Invariant Theory and Leptonic CP Violation

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Based on JHEP 09 (2021) 053, JHEP 10 (2021) 017, JHEP 08 (2022) 017, Phys.Rev.D 106 (2022) 5, L051701

In collaboration with Yilin Wang and Shun Zhou

Motivation

Physical observables can be calculated from the coupling parameters in Lagrangian using QFT

$$\mathcal{L}(c_1, c_2, \ldots) \xrightarrow{\text{QFT}} \mathcal{O}(c_1, c_2, \ldots) \xrightarrow[c_i \neq c_i']{\text{basis trans.}} \mathcal{O}(c_1', c_2', \ldots)$$

Couplings are usually *not* invariant under basis transformations (gauge trans., flavor trans., etc.), while physical observables are expected to be independent of the choice of basis.

Can we construct quantities from couplings directly that are invariant under basis transformations?

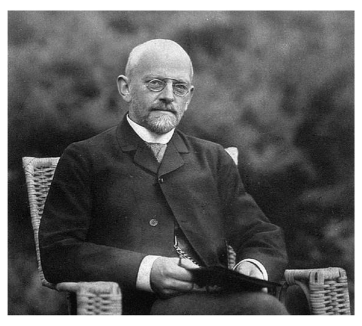
$$\mathcal{I}(c_1, c_2, \ldots) \xrightarrow[c_i \neq c_i']{\text{basis trans.}} \mathcal{I}(c_1', c_2', \ldots) = \mathcal{I}(c_1, c_2, \ldots)$$

Invariants can be used to describe physical observables in a basis-independent way.
How to construct invariants systematically, and establish their connections to physical observables?

Invariant Theory and Hilbert Series

Hilbert's fourteenth problem (David Hilbert, 1900)

Let G be a subgroup of the full linear group of the polynomial ring in indeterminates $x_1, \ldots, x_n$ over a field k, and let $\mathfrak{o}$ be the set of elements of $k[x_1, \ldots, x_n]$ which are invariant under G. Is $\mathfrak{o}$ finitely generated?



Invariants form a *polynomial ring*, which can be generated from a *finite* number of generators

$$\text{generators}: \{\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n\} \xrightarrow[\text{decomposition}]{\text{polynomial}} \forall \text{ invariant } \mathcal{I}, \quad \mathcal{I} = \mathcal{P}(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$$

Hilbert Series (HS): **generating function** of the invariant ring

$$\mathscr{H}(q) \equiv \sum_{k=1}^{\infty} c_k q^k = \frac{1 + a_1 q + \cdots + a_{l-1} q^{l-1} + a_l q^l}{\prod_{i=1}^{r} (1 - q^{d_i})}$$

q : $|q| < 1$, labeling degrees of the building blocks
c_k : # linearly-independent invariants at degree k
r : # independent physical parameters in the theory

Denominator of HS: encoding information about physical parameters of the model
Numerator of HS: encoding information about generators and syzygies of the invariant ring

How to calculate the HS: Molien-Weyl formula (Molien, 1897; Weyl, 1926; Benvenuti *et al.*, 2007; Hanany *et al.*, 2011)

$$\mathrm{PE}(z_1, \ldots z_{r_0}; q) \equiv \exp\left[\sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{\chi_{R_i}(z_1^k, \ldots, z_{r_0}^k) q^k}{k}\right] \qquad \mathscr{H}(q) = \int [\mathrm{d}\mu]_G \, \mathrm{PE}(z_1, \ldots, z_{r_0}; q)$$

$$\text{Representations of building blocks} \xrightarrow[\text{decomposition}]{\text{tensor}} \text{Plethystic Exponential (PE)} \xrightarrow[\text{formula}]{\text{Molien-Weyl}} \text{Hilbert Series}$$

Example: HS in the flavor space of the Standard Model (SM) (Jenkins & Manohar, 2009)

$$\mathscr{H}_{\text{quark}}(q) = \frac{1 + q^{12}}{(1-q^2)^2 (1-q^4)^3 (1-q^6)^4 (1-q^8)}$$

