Feynman Integrals and Geometry

↪*equal-mass banana integrals to all loops*

Xing Wang (王 星), Technische Universität München

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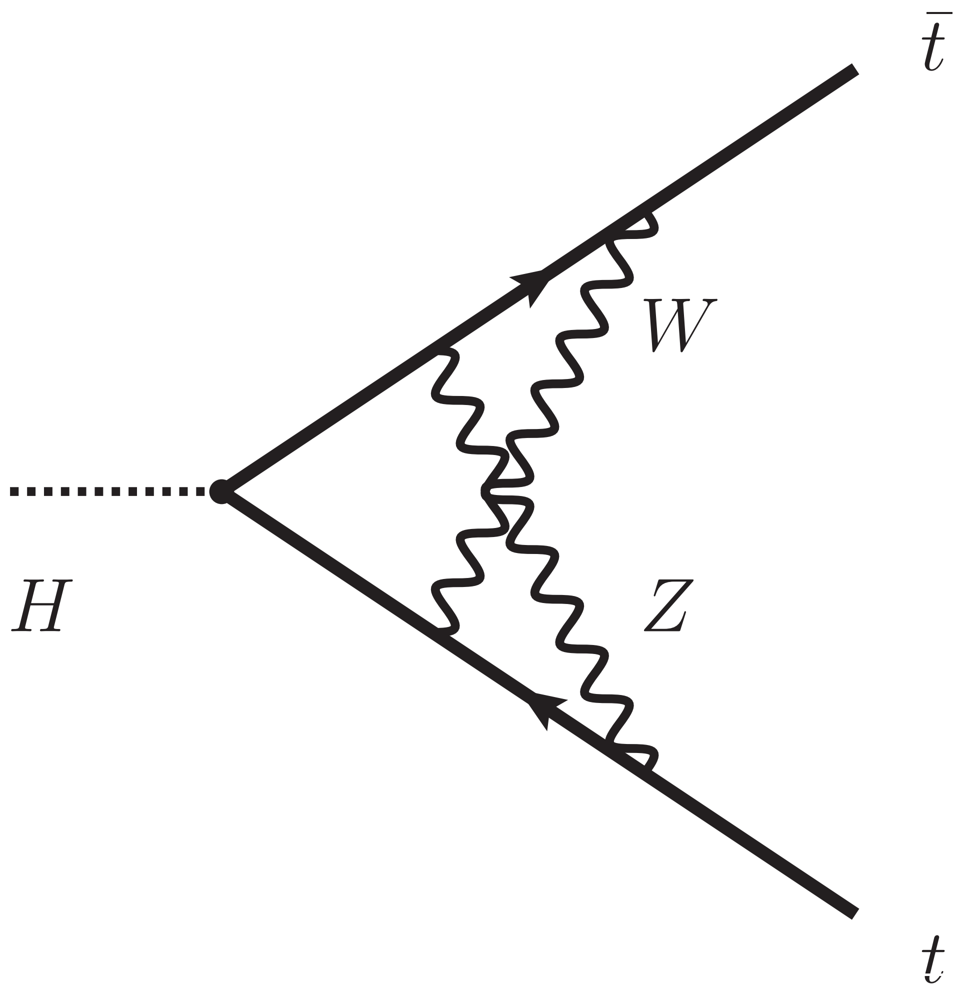
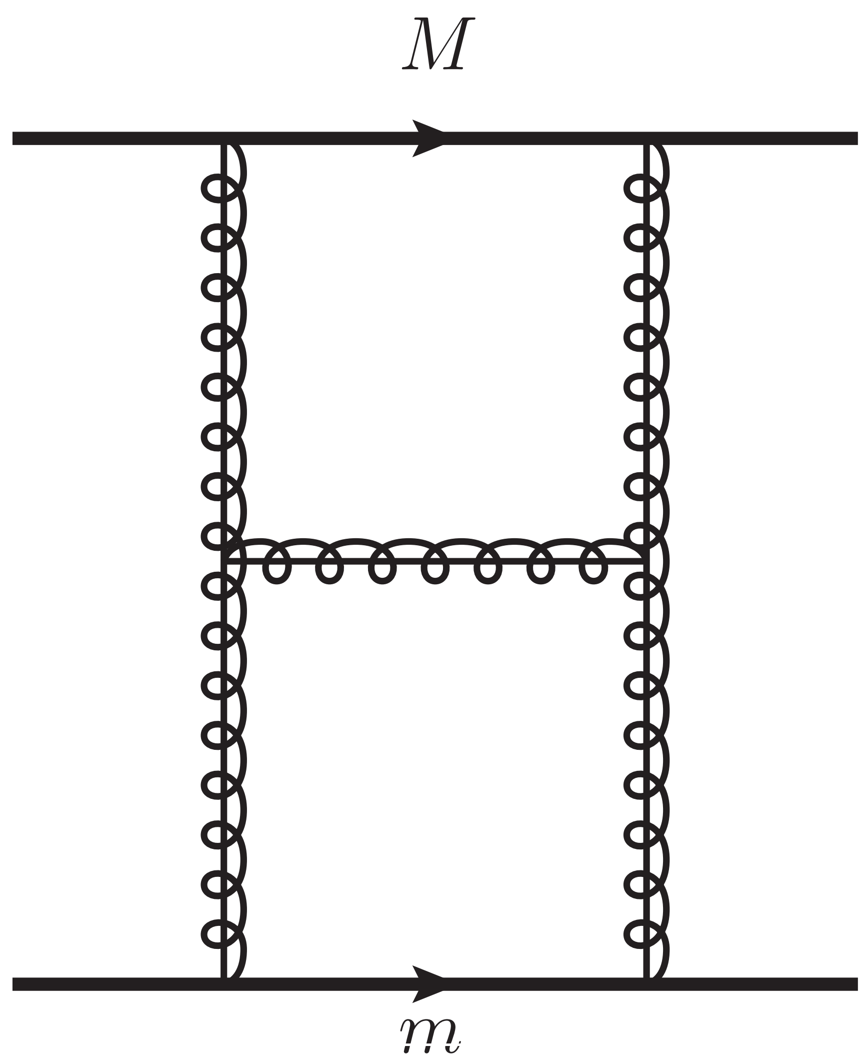
based on 2207.12893, 2211.04292 and 2212.08908, with **Sebastian Pögel** and **Stefan Weinzierl**

Why do we care about Feynman integrals*?



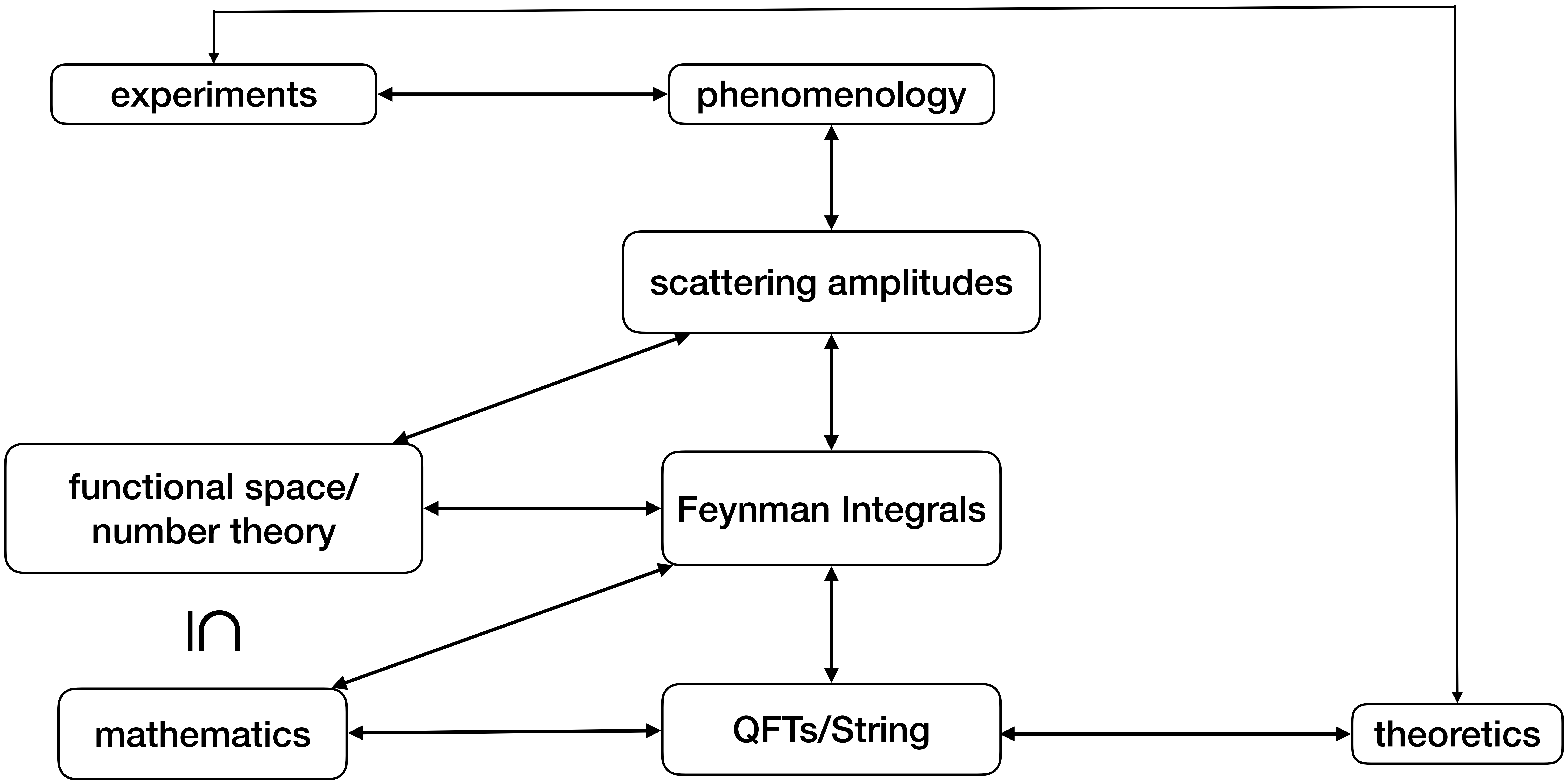
Higgs physics



Binary system of black holes

*Feynman integrals don't have to do with diagrams. Here it is for convenience.

