



Azimuthal asymmetries in photon induced dijet & dilepton productions in UPCs

Cheng Zhang 2023.5.28

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JHEP02(2023)002

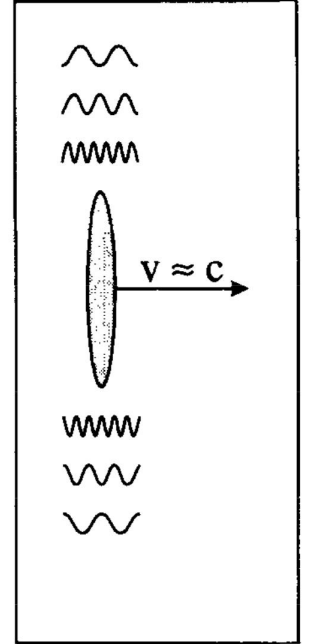
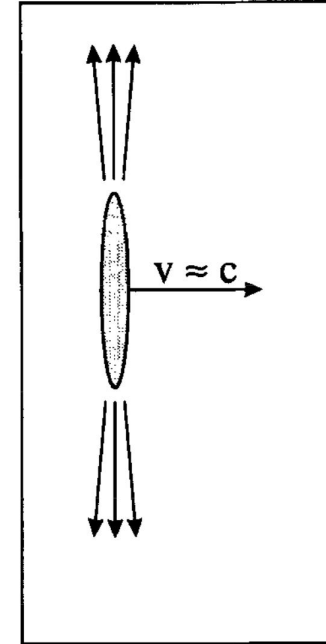
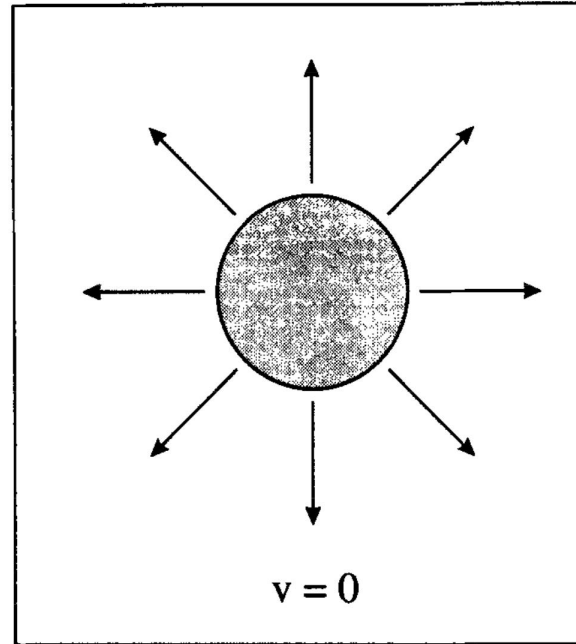
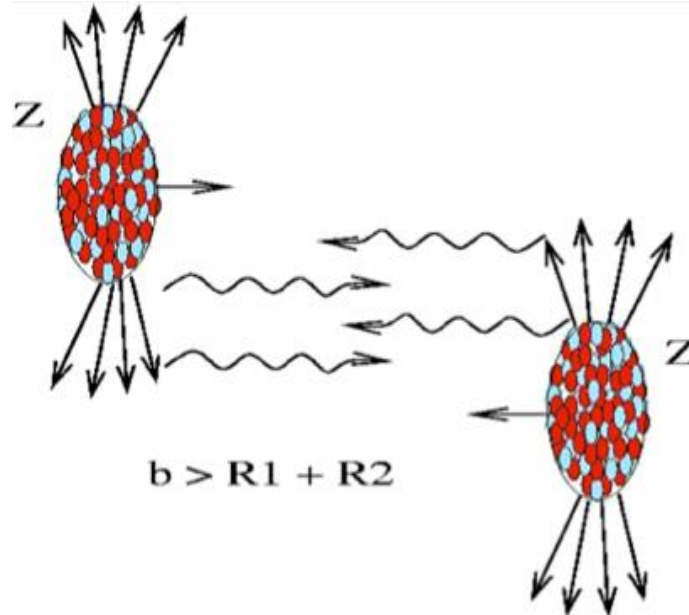
Phys.Rev.D107,036020(2023)

Outline

- Ultraperipheral collisions (UPCs)
- Acoplanarity (azimuthal decorrelation) of dijet productions from photon-photon fusions JHEP02(2023)002
- Acoplanarity & $\cos 2\varphi$, $\cos 4\varphi$ azimuthal asymmetries in di-lepton (e^+e^- & $\mu^+\mu^-$) productions Phys.Rev.D107,036020(2023)
- Summary and outlook

Ultrapерipheral collisions (UPCs)

Two nuclei miss each other,
interact electromagnetically



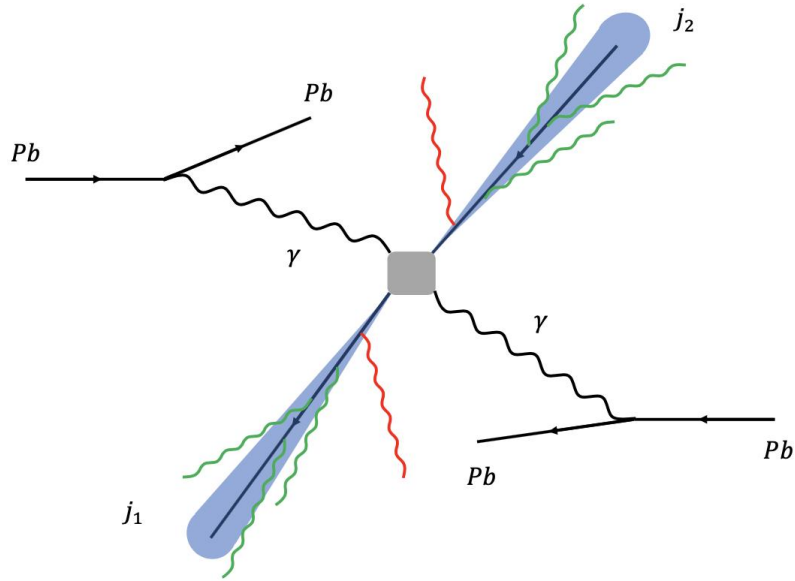
Different types: γ - γ , γ -A

Initial state coherent photons & gluons

Final state soft radiations & resummations

Fast moving nucleus, highly boosted and almost
transverse electromagnetic fields

$$\gamma(\omega_1, \mathbf{k}_{1T}) + \gamma(\omega_2, \mathbf{k}_{2T}) \rightarrow q(y_1, \mathbf{p}_{1T}) + \bar{q}(y_2, \mathbf{p}_{2T})$$



$$\begin{aligned} \frac{d^5\sigma_0}{d^2\mathbf{q}_T dp_T dy_1 dy_2} &= N_c \sum_q e_q^4 \frac{4Z^4 \alpha_{\text{em}}^4}{\pi^5 M^4} p_T \int d^2\mathbf{b}_T d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} d^2\mathbf{k}'_{1T} d^2\mathbf{k}'_{2T} \\ &\times \delta^{(2)}(\mathbf{k}_{1T} + \mathbf{k}_{2T} - \mathbf{q}_T) \delta^{(2)}(\mathbf{k}'_{1T} + \mathbf{k}'_{2T} - \mathbf{q}_T) e^{i(\mathbf{k}_{1T} - \mathbf{k}'_{1T}) \cdot \mathbf{b}_T} \\ &\times k_{1T} \frac{F(-k_1^2)}{-k_1^2} k_{2T} \frac{F(-k_2^2)}{-k_2^2} k'_{1T} \frac{F(-k_1'^2)}{-k_1'^2} k'_{2T} \frac{F(-k_2'^2)}{-k_2'^2} \frac{M^2 - 2p_T^2}{p_T^2} \\ &\times \cos(\phi_{\mathbf{k}_{1T}} - \phi_{\mathbf{k}'_{1T}} + \phi_{\mathbf{k}_{2T}} - \phi_{\mathbf{k}'_{2T}}), \end{aligned}$$

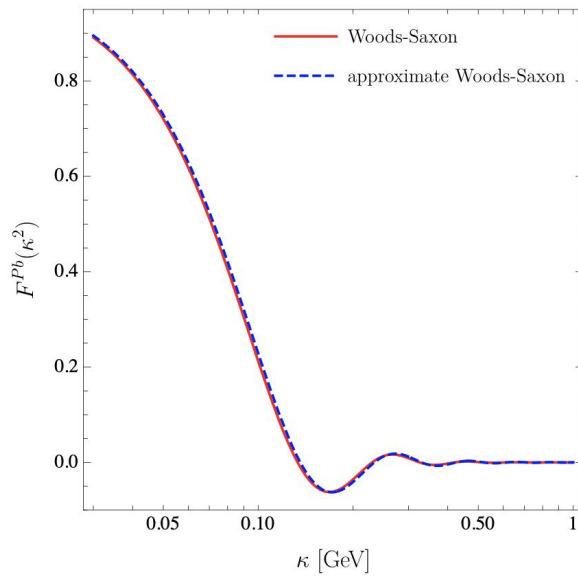
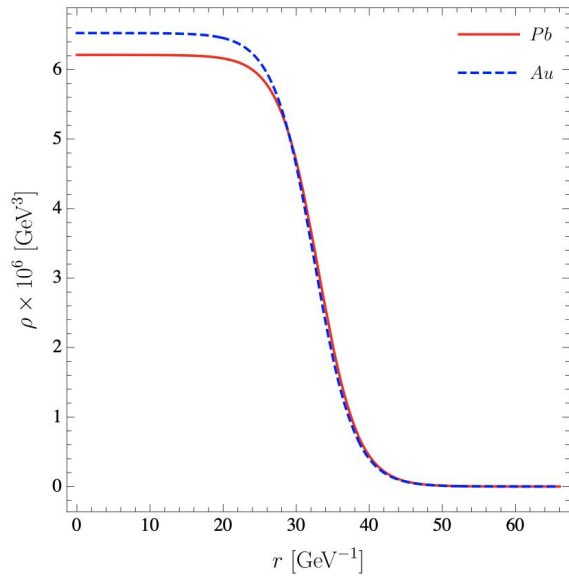
$$\sigma_{A_1 A_2 \rightarrow A_1 A_2 X}^{WW} = \iint d\omega_1 d\omega_2 n_{A_1}(\omega_1) n_{A_2}(\omega_2) \sigma_{\gamma\gamma \rightarrow X}(\omega_1, \omega_2)$$

