Majorana Phase and Matter Effects in Neutrino Chiral Oscillation

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1. Introduction to chiral oscillation

The dynamics of neutrinos in vacuum is described by Dirac equation

$$(i\partial\!\!\!/ - m)\psi = 0 \,. \tag{1}$$

It's worth noting that chiral eigenstates wouldn't evolve independently,

$$i\partial\!\!\!/\psi_L = \psi_R \,, \quad i\partial\!\!\!/\psi_R = \psi_L \,. \tag{2}$$

which indicates that there's the oscillation between chiral eigenstates.

To show it explicitly, firstly we can derive the Hamiltonian from eq.(1)

$$H = \gamma^0\boldsymbol{\gamma}\cdot\mathbf{p} + m\gamma^0 = \mathbf{p}\cdot\boldsymbol{\alpha} + m\beta = (\mathbf{p}\cdot\boldsymbol{\Sigma})\gamma^5 + m\beta \,. \tag{3}$$

And the evolution operator $U(t)$, defined by $\psi(t) = U(t)\psi(0)$, would be

$$U(t) = e^{-iHt} = \cos(Et) - i\frac{(\mathbf{p}\cdot\boldsymbol{\Sigma})\gamma^5 + m\beta}{E}\sin(Et) \tag{4}$$

From Hamiltonian we can deduce the energy eigenstate

$$\psi_1^h = \frac{1}{\sqrt{2E}}\begin{pmatrix}\sqrt{E - h\cdot p}\,u^h \\ \sqrt{E + h\cdot p}\,u^h\end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}} \,, \quad \psi_2^h = \frac{1}{\sqrt{2E}}\begin{pmatrix}\sqrt{E + h\cdot p}\,u^h \\ -\sqrt{E - h\cdot p}\,u^h\end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}} \,, \tag{5}$$

where $h = \pm 1$, u^h are the helicity eigenstates with $\mathbf{p}\cdot\boldsymbol{\sigma}u^h = (h\cdot p)u^h$. Then the evolution of energy eigenstate would be $\psi^h(t,\mathbf{x}) = U(t)\psi^h = \psi^h e^{-iEt}$.

While in the SM neutrinos are produced by W and/or Z interactions. At production $t = 0$ point, they are left-handed and normalized, ie.

$$\psi_L^h = \begin{pmatrix}u^h \\ 0\end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}} \,, \quad \psi_R^h = \begin{pmatrix}0 \\ u^h\end{pmatrix} e^{i\mathbf{p}\cdot\mathbf{x}} \,. \tag{6}$$

With the evolution, $\psi_L^h(t)$ would oscillate to ψ_R^h because of the $m\beta$ in $U(t)$,

$$\psi_L^h(t) = U(t)\psi_L^h = \left(\cos(Et) + i\frac{(\mathbf{p}\cdot\boldsymbol{\Sigma})}{E}\sin(Et)\right)\psi_L^h - i\frac{m}{E}\sin(Et)\psi_R^h \,. \tag{7}$$

So the oscillation probability would be

$$P(\nu_L^h \to \nu_L^h) = |\psi_L^{h\dagger}\psi_L^h(t)|^2 = 1 - \frac{m^2}{E^2}\sin^2(Et) \,, \quad P(\nu_L^h \to \nu_R^h) = |\psi_R^{h\dagger}\psi_L^h(t)|^2 = \frac{m^2}{E^2}\sin^2(Et) \,. \tag{8}$$

2. Dirac neutrino oscillation in matter

When active flavor neutrinos propagate in matter, the equation of motion is affected by effective potentials due to the interactions with the medium. We can write the effective Lagrangian for an active Dirac neutrino in matter as

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi - \rho\bar\psi\gamma_0\tfrac{1-\gamma_5}{2}\psi \,, \tag{9}$$

where $\rho = \sqrt{2}G_F\left(N_e\delta_{\alpha e} - \frac{1}{2}N_n\right)$. Then the equation of motion would be

$$\left(i\partial\!\!\!/ - m - \rho\gamma_0\tfrac{1-\gamma_5}{2}\right)\psi = 0 \,. \tag{10}$$