10 physical parameters: 6 quark masses, 3 mixing angles, 1 CP-violating phase

$$\mathrm{PE}^{-1}\left[\mathscr{H}_{\text{quark}}(q)\right] = 2q^2 + 3q^4 + 4q^6 + q^8 + q^{12} - q^{24}$$

11 generators in the invariant ring, 1 syzygy among the generators

Generators of the invariant ring in the SM ($H_{\rm u} \equiv M_{\rm u} M_{\rm u}^\dagger, H_{\rm d} \equiv M_{\rm d} M_{\rm d}^\dagger$)

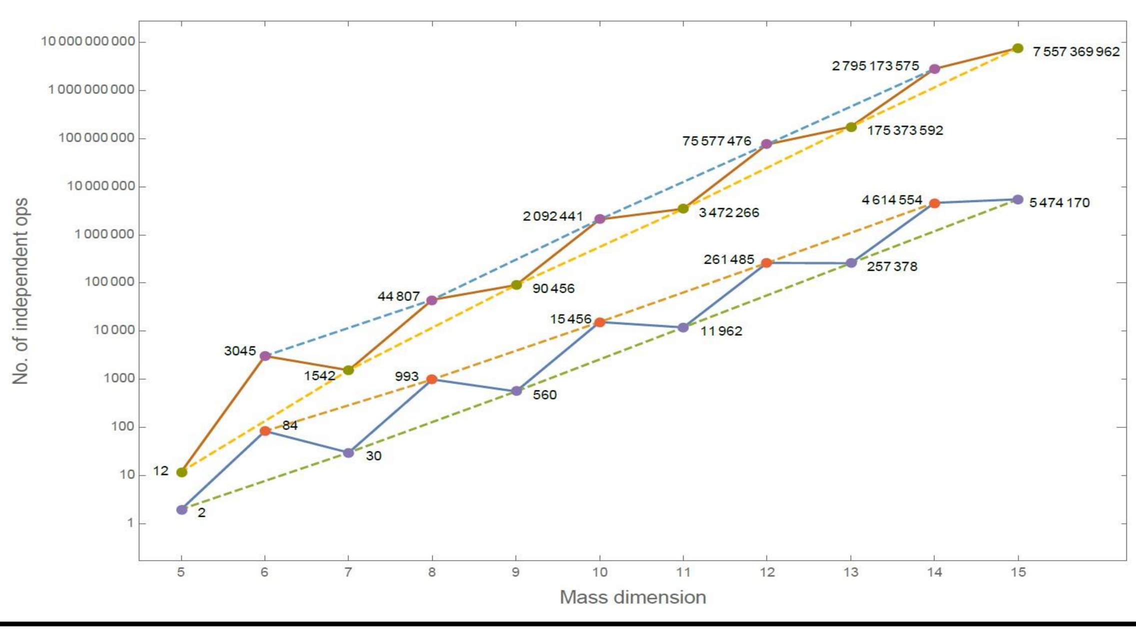
$I_{20} \equiv \mathrm{Tr}(H_{\rm u})$ $\quad I_{42} \equiv \mathrm{Tr}(H_{\rm u}^2 H_{\rm d})$
$I_{02} \equiv \mathrm{Tr}(H_{\rm d})$ $\quad I_{24} \equiv \mathrm{Tr}(H_{\rm u} H_{\rm d}^2)$
$I_{40} \equiv \mathrm{Tr}(H_{\rm u}^2)$ $\quad I_{06} \equiv \mathrm{Tr}(H_{\rm d}^3)$
$I_{22} \equiv \mathrm{Tr}(H_{\rm u} H_{\rm d})$ $\quad I_{44} \equiv \mathrm{Tr}(H_{\rm u}^2 H_{\rm d}^2)$
$I_{04} \equiv \mathrm{Tr}(H_{\rm d}^2)$ $\quad I_{66} \equiv \mathrm{Im}\ \mathrm{Tr}(H_{\rm u}^2 H_{\rm d}^2 H_{\rm u} H_{\rm d})$
$I_{60} \equiv \mathrm{Tr}(H_{\rm u}^3)$ $\quad = \mathrm{Det}\,[H_{\rm u}, H_{\rm d}]/(2\mathrm{i})$