Equivalent photon approximation(EPA):

$$n(\omega) = \frac{4Z^2 \alpha}{\omega} \int \frac{d^2 k_{\perp}}{(2\pi)^2} \left(\frac{F(\mathbf{k}_{\perp}^2 + \omega^2/\gamma^2)}{\mathbf{k}_{\perp}^2 + \omega^2/\gamma^2} \right)^2 |\mathbf{k}_{\perp}|^2$$

Woods-Saxon nuclear charge form factor:

$$F(\mathbf{k}_{\perp}) = \int d^3 \mathbf{x} e^{i\mathbf{k} \cdot \mathbf{x}} \frac{\rho}{1 + \exp[(r - R_{WS})/d]}$$



Energy modes in the factorized expression

$$n_i \text{ collinear} : p_{c_i}^\mu \sim p_T (R^2, 1, R)_{n_i \bar{n}_i},$$

$$n_i \text{ collinear-soft} : p_{cs_i}^\mu \sim \frac{p_T \delta\phi}{R} (R^2, 1, R)_{n_i \bar{n}_i}$$

$$\text{soft} : p_s^\mu \sim p_T (\delta\phi, \delta\phi, \delta\phi),$$

Factorization and Resummation formalism in Soft-Collinear Effective Theory (SCET):

$$\frac{d^4\sigma}{dq_x dp_T dy_1 dy_2} = \int dk_x d\lambda_x dl_{1,x} dl_{2,x} \delta(k_x + \lambda_x + l_{1,x} + l_{2,x} - q_x) B(k_x, p_T, y_1, y_2)$$

hard function

$$\times H(p_T, \Delta y, \mu) \boxed{S(\lambda_x, y_1, y_2, \mu, \nu) U_1(l_{1,x}, R, y_1, \mu, \nu) J_1(p_T, R, \mu)}$$

jet function

$$\times U_2(l_{2,x}, R, y_2, \mu, \nu) J_2(p_T, R, \mu),$$

acoplanarity

$$|q_x| \equiv p_T \sin(\pi - \Delta\phi_{jj}) = p_T \sin \delta\phi$$

RG equation of the hard function

$$\frac{d}{d \ln \mu} H(p_T, \Delta y, \mu) = \underbrace{\left[-2C_F \gamma_{\text{cusp}}(\alpha_s) \ln \frac{\mu^2}{M^2} + 4\gamma_q(\alpha_s) \right]}_{\equiv \Gamma_H(\alpha_s)} H(p_T, \Delta y, \mu)$$

RG equation of the hard function

$$\frac{d}{d \ln \mu} J_i(p_T, R, \mu) = \underbrace{\left[-C_F \gamma_{\text{cusp}}(\alpha_s) \ln \frac{p_T^2 R^2}{\mu^2} - 2\gamma_q(\alpha_s) \right]}_{\equiv \Gamma_J(\alpha_s)} J_i(p_T, R, \mu)$$

RG consistency relations

$$\frac{d}{d \ln \mu} \left[S U_1 U_2 H(p_T, \Delta y, \mu) J_1(p_T, R, \mu) J_2(p_T, R, \mu) \right] = 0$$

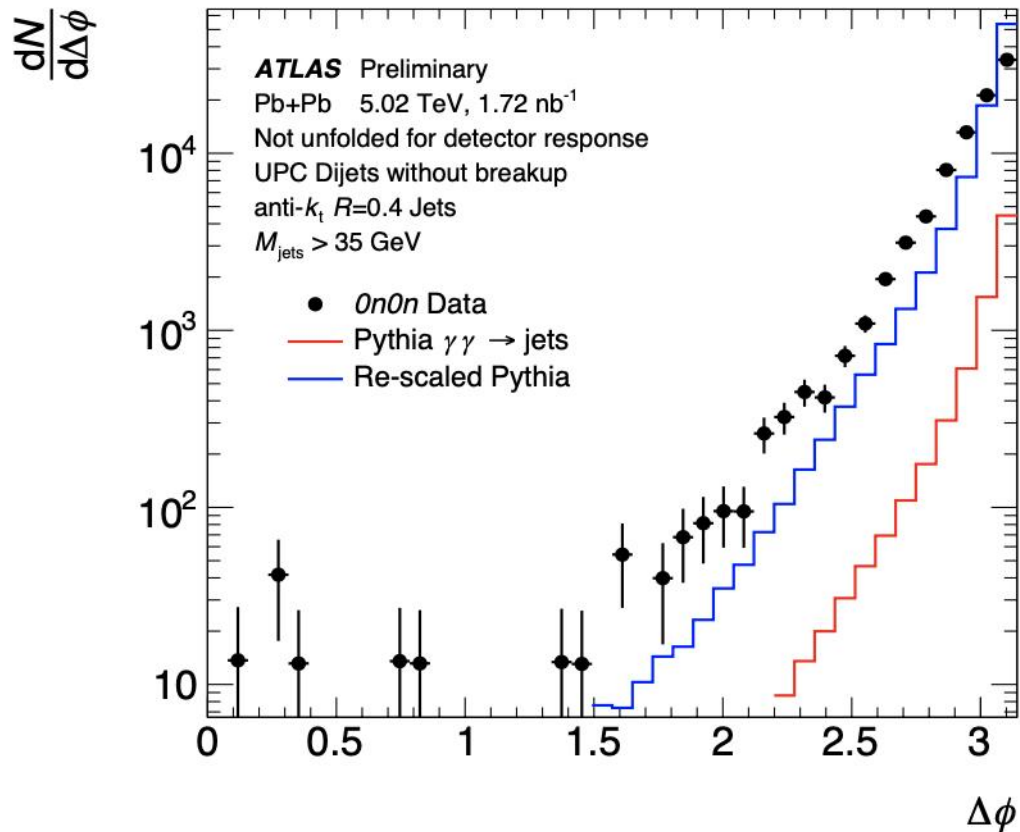
Evolve the hard and jet function at NLL accuracy

$$\frac{d^4 \sigma^{\text{NLL}}}{dq_x dp_T dy_1 dy_2} = \int_0^\infty \frac{db_x}{\pi} \cos(q_x b_x) \tilde{B}(b_x, p_T, y_1, y_2) \times \exp \left[\int_{\mu_h}^{\mu_b} \frac{d\mu}{\mu} \Gamma_H(\alpha_s) + 2 \int_{\mu_j}^{\mu_b} \frac{d\mu}{\mu} \Gamma_J(\alpha_s) \right]$$

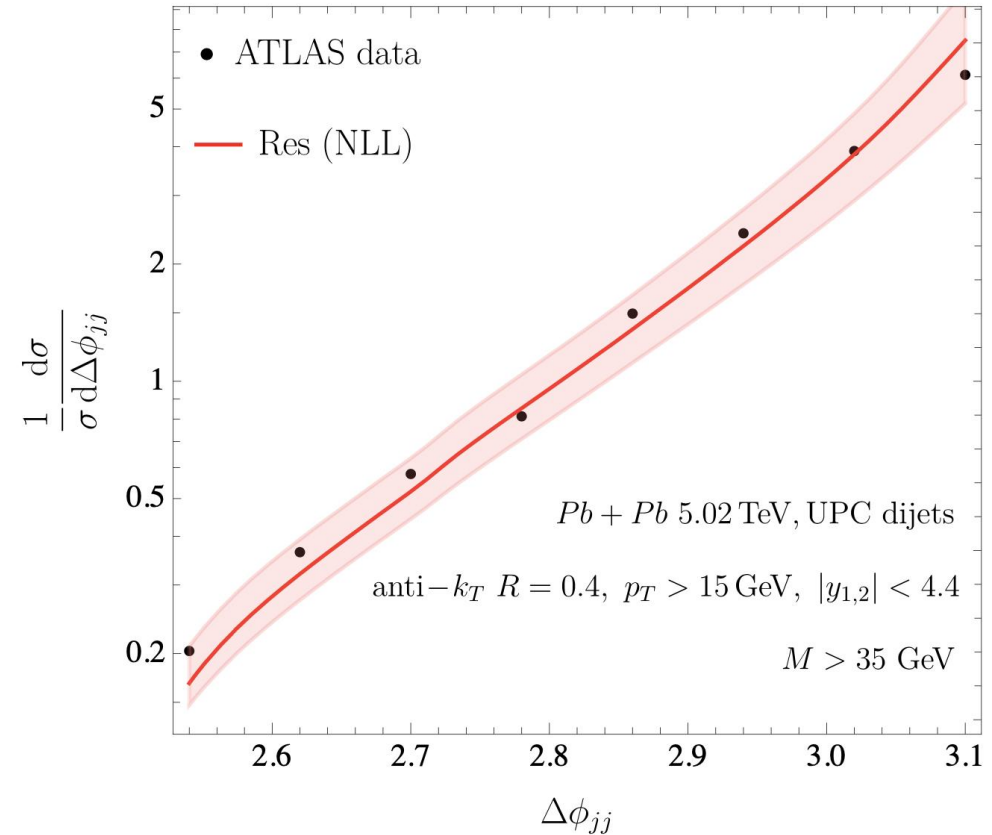
Acoplanarity

for $\gamma\gamma$ processes. Also, the data fall off more steeply with increasing H_T than the PYTHIA 8 and the measured $\Delta\phi$ distribution is noticeably wider than that in the PYTHIA 8 MC.