Why do we care about Feynman integrals?



Why do we push to such higher loops?

- Required by (more) precision phenomenology (?)
- Apart from experimentally excited datas (more precise datas or new particles...), we still need to accumulate theoretical datas

➡ theoretical "lab" for more general FIs/amplitudes:

✓ banana family is one of the hydrogens of FIs, like supersymmetric Yang-Mills to QCD

➡ a probe for new tools and methods

✓ two-loop banana family is the simplest FI beyond multiple polylogarithms (**MPL**);

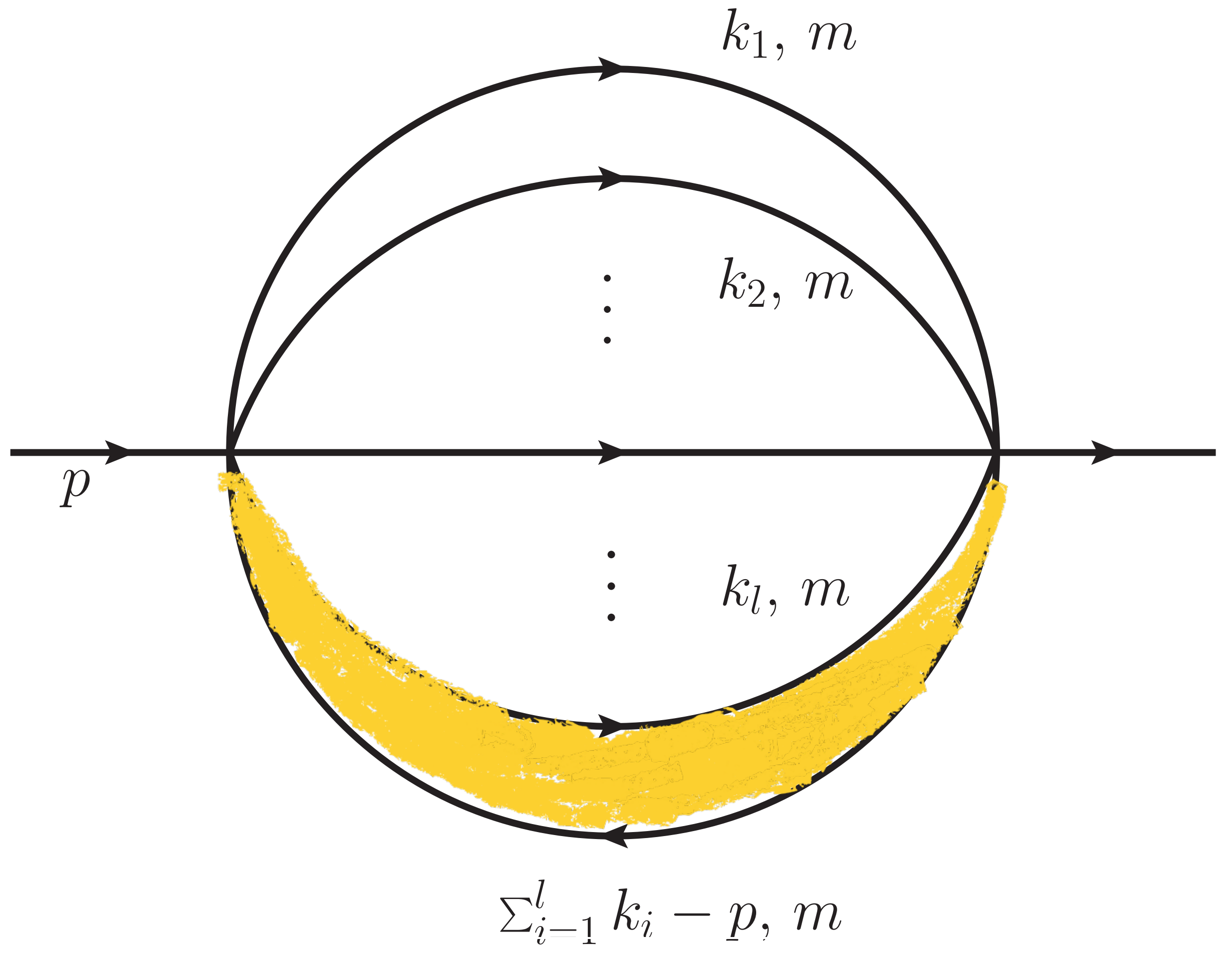
✓ three-loop banana case is the simplest FI beyond elliptic multiple polylogarithms (**eMPL**);

✓ four-loop banana case is the simplest FI involving a general Calabi-Yau manifold;

✓ geometry-inspired methods and tools apply to more general FIs!

What is the (equal-mass) banana family?

$$F^{(l)}_{\vec{\nu}}(d = 2 - 2\varepsilon, y = -m^2/p^2) :=$$

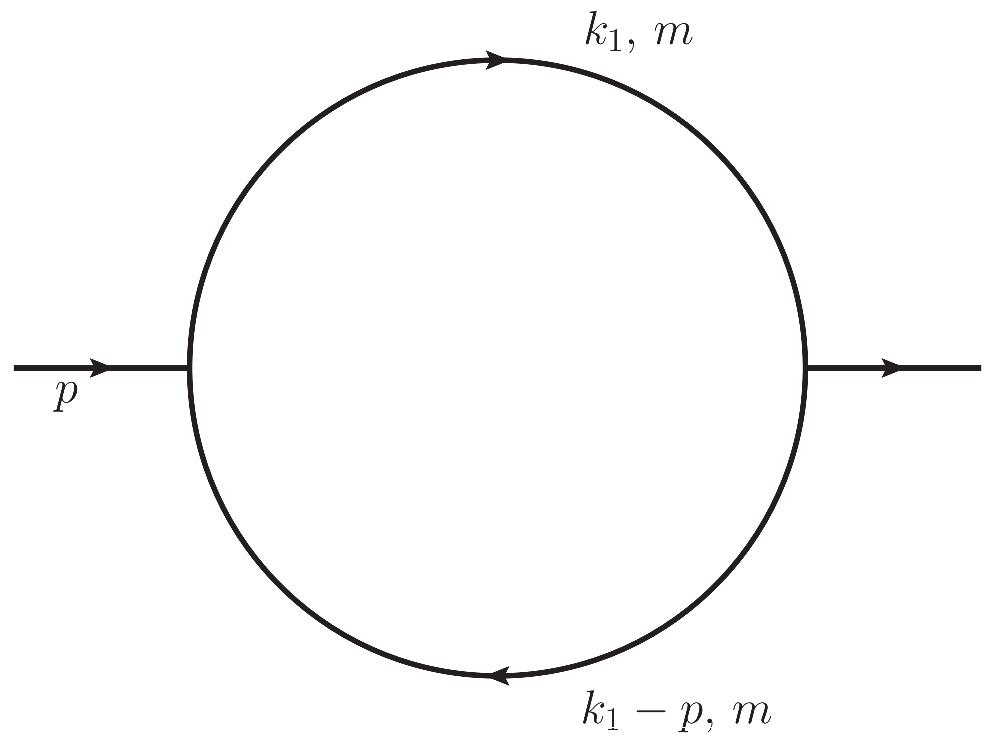


$$|\vec{\nu}| = \nu_1 + \nu_2 + \cdots \nu_{l+1}$$

$$\rotatebox{90}{$=$}$$

$$e^{l\varepsilon\gamma_E} \cdot (m^2)^{|\vec{\nu}| - ld/2} \cdot \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \cdots \frac{\mathrm{d}^d k_l}{i\pi^{d/2}} \frac{1}{\left[k_1^2 - m^2\right]^{\nu_1} \cdots \left[k_2^2 - m^2\right]^{\nu_l} \left[(k_1 + \cdots k_l - p)^2 - m^2\right]^{\nu_{l+1}}}$$

Warm-up: one-loop banana



$$F^{(1)}_{\nu_1,\nu_2}(2-2\varepsilon,y) = e^{\varepsilon\gamma_E}\cdot(m^2)^{|\vec{\nu}|-\frac{d}{2}}\cdot\int\frac{\mathrm{d}^dk_1}{i\pi^{d/2}}\frac{1}{\left[k_1^2-m^2\right]^{\nu_1}\left[(k_1-p)^2-m^2\right]^{\nu_2}}$$

✓ The vector basis has two independent components: $F^{(1)}_{1,0}$ and $F^{(1)}_{1,1}$:

$$\mathrm{e.g}:\ (4+1/y)\cdot F^{(1)}_{21} = (1+2\varepsilon)\cdot F^{(1)}_{11} + \varepsilon F^{(1)}_{10}\,.$$

✓ Dependence on $y = -m^2/p^2$ is controlled by the DE for the two MIs:

$$\frac{d}{dy}\begin{pmatrix}F^{(1)}_{10}\\ F^{(1)}_{11}\end{pmatrix} = \begin{pmatrix}0 & 0\\ \frac{2\varepsilon}{1+4y} & -\frac{1+\varepsilon+2y}{y(1+4y)}\end{pmatrix}\begin{pmatrix}F^{(1)}_{10}\\ F^{(1)}_{11}\end{pmatrix}$$