10 CP-even, 1 CP-odd

- All physical parameters in the quark sector can be extracted from invariants
- Only source of CP violation in the SM (Jarlskog) $\propto I_{66}$
- Syzygy relation $I_{66}^2 = P(I_{20}, I_{02}, \ldots, I_{44})$ *Calculating magnitude of CP violation from CP-conserving quantities*

Applications of Invariant Theory in Particle Physics

- Counting gauge-invariant chiral operators in supersymmetric gauge field theories (Pouliot, 1999; Romelsberger, 2006; Benvenuti *et al.*, 2007)
- Study flavor structure and CP violation in quark and leptonic sector (Jenkins & Manohar, 2009)
- Counting high-dimensional operators in EFTs (Lehman *et al.*, 2015; Henning *et al.*, 2017)
- Calculating high-loop Feynman integrals (Larsen & Zhang, 2015)
- Construction of scalar potential in multi-Higgs-doublet models (Trautner, 2018)
- Study CP violation in SMEFT (Bonnefoy *et al.*, 2022; Murayama *et al.*, 2023)



Henning *et al.*, JHEP 08 (2017) 016

Number of high-dimensional effective operators calculated using the invariant theory

Leptonic Invariant Ring with Majorana Neutrinos

Relevant Lagrangian in the leptonic sector

$$\mathcal{L}_{\text{lepton}} = -\overline{l_{\rm L}} M_l l_{\rm R} - \frac{1}{2} \overline{\nu_{\rm L}} M_\nu \nu_{\rm L}^{\rm c} + \frac{g}{\sqrt{2}} \overline{l_{\rm L}} \gamma^\mu \nu_{\rm L} W_\mu^- + \text{h.c.}$$

Building blocks to construct invariants: charged-lepton mass matrix M_l (adjoint rep. of U(3) group) and Majorana neutrino mass matrix M_ν (rank-two symmetric tensor rep. of U(3) group)

Calculating the HS using Molien-Weyl formula:

$$\mathscr{H}_{\text{lepton}}(q) = \frac{1 + q^6 + 2q^8 + 4q^{10} + 8q^{12} + 7q^{14} + 9q^{16} + 10q^{18} + 9q^{20} + 7q^{22} + 8q^{24} + 4q^{26} + 2q^{28} + q^{30} + q^{36}}{(1-q^2)^2 (1-q^4)^3 (1-q^6)^4 (1-q^8)^2 (1-q^{10})}$$

$$\mathrm{PE}^{-1}\left[\mathscr{H}_{\text{lepton}}(q)\right] = 2q^2 + 3q^4 + 5q^6 + 4q^8 + 5q^{10} + 7q^{12} + 5q^{14} + 2q^{16} - \mathcal{O}(q^{18})$$

12 factors $\Longleftrightarrow$ 12 physical parameters: 3 charged-lepton masses, 3 neutrino masses, 3 flavor mixing angles, 3 CP-violating phases

Constructing all generators of the invariant ring in the leptonic sector:

flavor invariants	(q_l, q_ν)	$q_l + q_\nu$	CP parity
$I_1 \equiv \mathrm{Tr}(H_l)$	$(2,0)$	2	+
$I_2 \equiv \mathrm{Tr}(H_\nu)$	$(0,2)$	2	+
$I_3 \equiv \mathrm{Tr}(H_l^2)$	$(4,0)$	4	+
$I_4 \equiv \mathrm{Tr}(H_l H_\nu)$	$(2,2)$	4	+
$I_5 \equiv \mathrm{Tr}(H_\nu^2)$	$(0,4)$	4	+
$I_6 \equiv \mathrm{Tr}(H_l^3)$	$(6,0)$	6	+
$I_7 \equiv \mathrm{Tr}(H_l^2 H_\nu)$	$(4,2)$	6	+
$I_8 \equiv \mathrm{Tr}(H_l G_{l\nu})$	$(4,2)$	6	+
$I_9 \equiv \mathrm{Tr}(H_l H_\nu^2)$	$(2,4)$	6	+
$I_{10} \equiv \mathrm{Tr}(H_\nu^3)$	$(0,6)$	6	+
$I_{11} \equiv \mathrm{Tr}(H_l^2 G_{l\nu})$	$(6,2)$	8	+
$I_{12} \equiv \mathrm{Tr}(\{H_l, H_\nu\} G_{l\nu})$	$(4,4)$	8	+
$I_{13} \equiv \mathrm{Tr}([H_l, H_\nu] G_{l\nu})$	$(4,4)$	8	−
$I_{14} \equiv \mathrm{Tr}(H_l^2 H_\nu^2)$	$(4,4)$	8	+
$I_{15} \equiv \mathrm{Tr}\left(H_l^2 G_{l\nu}^{(2)}\right)$	$(8,2)$	10	+
$I_{16} \equiv \mathrm{Tr}(\{H_l^2, H_\nu\} G_{l\nu})$	$(6,4)$	10	+
$I_{17} \equiv \mathrm{Tr}([H_l^2, H_\nu] G_{l\nu})$	$(6,4)$	10	−

flavor invariants	(q_l, q_ν)	$q_l + q_\nu$	CP parity
$I_{18} \equiv \mathrm{Tr}(\{H_l, H_\nu^2\} G_{l\nu})$	$(4,6)$	10	+
$I_{19} \equiv \mathrm{Tr}([H_l, H_\nu^2] G_{l\nu})$	$(4,6)$	10	−
$I_{20} \equiv \mathrm{Tr}\left(\{H_l^2, H_\nu\} G_{l\nu}^{(2)}\right)$	$(8,4)$	12	+
$I_{21} \equiv \mathrm{Tr}\left([H_l^2, H_\nu] G_{l\nu}^{(2)}\right)$	$(8,4)$	12	−
$I_{22} \equiv \mathrm{Tr}(H_l^2 H_\nu H_l G_{l\nu}) - \mathrm{Tr}(H_l^2 G_{l\nu} H_l H_\nu)$	$(8,4)$	12	−
$I_{23} \equiv \mathrm{Tr}\left(\{H_l, H_\nu^2\} G_{l\nu}^{(2)}\right)$	$(6,6)$	12	+
$I_{24} \equiv \mathrm{Tr}\left([H_l, H_\nu^2] G_{l\nu}^{(2)}\right)$	$(6,6)$	12	−
$I_{25} \equiv \mathrm{Tr}(H_l^2 H_\nu^2 H_l H_\nu) - \mathrm{Tr}(H_l^2 H_\nu H_l H_\nu^2)$	$(6,6)$	12	−
$I_{26} \equiv \mathrm{Tr}(H_l H_\nu^2 G_{l\nu} H_\nu) - \mathrm{Tr}(H_l H_\nu G_{l\nu} H_\nu^2)$	$(4,8)$	12	−
$I_{27} \equiv \mathrm{Tr}\left(H_l^2 H_\nu H_l G_{l\nu}^{(2)}\right) - \mathrm{Tr}\left(H_l^2 G_{l\nu}^{(2)} H_l H_\nu\right)$	$(10,4)$	14	−
$I_{28} \equiv \mathrm{Tr}(\{H_l^2, H_\nu\} G_{l\nu}^2)$	$(8,6)$	14	+
$I_{29} \equiv \mathrm{Tr}([H_l^2, H_\nu] G_{l\nu}^2)$	$(8,6)$	14	−
$I_{30} \equiv \mathrm{Tr}(H_l^2 H_\nu^2 H_l G_{l\nu}) - \mathrm{Tr}(H_l^2 G_{l\nu} H_l H_\nu^2)$	$(8,6)$	14	−
$I_{31} \equiv \mathrm{Tr}(H_l^2 H_\nu^2 G_{l\nu} H_\nu) - \mathrm{Tr}(H_l^2 H_\nu G_{l\nu} H_\nu^2)$	$(6,8)$	14	−
$I_{32} \equiv \mathrm{Tr}\left(H_l^2 G_{l\nu} H_l G_{l\nu}^{(2)}\right) - \mathrm{Tr}\left(H_l^2 G_{l\nu}^{(2)} H_l G_{l\nu}\right)$	$(12,4)$	16	−
$I_{33} \equiv \mathrm{Tr}(H_l^2 H_\nu H_l G_{l\nu}^2) - \mathrm{Tr}(H_l^2 G_{l\nu}^2 H_l H_\nu)$	$(10,6)$	16	−
$I_{34} \equiv \mathrm{Tr}(H_l^2 H_\nu^2 G_{l\nu}^2) - \mathrm{Tr}(H_l^2 G_{l\nu}^2 H_\nu^2)$	$(8,8)$	16	−