QM2022, ATLAS-CONF-2022-021

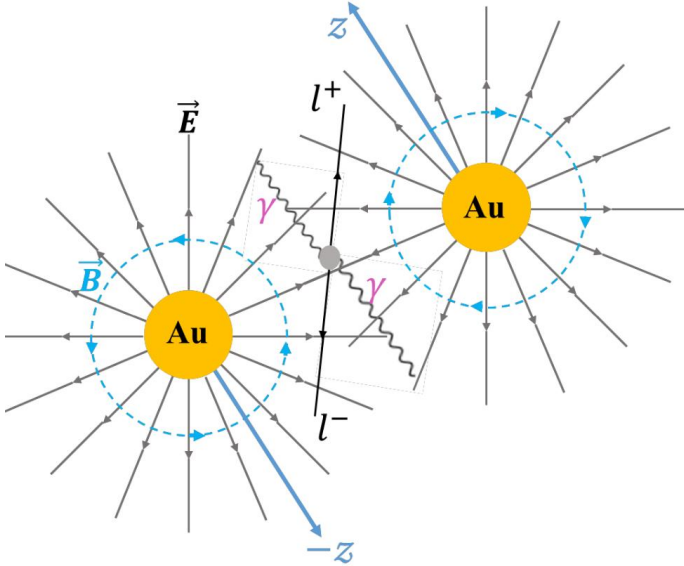


C.Z., Q.S. Dai and D.Y. Shao, JHEP02(2023)



Di-lepton (e^+e^-) production

$$\gamma(x_1P + k_{1\perp}) + \gamma(x_2\bar{P} + k_{2\perp}) \rightarrow l^+(p_1) + l^-(p_2) \quad \text{J.Adam et al. (STAR) 2021 Phys. Rev. Lett. 127 52302.}$$

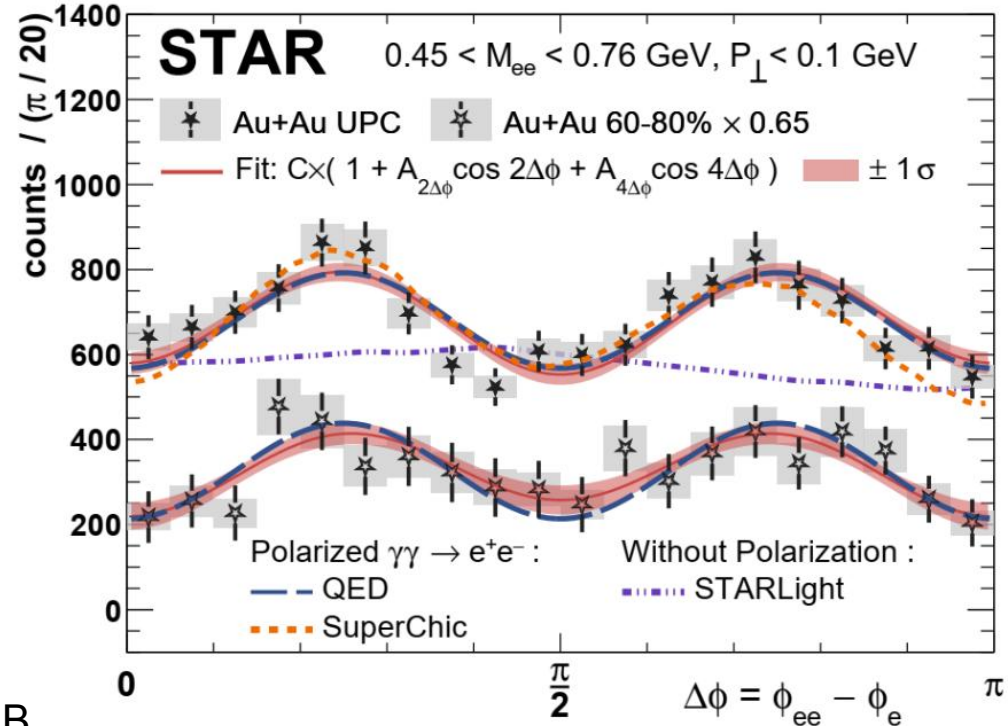


C. Li, J. Zhou and Y. Zhou, 2020

W. Zha, L. Ruan, Z. Tang, Z. Xu, S. Yang, 2018 Phys. Lett.B

BW. Xiao, and F. Yuan, J. Zhou, 2020 Phys.Rev.Lett

RJ. Wang, Shi Pu, and Qun Wang, 2021 Phys.Rev.D



Azimuthal asymmetry

$$\langle \cos(n\phi) \rangle = \frac{\int \frac{d\sigma}{d\mathcal{P}.S.} \cos(n\phi) d\mathcal{P}.S.}{\int \frac{d\sigma}{d\mathcal{P}.S.} d\mathcal{P}.S.}$$

$$\phi = \phi_{ee} - \phi_e$$

e^+e^- production in UPCs $M \gg q_\perp \gtrsim m$

Resummation formula that includes lepton mass resummation

$$\frac{d\sigma(q_\perp)}{d\mathcal{P}.\mathcal{S}} = H(M, \mu) J^2(m, \mu) \int d^2l_\perp d^2k_{1\perp} d^2k_{2\perp} \frac{d\sigma_0(q_\perp - l_\perp - k_{1\perp} - k_{2\perp})}{d\mathcal{P}.\mathcal{S}} \\ \times S(l_\perp, \Delta y, \mu) C_1(k_{1\perp}, P_\perp, y_1, m, \mu) C_2(k_{2\perp}, P_\perp, y_2, m, \mu),$$

Anomalous dimension

$$\Gamma_H = \frac{\alpha_e}{4\pi} \left(8 \ln \frac{M^2}{\mu^2} - 12 \right) \quad \Gamma_S = \frac{\alpha_e}{4\pi} \left(8 \ln \frac{\mu^2 r_\perp^2}{b_0^2} + 8 \ln \cos^2 \phi_r - 8 \ln \frac{1 + \cosh \Delta y}{2} \right) \\ \Gamma_J = \frac{\alpha_e}{4\pi} \left(4 \ln \frac{\mu^2}{m^2} + 2 \right) \quad \Gamma_{C_{1,2}} = \frac{\alpha_e}{4\pi} \left(-4 \ln \frac{4P_\perp^2 \mu^2 r_\perp^2}{b_0^2 m^2} + 4 - 4 \ln \cos^2 \phi_r \pm 4i\pi \right)$$

RG consistency relations $\Gamma_H + \Gamma_S + 2\Gamma_J + \Gamma_{C_1} + \Gamma_{C_2} = 0$.

Sudakov factor

$$\text{Sud}(r_\perp) = \int_{\mu_r}^M \frac{d\mu}{\mu} \Gamma_H + 2 \int_{\mu_r}^m \frac{d\mu}{\mu} \Gamma_J + \int_{\mu_r}^{\mu_r m / (2P_\perp)} \frac{d\mu}{\mu} \Gamma_{C_1} + \int_{\mu_r}^{\mu_r m / (2P_\perp)} \frac{d\mu}{\mu} \Gamma_{C_2} \\ \text{Sud}(r_\perp) \Big|_{\text{DL}, \Delta y=0} = \boxed{\frac{\alpha_e}{\pi} \ln \frac{M^2}{m^2} \ln \frac{P_\perp^2}{\mu_r^2}} + \frac{\alpha_e}{\pi} \ln \frac{M^2}{m^2} \ln 4 \cos^2 \phi_r$$

Soft photon resummed cross section

$$\frac{d\sigma}{d^2p_{1\perp}d^2p_{2\perp}dy_1dy_2d^2b_{\perp}} = \int \frac{d^2r_{\perp}}{(2\pi)^2} e^{ir_{\perp}\cdot q_{\perp}} e^{-\text{Sud}(r_{\perp})} \int d^2q'_{\perp} e^{-ir_{\perp}\cdot q'_{\perp}} \frac{d\sigma_0(q'_{\perp})}{d\mathcal{P}\cdot\mathcal{S}}$$

$$\text{Sud}(r_{\perp}) \Big|_{\text{DL}, \Delta y=0} = \frac{\alpha_e}{\pi} \ln \frac{M^2}{m^2} \ln \frac{P_{\perp}^2}{\mu_r^2} + \frac{\alpha_e}{\pi} \ln \frac{M^2}{m^2} \ln 4 \cos^2 \phi_r$$

Born cross section of dilepton

$$\frac{d\sigma_0}{d^2q_{\perp}d^2P_{\perp}dy_1dy_2d^2b_{\perp}} = A_0 + A_2 \cos 2\phi + A_4 \cos 4\phi$$

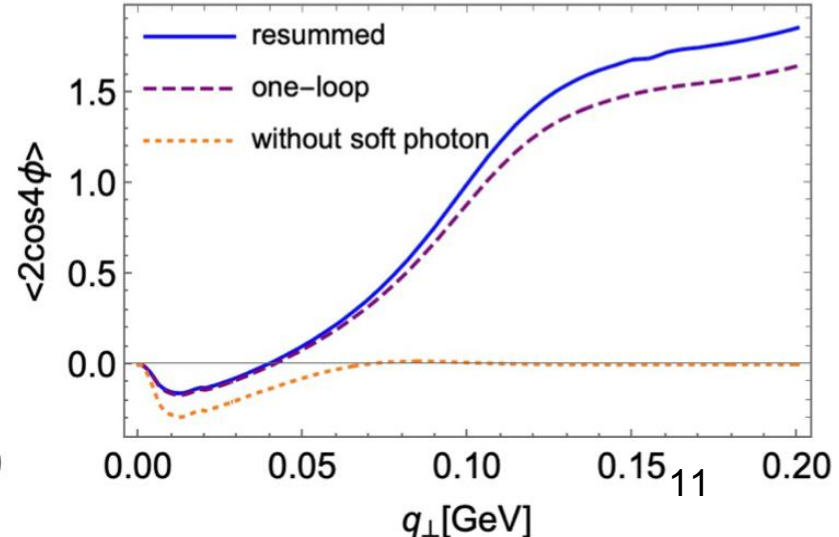
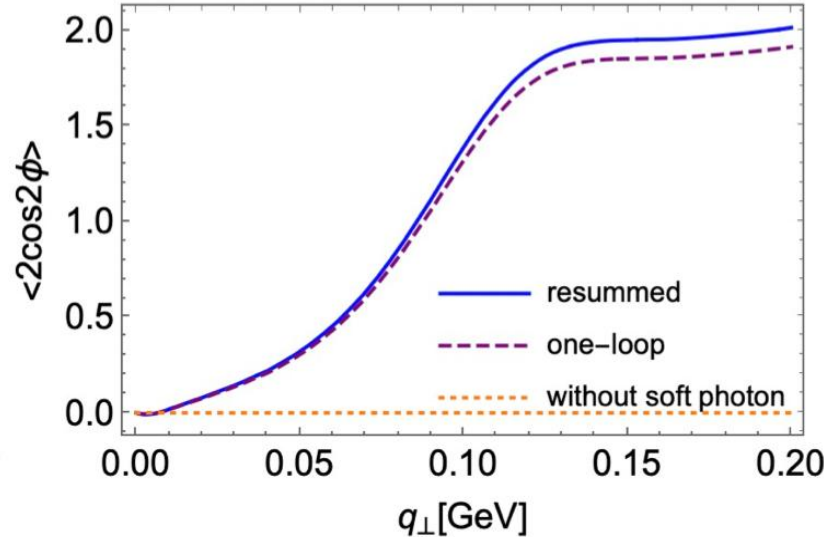
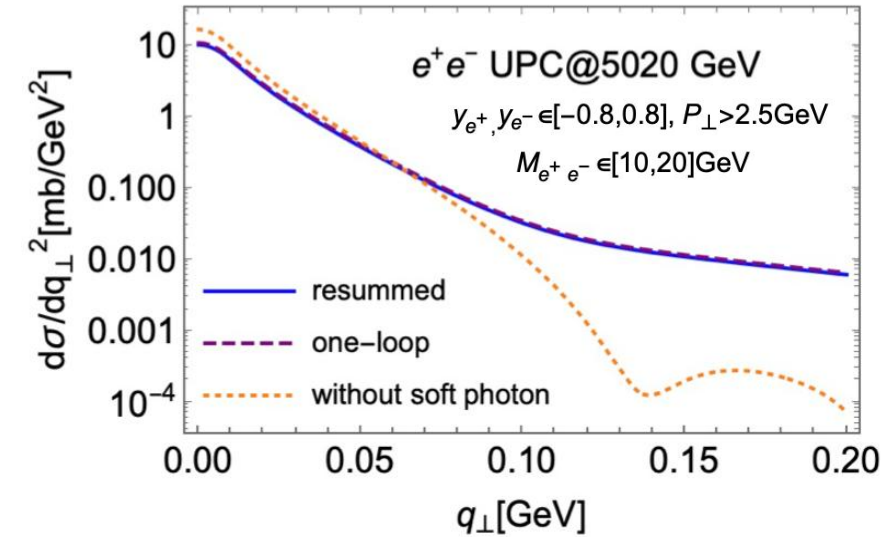
Azimuthal asymmetry

