

# Feynman integrals and elliptic curves

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*Li Lin Yang  
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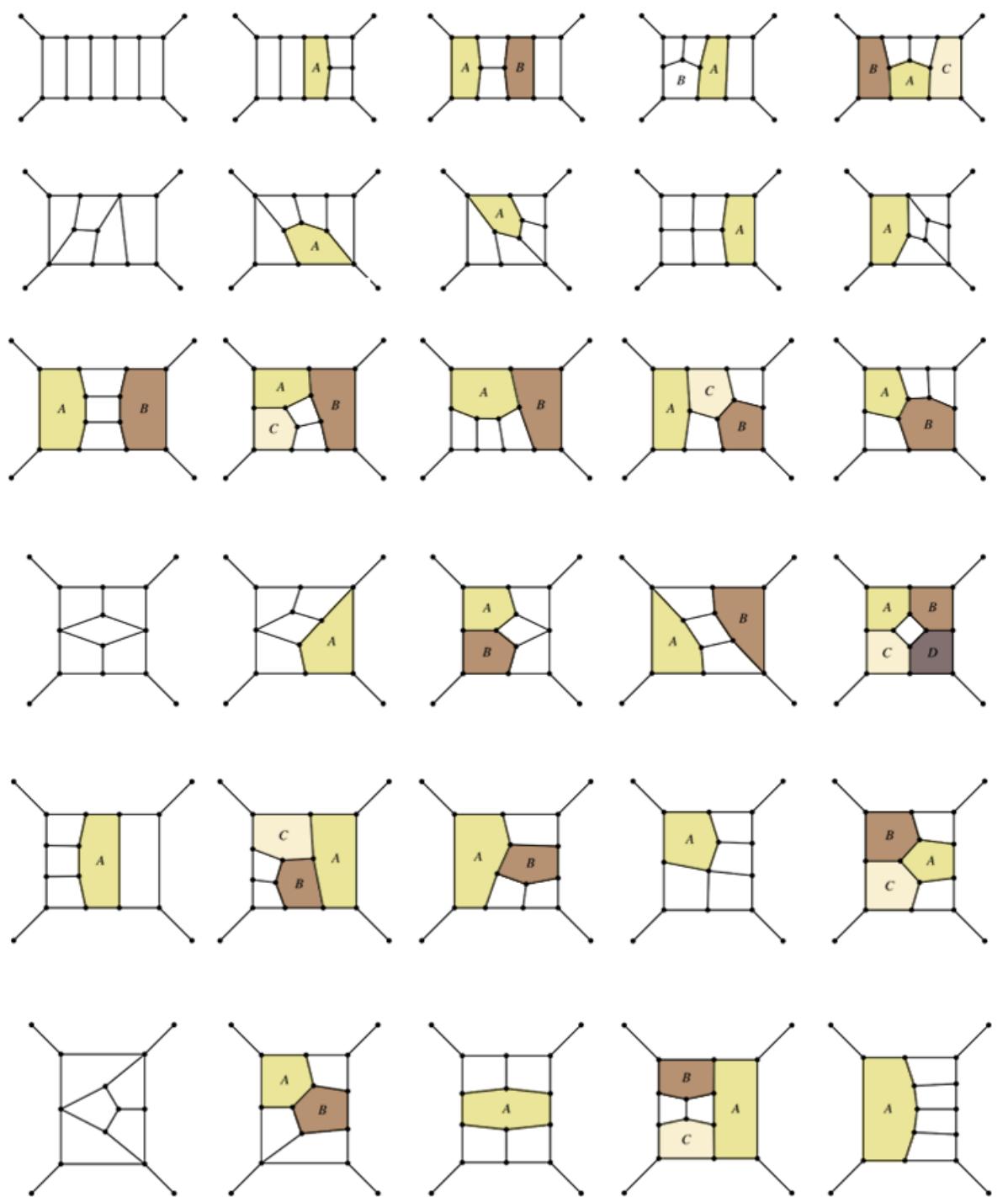
Based on collaborations with Xuhang Jiang, Xing Wang and Jing-bang Zhao

arXiv:2304.xxxxxx

# Multi-loop Feynman integrals

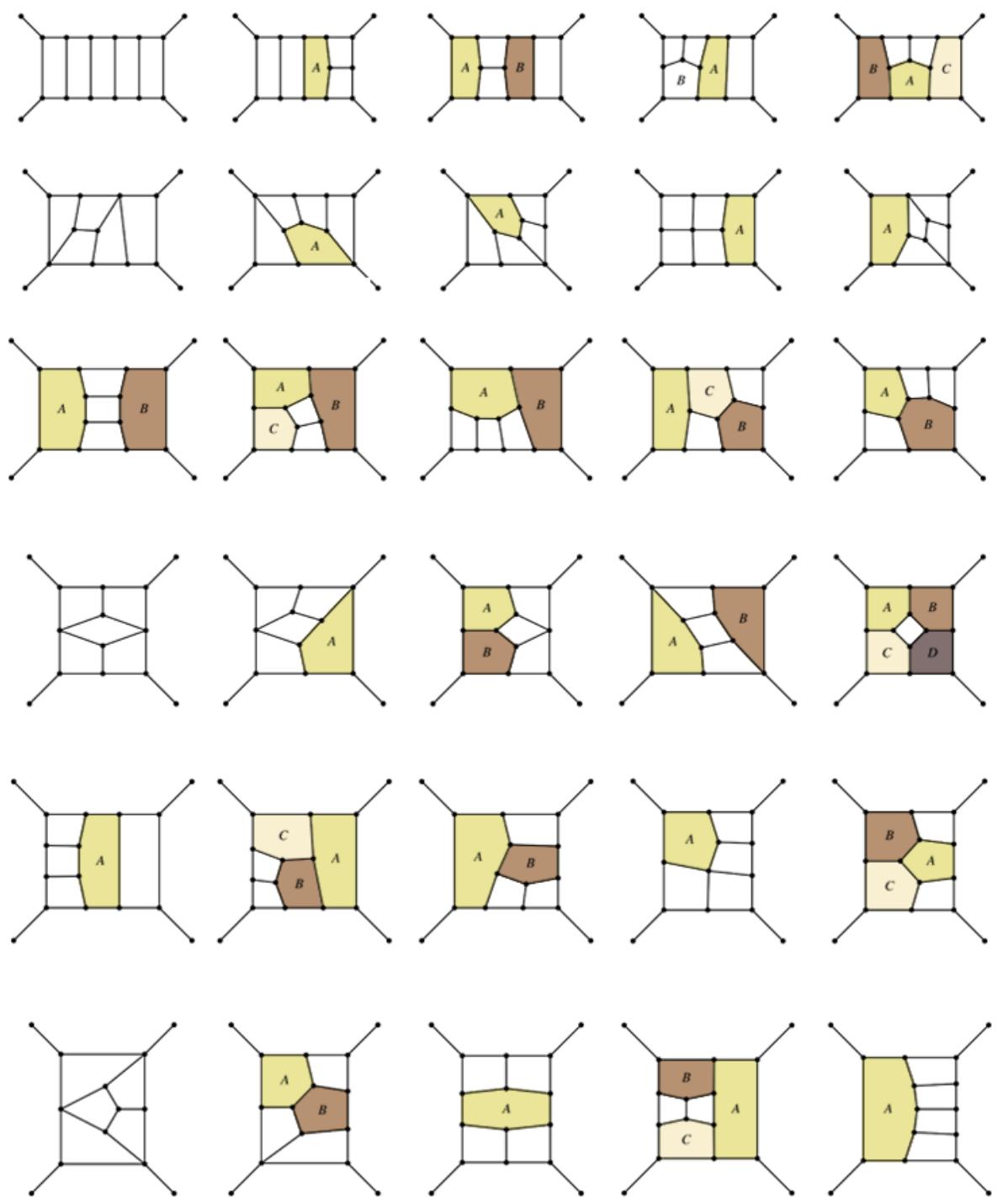
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One of the main bottlenecks towards higher precision predictions in perturbative QFTs



# Multi-loop Feynman integrals

One of the main bottlenecks towards higher precision predictions in perturbative QFTs



Numeric methods (e.g., SecDec, AMFlow, ...)

- General purpose
- Usually slow

[See talk by Y.-Q. Ma](#)

Analytic methods

- Limited applicability
- Fast numeric evaluation
- Algebraic and analytic properties
- Mathematical structure of QFTs

# What can be called “analytic expressions”?

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$$\begin{aligned}\Pi(k^2) = & \frac{1}{2}\alpha \int_0^1 dx D \ln(D/m^2) \\ & - \left\{ \frac{1}{6}\alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \frac{1}{2} \right] + A \right\} k^2 \\ & - \left\{ \alpha \left[ \frac{1}{\varepsilon} + \ln(\mu/m) + \frac{1}{2} \right] + B \right\} m^2 + O(\alpha^2)\end{aligned}$$

# What can be called “analytic expressions”?

---

$$\Pi(k^2) = \frac{1}{2}\alpha \int_0^1 dx D \ln(D/m^2)$$

$$\begin{aligned} & + 105(2G_{0,1/y} + G_{1/y,-1} + G_{1/y,1} - 2G_{1/y,1/y}) + 18(4G_{0,0,-1/y} + 2G_{0,-1/y,-1} + 2G_{0,-1/y,1} \\ & - 4G_{0,-1/y,-1/y} - G_{-1/y,-1,-1} + G_{-1/y,-1,1} + 2G_{-1/y,0,-1} + 2G_{-1/y,0,1} - 4G_{-1/y,0,-1/y} \\ & + G_{-1/y,1,-1} - G_{-1/y,1,1} - 2G_{-1/y,-1/y,-1} - 2G_{-1/y,-1/y,1} + 4G_{-1/y,-1/y,-1/y}) \\ & + 63(4G_{0,0,1/y} + 2G_{0,1/y,-1} + 2G_{0,1/y,1} - 4G_{0,1/y,1/y} - G_{1/y,-1,-1} + G_{1/y,-1,1} + 2G_{1/y,0,-1} \\ & + 2G_{1/y,0,1} - 4G_{1/y,0,1/y} + G_{1/y,1,-1} - G_{1/y,1,1} - 2G_{1/y,1/y,-1} - 2G_{1/y,1/y,1} + 4G_{1/y,1/y,1/y}) \end{aligned}$$

222    ε -2    7

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$$\begin{aligned}
& - z_{45}^{-1} \\
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& + 2\epsilon \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & 1-r_+ \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 0 & r_+ \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & 1-r_+ \end{smallmatrix}; 1\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1 & r_+ \end{smallmatrix}; 1\right) , \\
& - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 0 \end{smallmatrix}; 1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & 1 \end{smallmatrix}; 1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 0 \end{smallmatrix}; 1\right) - \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & 1 \end{smallmatrix}; 1\right) \\
& + \log(a) \left[ \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & 1-r_- & ; 1 \end{smallmatrix}\right) + \mathcal{E}_4\left(\begin{smallmatrix} 0 & -1 & 1 & 1 \\ 0 & \infty & r_- & ; 1 \end{smallmatrix}\right) \right] ,
\end{aligned}$$

# The multiple polylogarithms (MPLs)

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The simplest class of functions appearing in scattering amplitudes

- Easy to perform analytic continuation
- Rich algebraic properties to simplify expressions and find relations
- Fast numeric evaluation for phenomenological applications

GiNaC      [Vollinga, Weinzierl: hep-ph/0410259](#)

handyG      [Naterop, Signer, Ulrich: 1909.01656](#)

FastGPL      [Wang, LLY, Zhou: 2112.04122](#)

# Canonical differential equations and symbols

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Solutions in terms of MPLs often arise from canonical differential equations

$$d\vec{f}(\boldsymbol{x}, \epsilon) = \epsilon \operatorname{dlog}(\alpha_i(\boldsymbol{x})) A_i \vec{f}(\boldsymbol{x}, \epsilon)$$

Henn: 1304.1806

$$\vec{f}(\boldsymbol{x}, \epsilon) = \sum_n \epsilon^n \vec{f}^{(n)}(\boldsymbol{x})$$

Uniform Transcendentality (UT)

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$$\vec{f}^{(n)}(\mathbf{x}) \supset \int_{x_0}^{\mathbf{x}} \operatorname{dlog}(\alpha_{i_n}(x_n)) \cdots \int_{x_0}^{x_3} \operatorname{dlog}(\alpha_{i_2}(x_2)) \int_{z_0}^{x_2} \operatorname{dlog}(\alpha_{i_1}(x_1))$$

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Symbol map

Goncharov et al.: 1006.5703

Duhr et al.: 1110.0458

$$\alpha_{i_1} \otimes \alpha_{i_2} \otimes \cdots \otimes \alpha_{i_n}$$



Symbols: encode algebraic and analytic information about the integrals

# From Feynman integrals to MPLs

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# The (generalized) Baikov representations