Warm-up: one-loop banana

$$\begin{aligned} I_0 &= \varepsilon^2 F_{10}^{(1)}, \\ I_1 &= \varepsilon^2 \frac{F_{11}^{(1)}}{\psi_0} \end{aligned} \longrightarrow \frac{d}{dy}\begin{pmatrix} I_0 \\ I_1 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 \\ \frac{2}{y\sqrt{1+4y}} & \frac{1}{y(1+4y)} \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \end{pmatrix}$$

$$\psi_0 = \frac{y}{\sqrt{1+4y}}, \qquad y = \frac{q}{(1-q)^2} \left(q = \frac{\sqrt{1+4y}-1}{\sqrt{1+4y}+1} \right).$$

$$\sqrt{1+4y} = \frac{1+q}{1-q}.$$

Toy model for ε-form

$$\frac{d}{dy}\begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} = \varepsilon \begin{pmatrix} \frac{1}{y} & 0 \\ \frac{1}{y} & \frac{1}{1-y} \end{pmatrix} \begin{pmatrix} f_0(y) \\ f_1(y) \end{pmatrix} \quad \text{with boundary} \quad f_0\,|_{y\to 0} = y^{\varepsilon}, f_1\,|_{y\to 0} = y^{\varepsilon} - \frac{\Gamma(1+\varepsilon)^2}{\Gamma(1+2\varepsilon)}$$

$$\left(f_1(y) = \frac{\varepsilon(y-1)}{1+\varepsilon}\, {}_2F_1(1-\varepsilon, 1+\varepsilon; 2+\varepsilon; y)\,. \right)$$

$$\begin{aligned} f_0(y) &= \sum_{n\geq 0} \varepsilon^n f_0^{(n)}(y) \\ f_1(y) &= \sum_{n\geq 0} \varepsilon^n f_1^{(n)}(y) \end{aligned} \longrightarrow \frac{d}{dy}\begin{pmatrix} f_0^{(n)}(y) \\ f_1^{(n)}(y) \end{pmatrix} = \begin{pmatrix} \frac{1}{y} & 0 \\ \frac{1}{y} & \frac{1}{1-y} \end{pmatrix} \begin{pmatrix} f_0^{(n-1)}(y) \\ f_1^{(n-1)}(y) \end{pmatrix}$$

$$\frac{d}{dy}\begin{pmatrix} f_0^{(0)}(y) \\ f_1^{(0)}(y) \end{pmatrix} = 0 \longrightarrow \begin{pmatrix} f_0^{(0)}(y) \\ f_1^{(0)}(y) \end{pmatrix} = \begin{pmatrix} f_0^{(0)}(0) \\ f_1^{(0)}(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$\frac{d}{dy}\begin{pmatrix} f_0^{(1)}(y) \\ f_1^{(1)}(y) \end{pmatrix} = \begin{pmatrix} \frac{1}{y} \\ \frac{1}{y} \end{pmatrix} \longrightarrow \begin{pmatrix} f_0^{(1)}(y) \\ f_1^{(1)}(y) \end{pmatrix} = \begin{pmatrix} \ln y \\ \ln y \end{pmatrix}.$$

Toy model for ε-form and iterated integrals

$$f_1^{(n)}(y) = \int_0^y dy' \left[\frac{1}{y'} f_0^{(n-1)}(y') + \frac{1}{1-y'} f_1^{(n-1)}(y')\right] + f_1^{(n)}(y)\,|_{y\to 0}$$

$$\begin{aligned} I(l_1, l_2, \cdots, l_n; z_0, z) &= \int_{z_0}^{z} dz_1\, l_1(z_1)\, I(l_2, \cdots, l_n; z_0, z_1) \\ &= \int_{z_0}^{z} dz_1\, l_1\left(z_1\right) \int_{z_0}^{z_1} dz_2\, l_2\left(z_2\right) \cdots \int_{z_0}^{z_{n-1}} dz_n\, l_n\left(z_n\right). \end{aligned}$$

✓ Iterated integrals have some nice algebraic and geometric properties;

✓ They suit perfectly as solutions of differential equations.

Take-home message

- Differential equations are very powerful to calculate Feynman integrals. Once ε form is obtained, the problem is fairly solved;
- The ε form is achieved by variable change + rotation of the basis.

What follows

- Essential ingredients to ε form is fixed by geometry;
- The methodology holds to all loops of banana integrals!

Geometry behind l-loop equal-mass banana

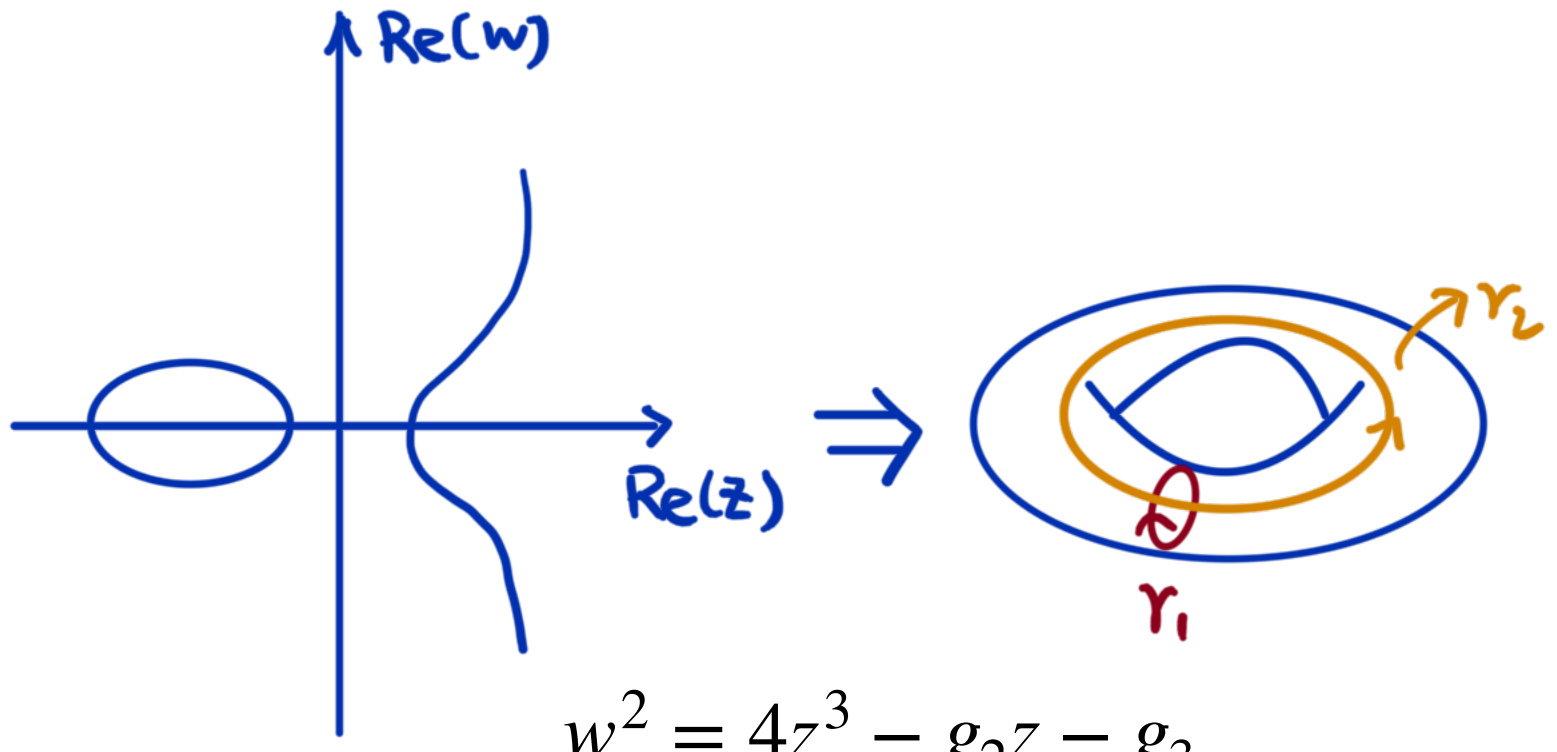
$$F^{(l)}_{\underbrace{111\cdots1}_{l+1}} = e^{l\varepsilon\gamma_E}\cdot\Gamma(1+l\varepsilon)\cdot\int\limits_{\alpha_i\geq0} d^{l+1}\alpha\,\delta\left(1-\sum_{i=1}^{l+1}\alpha_i\right)\frac{\mathcal{U}(\alpha)^{(l+1)\varepsilon}}{\mathcal{F}(\alpha)^{1+l\varepsilon}}$$

$$\mathcal{U}(\alpha) = \left(\prod_i^{l+1}\alpha_i\right)\sum_j^{l+1}\frac{1}{\alpha_j},\qquad \mathcal{F}(\alpha) = -\frac{1}{y}\left(\prod_i^{l+1}\alpha_i\right)+\mathcal{U}(\alpha)\sum_i^{l+1}\alpha_i\,.$$

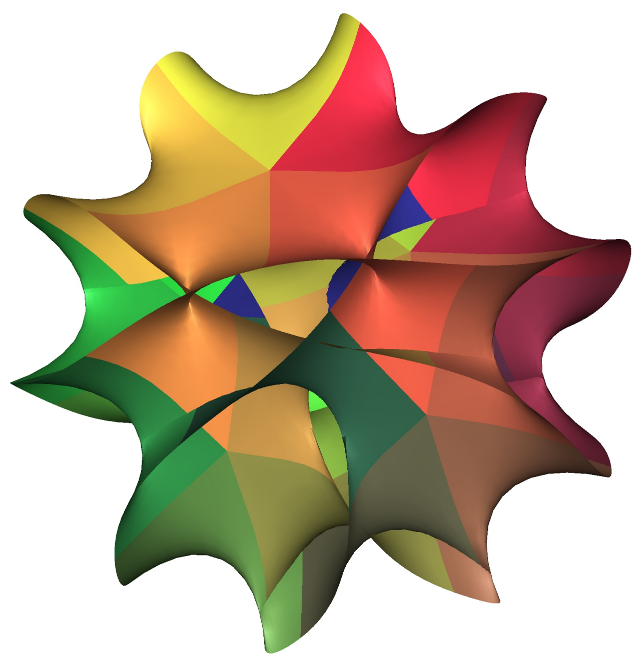
The hypersurface, as variety from the second graph polynomial $X = \left\{[\alpha_1:\alpha_2:\cdots:\alpha_{l+1}]\in\mathbb{CP}^l \mid \mathcal{F}(\alpha)=0\right\}$,

is smooth and defined as Calabi-Yau $(l-1)$-folds.