$H_l \equiv M_l M_l^\dagger$
$H_\nu \equiv M_\nu M_\nu^\dagger$
$G_{l\nu} \equiv M_\nu H_l^* M_\nu^\dagger$
$G_{l\nu}^{(2)} \equiv M_\nu (H_l^*)^2 M_\nu^\dagger$

34 generators
19 CP-even
15 CP-odd

12 algebraically-independent invariants (primary invariants)

$$\{\underbrace{I_1, I_2, I_3, I_4, I_5, I_7, I_9, I_{10}, I_{14}}_{\text{CP-even}}, \underbrace{I_{13}, I_{19}, I_{26}}_{\text{CP-odd}}\}$$

All physical parameters in the leptonic sector can be extracted from primary invariants

extract

$$\{m_1, m_2, m_3, m_e, m_\mu, m_\tau, \theta_{12}, \theta_{13}, \theta_{23}, \delta, \rho, \sigma\}$$

Y. Wang, **BY**, S. Zhou, JHEP 09 (2021) 053

Conditions for CP conservation in the Leptonic Sector

Phases in the PMNS matrix describe CP violation in the leptonic sector:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{\mathrm{i}\delta} & +c_{12}c_{23} - s_{12}s_{13}s_{23}e^{\mathrm{i}\delta} & c_{13}s_{23} \\ +s_{12}s_{23} - c_{12}s_{13}c_{23}e^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{\mathrm{i}\delta} & c_{13}c_{23} \end{pmatrix} \cdot \begin{pmatrix} e^{\mathrm{i}\rho} & 0 & 0 \\ 0 & e^{\mathrm{i}\sigma} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

CP conservation $\Longleftrightarrow$ All phases take trivial values $\Longleftrightarrow$ $\delta = 0$ or π and $\rho, \sigma = 0$ or $\pi/2$

Can one construct the necessary and sufficient conditions for CP conservation *without* diagonalizing the lepton mass matrices? Yes!

$$\boxed{\begin{aligned} \mathcal{I}_1 &\equiv \mathrm{Im}\ \mathrm{Tr}\left(H_l^2 H_\nu^2 H_l H_\nu\right) = 0 \\ \mathcal{I}_2 &\equiv \mathrm{Im}\ \mathrm{Tr}\left(H_l H_\nu G_{l\nu}\right) = 0 \\ \mathcal{I}_3 &\equiv \mathrm{Im}\ \mathrm{Tr}\left(H_l H_\nu^2 G_{l\nu}\right) = 0 \end{aligned}}$$

BY, S. Zhou, PLB 800 (2020) 135085

$$\mathcal{I}_1 = \Delta_{\mu e}\Delta_{\tau\mu}\Delta_{\tau e}\Delta_{21}\Delta_{31}\Delta_{32} s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^2 \sin\delta = 0$$
$$\mathcal{I}_2 = f_1 \sin 2\rho + f_2 \sin 2\sigma + f_3 \sin(2\rho - 2\sigma) = 0$$
$$\mathcal{I}_3 = h_1 \sin 2\rho + h_2 \sin 2\sigma + h_3 \sin(2\rho - 2\sigma) = 0$$