Standard Baikov rep.

$$\int_{\mathcal{C}} u_{\text{Std}}(z) \frac{dz_1 \wedge \cdots \wedge dz_N}{z_1^{a_1} \cdots z_N^{a_N}}$$

Baikov variables (propagators  
and irreducible scalar products)

Loop-by-loop (LBL) Baikov rep.

$$\int_{\mathcal{C}} u_{\text{LBL}}(z) \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n}}$$

Baikov: hep-ph/9611449

Lee: 1007.2256

Frellesvig, Papadopoulos: 1701.07356

Chen, Jiang, Xu, LLY: 2008.03045

Chen, Jiang, Ma, Xu, LLY: 2202.08127

Generalized LBL Baikov rep.

$$\int_{\mathcal{C}} u_{\text{LBL}}(z) \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}}$$

A multi-valued function defined  
by a given integral family

Polynomial factors in  $u_{\text{LBL}}(z)$

# Algebraic structure of generalized Baikov integrals

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Generalized Baikov integrals for a given integral family form a **vector space**

Equipped with an “inner-product” called “**intersection number**”

[Frellesvig et al.: 1901.11510, 1907.02000, 2008.04823](#)

$$\langle e_i | d_j \rangle = \delta_{ij}$$

Orthonormal basis

$$\langle \varphi | = \sum_{i=1}^{\nu} \langle \varphi | d_i \rangle \langle e_i |$$

Decomposition of a vector to a basis (IBP reduction)

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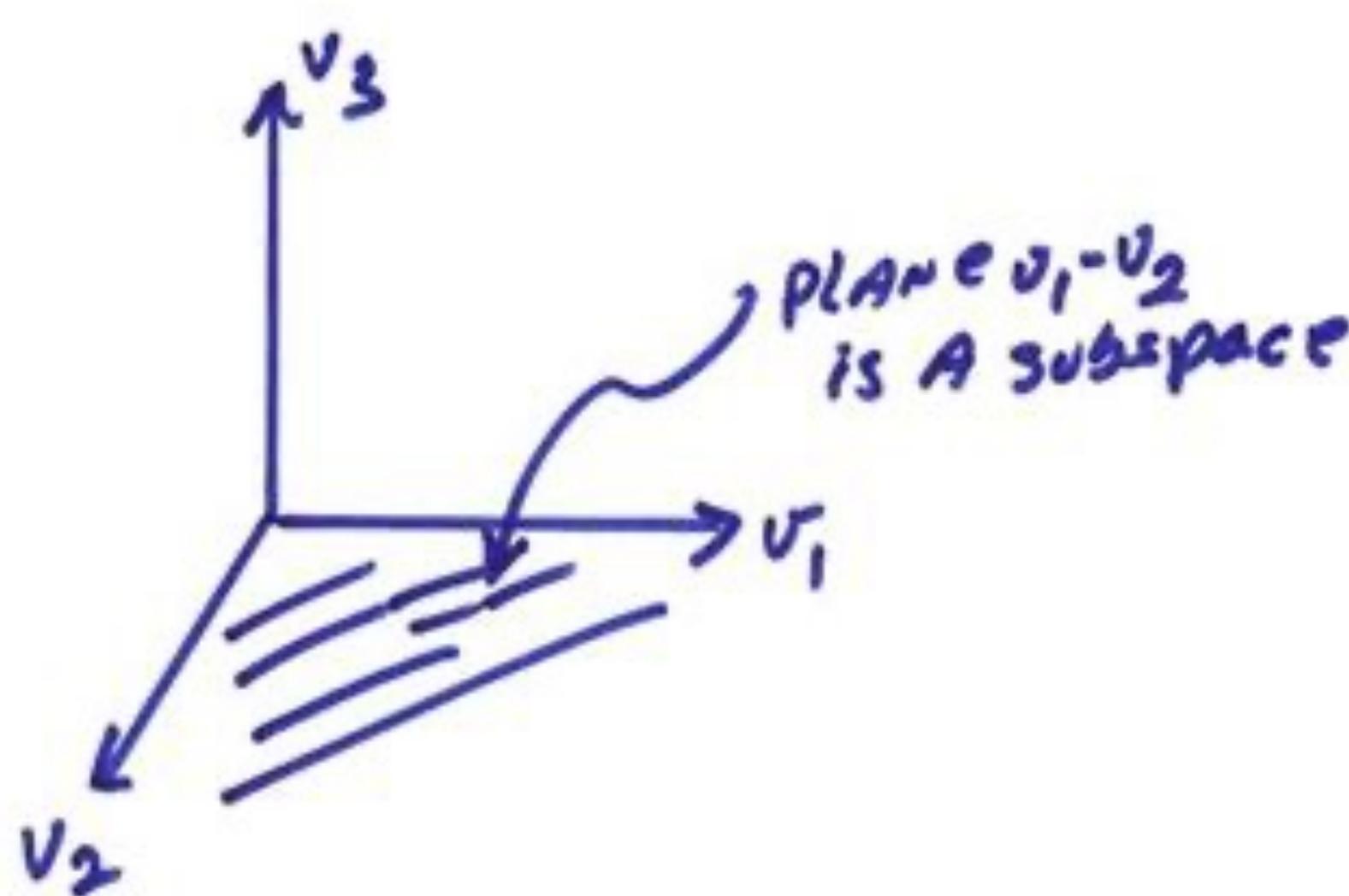
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Decomposition of a vector to a basis (IBP reduction)

# of dimensions = # of independent Baikov integrals

$\geq$  # of master Feynman integrals

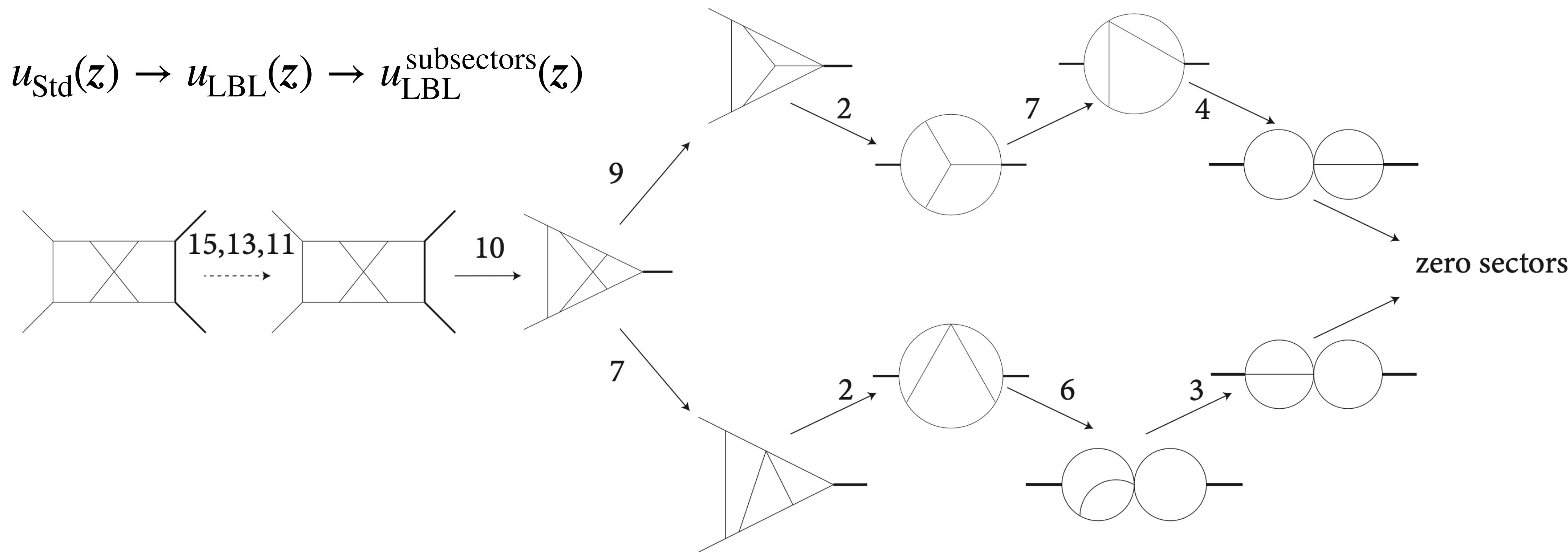
Feynman integrals (in general) live in a subspace!



# Algebraic structure of generalized Baikov integrals

Baikov representations exhibit a recursive structure

Jiang, LLY: 2303.11657



Helps to relate different sub-sectors within an integral family

# UT integrals in Baikov representations

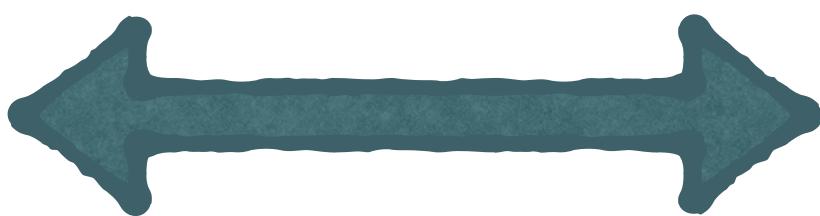
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Generalized d-log integrands

Chen, Jiang, Xu, LLY: 2008.03045

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$$\int_{\mathcal{C}} u(z) \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}} = \int_{\mathcal{C}} [G(z)]^\epsilon \bigwedge_{j=1}^n d \log f_j(z)$$



UT integrals

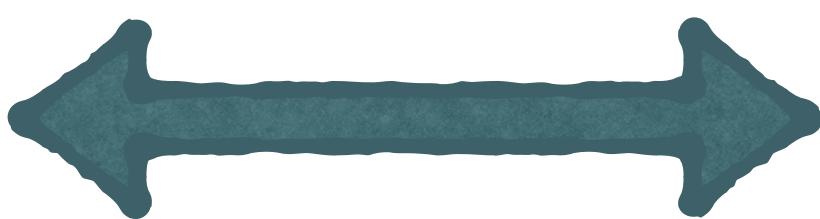
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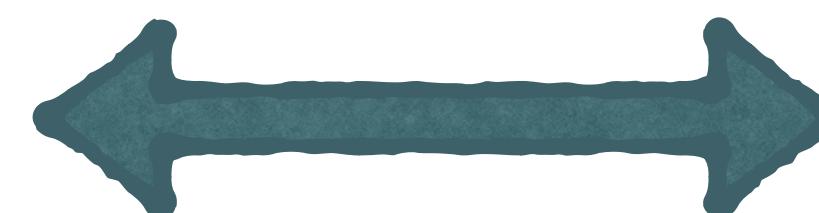


An algorithm to construct such integrands

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UT integrals

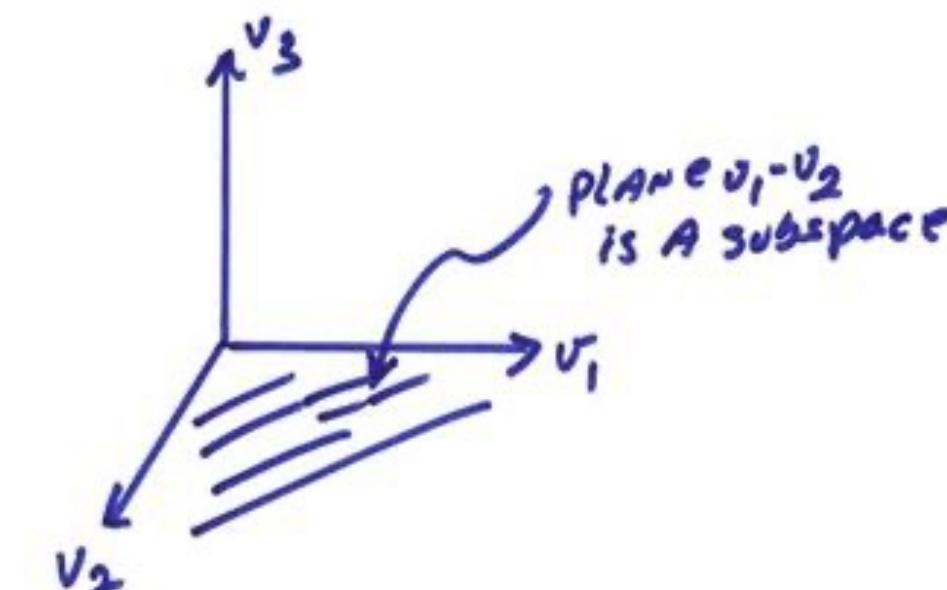
Chen, Jiang, Xu, LLY: 2008.03045

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Projection to Feynman integrals  
via linear decomposition

An algorithm to construct such integrands



# One-loop symbols from UT Baikov integrals

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Abreu et al.: 1704.07931  
Chen, Ma, LLY: 2201.12998  
Jiang, LLY: 2303.11657

UT Baikov integrals give us a way to study their singularity-structures

One-loop symbol letters can be generically constructed...