Born cross section

$$A_0 = \int [d\mathcal{K}_{\perp}] \frac{1}{(P_{\perp}^2 + m^2)^2} \left[-2m^4 \cos(\phi_{k_{1\perp}} + \phi_{k'_{1\perp}} - \phi_{k_{2\perp}} - \phi_{k'_{2\perp}}) + P_{\perp}^2 (M^2 - 2P_{\perp}^2) \cos(\phi_{k_{1\perp}} - \phi_{k'_{1\perp}} + \phi_{k_{2\perp}} - \phi_{k'_{2\perp}}) \right],$$

$$A_2 = \int [d\mathcal{K}_{\perp}] \frac{8m^2 P_{\perp}^2}{(P_{\perp}^2 + m^2)^2} \cos(\phi_{k_{1\perp}} - \phi_{k_{2\perp}}) \cos(\phi_{k'_{1\perp}} + \phi_{k'_{2\perp}} - 2\phi)$$

$$A_4 = \int [d\mathcal{K}_{\perp}] \frac{-2P_{\perp}^4}{(P_{\perp}^2 + m^2)^2} \cos(\phi_{k_{1\perp}} + \phi_{k'_{1\perp}} + \phi_{k_{2\perp}} + \phi_{k'_{2\perp}} - 4\phi)$$



Factorization formula at low α

$$\frac{d\sigma(\alpha)}{d\mathcal{P}.\mathcal{S}.} = 2P_{\perp} H(M, \mu) J^2(m, \mu) \int dl_x dk_{1,x} dk_{2,x} \frac{d\sigma_0(q_x - l_x - k_{1x} - k_{2x})}{d\mathcal{P}.\mathcal{S}.}$$

$$\times S(l_x, \Delta y, \mu, \nu) C_1(k_{1x}, P_{\perp}, y_1, m, \mu, \nu) C_2(k_{2x}, P_{\perp}, y_2, m, \mu, \nu)$$

Sudakov factor for α distribution

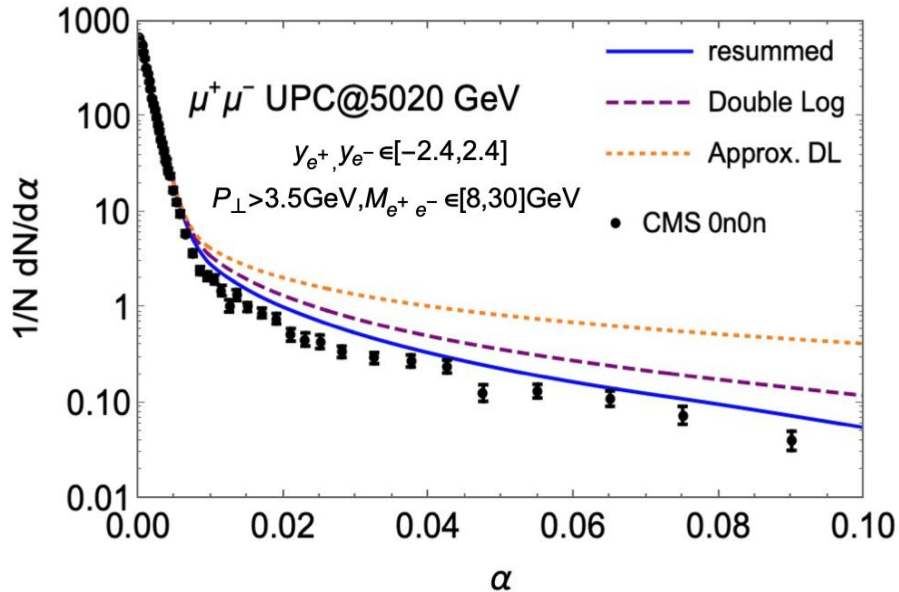
$$\text{Sud}_\alpha(r_x) = \int_{\mu_{rx}}^M \frac{d\mu}{\mu} \Gamma_H + 2 \int_{\mu_{rx}}^m \frac{d\mu}{\mu} \Gamma_J \theta(m - \mu_{rx})$$

$$= \frac{\alpha_e}{2\pi} \left[\left(\ln^2 \frac{M^2}{\mu_{rx}^2} - 3 \ln \frac{M^2}{\mu_{rx}^2} \right) \right]$$

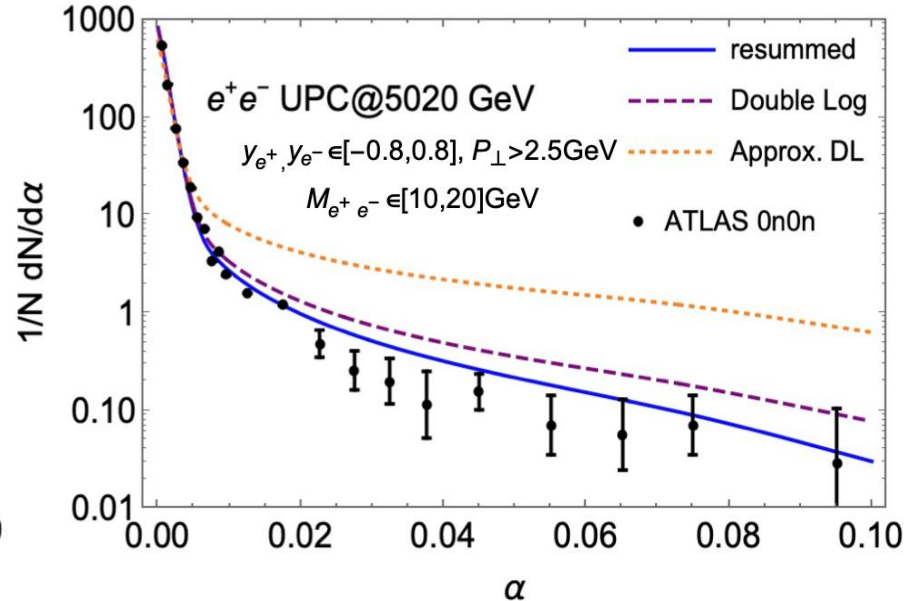
$$|q_x| \equiv p_T \sin(\pi - \Delta\phi_{jj}) = p_T \sin \alpha$$

Acoplanarity

CMS, 2021 Phys.Rev.Lett.



ATLAS, 2207.12781

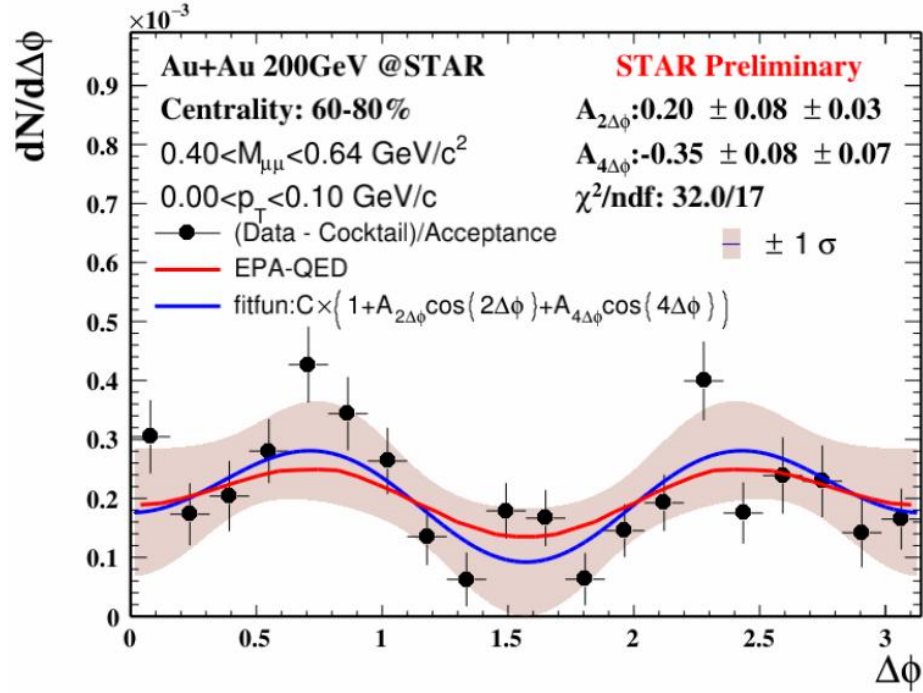
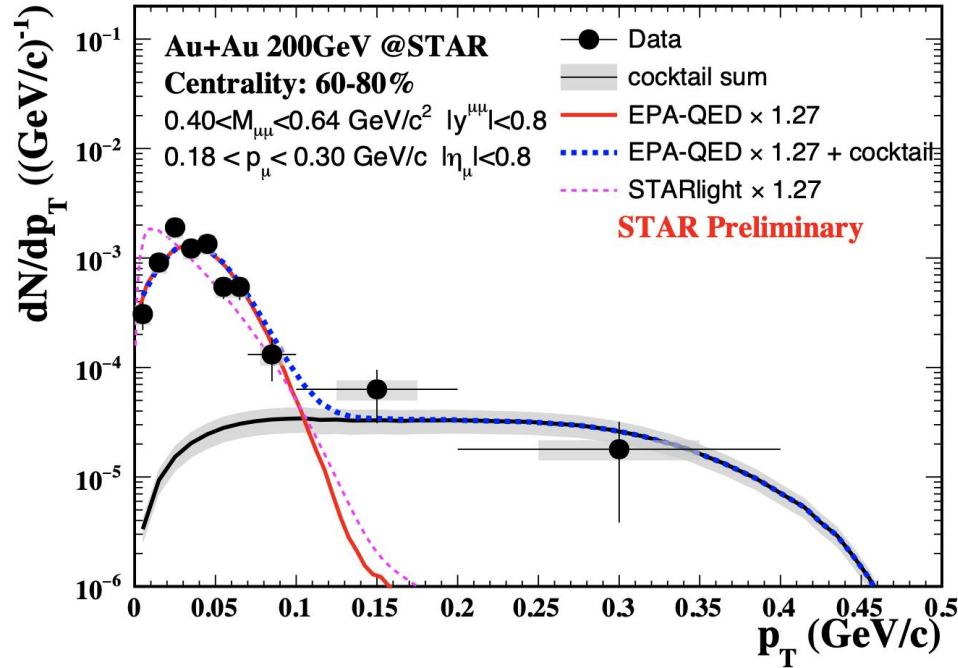