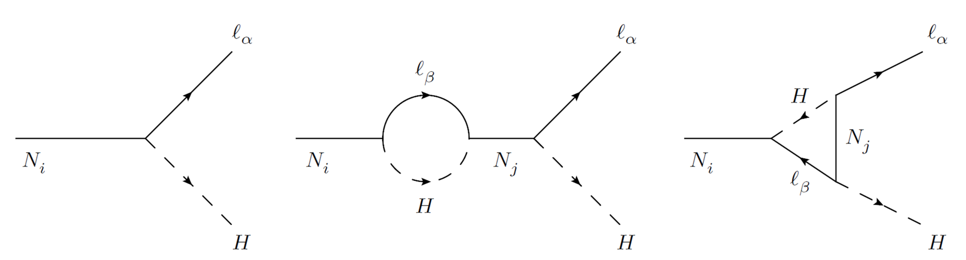
$$\Delta_{\alpha\beta} \equiv m_\alpha^2 - m_\beta^2 \qquad \Delta_{ij} \equiv m_i^2 - m_j^2$$

f_i and h_i are only functions of masses and flavor mixing angles

These three CP-odd invariants are exactly I_{25}, I_{13}, and I_{19} in the generating set above !

Invariants and Leptogenesis

Leptogenesis provides an elegant explanation for cosmological matter-antimatter asymmetry



$$\mathcal{L}_{\text{seesaw}} = -\overline{\ell_{\rm L}} Y_l H l_{\rm R} - \overline{\ell_{\rm L}} Y_\nu \widetilde{H} N_{\rm R} - \frac{1}{2} \overline{N_{\rm R}^c} M_{\rm R} N_{\rm R} + \text{h.c.}\ .$$

Fukugita & Yanagida, 1986; Luty, 1992

CP violation in leptogenesis comes from phases in Yukawa couplings, i.e., Y_ν

$$\epsilon_1 \equiv \frac{\sum_\alpha \left[\Gamma(N_1 \to \ell_\alpha + H) - \Gamma\left(N_1 \to \overline{\ell_\alpha} + \overline{H}\right)\right]}{\sum_\alpha \left[\Gamma(N_1 \to \ell_\alpha + H) + \Gamma\left(N_1 \to \overline{\ell_\alpha} + \overline{H}\right)\right]} = -\frac{3}{16\pi \left(Y_\nu^\dagger Y_\nu\right)_{11}} \sum_{j \neq 1} \left(\frac{M_1}{M_j}\right) \mathrm{Im}\left[\left(Y_\nu^\dagger Y_\nu\right)_{1j}^2\right] \qquad M_1 \ll M_j$$

CP violation in leptogenesis can be well described by CP-odd invariants

$$\epsilon_1 = \frac{3\mathrm{i}}{16\pi v^2} \frac{I_{044}}{I_{002}\left(I_{022} - I_{002} I_{020}\right)}$$

explicitly independent of the flavor basis

BY, S. Zhou, JHEP 10 (2021) 017

Through the *matching procedure of invariants*, CP violation in leptogenesis can be connected to CP violation at low energies

Rational functions of CP-even invariants

$$\underbrace{\epsilon_1}_{\text{CP violation in leptogenesis}} = \mathcal{R}_1\left[\mathcal{I}_{\text{even}}\right] \underbrace{\mathcal{I}_{121}^{(2)}}_{\text{CP violation in neutrino oscillation}} + \mathcal{R}_2\left[\mathcal{I}_{\text{even}}\right] \underbrace{\mathcal{I}_{240}}_{\text{CP violation in neutrino-antineutrino oscillation}}$$

BY, S. Zhou, JHEP 08 (2022) 017; Phys.Rev.D 106 (2022) 5, L051701

Summary

- Physical observables are independent of the choice of basis. It is helpful and more natural to describe physical observables in terms of invariants.
- Invariant theory provides a systematic way to construct the whole invariant ring, once the symmetries in the model are given.
- Invariants are useful to describe CP violation. They also establish the connections between observables at high and low energies, in a basis- and parametrization-independent way.