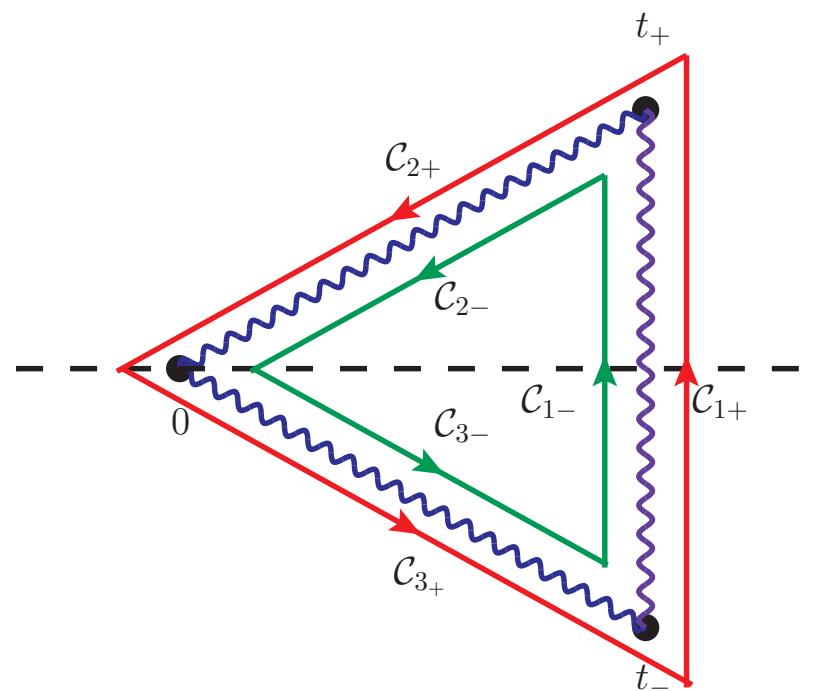
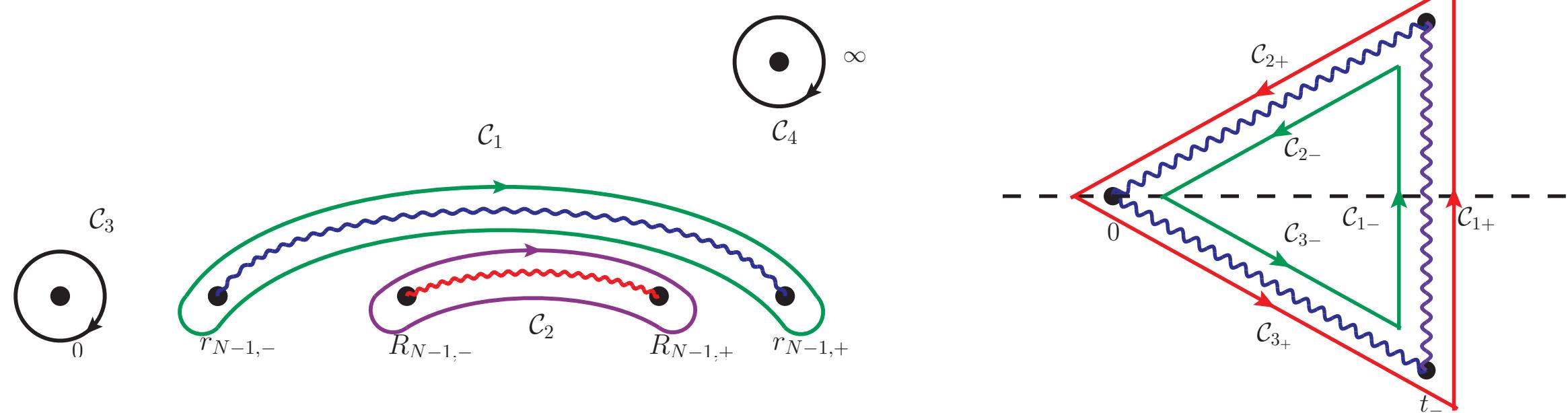
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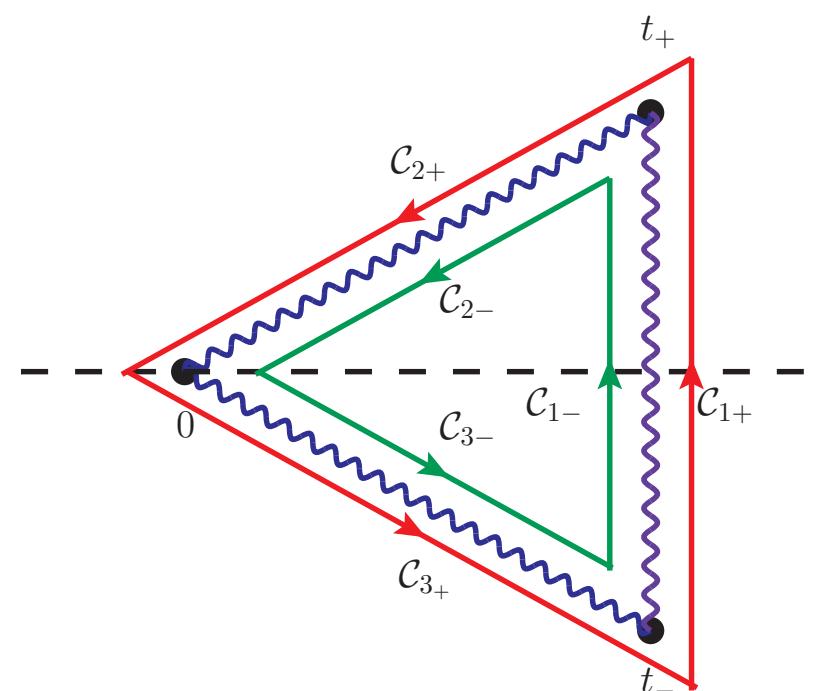
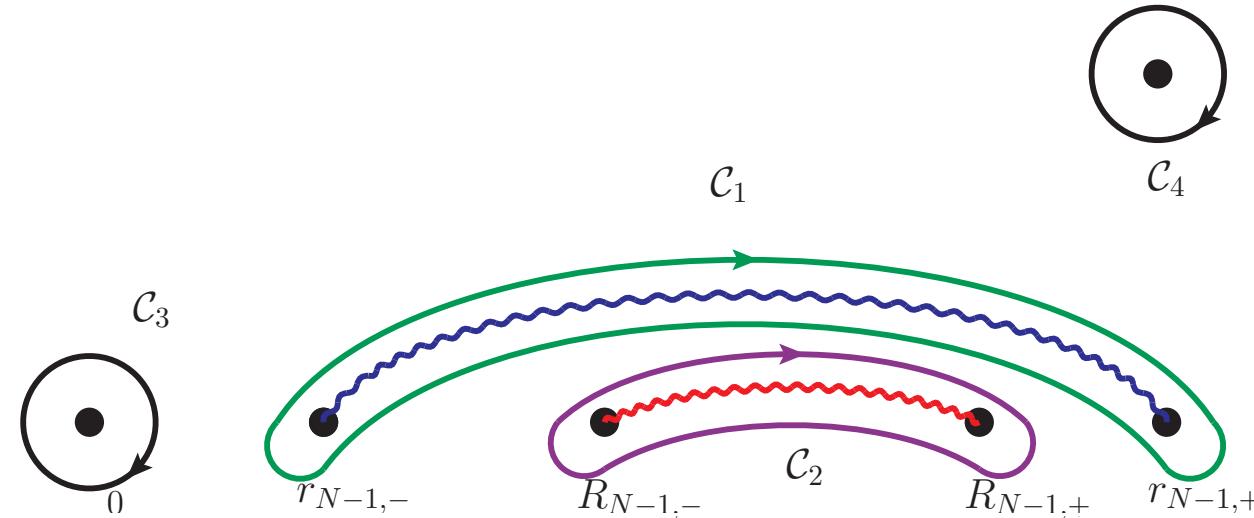
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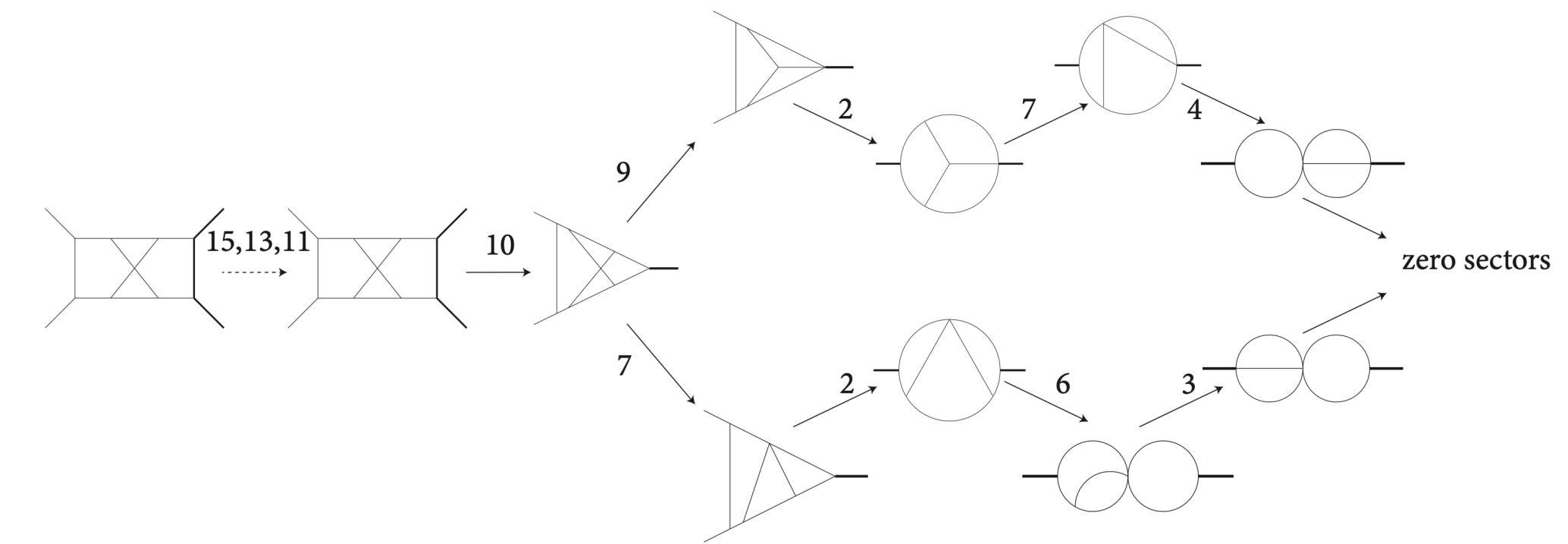
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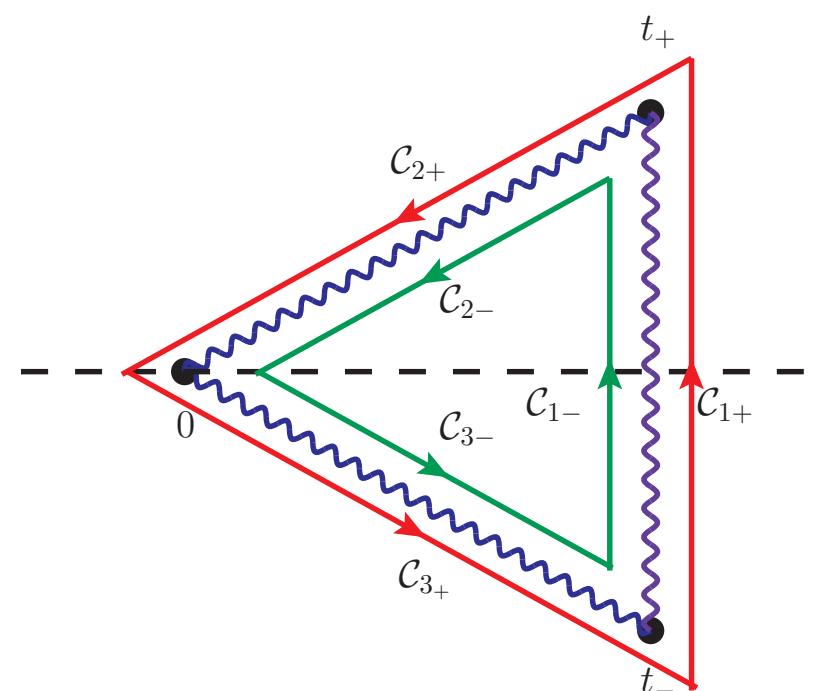
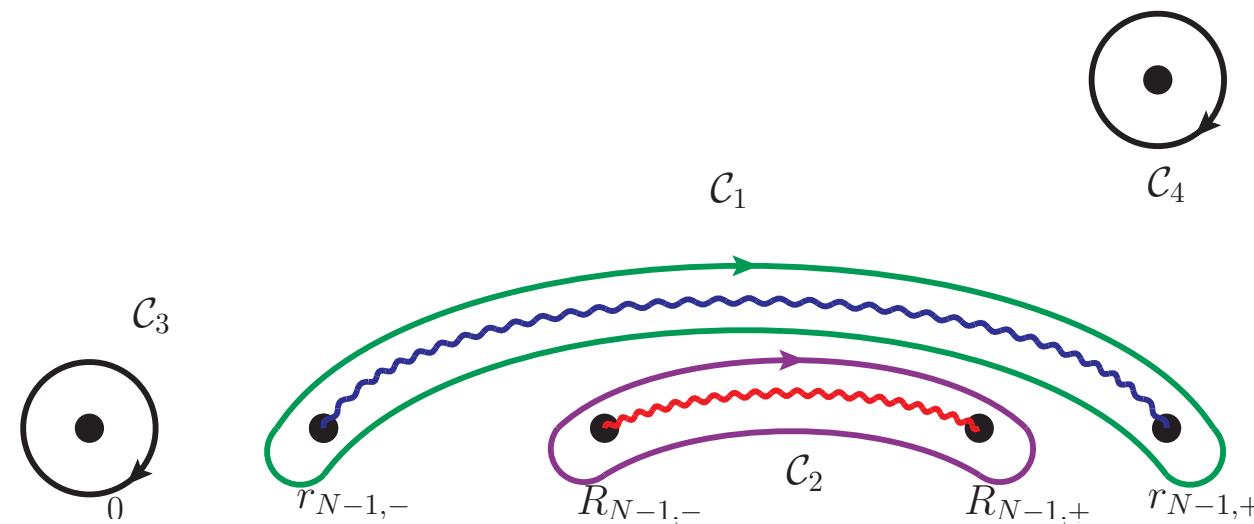
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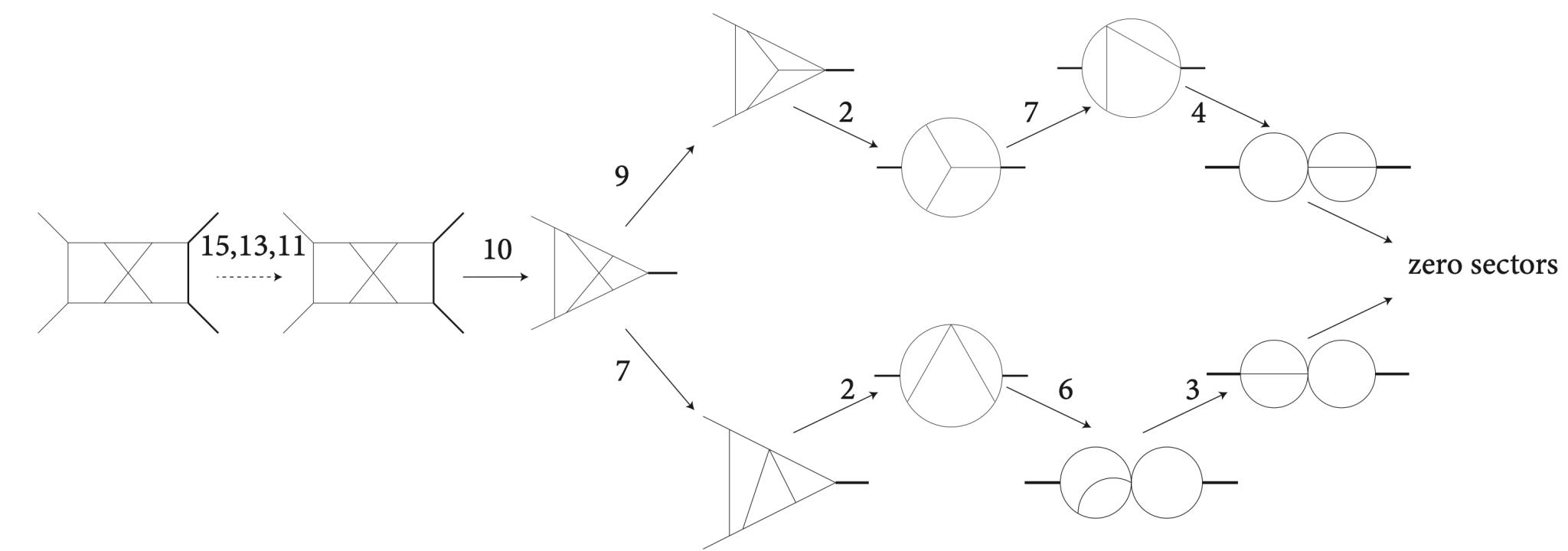
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One-loop symbol letters can be generically constructed...