Di-lepton ($\mu^+\mu^-$) production

$$M \sim m \gg q_{\perp}$$

$\mu^+\mu^-$ pairs

STAR QM 2022



Azimuthal asymmetry

$$\langle \cos(n\phi) \rangle = \frac{\int \frac{d\sigma}{d\mathcal{P}.S.} \cos(n\phi) d\mathcal{P}.S.}{\int \frac{d\sigma}{d\mathcal{P}.S.} d\mathcal{P}.S.}$$

$$\phi = \phi_{\mu\mu} - \phi_{\mu}$$

Azimuthal asymmetries of $\mu^+\mu^-$ production $M \sim m \gg q_\perp$

$$\gamma(x_1 P + k_{1\perp}) + \gamma(x_2 \bar{P} + k_{2\perp}) \rightarrow l^+(p_1) + l^-(p_2)$$

$$\frac{d\sigma_0}{d^2q_\perp d^2P_\perp dy_1 dy_2 d^2b_\perp} = A_0 + A_2 \cos 2\phi + A_4 \cos 4\phi, \quad \frac{d\sigma(q_\perp)}{d\mathcal{P}.S.} = \int d^2l_\perp \frac{d\sigma_0(q_\perp - l_\perp)}{d\mathcal{P}.S.} S(l_\perp)$$

Born cross section

$$A_0 = \int [d\mathcal{K}_\perp] \frac{1}{(P_\perp^2 + m^2)^2} \left[-2m^4 \cos(\phi_{k_{1\perp}} + \phi_{k'_{1\perp}} - \phi_{k_{2\perp}} - \phi_{k'_{2\perp}}) + m^2 (M^2 - 2m^2) \cos(\phi_{k_{1\perp}} - \phi_{k'_{1\perp}} - \phi_{k_{2\perp}} + \phi_{k'_{2\perp}}) \right. \\ \left. + P_\perp^2 (M^2 - 2P_\perp^2) \cos(\phi_{k_{1\perp}} - \phi_{k'_{1\perp}} + \phi_{k_{2\perp}} - \phi_{k'_{2\perp}}) \right], \quad (3)$$

$$A_2 = \int [d\mathcal{K}_\perp] \frac{8m^2 P_\perp^2}{(P_\perp^2 + m^2)^2} \cos(\phi_{k_{1\perp}} - \phi_{k_{2\perp}}) \cos(\phi_{k'_{1\perp}} + \phi_{k'_{2\perp}} - 2\phi), \quad (4)$$

$$A_4 = \int [d\mathcal{K}_\perp] \frac{-2P_\perp^4}{(P_\perp^2 + m^2)^2} \cos(\phi_{k_{1\perp}} + \phi_{k'_{1\perp}} + \phi_{k_{2\perp}} + \phi_{k'_{2\perp}} - 4\phi). \quad (5)$$

Resummed cross section

$$\frac{d\sigma(q_\perp)}{d\mathcal{P}.S.} = \int \frac{d^2r_\perp}{(2\pi)^2} \left[1 - \frac{2\alpha_e c_2}{\pi} \cos 2\phi_r + \frac{\alpha_e c_4}{\pi} \cos 4\phi_r \right] e^{ir_\perp \cdot q_\perp} e^{-\text{Sud}(r_\perp)} \int d^2q'_\perp e^{ir_\perp \cdot q'_\perp} \frac{d\sigma_0(q'_\perp)}{d\mathcal{P}.S.}$$

Soft function

$$S(l_{\perp}, m, M) = \sum_{i,j} \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2) \theta(k^0) \frac{e^{2v_i \cdot v_j}}{v_i \cdot k v_j \cdot k} \times \delta^{(2)}(l_{\perp} - k_{\perp})$$

Massless limit $M \gg m$

$$S(l_{\perp}) = \delta(l_{\perp}) + \frac{\alpha_e}{\pi^2 l_{\perp}^2} \{c_0 + 2c_2 \cos 2\phi_l + 2c_4 \cos 4\phi_l + \dots\}$$

$$c_0 \approx \ln \frac{M^2}{m^2}, \quad c_2 \approx \ln \frac{M^2}{m^2} - 4 \ln 2 \quad \text{and} \quad c_4 \approx \ln \frac{M^2}{m^2} - 4$$

$$\text{Sud}(r_{\perp}) = \frac{\alpha_e}{\pi} \ln \frac{M^2}{m^2} \ln \frac{P_{\perp}^2}{\mu_r^2}$$

Y. Hatta, B.W. Xiao, F. Yuan, and J. Zhou,
2021 Phys. Rev. Lett

Massive soft function $M \sim m$

S. Catani, M. Grazzini, A. Torre, 2014 Nucl.Phys. B

$$e^{-\text{Sud}(r_{\perp})} \left[1 + \frac{\alpha_e}{4\pi} (s_{11} + s_{22} + 2s_{12}) \right]$$

ere the one-loop Sudakov factor is given by

$$\text{Sud}(r_{\perp}) = \frac{\alpha_e}{\pi} \ln \frac{P_{\perp}^2}{\mu_r^2} \left(1 - \frac{1 + \beta^2}{2\beta} \ln \frac{1 - \beta}{1 + \beta} \right)$$

$$s_{11} = s_{22} = -\frac{4c_r}{\sqrt{c_r^2 + 1}} \ln \left(\sqrt{c_r^2 + 1} + c_r \right), \quad (13)$$

$$s_{12} = -\frac{1 + \beta^2}{2\beta} \text{sign}(c_r) \left[L_{\zeta} [\zeta(c_r, \alpha_r), \alpha_r] - L_{\zeta} [\zeta(-c_r, \alpha_r), \alpha_r] \right], \quad (14)$$

with

$$c_r = \cos \phi_r P_{\perp} / m, \quad \beta = \sqrt{1 - 4m^2 / M^2},$$

$$\alpha_r = \frac{2P_{\perp}^2 \cos^2 \phi_r}{-m^2 + P_{\perp}^2 + (m^2 + P_{\perp}^2) \cosh(y_1 - y_2)},$$

$$\zeta(a, b) = \left(a + \sqrt{1 + a^2} \right) \left(a + \sqrt{a^2 + b} \right),$$

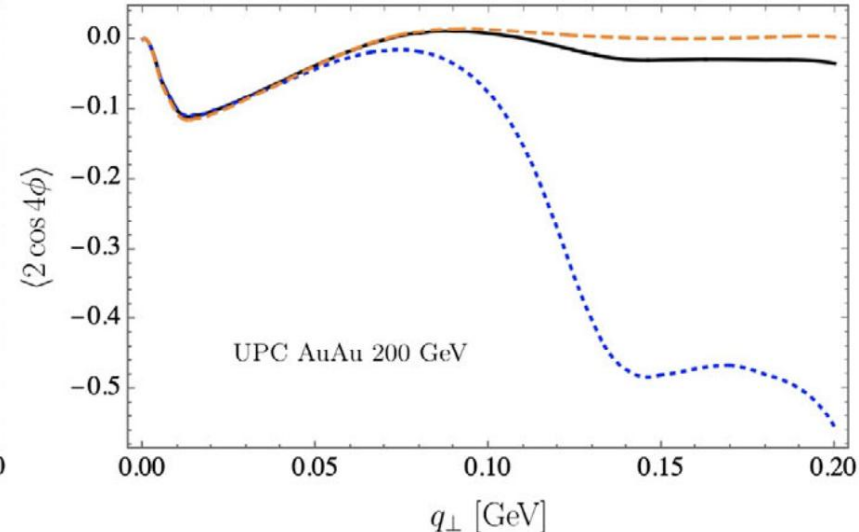
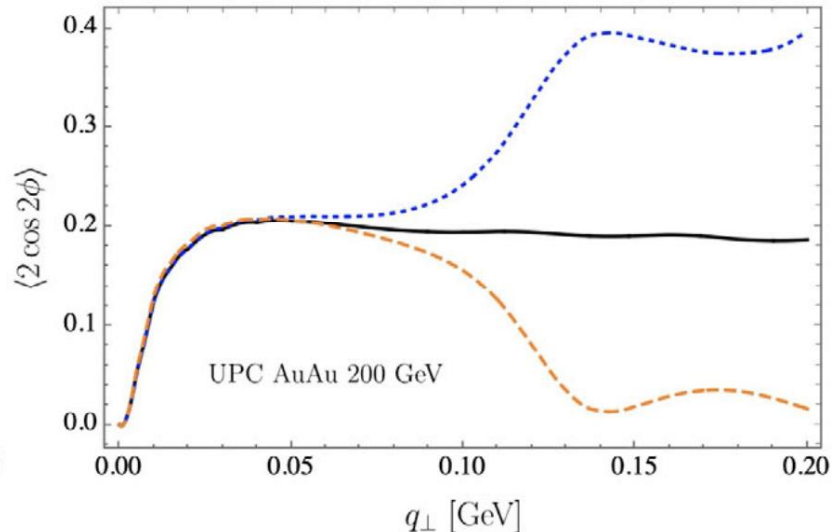
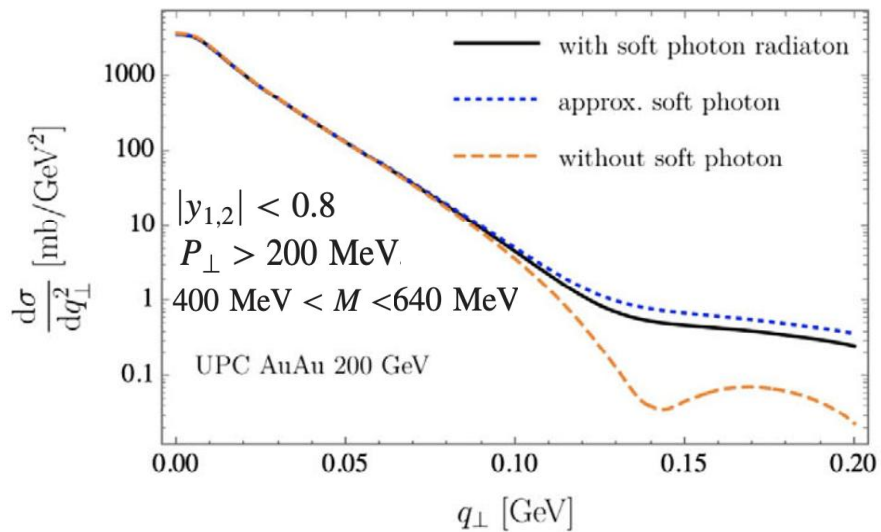
$$L_{\zeta}(a, b) = 2 \left[-\text{Li}_2 \left(\frac{a+b}{b-1} \right) + \text{Li}_2(-a) \right.$$

$$\left. + \ln(a+b) \ln(1-b) \right] - \ln^2 \left(\frac{a}{a+b} \right) + \frac{1}{2} \ln^2 \left[\frac{a(a+1)}{a+b} \right]. \quad 15$$