So the Hamiltonian would be

$$H = (\mathbf{p}\cdot\boldsymbol{\Sigma})\gamma^5 + m\beta + \rho\tfrac{1-\gamma_5}{2} \,, \tag{11}$$

with energy eigenvalues given by $E_1 = \frac{\rho}{2} + E_h$ and $E_2 = \frac{\rho}{2} - E_h$, where $E_h = \sqrt{m^2 + \left(h\cdot p - \frac{\rho}{2}\right)^2}$, and the corresponding eigenstates

$$\psi_1 = \frac{1}{\sqrt{2E_h}}\begin{pmatrix}\sqrt{E_h - (h\cdot p - \frac{\rho}{2})}\,u^h \\ \sqrt{E_h + (h\cdot p - \frac{\rho}{2})}\,u^h\end{pmatrix} \,, \quad \psi_2 = \frac{1}{\sqrt{2E_h}}\begin{pmatrix}\sqrt{E_h + (h\cdot p - \frac{\rho}{2})}\,u^h \\ -\sqrt{E_h - (h\cdot p - \frac{\rho}{2})}\,u^h\end{pmatrix} \,. \tag{12}$$

Then with the expression of H, we can calculate the $U(t)$ in matter,

$$U(t) = e^{-iHt} = e^{-\frac{i}{2}\rho t}\begin{pmatrix}\cos(E_ht) + i\frac{h\cdot p - \frac{\rho}{2}}{E_h}\sin(E_ht) & -i\frac{m}{E_h}\sin(E_ht) \\ -i\frac{m}{E_h}\sin(E_ht) & \cos(E_ht) - i\frac{h\cdot p - \frac{\rho}{2}}{E_h}\sin(E_ht)\end{pmatrix} . \tag{13}$$

So the oscillation probability would be

$$P(\nu_L^h \to \nu_L^h) = \left|\psi_L^{h\dagger}U(t)\psi_L^h\right|^2 = 1 - \frac{m^2}{E_h^2}\sin^2(E_ht) \,, \quad P(\nu_L^h \to \nu_R^h) = \left|\psi_R^{h\dagger}U(t)\psi_L^h\right|^2 = \frac{m^2}{E_h^2}\sin^2(E_ht) \,. \tag{14}$$

We can see different helicity with the different oscillation probability.

3. Majorana neutrino oscillation in matter

Though the Lagrangian would be same as Dirac case in eq.(9), ψ and ψ^* wouldn't be independent because $\psi = \begin{pmatrix}\nu_L \\ \nu_L^c\end{pmatrix}$. So the equation of motion would be modified as

$$\left(i\partial\!\!\!/ - m - \rho\gamma_0\tfrac{1-\gamma_5}{2} + \rho^*\gamma_0\tfrac{1+\gamma_5}{2}\right)\psi = (i\partial\!\!\!/ - m + \rho\gamma_0\gamma_5)\psi = 0 \,. \tag{15}$$

The corresponding Hamiltonian would be

$$H = (\mathbf{p}\cdot\boldsymbol{\Sigma})\gamma^5 + m\beta - \rho\gamma_5 \,, \tag{16}$$

whose eigenvalues would be $E_h = \pm\sqrt{m^2 + (h\cdot p - \rho)^2}$. Then we can calculate the oscillation probability in analogy to Dirac case,

$$P(\nu_L^h \to \nu_L^h) = \left|\psi_L^{h\dagger}U(t)\psi_L^h\right|^2 = 1 - \frac{m^2}{E_h^2}\sin^2(E_ht) \,, \quad P(\nu_L^h \to (\nu_L^h)^c) = \left|\psi_R^{h\dagger}U(t)\psi_L^h\right|^2 = \frac{m^2}{E_h^2}\sin^2(E_ht) \,. \tag{17}$$

where we can see that compared to the Dirac case the ρ contribution would be twice.