Either by dedicated contour-integration...



Or directly from the recursive structure



🔑 Universal formulae for any scattering processes

🔑 Easy to compute in terms of Gram determinants or minors of a single matrix

🔑 Possible extensions to higher loop orders

# Beyond MPLs

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Simplest class of functions beyond MPLs: elliptic integrals and iterated integrals over them

$$F(x; k) = \int_0^x \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}}$$

$$E(x; k) = \int_0^x \frac{\sqrt{1 - k^2 t^2}}{\sqrt{1 - t^2}} dt$$

$$\Pi(n; \varphi \mid m) = \int_0^{\sin \varphi} \frac{1}{1 - nt^2} \frac{dt}{\sqrt{(1 - mt^2)(1 - t^2)}}$$

# Beyond MPLs

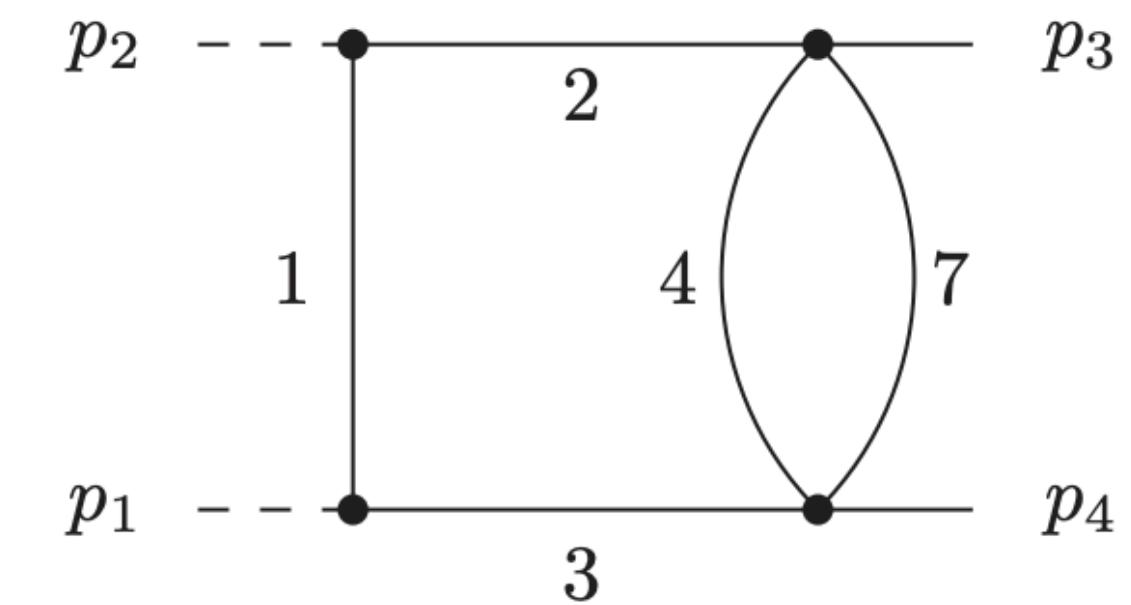
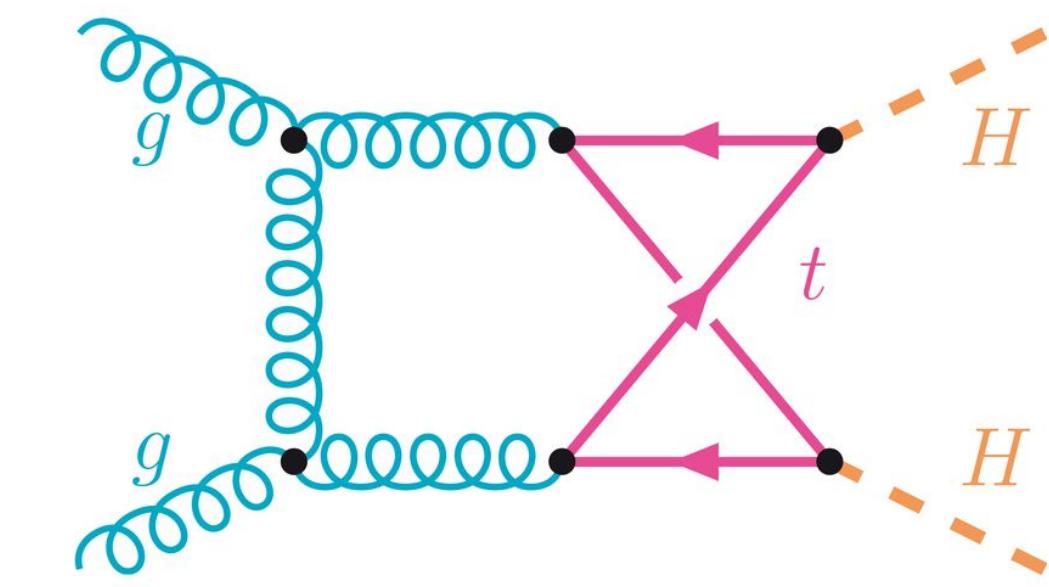
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Appearing in cutting-edge calculations



# Beyond MPLs

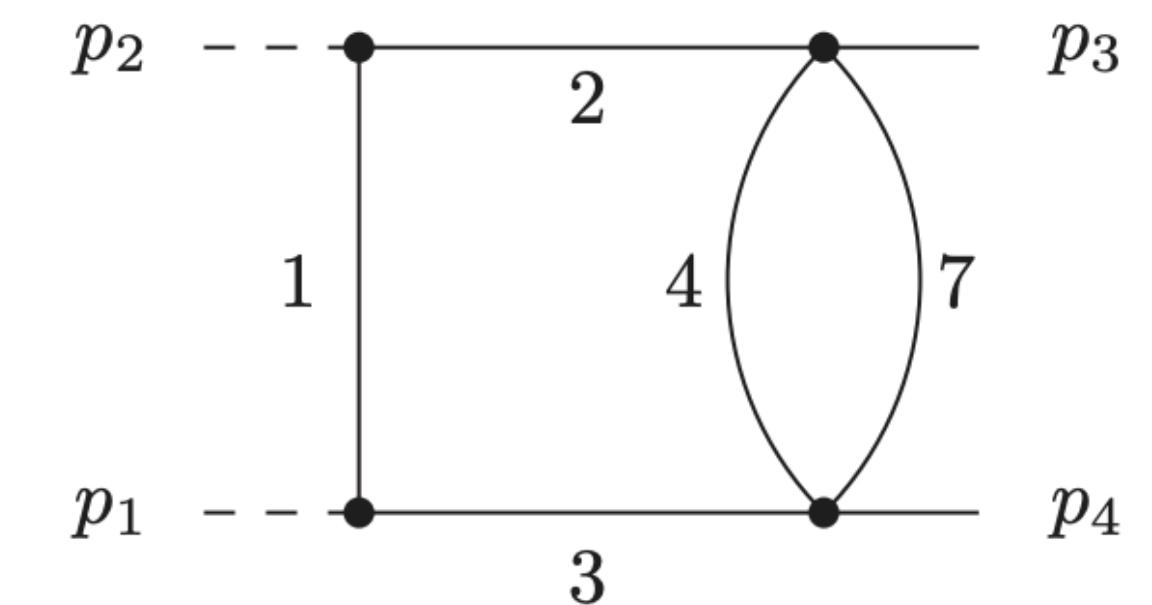
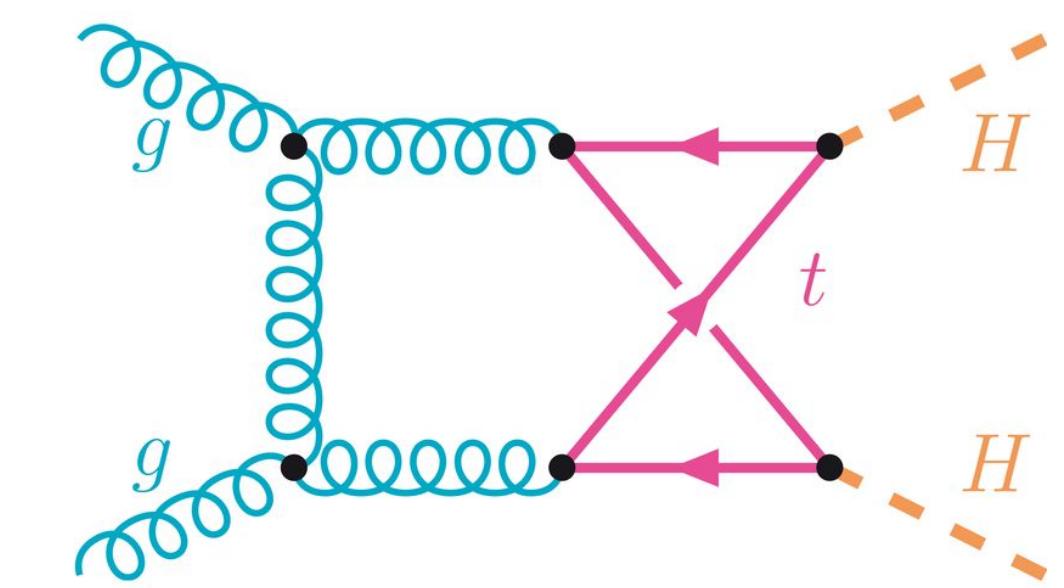
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Appearing in cutting-edge calculations



10 years ago we wouldn't call these functions “analytic results” ...