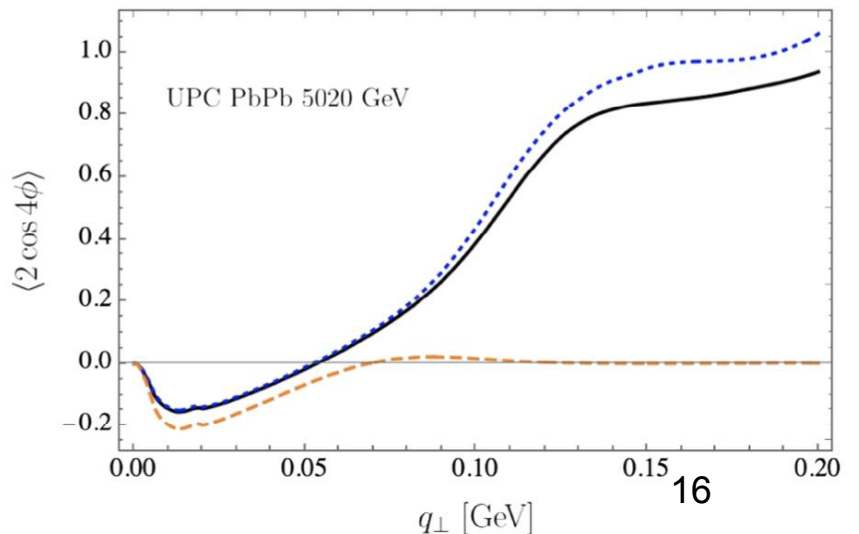
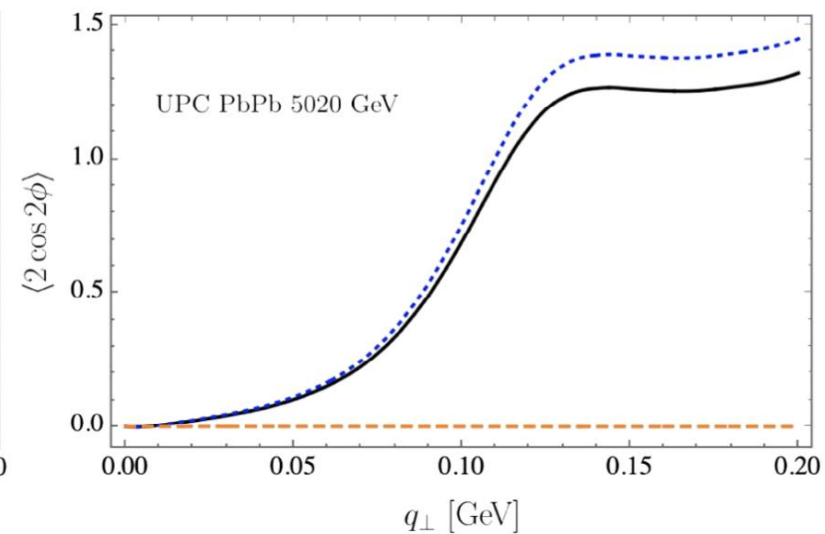
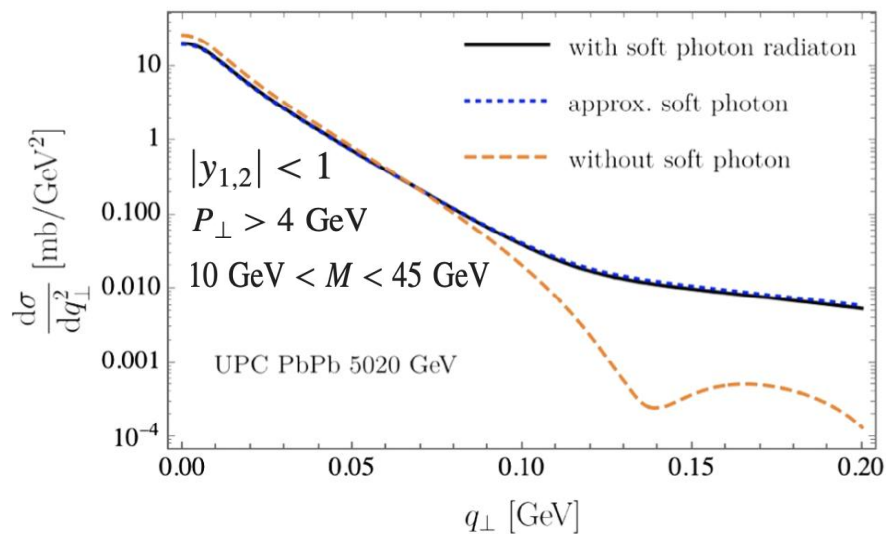
Azimuthal asymmetry

RHIC

D.Y. Shao, C.Z., J. Zhou, Y.j. Zhou, Phys.Rev.D(2023)



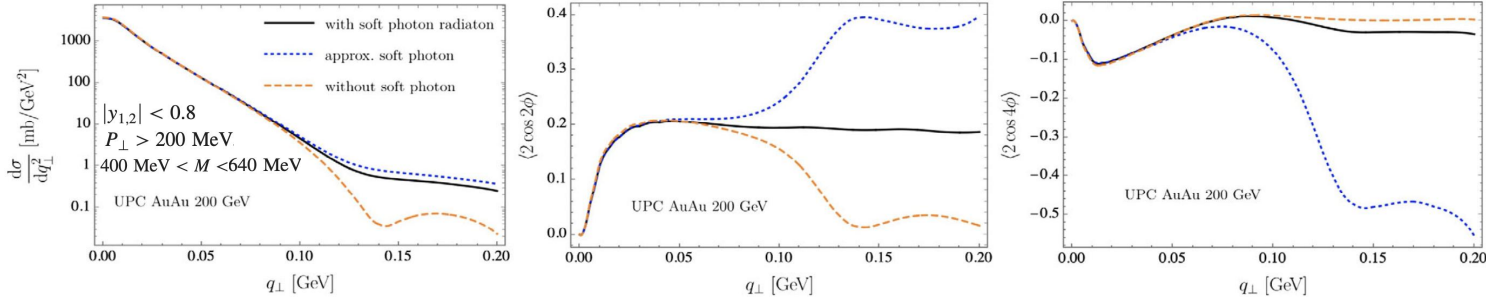
LHC



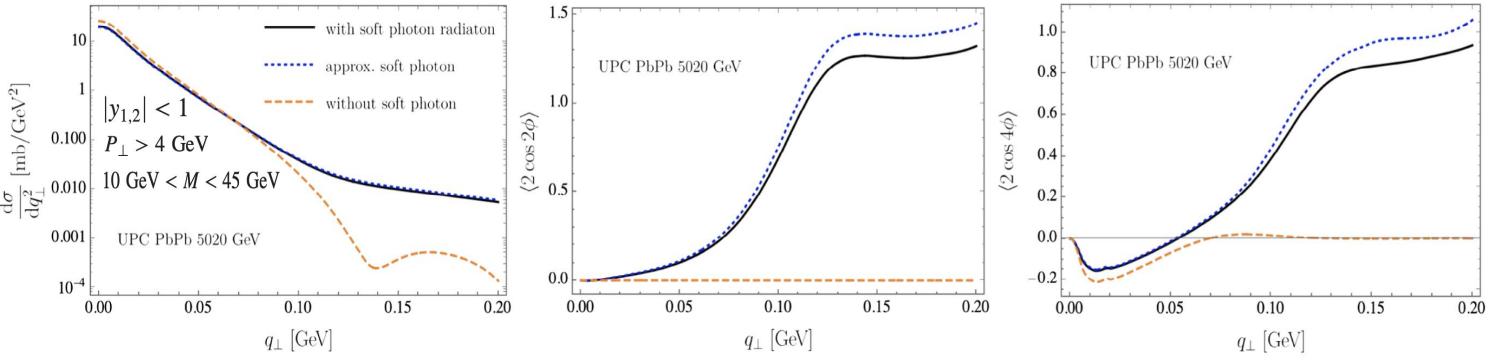
Azimuthal asymmetry

D.Y. Shao, C.Z., J. Zhou, Y.j. Zhou, Phys.Rev.D(2023)