4. General mass term neutrino oscillation in vacuum and matter

The most general the effective Lagrangian for these Seesaw neutrinos in matter will be given by

$$\begin{aligned}\mathcal{L} =& \bar\nu_L i\partial\!\!\!/\nu_L + \bar N_R i\partial\!\!\!/ N_R - \frac{1}{2}\left(\begin{pmatrix}\bar\nu_L^c & \bar N_R\end{pmatrix}\begin{pmatrix}M_L & M_D^T \\ M_D & M_R\end{pmatrix}\begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix} + \text{h.c.}\right) - \begin{pmatrix}\bar\nu_L & \bar N_R^c\end{pmatrix}\begin{pmatrix}j_L^\mu & j_{RL}^\mu \\ j_{RL}^{\mu\dagger} & j_R\end{pmatrix}\gamma_\mu\begin{pmatrix}\nu_L \\ N_R^c\end{pmatrix} \\ =& \bar\psi_L i\partial\!\!\!/\psi_L - \frac{1}{2}\left(\bar\psi_L^c\mathcal{M}\psi_L + \text{h.c.}\right) - \bar\psi_L J^\mu\gamma_\mu\psi_L \,,\end{aligned} \tag{18}$$

where the mass matrix $\mathcal{M}$ can be diagonalized as $V^T\mathcal{M}V = \widehat{M} = \text{diag}\{m_1, m_2, m_3, M_1, M_2, M_3\} = \text{diag}\{\widehat{M}_l, \widehat{M}_h\}$ with $\psi_L = V\psi_L^m$. Then the expression of Lagrangian would be

$$\mathcal{L} = \frac{1}{2}\left(\bar\psi^m(i\partial\!\!\!/ - \widehat{M})\psi^m\right) - \bar\psi^m\widetilde{J}^\mu\gamma_\mu\tfrac{1-\gamma_5}{2}\psi^m \,, \tag{19}$$

and the equation of motion is

$$(i\partial\!\!\!/ - \widehat{M})\psi^m - \widetilde{J}^\mu\gamma_\mu\tfrac{1-\gamma_5}{2}\psi^m + (\widetilde{J}^\mu)^*\gamma_\mu\tfrac{1+\gamma_5}{2}\psi^m = 0 \,. \tag{20}$$

Here $\psi^m = \psi_L^m + (\psi_L^m)^c$ and $\widetilde{J}^\mu = V^\dagger J^\mu V$. So the Hamiltonian H for the general case is

$$H = (\mathbf{p}\cdot\boldsymbol{\Sigma})\gamma_5 + \hat{M}\beta + \gamma_0\widetilde{J}^\mu\gamma_\mu\tfrac{1-\gamma_5}{2} - \gamma_0(\widetilde{J}^\mu)^*\gamma_\mu\tfrac{1+\gamma_5}{2} \,. \tag{21}$$

To explicit the chiral oscillation, let us consider the simple case of just one light and one heavy neutrinos with seesaw mass matrix $\widehat{M} = \text{diag}\{m, M\}$, set $j_{R,RL}^\mu = 0$ for the case of having just SM interactions in matter and parameterize the mixing matrix as $V = \begin{pmatrix}\cos\theta & e^{i\eta}\sin\theta \\ -\sin\theta & e^{i\eta}\cos\theta\end{pmatrix}$. Then we can show the expression of Hamiltonian explicitly,

$$H = \begin{pmatrix}\frac{\rho}{2}(1+\cos 2\theta) - \mathbf{p}\cdot\boldsymbol{\sigma} & m & \frac{\rho}{2}e^{i\eta}\sin 2\theta & 0 \\ m & \mathbf{p}\cdot\boldsymbol{\sigma} - \frac{\rho}{2}(1+\cos 2\theta) & 0 & -\frac{\rho}{2}e^{-i\eta}\sin 2\theta \\ \frac{\rho}{2}e^{-i\eta}\sin 2\theta & 0 & \frac{\rho}{2}(1-\cos 2\theta) - \mathbf{p}\cdot\boldsymbol{\sigma} & M \\ 0 & -\frac{\rho}{2}e^{i\eta}\sin 2\theta & M & \mathbf{p}\cdot\boldsymbol{\sigma} - \frac{\rho}{2}(1-\cos 2\theta)\end{pmatrix} . \tag{22}$$