But we now have much better understanding!

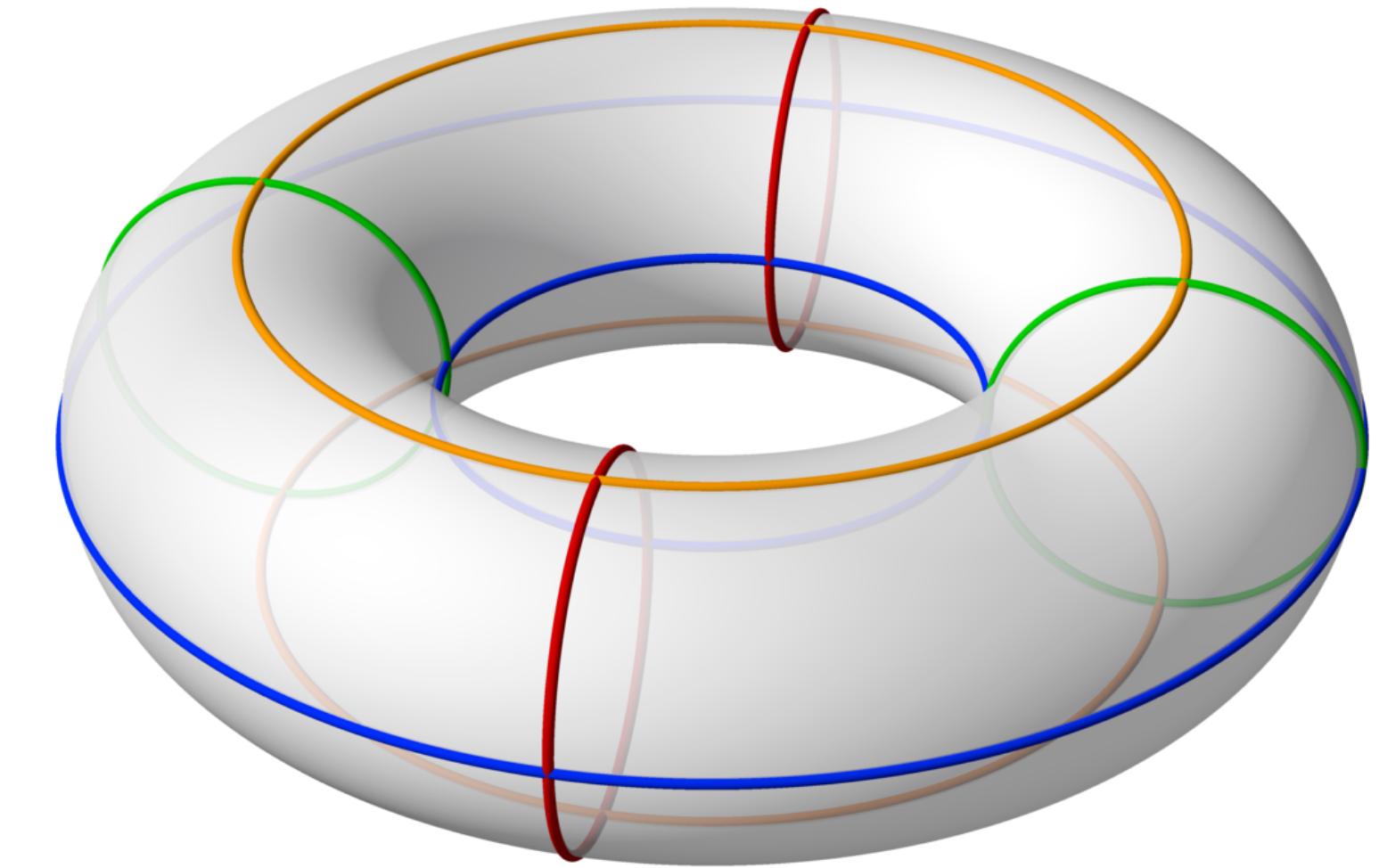
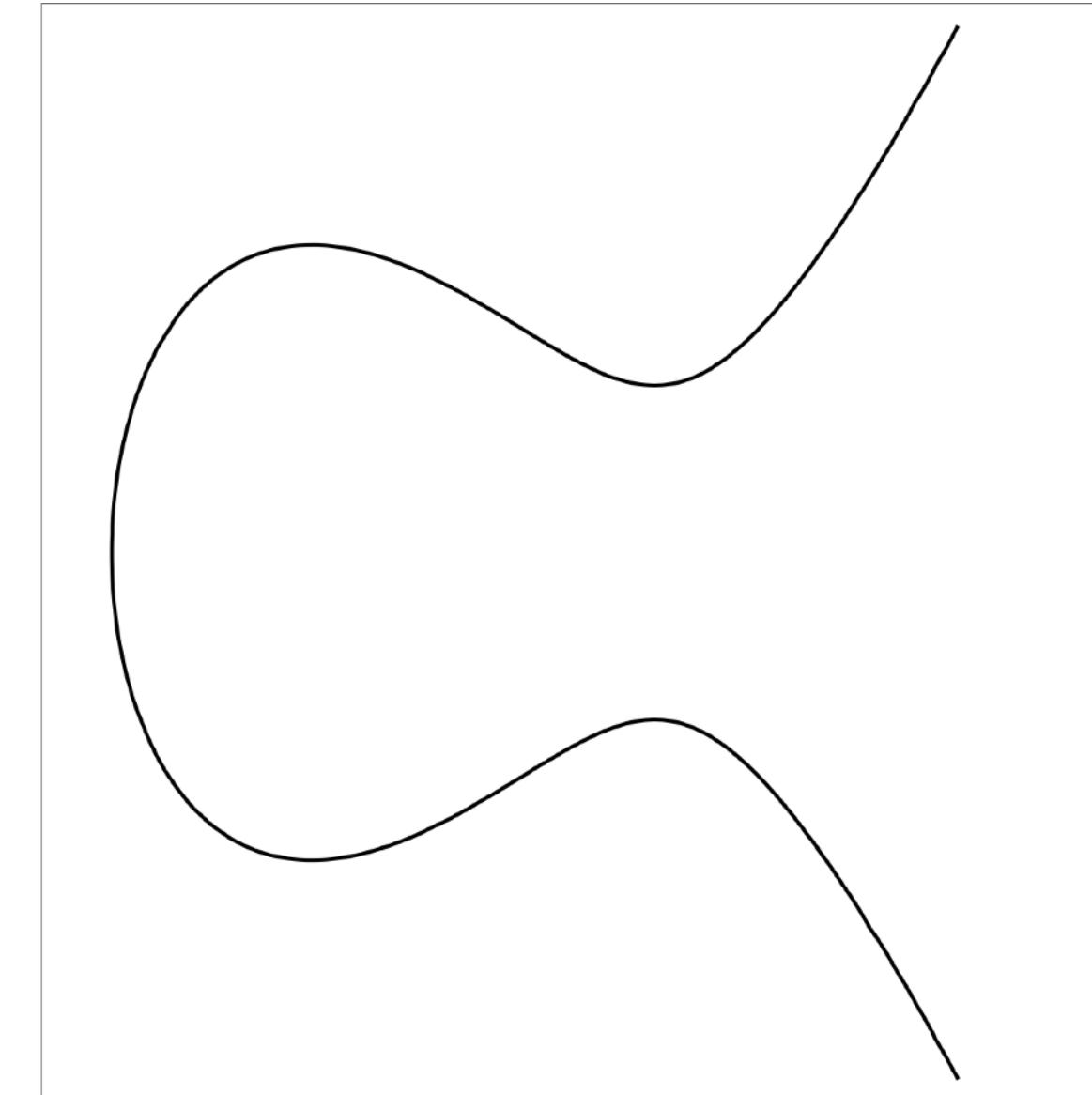
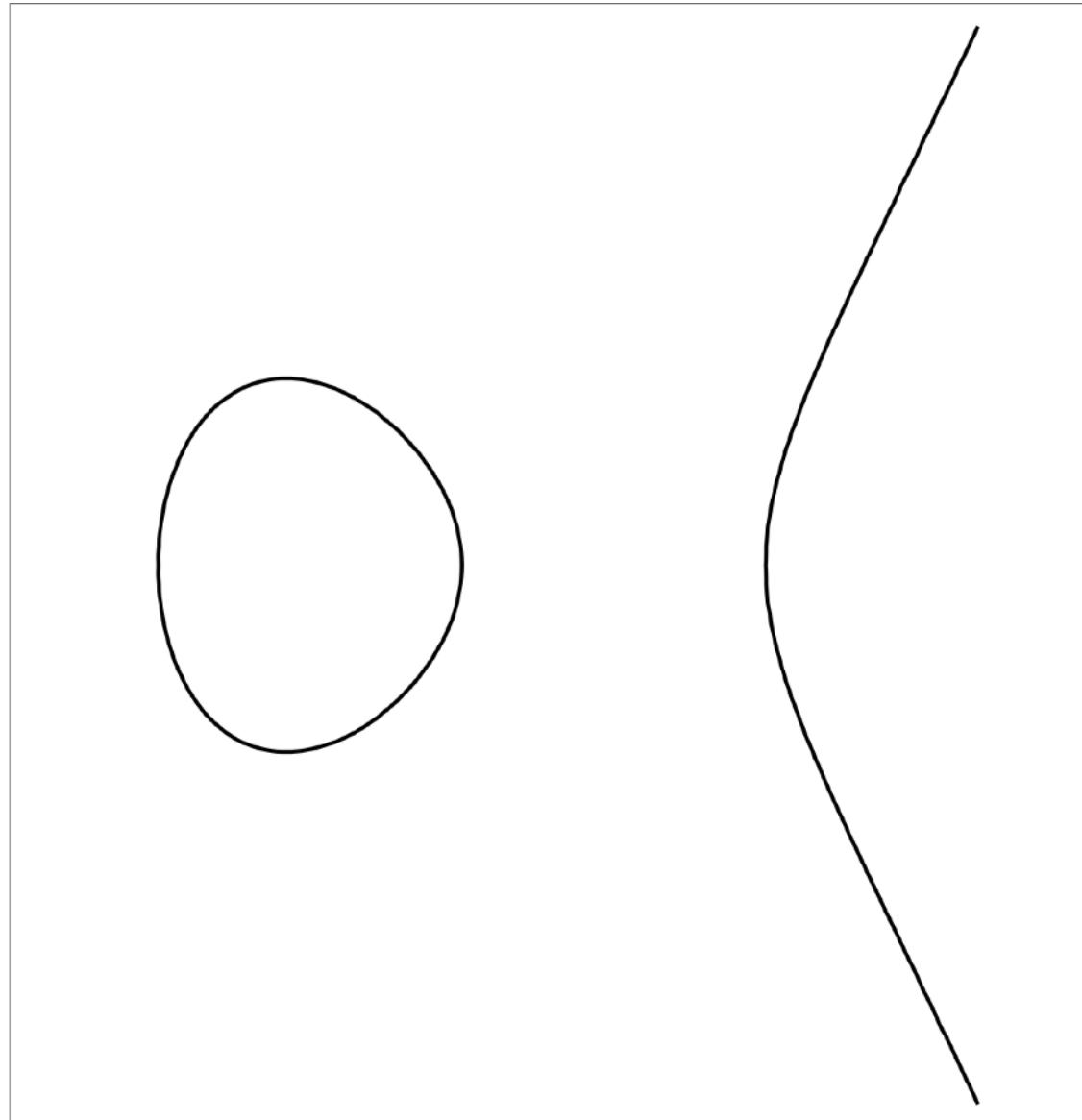
# Elliptic integrals and elliptic curves