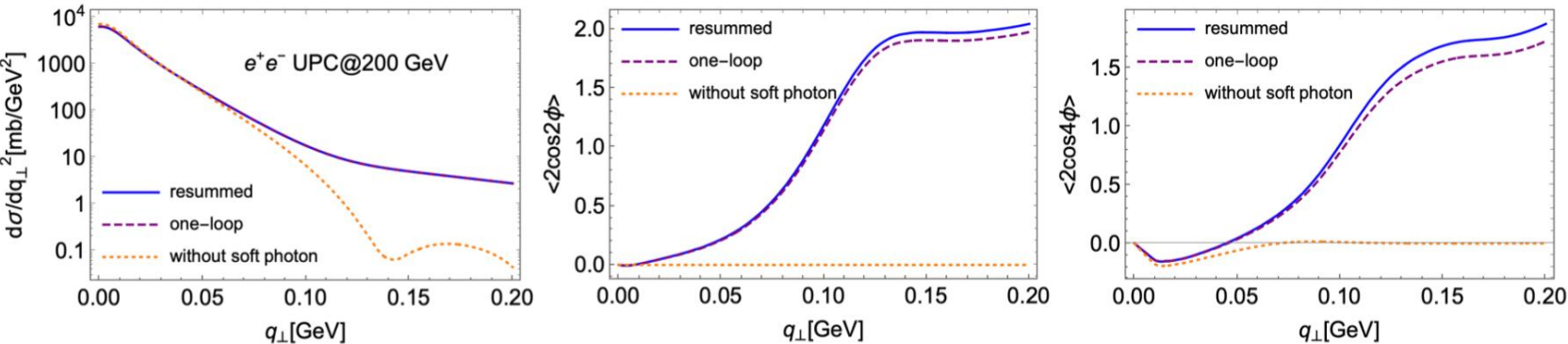
RHIC $\mu+\mu-$



LHC $\mu+\mu-$



RHIC $e+e-$

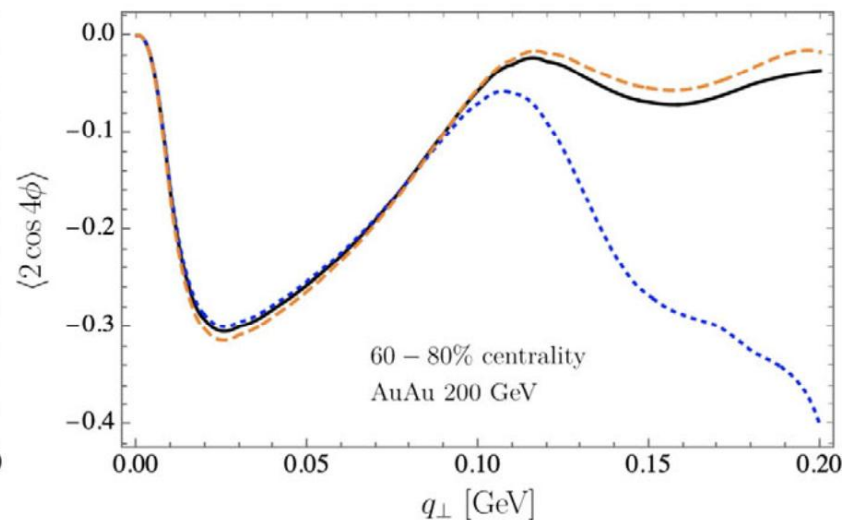
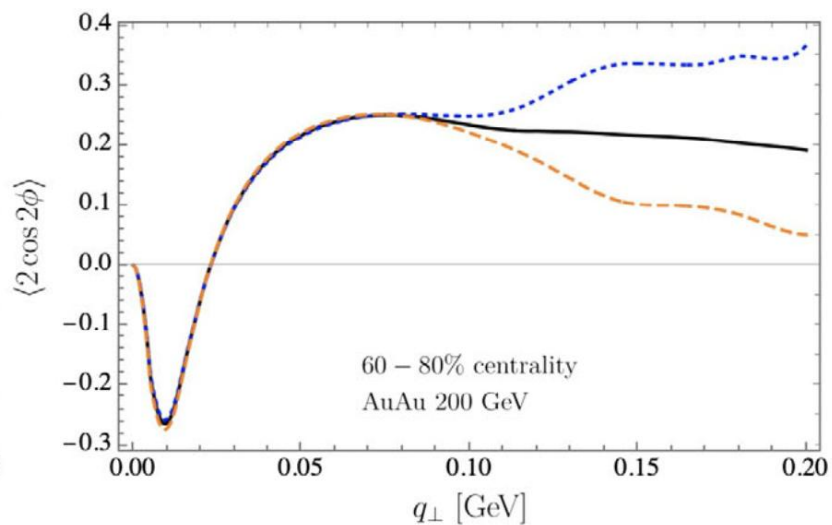
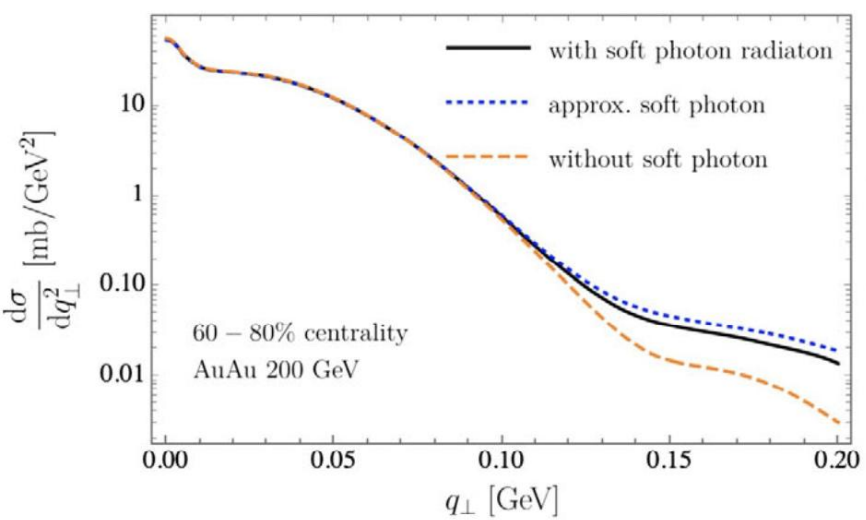


$$\text{Sud}(r_{\perp}) = \frac{\alpha_e}{\pi} \ln \frac{M^2}{m^2} \ln \frac{P_{\perp}^2}{\mu_r^2}$$

Summary

- Dijet productions in UPCs: initial EPA photon & final QCD radiations leads to the factorization and resummation formula at NLL accuracy in SCET. Good agreement with the ATLAS acoplanarity, wider than Pythia8.
- e^+e^- productions: single logarithm accuracy of resummation formula is essential to describe the acoplanarity data from ATLAS and CMS.
- $\mu^+\mu^-$ productions: muon mass corrections to the asymmetries is quite sizable at large q_\perp . At low q_\perp , the asymmetries are mainly induced by the linearly-polarized coherent photons.

RHIC $\mu+\mu^-$



$$\begin{aligned}
\tilde{S}(b_x, y_1, y_2, \mu, \nu) &= 1 + 2 C_F g_s^2 \tilde{\mu}^{2\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2) \theta(k^0) \left(\frac{\nu}{2k^0} \right)^\eta \frac{n_1 \cdot n_2}{n_1 \cdot k k \cdot n_2} e^{-i b_x k_x} \\
&= 1 + \frac{\alpha_s}{4\pi} C_F \left[- \left(\frac{2}{\eta} + \ln \frac{\nu^2 n_1 \cdot n_2}{2\mu^2} \right) \left(\frac{4}{\epsilon} + 4 \ln \frac{\mu^2 b_x^2}{b_0^2} \right) + \frac{4}{\epsilon^2} - 2 \ln^2 \frac{\mu^2 b_x^2}{b_0^2} - \frac{\pi^2}{3} \right],
\end{aligned}$$

with $\tilde{\mu}^2 = \mu^2 e^{\gamma_E} / (4\pi)$ and $b_0 = 2e^{-\gamma_E}$. Here the rapidity regulator is given in (3.9).

$$\begin{aligned}
\tilde{U}_i(b_\perp, y_i, \mu, \nu) &= 1 + 2 C_F g_s^2 \tilde{\mu}^{2\epsilon} \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2) \theta(k^0) \left(\frac{\nu}{\bar{n}_i \cdot k} \right)^\eta \frac{n_i \cdot \bar{n}_i}{n_i \cdot k k \cdot \bar{n}_i} \\
&\times \left\{ e^{-i b_\perp \cdot k_\perp} \theta \left[\frac{n_i \cdot k}{\bar{n}_i \cdot k} - \left(\frac{R}{2 \cosh y_i} \right)^2 \right] + \theta \left[\left(\frac{R}{2 \cosh y_i} \right)^2 - \frac{n_i \cdot k}{\bar{n}_i \cdot k} \right] \right\} \\
&= 1 + \frac{\alpha_s}{4\pi} C_F \left[\left(\frac{2}{\eta} + \ln \frac{\nu^2 R^2}{4\mu^2 \cosh^2 y_i} \right) \left(\frac{2}{\epsilon} + 2 \ln \frac{\mu^2 b_\perp^2}{b_0^2} \right) - \frac{2}{\epsilon^2} + \ln^2 \frac{\mu^2 b_\perp^2}{b_0^2} + \frac{\pi^2}{6} \right]
\end{aligned}$$

$$\begin{aligned}
\tilde{S}(b_x, y_1, y_2, \mu, \nu) \tilde{U}_1(b_x, y_1, \mu, \nu) \tilde{U}_2(b_x, y_2, \mu, \nu) \\
= 1 + C_F \frac{\alpha_s}{\pi} \left[\ln R^2 - \ln(2 + 2 \cosh \Delta y) \right] \left(\frac{1}{\epsilon} + \ln \frac{b_x^2 \mu^2}{b_0^2} \right) + \mathcal{O}(\alpha_s^2)
\end{aligned}$$

$$J^{\text{NLO}}(m, \mu) = 1 + \frac{\alpha_e}{4\pi} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left(1 + 2 \ln \frac{\mu^2}{m^2} \right) + \left(1 + \ln \frac{\mu^2}{m^2} \right) \ln \frac{\mu^2}{m^2} + 4 + \frac{\pi^2}{6} \right]$$

the anomalous dimension associated with this jet function is given by

$$\Gamma_J = \frac{\alpha_e}{4\pi} \left(4 \ln \frac{\mu^2}{m^2} + 2 \right).$$

$$S_{n_i}(x) = \exp \left[-ie \int_{-\infty}^0 ds n_i \cdot A(x + s n_i) \right],$$

describe a point-like source traveling along the path $x^\mu + s n_i^\mu$ with the light-like vector n_i . The soft function

$$\tilde{S}(r_\perp, \Delta y) = \langle 0 | \bar{T} [S_{n_1}^\dagger(r_\perp) S_{n_2}(r_\perp)] T [S_{n_2}^\dagger(0) S_{n_1}(0)] | 0 \rangle,$$

denote the directions of final-state leptons. Expanding the Wilson line in the coefficient function is obtained as

$$\tilde{S}^{\text{NLO}}(r_\perp, \Delta y) = 1 + e_0^2 \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2) \theta(k^0) \frac{2n_1 \cdot n_2}{n_1 \cdot k k \cdot n_2} e^{i k_\perp \cdot r_\perp},$$

where e_0 is the bare electric charge, and k is the momentum of the final-state photon. Note that k is the momentum with the beam directions which is different from the direction of the integral, we obtain

$$\tilde{S}^{\text{NLO}}(r_\perp, \Delta y) = 1 + \frac{\alpha_e}{4\pi} \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \ln \frac{\mu^2 r_\perp^2}{b_0^2 A_r} + 2 \ln^2 \frac{\mu^2 r_\perp^2}{b_0^2 A_r} + \pi^2 - 4 \ln A_r \ln(1 - A_r) \right]$$

$$A_r = M^2 / (4P_\perp^2 \cos^2 \phi_r)$$

$$\Gamma_S = \frac{\alpha_e}{4\pi} \left(8 \ln \frac{\mu^2 r_\perp^2}{b_0^2} + 8 \ln \cos^2 \phi_r - 8 \ln \frac{1 + \cosh \Delta y}{2} \right)$$

$$\tilde{C}_i(r_\perp, P_\perp, y_i, m) = \langle 0 | \bar{T} [S_{v_i}^\dagger(r_\perp) S_{\bar{n}_i}(r_\perp)] T [S_{\bar{n}_i}^\dagger(0) S_{v_i}(0)] | 0 \rangle,$$

the soft Wilson line S_{v_i} is defined in analogy with S_{n_i} in Eq. (13), but with the light-like vector v_i , which is

$$v_i^\mu = \frac{\omega_i}{m} \frac{n_i^\mu}{2} + \frac{m}{\omega_i} \frac{\bar{n}_i^\mu}{2}, \quad \text{with } \omega_i = 2P_\perp \cosh y_i.$$

