^{\frac{3}{2}} = a_{0+}^{+} - a_{0+}^{-}$ **the results satisfy**

$$a_{0+}^{I} = -\frac{M_\pi}{8\pi(1+\mu)F_\pi^2}\left[I(I+1)-\frac{3}{4}-2\right]$$

Thanks!