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Functions can be categorized by the underlying geometry

- The geometric object underlying MPLs is a sphere
- The geometric object underlying iterated elliptic integrals is an elliptic curve (a torus)

$$y^2 = P(x) \quad (\text{Degree-3 or 4 polynomial with distinct roots})$$



# Uniform transcendentality in elliptic sectors

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The concept of UT basis extremely powerful in MPL sectors

Natural to think about extending to elliptic sectors

Many works! see, e.g. [Weinzierl: 2201.03593](#) and references therein

- Functional basis: elliptic multiple polylogarithms (eMPLs)?
- Canonical differential equations
- Concepts of transcendentality and symbology
- Numerical evaluation

# Uniform transcendentality in elliptic sectors

---

There are two things going into the UT basis

Frellesvig, Weinzierl: 2301.02264

- $\epsilon$ -factorized differential equations

$$d\vec{f}(\mathbf{x}, \epsilon) = \epsilon dA_i \vec{f}(\mathbf{x}, \epsilon)$$

- UT boundary condition (what's the proper definition of UT in elliptic case?)

$$\vec{f}(0, \epsilon) = \sum_n \epsilon^n \vec{f}^{(n)}(0)$$

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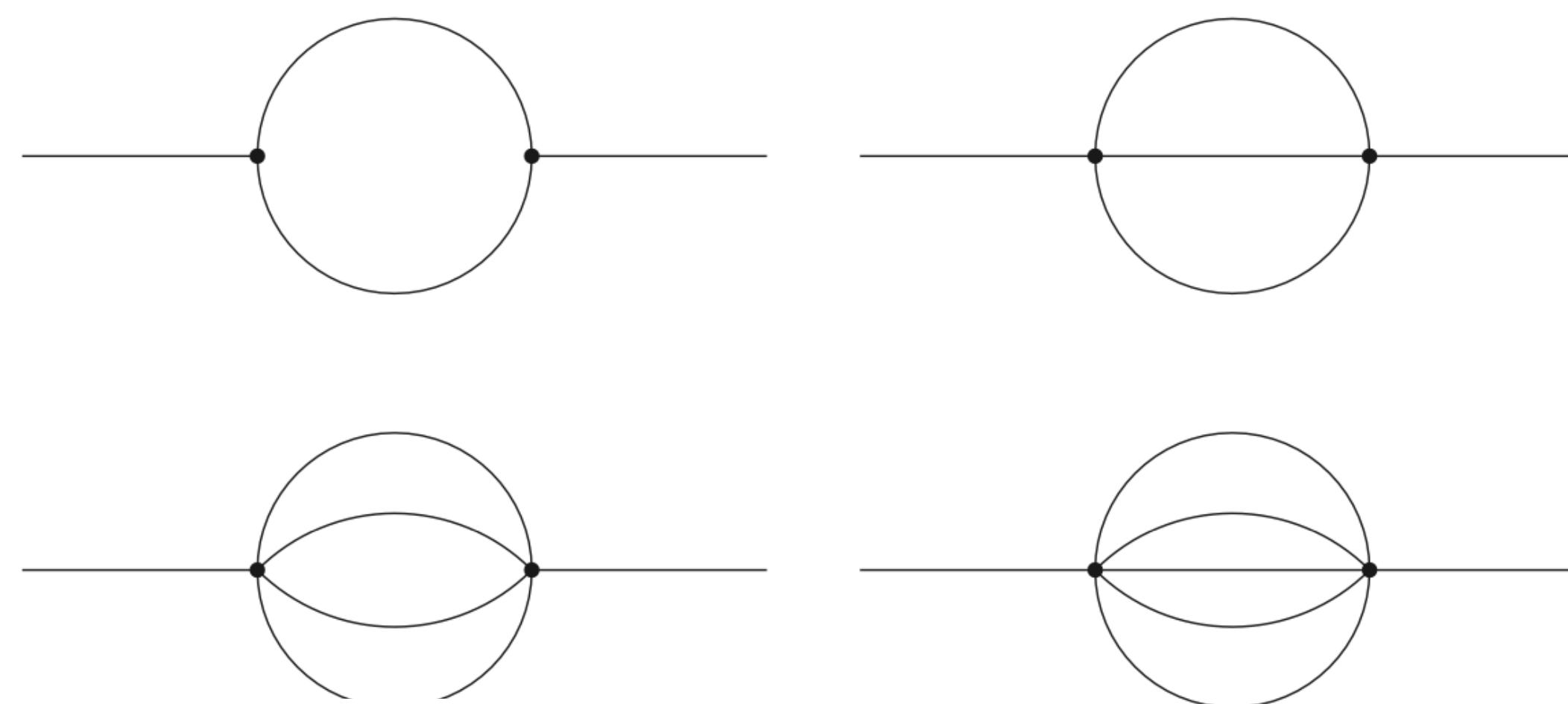
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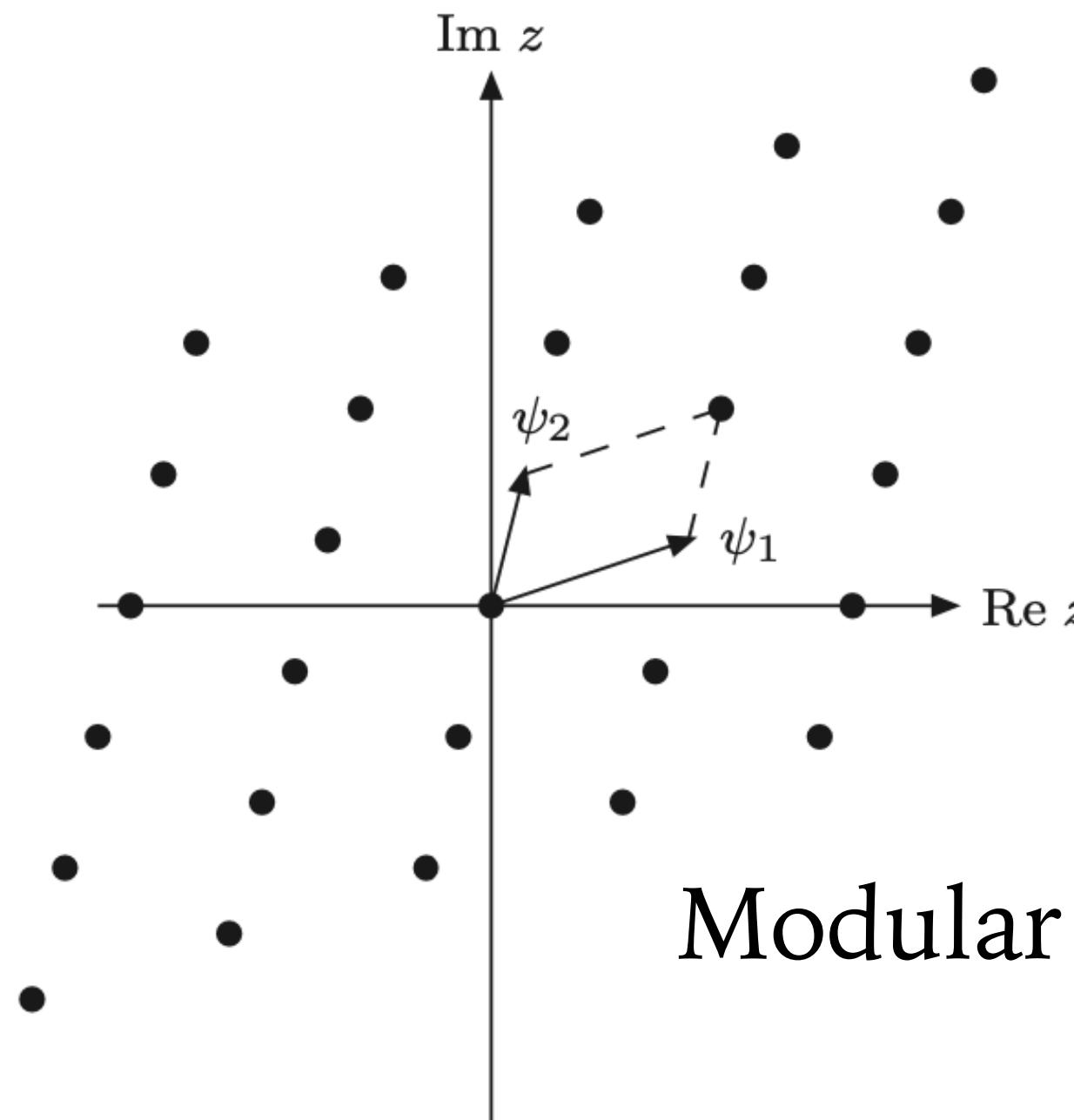
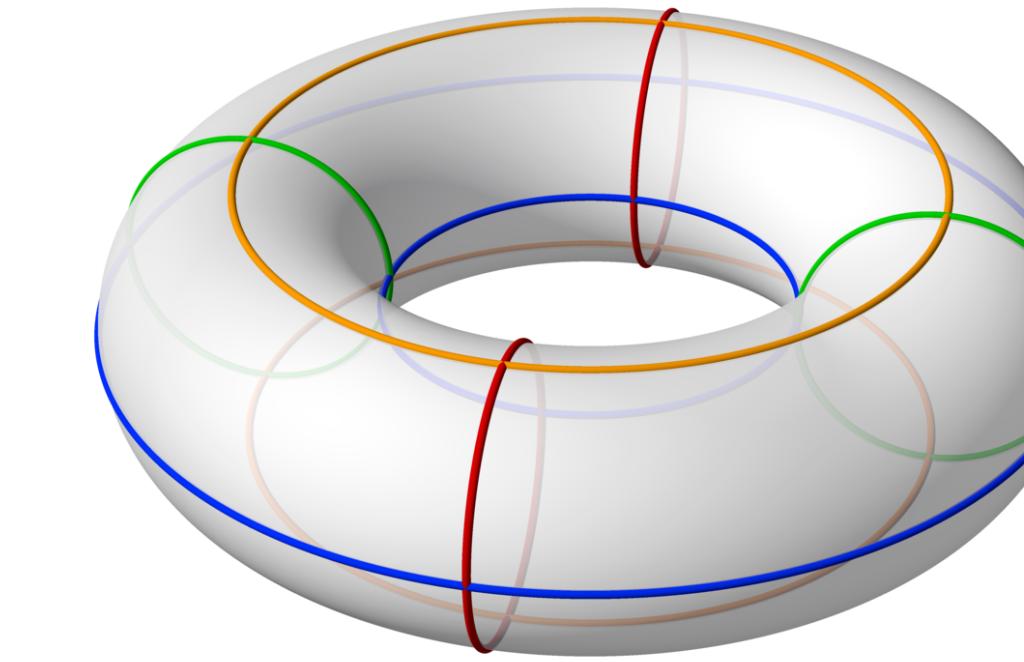
$$\vec{f}(0, \epsilon) = \sum_n \epsilon^n \vec{f}^{(n)}(0)$$

How to generically find such a basis and how to represent the results?