In the next loop, the perturbative expansion of collinear-soft function gives us

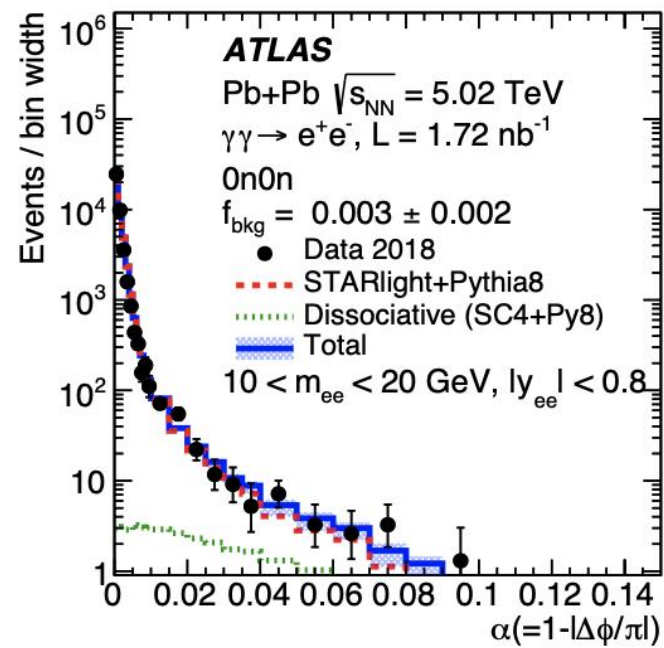
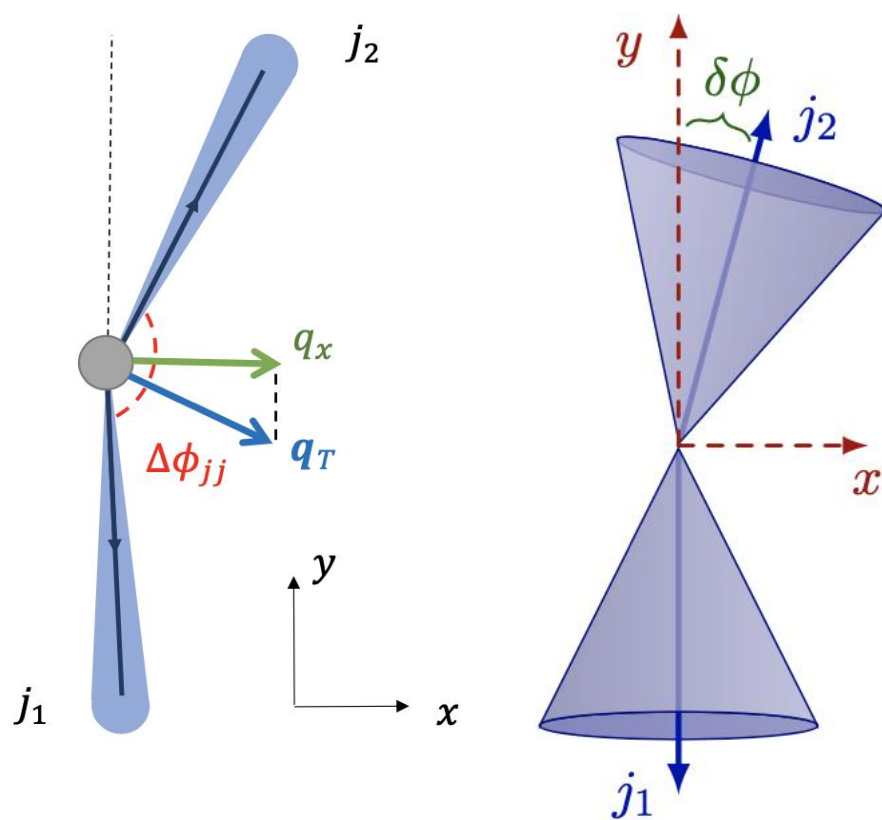
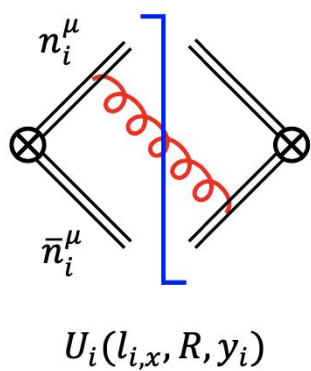
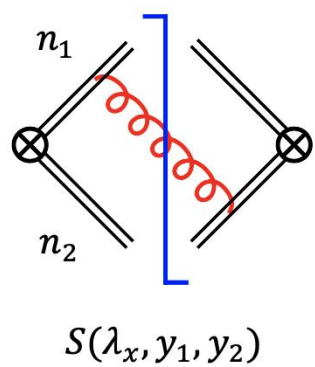
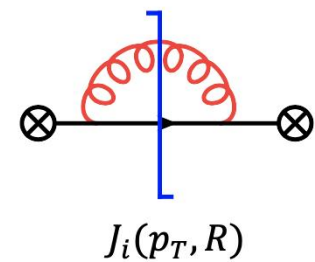
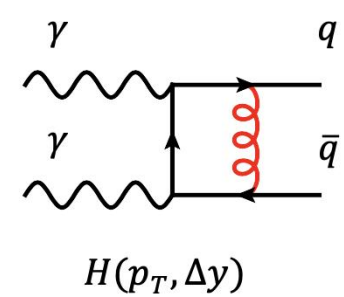
$$\tilde{C}_i^{\text{NLO}}(r_\perp, P_\perp, y_i, m) = 1 + e_0^2 \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2) \theta(k^0) \left(\frac{2v_i \cdot \bar{n}_i}{v_i \cdot k k \cdot \bar{n}_i} - \frac{v_i \cdot v_i}{v_i \cdot k k \cdot v_i} \right)$$

we obtain

$$\tilde{C}_i^{\text{NLO}}(r_\perp, P_\perp, y_i, m) = 1 + \frac{\alpha_e}{4\pi} \left[-\frac{2}{\epsilon^2} + \frac{2}{\epsilon} (1 - 2 \ln \mu R) - 4 \ln^2 \mu R + 4 \ln \mu R \right]$$

where $R = -iP_\perp e^{\gamma_E} n_i \cdot r_\perp / (m r_\perp)$, and the anomalous dimension is

$$\Gamma_{C_{1,2}} = \frac{\alpha_e}{4\pi} \left(-4 \ln \frac{4P_\perp^2 \mu^2 r_\perp^2}{b_0^2 m^2} + 4 - 4 \ln \cos^2 \phi_r \pm 4i\pi \right).$$



$$\begin{aligned}
\frac{d\sigma}{d^2k_{1\perp}d^2k_{2\perp}dy_1dy_2d^2\tilde{b}_\perp} &= \frac{2N_c\alpha_e e_q^2}{(2\pi)^2} \int d^2\Delta_\perp d^2k_\perp d^2k'_\perp \delta^2(k_\perp + \Delta_\perp - l_\perp) \int d^2q_\perp d^2q'_\perp \\
&\times 2z(1-z)[z^2 + (1-z)^2] \left[\left(\frac{\vec{P}_\perp}{P_\perp^2 + e_f^2} - \frac{\vec{P}_\perp - \vec{q}_\perp}{(P_\perp - q_\perp)^2 + e_f^2} \right) \cdot \hat{k}_\perp \right] \cdot \left[\left(\frac{\vec{P}_\perp}{P_\perp^2 + e_f^2} - \frac{\vec{P}_\perp - \vec{q}'_\perp}{(P_\perp - q'_\perp)^2 + e_f^2} \right) \cdot \hat{k}'_\perp \right] \\
&\left\{ \left[e^{i\tilde{b}_\perp \cdot (k'_\perp - k_\perp)} G(Y, q_\perp, \Delta_\perp) G(Y, q'_\perp, \Delta'_\perp) \mathcal{F}(Y, k_\perp) \mathcal{F}(Y, k'_\perp) \right] \right. \\
&+ \left[e^{i\tilde{b}_\perp \cdot (\Delta'_\perp - \Delta_\perp)} G(-Y, q_\perp, \Delta_\perp) G(-Y, q'_\perp, \Delta'_\perp) \mathcal{F}(-Y, k_\perp) \mathcal{F}(-Y, k'_\perp) \right] \\
&+ \left[e^{i\tilde{b}_\perp \cdot (\Delta'_\perp - k_\perp)} G(Y, q_\perp, \Delta_\perp) G(-Y, q'_\perp, \Delta'_\perp) \mathcal{F}(Y, k_\perp) \mathcal{F}(-Y, k'_\perp) \right] \\
&\left. + \left[e^{i\tilde{b}_\perp \cdot (k'_\perp - \Delta_\perp)} G(-Y, q_\perp, \Delta_\perp) G(Y, q'_\perp, \Delta'_\perp) \mathcal{F}(-Y, k_\perp) \mathcal{F}(Y, k'_\perp) \right] \right\},
\end{aligned}$$

The gluon distribution is given by,

$$G(q_\perp, \Delta_\perp) = \int \frac{d^2r_\perp}{(2\pi)^2} \frac{d^2b_\perp}{(2\pi)^2} e^{-iq_\perp \cdot r_\perp - i\Delta_\perp \cdot b_\perp} \frac{1}{N_c} \text{Tr} \left[U(b_\perp + \frac{r_\perp}{2}) U^\dagger(b_\perp - \frac{r_\perp}{2}) \right]$$

Spencer Klein, A. H. Mueller, Bo-Wen Xiao, and Feng Yuan, 2019 Phys.Rev.Lett