- two-loop: Calabi-Yau 1-fold $\longrightarrow$ an elliptic curve
- three-loop: Calabi-Yau 2-fold $\longrightarrow$ a K3 surface
- four-loop: Calabi-Yau 3-fold



$$w^2 = 4z^3 - g_2 z - g_3$$



Geometry behind l-loop equal-mass banana

- A Calabi-Yau manifold, M, of complex dimension n is a compact Kähler manifold with vanishing Ricci curvature. It is uniquely characterised by a triple:

define complex structure $\quad (M, \boxed{\Omega}, \boxed{\omega}) \quad$ Kähler form: metric

- Example: $(\mathscr{E}, dz/w, dz \wedge dw)$;
- For a Calabi-Yau manifold M, there exists a mirror manifold W, whose complex structure and Kähler structure are exchanged.

For our purpose, it is defined by the hypersurface from the homogeneous (second) graph polynomials $\mathscr{F}(\alpha_1, \alpha_2, \cdots, \alpha_{l+1})$.

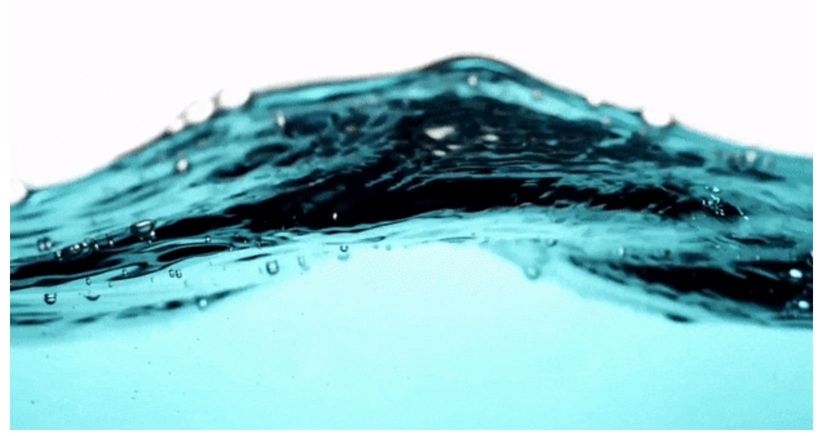
Basic concepts of (banana) Feynman integrals

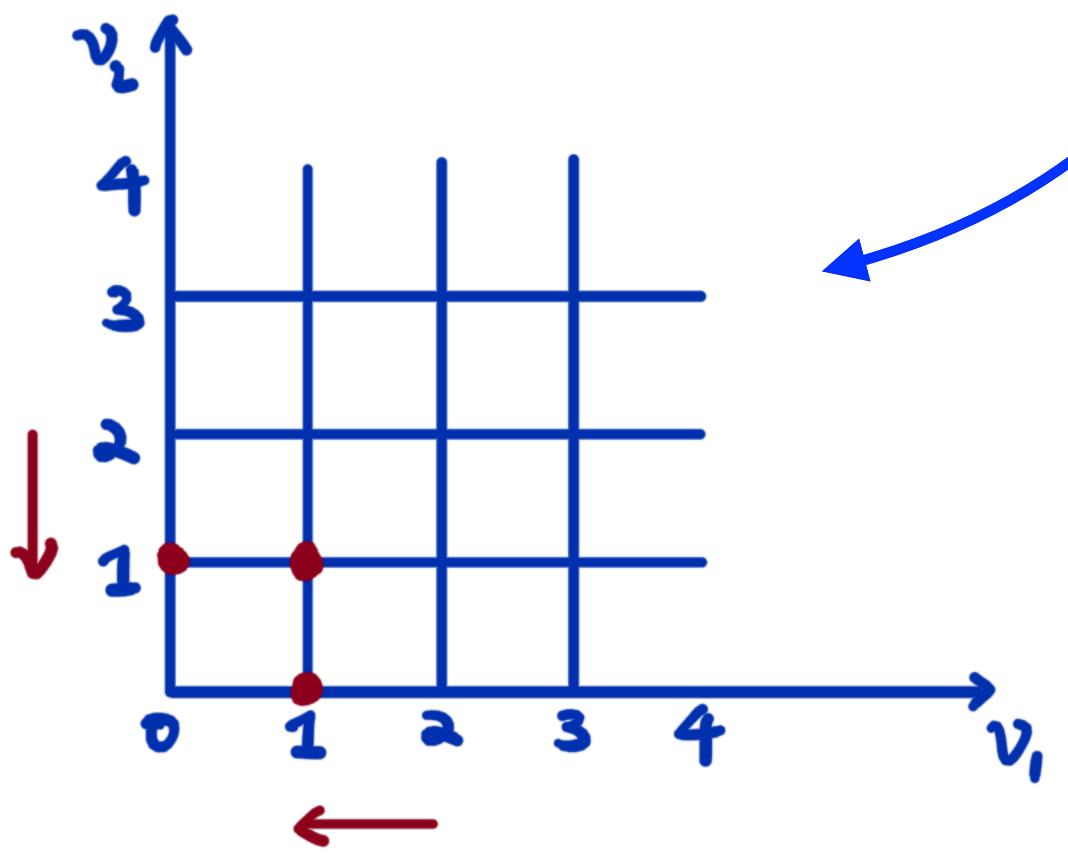
- Dependence in ε is at most meromorphic;
- Around $d = 2$ dimensions, no UV/IR divs;
- Dimension shift relations.

- Dependence in y is the most complicated, without any all-order claim;
- The dependence is controlled by the **differential equation** w.r.t. y;
- Normally, DE is built with MIs, i.e., projected upon a basis.

$$\frac{d\vec{F}^{(l)}}{dy} = \mathbf{A}(\varepsilon, y)\, \vec{F}^{(l)}$$

$$F^{(l)}_{\boxed{\vec{\nu}}}\left(\boxed{d = 2 - 2\varepsilon}, \boxed{y = -m^2/p^2}\right)$$





- Integration by parts (IBP): given a family, there are only finite number of integrals, called ***master integrals (MIs)***, constituting a basis in the lattice space;
- Intersection theory, GKZ, Griffiths reduction method, etc.

See 2201.03593 for a review of these aspects and references therein.

Basic concepts of (banana) Feynman integrals

$$\frac{d\vec{F}^{(l)}(\varepsilon,y)}{dy} = \mathbf{A}(\varepsilon,y)\vec{F}^{(l)}(\varepsilon,y)$$

rotate the basis (gauge transformation)

$$\vec{F}' = \mathbf{U}\vec{F}$$

$$\downarrow$$

$$\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} + \mathbf{U}\frac{d}{dy}\mathbf{U}^{-1}$$

variable change (coordinate transformation)

$$y' = \eta(y)$$

$$\downarrow$$

$$\mathbf{A}' = \mathbf{A}\frac{dy}{dy'}$$

$$\frac{d\vec{F}^{(l)}(\varepsilon,y)}{dy} = \varepsilon\,\mathbf{A}(y)\vec{F}^{(l)}(\varepsilon,y)$$

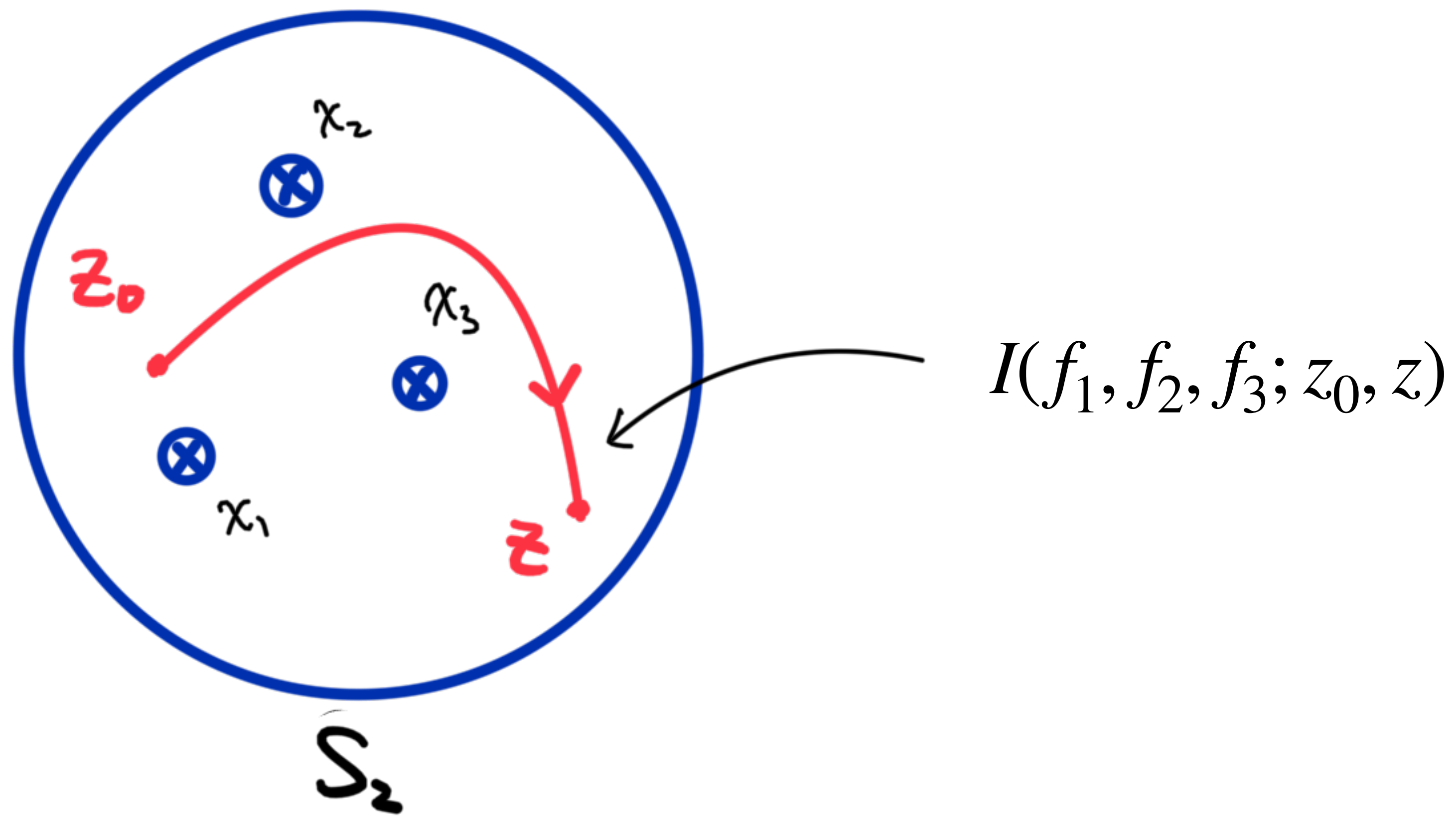
ε-form

$$\vec{F}^{(l)}(\varepsilon,y) = \sum_n \varepsilon^n \vec{F}^{(l),n}(y) \longrightarrow \frac{d\vec{F}^{(l),n}(y)}{dy} = \mathbf{A}(y)\vec{F}^{(l),n-1}(y)$$