The eigenvalues would be $E_{1,2} = \mp\sqrt{A_1 - A_2}$, $E_{3,4} = \mp\sqrt{A_1 + A_2}$, where

$$A_1 = \frac{m^2+M^2+2p^2-2(h\cdot p)\rho+\rho^2}{2} \,, \quad A_2 = \frac{\sqrt{\left((m^2-M^2)\cos 2\theta - 2\rho(h\cdot p - \frac{\rho}{2})\right)^2 + \left((m^2-M^2)^2 + \rho^2(m^2+M^2-2mM\cos 2\eta)\right)\sin^2 2\theta}}{2} \,. \tag{23}$$

Then after calculation of evolution operator $U(t) = e^{-iHt}$, we can derive the oscillation probability. But the general expression is so complicated, so let's see some special case first.

Setting $\rho = 0$, oscillation probabilities in vacuum would be

$$\begin{aligned}P(\nu_L^h \to \nu_L^h) &= \left(\cos^2\theta\cos(E_mt) + \sin^2\theta\cos(E_Mt)\right)^2 + p^2\left(\frac{\cos^2\theta}{E_m}\sin(E_mt) + \frac{\sin^2\theta}{E_M}\sin(E_Mt)\right)^2 \\ P(\nu_L^h \to (N_R^h)^c) &= \frac{\sin^2 2\theta}{4}\left((\cos(E_mt) - \cos(E_Mt))^2 + p^2\left(\frac{\sin(E_mt)}{E_m} - \frac{\sin(E_Mt)}{E_M}\right)^2\right) \\ P(\nu_L^h \to (\nu_L^h)^c) &= \frac{m^2}{(E_m)^2}\cos^4\theta\sin^2(E_mt) + \frac{mM}{2E_mE_M}\sin^2 2\theta\cos 2\eta\sin(E_mt)\sin(E_Mt) + \frac{M^2}{(E_M)^2}\sin^4\theta\sin^2(E_Mt) \\ P(\nu_L^h \to N_R^h) &= \frac{\sin^2 2\theta}{4}\left(\frac{m^2}{(E_m)^2}\sin^2(E_mt) - \frac{2mM}{E_mE_M}\cos 2\eta\sin(E_mt)\sin(E_Mt) + \frac{M^2}{(E_M)^2}\sin^2(E_Mt)\right) \,,\end{aligned} \tag{24}$$

We can see that though the Majorana phase would give the contribution to oscillation probability, it would dis appear after taking the time average.

Besides, in ultrarelativistic limit, ie. $p \gg M > m \gg \rho$, after taking the time average, the oscillation probability would be

$$\begin{aligned}P(\nu_L^h \to (\nu_L^h)^c) &= \frac{(m^2-M^2)^2}{8A_2^2}\left(\frac{(m^2\cos^4\theta + M^2\sin^4\theta)}{p^2} - \frac{\rho(4m^2\cos^6\theta - 4M^2\sin^6\theta + mM\cos 2\eta\cos 2\theta\sin^2 2\theta)}{p(m^2-M^2)} + \frac{2\rho^2(2m^2\cos^4\theta + 2M^2\sin^4\theta + mM\cos 2\eta\sin^2 2\theta)}{(m^2-M^2)^2}\right) , \\ P(\nu_L^h \to N_R^h) &= \frac{(m^2-M^2)^2\sin^2 2\theta}{32A_2^2}\left(\frac{m^2+M^2}{p^2} - \frac{2\rho(m^2+M^2-2mM\cos 2\eta)\cos 2\theta}{p(m^2-M^2)} + \frac{4\rho^2\left(m^2+M^2-2mM\cos 2\eta\right)}{(m^2-M^2)^2}\right) .\end{aligned} \tag{25}$$