# Sunrise and Banana families



See, e.g., Weinzierl: 2201.03593 and references therein  
and Pogel, Wang, Weinzierl: 2207.12893, 2211.04292, 2212.08908



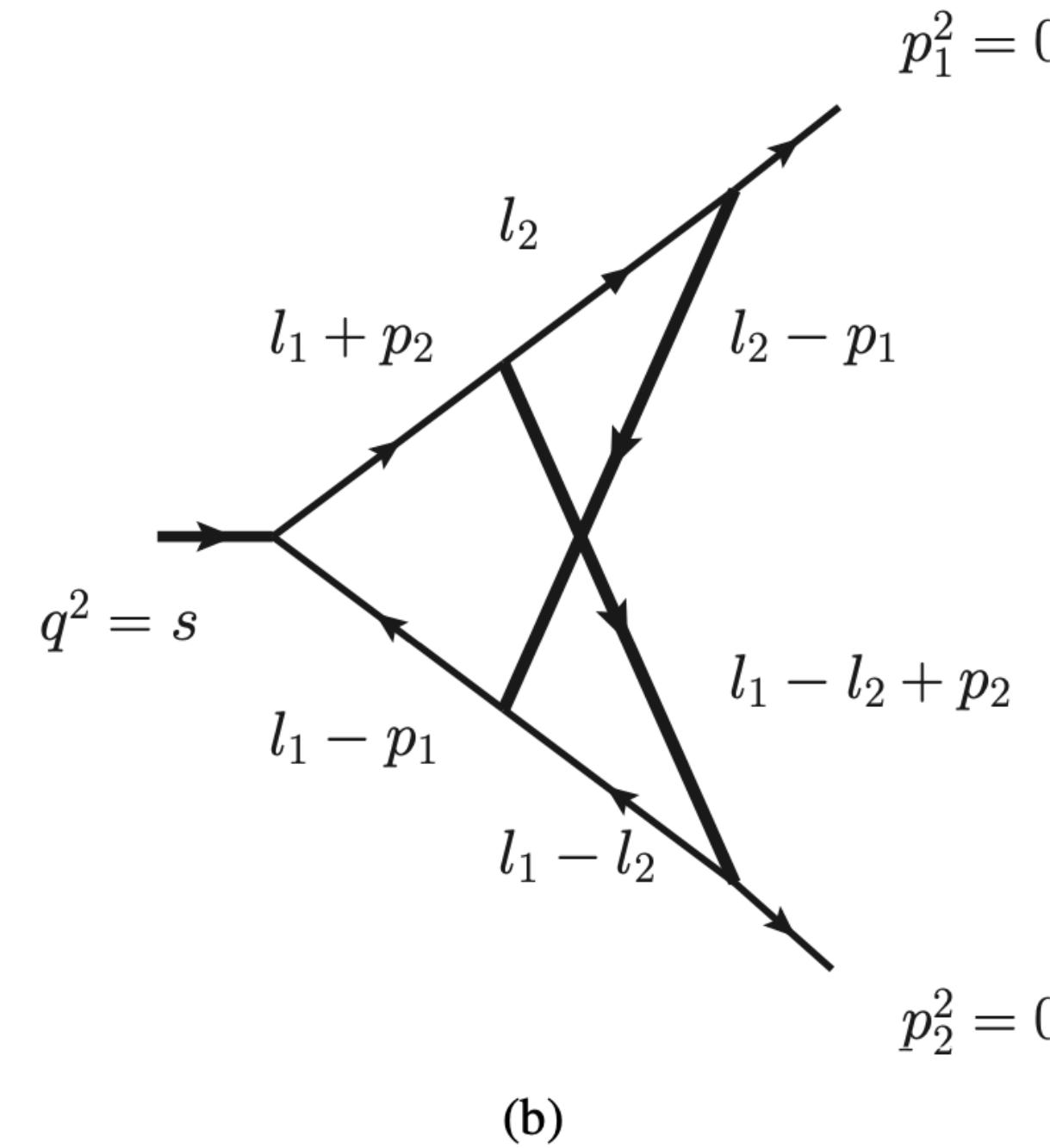
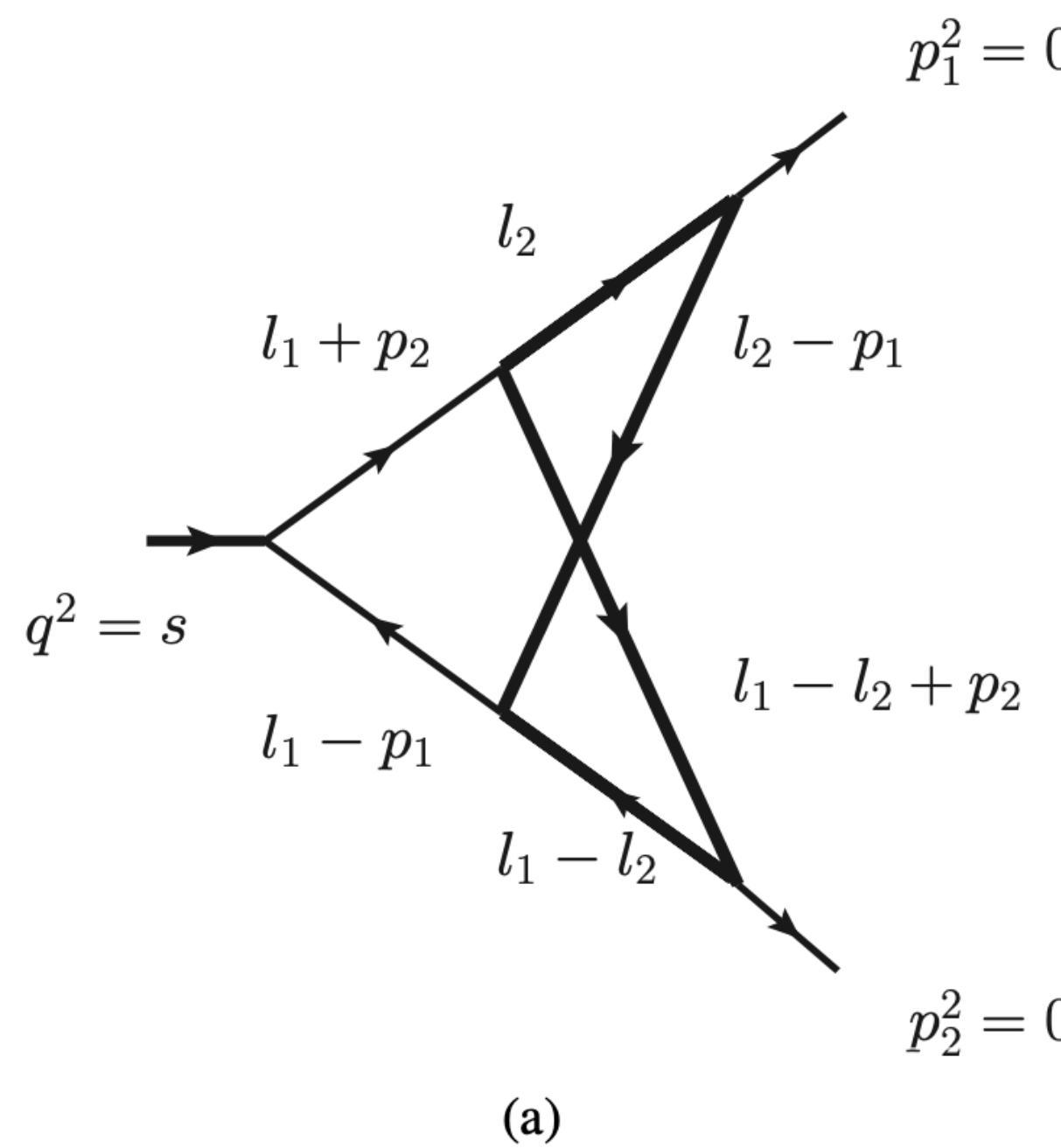
Lessons from studies on sunrise and banana families:  
we should utilize modular transformations and modular forms  
associated with the elliptic curves

Modular variable  $\tau = \frac{\psi_2}{\psi_1}$

Modular transformation  $\tau \rightarrow \frac{a\tau + b}{c\tau + d}$

# Single parameter elliptic families with non-trivial sub-sectors

Jiang, Wang, LLY, Zhao: 2304.xxxxx



$$y = -\frac{m^2}{s}$$

Appearing in, e.g., HH&ZH production and Higgs decays

Non-trivial sub-sectors: 2 top-sector MIs + 9 sub-sector MIs for family (a)

3 top-sector MIs + 15 sub-sector MIs for family (b)

# UT bases: construction

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The sub-sectors are non-elliptic, and we construct d-log bases

$$\int_{\mathcal{C}} u(z) \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}} = \int_{\mathcal{C}} [G(z)]^\epsilon \bigwedge_{j=1}^n d \log f_j(z)$$

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The top-sectors are analyzed by investigating the associated Picard-Fuchs operators, e.g.

$$L_3^{(0)}(y) = \left[ \frac{d}{dy} + \frac{8}{8y-1} \right] \left[ \frac{d^2}{dy^2} + \left( \frac{1}{y+1} + \frac{8}{8y-1} - \frac{3}{y} \right) \frac{d}{dy} + \frac{8y(y+2)-4}{y^2(y+1)(8y-1)} \right]$$

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The sub-sectors are non-elliptic, and we construct d-log bases

$$\int_{\mathcal{C}} u(z) \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{a_1} \cdots z_n^{a_n} P_1^{b_1} \cdots P_m^{b_m}} = \int_{\mathcal{C}} [G(z)]^\epsilon \bigwedge_{j=1}^n d \log f_j(z)$$

The top-sectors are analyzed by investigating the associated Picard-Fuchs operators, e.g.

$$L_3^{(0)}(y) = \left[ \frac{d}{dy} + \frac{8}{8y-1} \right] \left[ \frac{d^2}{dy^2} + \left( \frac{1}{y+1} + \frac{8}{8y-1} - \frac{3}{y} \right) \frac{d}{dy} + \frac{8y(y+2)-4}{y^2(y+1)(8y-1)} \right]$$



$$\frac{d}{dy} \vec{f}(y, \epsilon) = \epsilon A_i \vec{f}(y, \epsilon)$$

# UT bases: solution

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Boundary conditions obtained using, e.g., Mellin-Barnes representation

$$\begin{aligned} \left. \frac{I_1}{y^2} \right|_{y \rightarrow 0} = & \frac{7}{12} \log^4 y - \frac{\pi^2}{6} \log^2 y - 20\zeta_3 \log y - \frac{31\pi^4}{180} + \varepsilon \left[ \frac{3}{20} \log^5 y - \frac{3\pi^2}{2} \log^3 y \right. \\ & \left. - 34\zeta_3 \log^2 y - \frac{25\pi^4}{36} \log y + \frac{20\pi^2}{3} \zeta_3 + 8\zeta_5 \right] + \varepsilon^2 \left[ \frac{37}{360} \log^6 y - \frac{\pi^2}{3} \log^4 y \right. \\ & \left. - \frac{88\zeta_3}{3} \log^3 y + \frac{13\pi^4}{360} \log^2 y + \left( \frac{50\pi^2\zeta_3}{3} - 112\zeta_5 \right) \log y + \frac{41\pi^6}{540} + 188\zeta_3^2 \right] + O(\varepsilon^3) \end{aligned}$$

Solutions expressed as iterated integrals with kernels being (mostly) modular forms

$$I(f_1, f_2, \dots, f_n; \tau, \tau_0) = (2\pi i)^n \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau_1} d\tau_2 \cdots \int_{\tau_0}^{\tau_{n-1}} d\tau_n f_1(\tau_1) f_2(\tau_2) \dots f_n(\tau_n)$$