[Remiddi '97; Gehrmann and Remiddi '00; Henn '13] + so many applications: a revolution in precision era

Multiple polylogarithms (Generalised polylogarithm)

- ✓ MPLs are iterated integrals with all letters a simple rational function: $f_i(z) = \dfrac{a_i}{z - x_i}$;
- ✓ Geometrically, they can be seen as iterated integrals on a (punctured) Riemann sphere.



$$G(;z) = 1,$$

$$G\left(x_1, x_2 \ldots, x_n; z\right) = \int_0^z \frac{dz_1}{z_1 - x_1} G\left(x_2 \ldots, x_n; z_1\right).$$

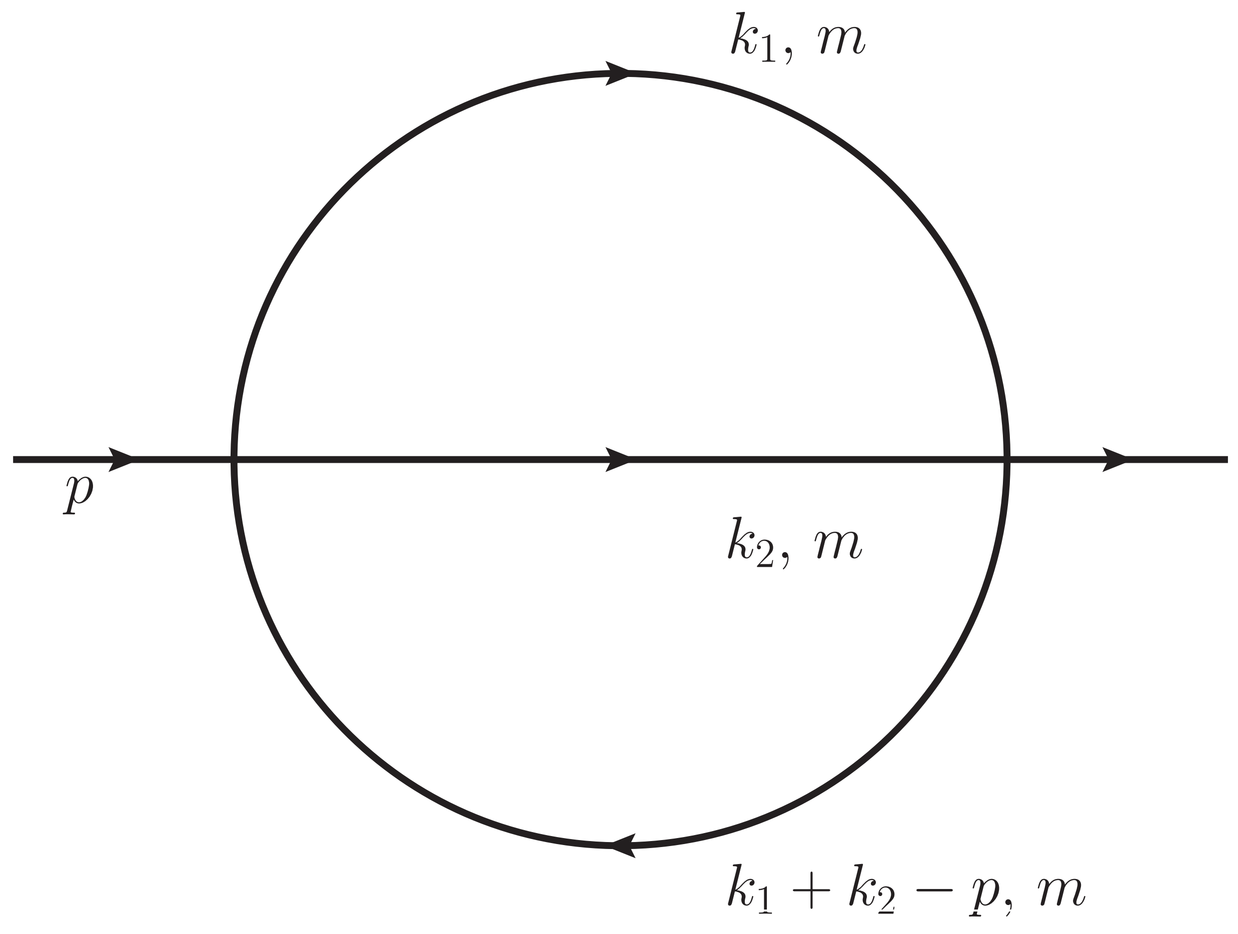
for example, $G(0;z) = \ln z,\ G(0{,}1;z) = -\,\mathrm{Li}_2(z)\,.$

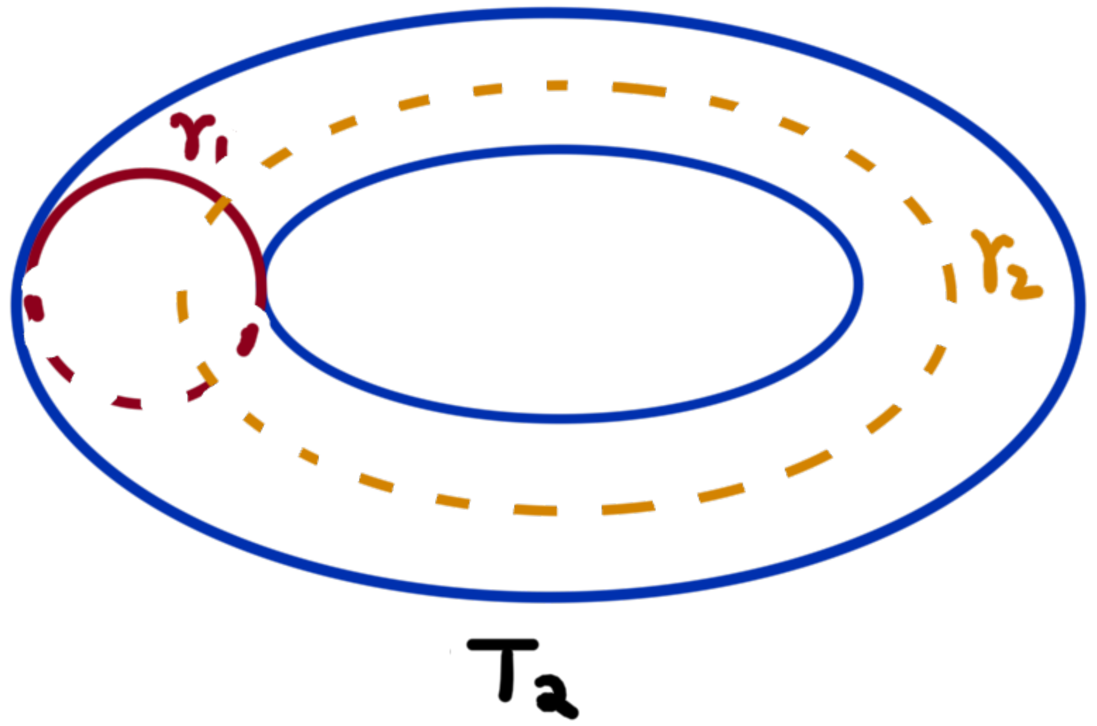
Goncharov, '98, 01'

Two-loop banana: sunrise

$$F^{(2)}_{\nu_1,\nu_2,\nu_3}(2-2\varepsilon,y) = e^{2\varepsilon\gamma_E}\cdot(m^2)^{|\vec{\nu}|-d}\cdot\int\frac{\mathrm{d}^dk_1}{i\pi^{d/2}}\frac{\mathrm{d}^dk_2}{i\pi^{d/2}}\frac{1}{\left[k_1^2-m^2\right]^{\nu_1}\left[k_2^2-m^2\right]^{\nu_2}\left[(k_1+k_2-p)^2-m^2\right]^{\nu_3}}$$

- The basis constitutes of three MIs: $F^{(2)}_{1,1,0}$, $F^{(2)}_{1,1,1}$, $F^{(2)}_{2,1,1}$, where the last two are not MPLs;
- The geometry behind is an elliptic curve (equivalent to a torus, Calabi-Yau 1-fold), which has genus 1;
- The solution space of the DE of this family is iterated integrals on (punctured) torus, i.e., eMPL;
- In the equal-mass case, they reduce to iterated integrals of modular forms.





$$\psi_0 = \int_{\gamma_1}\frac{dz}{w},\quad \psi_1 = \int_{\gamma_2}\frac{dz}{w},$$

$$\mathscr{E}:\ \ w^2 = 4z^3 + g_2 z + g_3\,.$$

Three-loop banana

- Leading term in ε [Bloch, Kerr, Vanhove, 1406.2664]
- ε-factorised form by maximal cuts [Primo, Tancredi, 1704.05465]
- Master integrals in d=2 in terms of eMPLs [Broedel, Duhr, Dulat, Marzucca, Penante, 1907.03787]
- DEQ with meromorphic modular forms [Broedel, Duhr, Matthes, 2109.15251]
- Part of larger l-Loop banana program [Bönisch, Duhr, Klemm, Nega, Safari; Kreimer; Forum, von Hippel]
- ε-factorised form with meromorphic modular forms [Pögel, **XW** and Weinzierl, 2207.12893].