The probabilities for the other two oscillation modes are

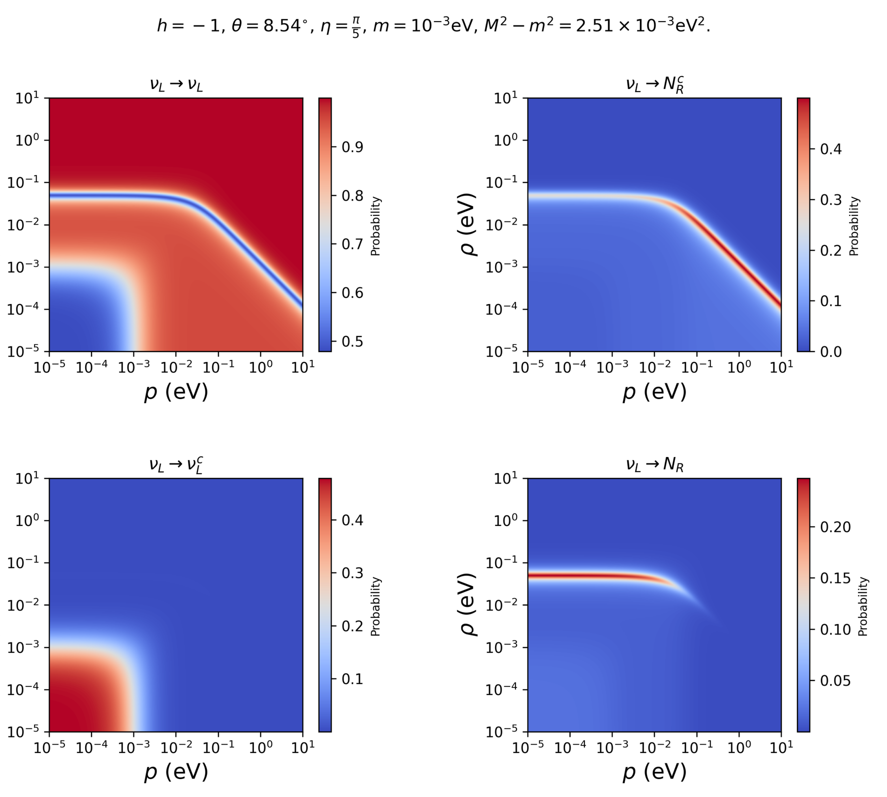
$$P(\nu_L^h \to \nu_L^h) = \frac{1}{2} + \frac{\cos^2 2\theta_{\text{eff}}}{2} - P(\nu_L^h \to (\nu_L^h)^c) \,, \quad P(\nu_L^h \to (N_R^h)^c) = \frac{\sin^2 2\theta_{\text{eff}}}{2} - P(\nu_L^h \to N_R^h) \,. \tag{26}$$

where $\cos 2\theta_{\text{eff}} = \frac{(M^2-m^2)\cos 2\theta + 2\rho(h\cdot p - \frac{\rho}{2})}{2A_2}$.

So we can see that after taking the time average the Majorana phase still have the contribution to oscillation probability in matter. By the way, it's consistent with MSW effect when we just consider the leading order in eq.(26).

It's obvious that the chiral oscillation would be significant in non-relativistic limit, such as cosmic background neutrinos. To explicit this fact, we draw the time average oscillation probability.

We can see that in low energy region, the MSW effect would be suppressed, and the matter effect would occur in chiral oscillation.



5. Conclusion

1. Once we introduce the neutrino mass, there is oscillation between left and right chiral states.
2. Chiral oscillation probability in matter would depend on the helicity.
3. Considering the matter effect, there's a chiral oscillation resonant density.
4. The chiral oscillation matter effect in Majorana case would twice as Dirac case.
5. The Majorana phase would influence the chiral oscillation probability. But after taking time average, only considering the matter effect, can Majorana phase contributes to oscillation probability.

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