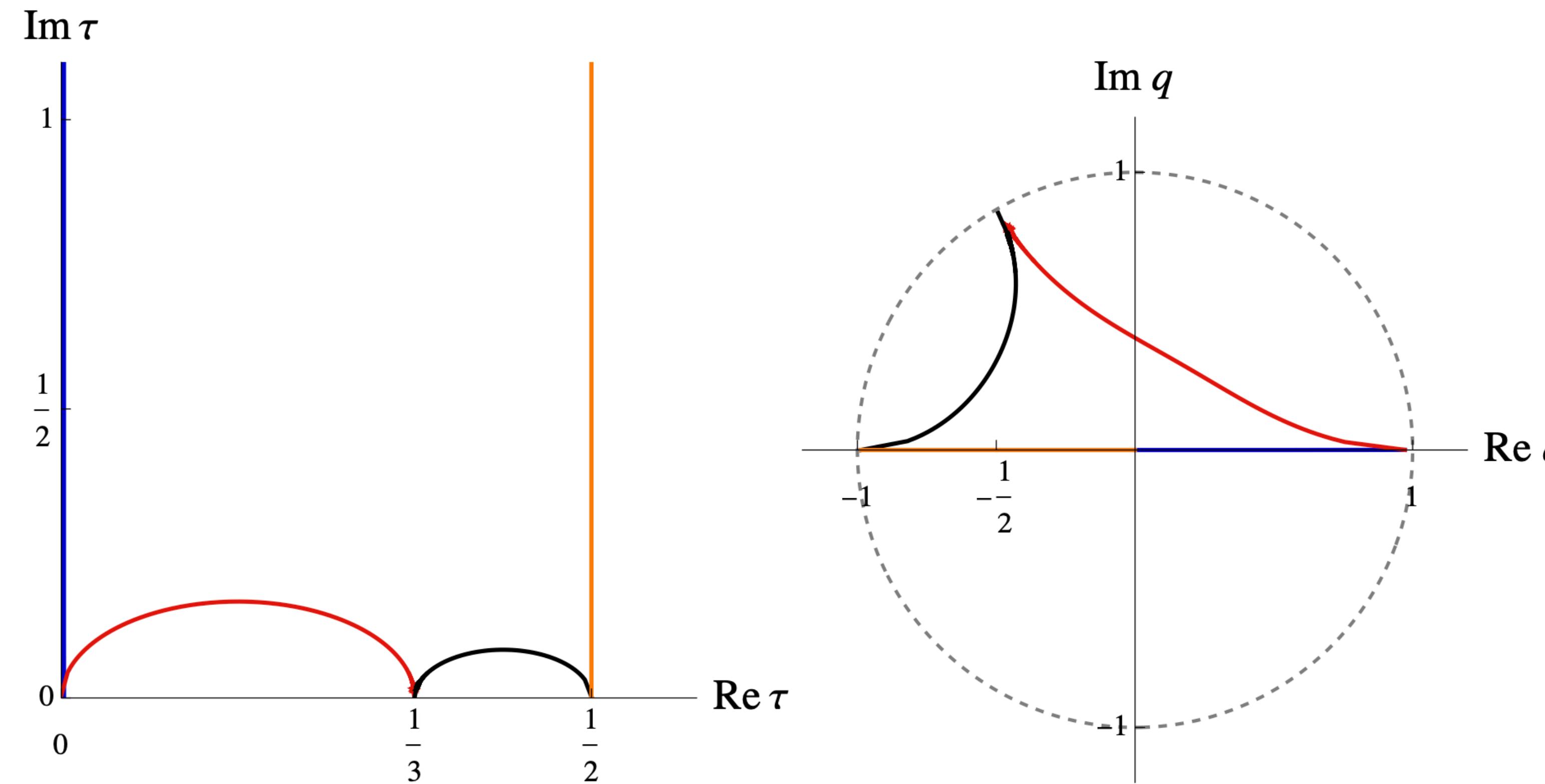
Easy to evaluate using the q-expansion

$$q = e^{2\pi i \tau}$$

# Analytic continuation

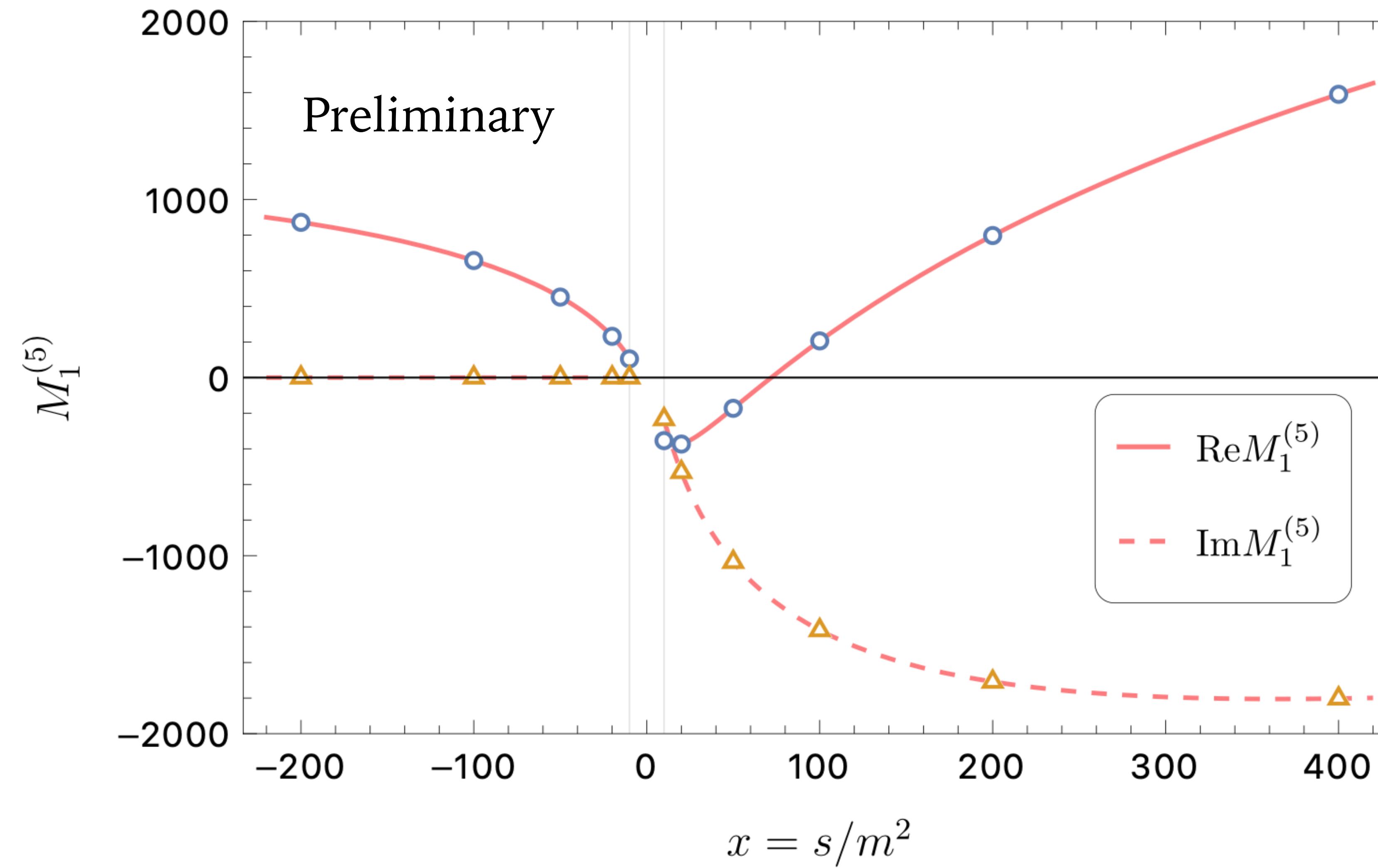
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Employ modular transformations to analytically continue to whole kinematic space



# Numeric results

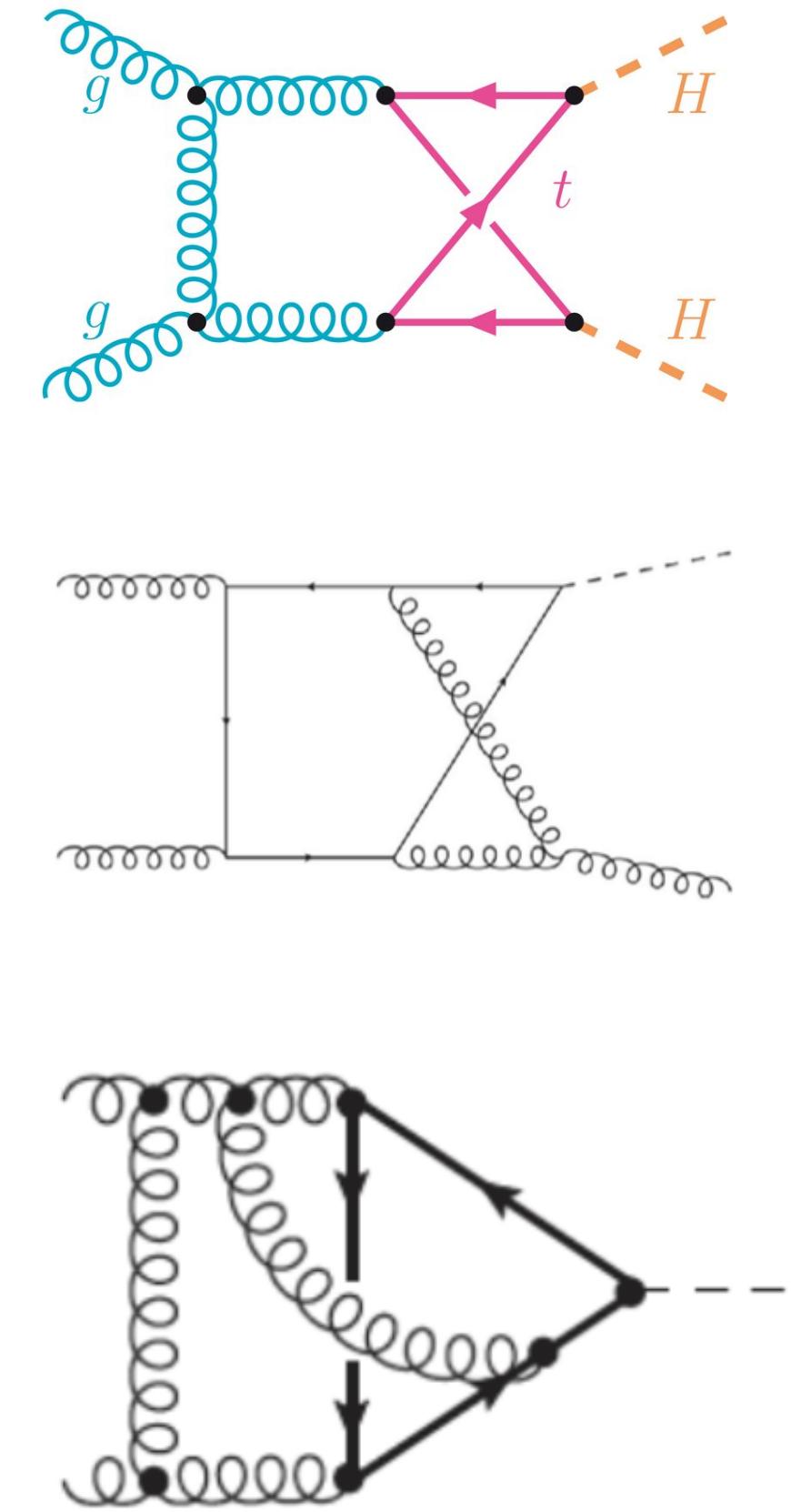
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# Outlook

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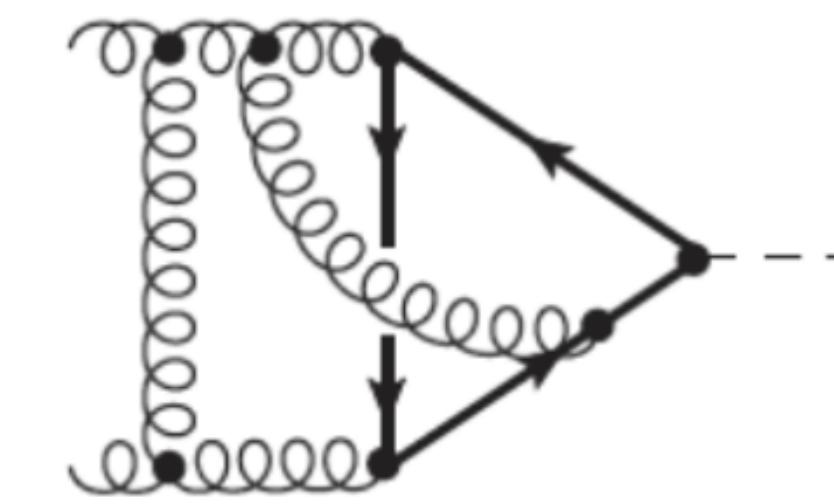
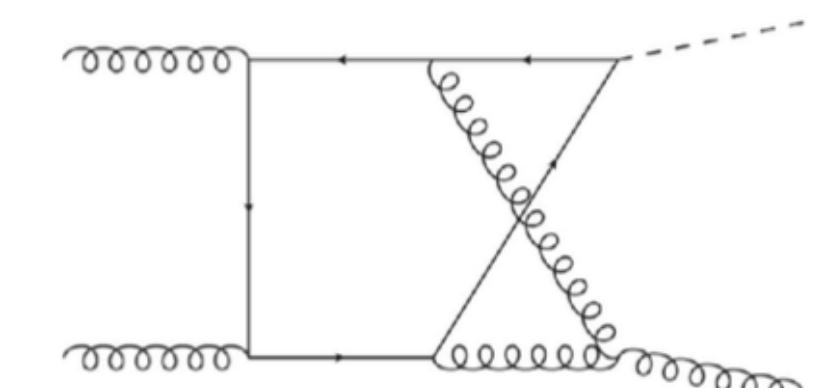
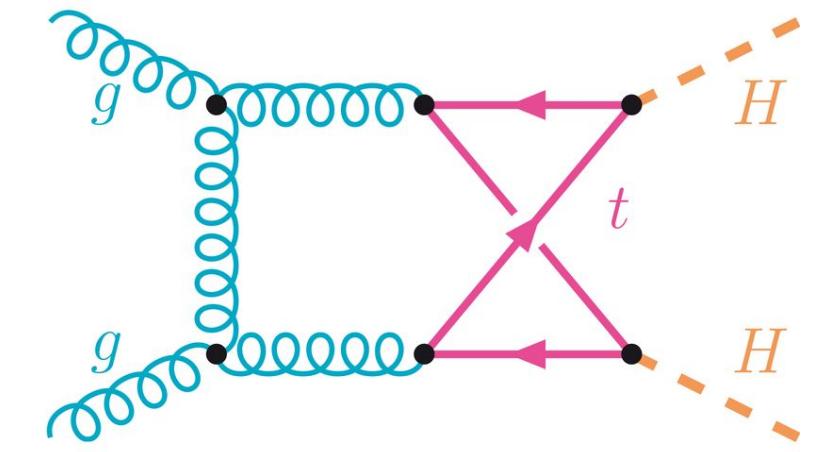
- There are more open questions than answers
- Extension to more loops and/or multiple parameter cases (scattering amplitudes involving 3 or more mass scales)
- Systematic integrand construction of UT bases (extension of d-log form integrands?)
- Generalization of symbol techniques to elliptic cases (modular forms as new “letters”?)
- Systematic reconstruction of analytic expressions from symbols (open questions in both MPL sectors and elliptic sectors)



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*Thank you!*