Four-loop banana

- Calculation in position space, at special points, numerics[Groote, Körner and Pivovarov, hep-ph/0506286]
- Part of larger l-Loop banana program, details about Calabi-Yau geometry behind[Bönisch, Duhr, Klemm, Nega, Safari, 1912.06201, 2008.10574, 2108.05310]
- On the cut banana graphs [Kreimer, 2202.05490]
- On symbols and coaction [Forum, von Hippel, 2209.03922]
- ε-factorised form [Pögel, **XW** and Weinzierl, 2211.04292].
- [Candelas, De La Ossa, Green, Parkes '91]
- [Morrison '91]
- Batyrev and van Straten '93]
- [Almkvist '06]
- **[Bogner, 1304.5434]**
- **Calabi-Yau operators, [van Straten, 1704.00164]**
- Calabi-Yau operators of degree two [Almkvist, van Straten, 2103.08651]

l-loop banana

- Part of larger l-Loop banana program, details about Calabi-Yau geometry behind[Bönisch, Duhr, Klemm, Nega, Safari, 1912.06201, 2008.10574, 2108.05310]
- Bananas of equal mass: any loop, any order in the dimensional regularisation parameter [Pögel, **XW** and Weinzierl, 2212.08908]
- The ice cone family and iterated integrals for Calabi-Yau varieties [Duhr, Klemm, Nega and Tancredi, 2212.09550]

- **[Bogner, 1304.5434]**

l-loop banana

$$F^{(l)}_{\vec{\nu}}(d=2-2\varepsilon, y) = e^{l\varepsilon\gamma_E} \cdot (m^2)^{|\vec{\nu}|-ld/2} \cdot \int \frac{\mathrm{d}^d k_1}{i\pi^{d/2}} \cdots \frac{\mathrm{d}^d k_l}{i\pi^{d/2}} \frac{1}{\left[k_1^2 - m^2\right]^{\nu_1} \cdots \left[k_2^2 - m^2\right]^{\nu_l} \left[(k_1 + \cdots k_l - p)^2 - m^2\right]^{\nu_{l+1}}}$$

- There are $l+1$ master integrals, upon which any integral in this family can be projected: $F^{(l)}_{11\cdots10}, F^{(l)}_{11\cdots1}, F^{(l)}_{21\cdots1}, F^{(l)}_{31\cdots1}, \cdots, F^{(l)}_{l1\cdots1}$;
- These MIs constitute a $(l+1)\times(l+1)$ linearly coupled differential equation.
- The geometry behind is a Calabi-Yau $(l-1)$-folds, l periods.
- As a a Calabi-Yau $(l-1)$-folds, there exists a no-where vanishing holomorphic differential form Ω, such that periods are related to $\psi_i = \int_{\gamma_i} \Omega$, but it is not even simple to explicitly define independent cycles in general.
- Instead, we resort to Picard-Fuchs differential equation of periods.

l-loop banana

$$\frac{d}{dy}\begin{pmatrix} F^{(l)}_{11\cdots 0} \\ F^{(l)}_{11\cdots 1} \\ F^{(l)}_{21\cdots 1} \\ F^{(l)}_{31\cdots 1} \\ \vdots \\ F^{(l)}_{l1\cdots 1} \end{pmatrix} = \mathbf{A}(\varepsilon, y)\begin{pmatrix} F^{(l)}_{11\cdots 0} \\ F^{(l)}_{11\cdots 1} \\ F^{(l)}_{21\cdots 1} \\ F^{(l)}_{31\cdots 1} \\ \vdots \\ F^{(l)}_{l1\cdots 1} \end{pmatrix} \longleftrightarrow$$

$$\boxed{\begin{aligned}
& & I_0 &= \varepsilon^l F^{(l)}_{11\cdots 0} = \left[e^{\varepsilon\gamma_E}\Gamma(1+\varepsilon)\right]^l \\
F^{(l)}_{11\cdots 1} &\sim & I_1 &= \varepsilon^l \frac{F^{(l)}_{11\cdots 1}}{\psi_0} \\
\frac{d}{dy}F^{(l)}_{11\cdots 1} &\sim & I_2 &= \frac{1}{Y_1}\left[\frac{1}{\varepsilon}J(y)\frac{d}{dy}I_1 - F_{11}I_1\right] \\
\frac{d^2}{dy^2}F^{(l)}_{11\cdots 1} &\sim & I_3 &= \frac{1}{Y_2}\left[\frac{1}{\varepsilon}J(y)\frac{d}{dy}I_2 - F_{21}I_1 - F_{22}I_2\right] \\
& & &\vdots \\
\frac{d^{l-1}}{dy^{l-1}}F^{(l)}_{11\cdots 1} &\sim & I_l &= \frac{1}{Y_{l-1}}\left[\frac{1}{\varepsilon}J(y)\frac{d}{dy}I_{l-1} - \sum_{k=1}^{l-1} F_{(l-1)k}\, I_k\right].
\end{aligned}}$$

$$\Updownarrow$$

Picard-Fuchs operator: $\hat{L}^{(l)}(y,\varepsilon)F^{(l)}_{11\cdots 1} = \dfrac{(l+1)!}{y^{l-1}\prod_{a\in S^{(l)}}(1+ay)}\varepsilon^l F^{(l)}_{11\cdots 0}$, with $\hat{L}^{(l)}(y,\varepsilon) = \sum_{j=0}^{l} r_j(y,\varepsilon)\dfrac{d^j}{dy^j}$.

l-loop banana: Picard-Fuchs operator

- The period integrals $\psi_i = \int_{\gamma_i} \Omega$ related to the geometry are annihilated by the Picard-Fuchs operator $\hat{L}^{(l)}(y, \varepsilon = 0)\, \psi_i = 0,\ i = 0,1,\cdots, l-1;$
- This operator $\hat{L}^{(l)}(y, \varepsilon = 0)$ is a Calabi-Yau operator [Bogner,'13], which has very nice properties:
  1. Around $y = 0$, the solutions has maximal logarithmic series as:
  $$\psi_i = \frac{1}{(2\pi i)^i} \sum_{j=0}^{i} \frac{\ln^j y}{j!} \sum_{n=0}^{\infty} a_{k-j,n} y^{n+\rho},\ \rho \in \mathbb{N}_+$$
  2. Mirror map (canonical variable change): $q = \exp\left[2\pi i\, \psi_1(y)/\psi_0(y)\right],\ q(y)\ \&\ y(q)$ are N-integral;
  3. The operator is self-dual: There exists a function $\alpha(y)$ such that:
  $$\alpha \hat{L}^* = \hat{L}\alpha, \text{ with } L^* = \sum_{j=0}^{l} (-1)^{l-j} \frac{d^j}{dy^j} r_j(y,0).$$

Mirror map and operator factorization

- Mirror map is related to mirror symmetry of Calabi-Yau manifolds. Calabi-Yau l-folds come in pair, whose complex structure and Kähler structure are exchanged.

$$
\begin{aligned}
&y = q^{(0)} \\
&y = q^{(1)} + 2\left(q^{(1)}\right)^2 + 3\left(q^{(1)}\right)^3 + 4\left(q^{(1)}\right)^4 + O\left(\left(q^{(1)}\right)^5\right) \\
&y = q^{(2)} + 4\left(q^{(2)}\right)^2 + 10\left(q^{(2)}\right)^3 + 20\left(q^{(2)}\right)^4 + O\left(\left(q^{(2)}\right)^5\right) \\
&y = q^{(3)} + 6\left(q^{(3)}\right)^2 + 21\left(q^{(3)}\right)^3 + 68\left(q^{(3)}\right)^4 + O\left(\left(q^{(3)}\right)^5\right) \\
&y = q^{(4)} + 8\left(q^{(4)}\right)^2 + 36\left(q^{(4)}\right)^3 + 168\left(q^{(4)}\right)^4 + O\left(\left(q^{(4)}\right)^5\right) \\
&y = q^{(5)} + 10\left(q^{(5)}\right)^2 + 55\left(q^{(5)}\right)^3 + 340\left(q^{(5)}\right)^4 + O\left(\left(q^{(5)}\right)^5\right) \\
&y = q^{(6)} + 12\left(q^{(6)}\right)^2 + 78\left(q^{(6)}\right)^3 + 604\left(q^{(6)}\right)^4 + O\left(\left(q^{(6)}\right)^5\right) \\
&\vdots
\end{aligned}
$$

$$\xrightarrow{\theta_q = q\frac{d}{dq} = \frac{d}{2\pi i d\tau}}$$

$$
\begin{aligned}
&\hat{L}^{(0)}(y, \varepsilon = 0) \propto 1 \\
&\hat{L}^{(1)}(y, \varepsilon = 0) \propto \theta_q \\
&\hat{L}^{(2)}(y, \varepsilon = 0) \propto \theta_q \cdot \theta_q \\
&\hat{L}^{(3)}(y, \varepsilon = 0) \propto \theta_q \cdot \theta_q \cdot \theta_q \\
&\hat{L}^{(4)}(y, \varepsilon = 0) \propto \theta_q \cdot \theta_q \cdot \frac{1}{Y_2} \cdot \theta_q \cdot \theta_q
\end{aligned}
$$

[van Straten, 1704.00164]

$$\hat{L}^{(l)}(y, \varepsilon = 0) = \frac{\alpha}{J\psi_0}\theta_q^2 \frac{1}{Y_2}\theta_q \frac{1}{Y_3} \cdots \frac{1}{Y_3}\theta_q \frac{1}{Y_2}\theta_q^2 \frac{1}{\psi_0}$$

l-loop banana: Y invariants

- Define recursively operators N_j by:

$$\hat{N}_0 = 1, \quad \hat{N}_{j+1} = y\frac{d}{dy}\frac{1}{(2\pi i)^j \hat{N}_j(\psi_j)}\hat{N}_j, \ \left(\hat{N}_j(\psi_i) = 0,\ i < j\right);$$

- Structure series $(\alpha_1, \alpha_2, \cdots, \alpha_{l-1})$:

$$\alpha_j = \frac{1}{(2\pi i)^j}\frac{1}{\hat{N}_j(\psi_j)};$$

- Y-invariants:

$$Y_j = \frac{\alpha_1}{\alpha_j},\ 1 \le j \le l-1;$$

- Self-dual implication:

$$Y_j = Y_{l-j}.$$

- Explicitly, one can verify that

[Bogner,'13]

$$\alpha_1^{-1} = \frac{y}{J(y)},\ \alpha_2^{-1} = \alpha_1^{-1}\frac{d^2}{d\tau^2}\frac{\psi_2}{\psi_0}, \alpha_3^{-1} = \alpha_1^{-1}\frac{d}{d\tau}\left(\alpha_2\alpha_1^{-1}\frac{d^2}{d\tau^2}\frac{\psi_3}{\psi_0}\right),\ \cdots$$

l-loop banana: ε-form Ansatz

- $\hat{L}^{(l)}(y,0)$ annihilates $F^{(l)}_{11\cdots 1}$ modulo ε and modulo tadpole;

$$\hookrightarrow I_1 = \varepsilon^l \frac{F_{11\cdots 1}}{\psi_0}$$

- Mirror map tells us to use the canonical variable τ or q:

$$\hookrightarrow J(y)\frac{d}{dy} = \frac{d}{2\pi i d\tau} = \theta_q;$$

- Factorization of $\hat{L}^{(l)}(y,0)$ suggests we include pre-factors for all the others apart from some unknown rotations coefficients:

$$\hookrightarrow I_l = \frac{1}{Y_{l-1}}\left[\frac{1}{\varepsilon}J(y)\frac{d}{dy}I_{l-1} - \sum_{k=1}^{l-1} F_{(l-1)k}\, I_k\right]$$

- By requiring $d\vec{I} = \varepsilon A(\tau)\vec{I}$, we have some constraints:
  - ☑ Constraints on ψ_0, Y_j and J are automatically satisfied;
  - ☑ Constraints for F_{ij} can be solved systematically by considering self-dual symmetry.

l-loop banana: ε-form Ansatz

$$\frac{1}{2\pi i}\frac{d}{d\tau}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ \vdots \\ I_{l-1} \\ I_l \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_{11} & 1 & 0 & 0 & 0 & 0 \\ 0 & F_{21} & F_{22} & Y_2 & 0 & 0 & 0 \\ 0 & F_{31} & F_{32} & F_{33} & Y_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & F_{(l-2)1} & F_{(l-2)2} & F_{(l-2)3} & \cdots & Y_{l-2} & 0 \\ 0 & F_{(l-1)1} & F_{(l-1)2} & F_{(l-1)3} & \cdots & F_{(l-1)(l-1)} & 1 \\ f_l & * & * & * & \cdots & * & * \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \\ I_4 \\ \vdots \\ I_{l-1} \\ I_l \end{pmatrix}$$

- Removing ε-dependent terms in the last row results in some constraints for F_{ij};
- The constraints can be dramatically simplified by observing that self-duality requires that coloured elements are the same respectively, just like Y invariants, resulting in some algebraic relations:

$$\hookrightarrow A_{ij} = A_{(l+1-j)(l+1-i)},\ 1 \leq i,j \leq l.$$

l-loop banana: Boundary and final results

- The boundary terms to all orders are easy to obtain with help of Mellin-Barnes techniques:

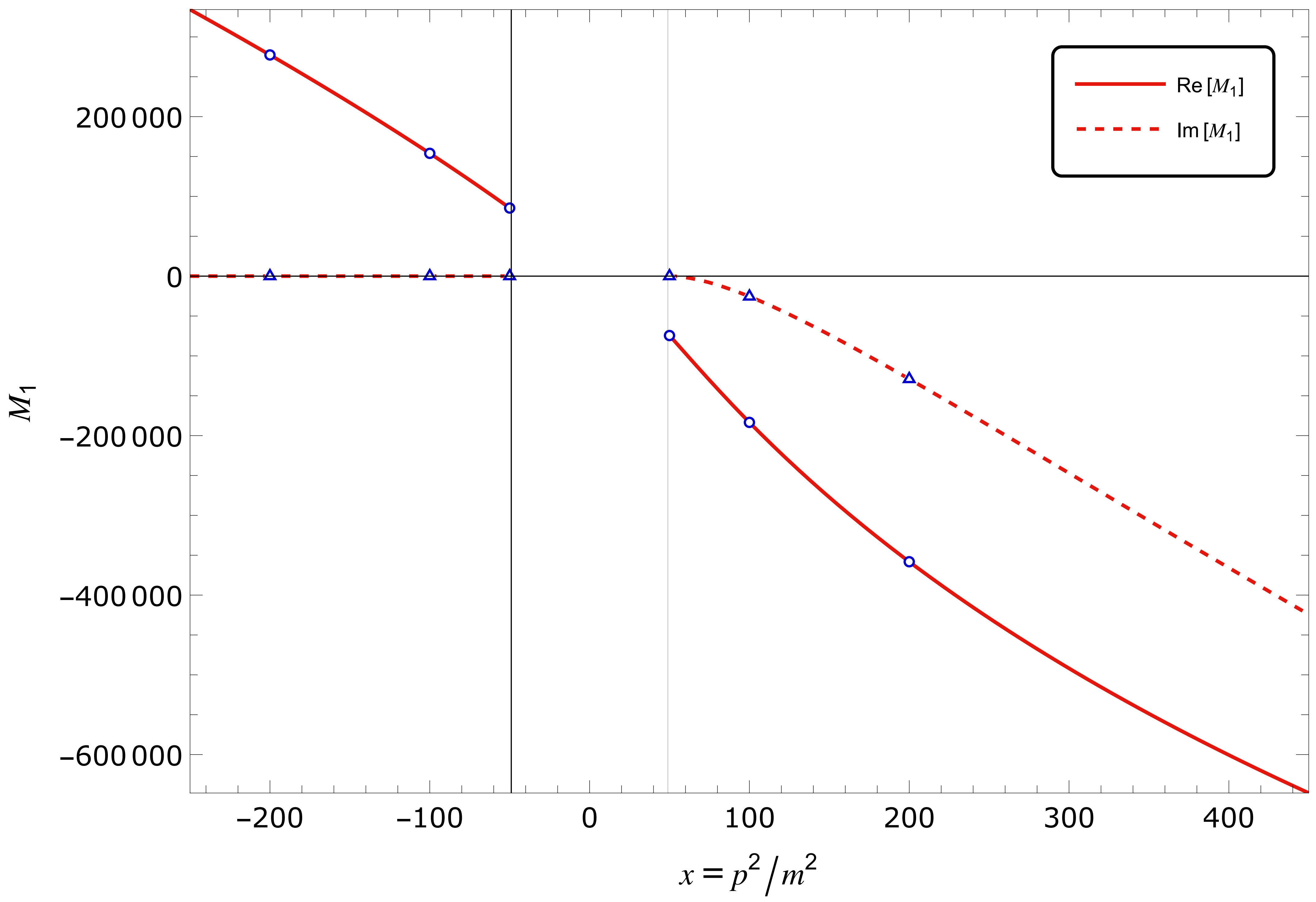
$$I_1\big|_{y\to 0} = (-1)^{l+1}(l+1)e^{l\varepsilon\gamma_E}\sum_{j=0}^{l}(-1)^j\binom{l}{j}\,y^{l\varepsilon}\frac{\Gamma(1+\varepsilon)^{l-j}\Gamma(1-\varepsilon)^{1+j}\Gamma(1+j\varepsilon)}{\Gamma(1-(j+1)\varepsilon)};$$

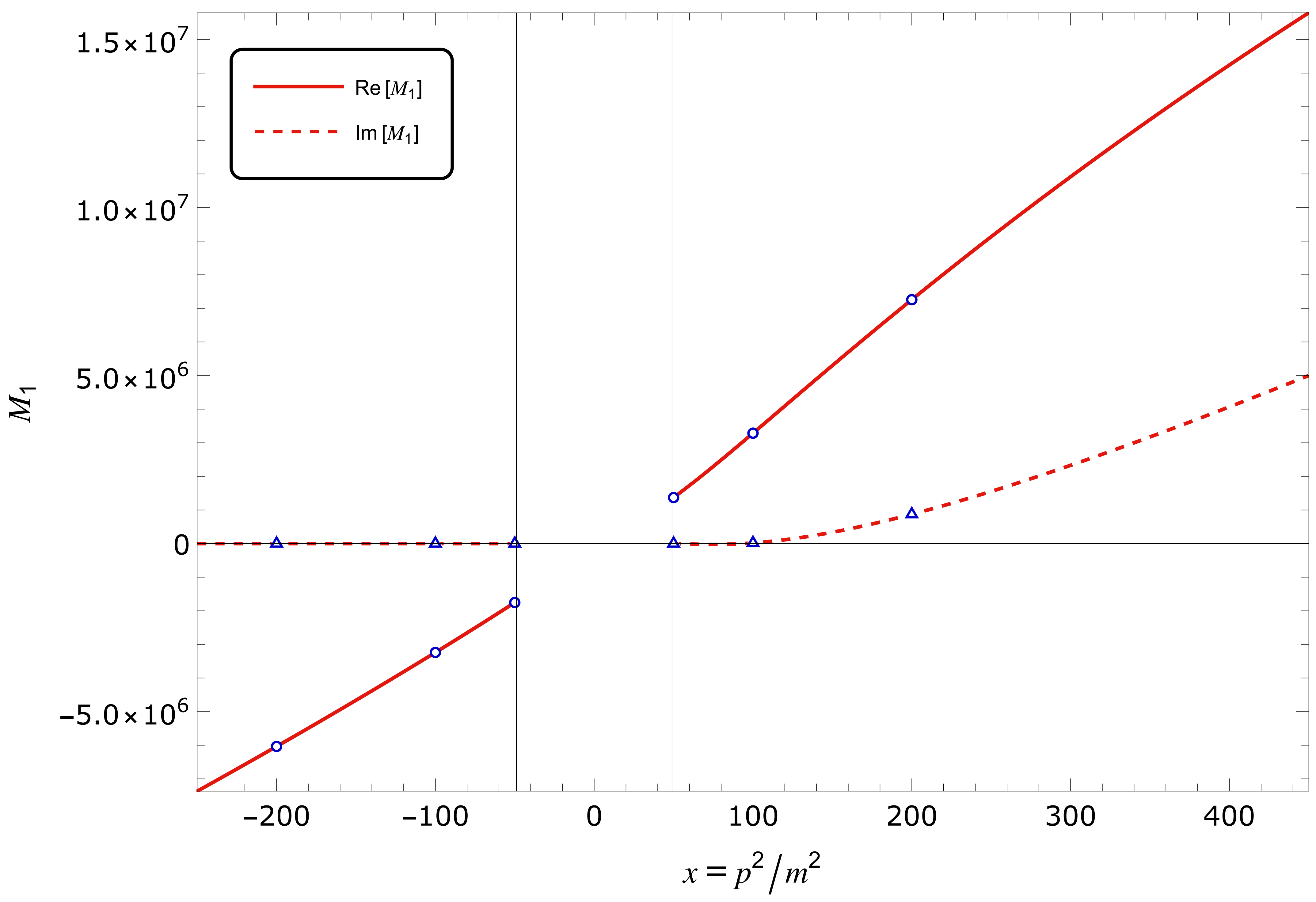
- Final results are expressed in terms of iterated integrals, e.g., $I_1^{(6)} = \varepsilon^6 I_1^{(6,6)} + \varepsilon^7 I_1^{(6,7)} + \cdots$:

$$\begin{aligned}
I_1^{(6,6)} &= 1120\zeta_3^2 - 2016\zeta_5 L_q - 3360\zeta_3 I\left(1,Y_2,Y_3\right) + I\left(1,Y_2,Y_3,Y_2,1,f_{7,0}\right)\\
&= 1120\zeta_3^2 - 560\zeta_3 L_q^3 - 2016\zeta_5 L_q + 7L_q^6 + 210q\left(-32\zeta_3 + 48\zeta_3 L_q - 3L_q^4 + 8L_q^3\right).\\
&+\frac{105}{2}q^2\left(208\zeta_3 - 1392\zeta_3 L_q + 87L_q^4 - 52L_q^3 - 180L_q^2 - 72L_q + 192\right) + \mathcal{O}(q^3)
\end{aligned}$$

- The geometry for six loops is a Calabi-Yau 5-folds!

l-loop banana: comparison





- Agree with pySecDec. Our evaluation of the curves takes seconds;
- The convergent range around $y = 0$ at six loops is $|p^2| > 49m^2$.

Conclusion and outlook

- Feynman integrals related to Calabi-Yau geometry interest the community from several aspects;
- We solve l-loop equal-mass banana integrals, whose geometry is Calabi-Yau $(l-1)$-folds;
- Geometry-inspired methods play an essential role, and we believe geometry should be guidance for other Feynman integrals as well;
- We put one more highly non-trivial evidence to the conjecture that all Feynman integrals may have an ε form.

- Analytical continuation to other singular points;
- Push to multi-parameter cases, e.g., unequal-mass banana integrals;
- What exactly are the letters beyond three loops? Automorphic forms?

Thank you !

Backup

First two periods to all orders

$$\psi_0^{(l)} = \sum_{n=0}^{\infty} a_{0,n}^{(l)} y^{n+1}, \quad a_{0,n}^{(l)} = (-1)^n \sum_{n_1+\ldots+n_{l+1}=n} \left(\frac{n!}{n_1!\cdots n_{l+1}!}\right)^2$$

$$S_1(n) = \sum_{j=1}^{n} \frac{1}{j}$$

$$\psi_1^{(l)} = \frac{1}{(2\pi i)} \sum_{n=0}^{\infty} \left[a_{1,n}^{(l)} + a_{0,n}^{(l)} \ln y\right] y^{n+1}, \quad a_{1,n}^{(l)} = 2(-1)^n \sum_{n_1+\ldots+n_{l+1}=n} \left(\frac{n!}{n_1!\cdots n_{l+1}!}\right)^2 \left[S_1(n) - S_1\left(n_1\right)\right]$$

Constraints @ 4 loops

There is extra non-linear constraint for the period at 4 loops, but it will show u again at 5 loops!

$$\frac{g_3''}{g_3} - \frac{4}{3}\left(\frac{g_3'}{g_3}\right)^2 - 10\left[\frac{\omega_1''}{\omega_1} - \frac{4}{3}\left(\frac{\omega_1'}{\omega_1}\right)^2\right] + \frac{1}{3}\left(\frac{g_3'}{g_3} - 10\frac{\omega_1'}{\omega_1}\right)\left[2\frac{\omega_1'}{\omega_1} + \frac{2}{x} + \frac{1}{x-1} + \frac{1}{x-9} + \frac{1}{x-25}\right] = \frac{1}{3}\left[\frac{1}{(x-1)^2} + \frac{1}{(x-9)^2} + \frac{1}{(x-25)^2} - \frac{2}{x^2} + \frac{(x-17)(9x-29)}{(x-25)(x-9)(x-1)x}\right].$$

$$F_{21}''' + \left(3\frac{J'}{J} - 2\frac{g_3'}{g_3}\right)F_{21}'' + \left[-6\frac{\omega_1''}{\omega_1} + 6\left(\frac{\omega_1'}{\omega_1}\right)^2 - \frac{J'}{J}\left(3\frac{J'}{J} - \frac{g_3'}{g_3}\right) + \frac{2\left(5x^3 - 112x^2 + 492x - 225\right)}{(x-25)(x-9)(x-1)x^2}\right]F_{21}' + \frac{J}{2\pi i}\left[2\left(\frac{1}{x} - \frac{2}{x-1} - \frac{2}{x-9} - \frac{2}{x-25}\right)\frac{\omega_1'''}{\omega_1} - \frac{3\left(20x^3 - 315x^2 + 518x + 225\right)}{(x-25)(x-9)(x-1)x^2}\frac{\omega_1''}{\omega_1} - \frac{2\left(35x^2 - 357x + 246\right)}{(x-25)(x-9)(x-1)x^2}\frac{\omega_1'}{\omega_1}\right] = 0.$$

Calabi-Yau manifold

Definition:

A Calabi-Yau n-fold is a compact Kähler manifold X of complex dimension n, equiped with a Kähler (1,1)-form ω, and satisfying one of the following equivalent conditions:

1. *The first Chern class of X vanishes (over $\mathbb{R}$)*
2. *X has a Kähler metric with vanishing Ricci curvature.*
3. *X has a holomorphic (n,0)-form Ω that vanishes nowhere.*
4. *The holonomy group of X is $SU(N)$.*
5. *A positive power of the canonical bundle of X is trivial.*
6. *X has a finite cover that is a product of a torus and a simply connected manifold with trivial canonical bundle.*
7. *The canonical bundle of X is trivial.*

- Forms Ω and ω are both **characteristics** for X, such that they are normally put together: (X, Ω, ω), e.g., an elliptic curve: $(\mathscr{E}, dx/y, dx \wedge dy)$.
- Roughly speaking, CY manifolds live between simple and general varities.

Yukawa coupling

Definition 1:

CY $(l-1)$-fold: from

$$\hat{L}_l(\varepsilon, x) = \frac{d^l}{dx^l} + p_{l-1}(x)\frac{d^{l-1}}{dx^{l-1}} + \cdots, \quad \longrightarrow \frac{dC_{l-1}(x)}{dx} = -\frac{2}{l}p_{l-1}(x)C_{l-1}(x)\,.$$

Definition 2:

$$W_k = \int_{X_x} \Omega(x) \wedge \frac{d^k}{dx^k}\Omega(x) = \Pi^T\Sigma\frac{d^k}{dx^k}\Pi = \begin{cases} 0, & k < l-1, \\ C_{l-1}, & k = l-1. \end{cases} \longrightarrow \frac{W_3}{\omega_1^2} = K(x) \longrightarrow K(q) = K(x)J(x)^3\,.$$

Elliptic curve from graph polynomial

$$F_{111}^{(2)} = e^{2\varepsilon\gamma_E} \cdot \Gamma(1+2\varepsilon) \cdot \int_{\alpha_i \geq 0} d^3\alpha\, \delta\left(1 - \sum_{i=1}^{3} \alpha_i\right) \frac{\mathcal{U}(\alpha)^{3\varepsilon}}{\mathcal{F}(\alpha)^{1+2\varepsilon}}$$

$$\mathcal{U}(\alpha) = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$$

$$\mathcal{F}(\alpha) = x\alpha_1\alpha_2\alpha_3 + (\alpha_1 + \alpha_2 + \alpha_3)\mathcal{U}(\alpha)$$

$$\mathcal{E} = \left\{ [\alpha_1 : \alpha_2 : \alpha_3] \in \mathbb{CP}^2 \middle| x\alpha_1\alpha_2\alpha_3 + \left(\alpha_1 + \alpha_2 + \alpha_3\right)\left(\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1\right) = 0 \right\}$$

Elliptic curve from maximal cut

Maximal cut:

- By sending all propagators on-shell, i.e., replacing with delta functions.
- Usually, maximal cut contains the most essential information of the integral family.
- Around 2-dimension, there are three delta functions fixing three integration components, leaving one unintegrated.

$$\mathrm{MaxCut}\, F_{111}^{(2)} = \frac{(2\pi i)^3}{\pi^2} \int_{\mathscr{C}_{\mathrm{MaxCut}}} \frac{dz}{\sqrt{z(z+4)\left[z^2 + 2(1-x)z + (1+x)^2\right]}} + O(\varepsilon)$$

$$\mathscr{E} : v^2 = u(u+4)\left[u^2 + 2(1-x)u + (1+x)^2\right] .$$

ε-form for the sunrise

$$
\begin{aligned}
I_0 &= \varepsilon^2 F_{110}^{(2)} \\
I_1 &= \varepsilon^2 \frac{F_{111}^{(2)}}{\psi_1}, \\
I_2 &= \frac{1}{\varepsilon}\frac{d}{d\tau} I_2 + F_{21} I_1
\end{aligned}
$$

$$
\frac{d}{d\tau}\begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta_2 & 1 \\ \eta_3 & \eta_4 & \eta_2 \end{pmatrix} \begin{pmatrix} I_0 \\ I_1 \\ I_2 \end{pmatrix}
$$

Adams, Weinzierl '17