

微扰论计算新方法及新程序

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北京大学



Outline

I. Introduction

II. Auxiliary mass flow

III. Block-triangular form

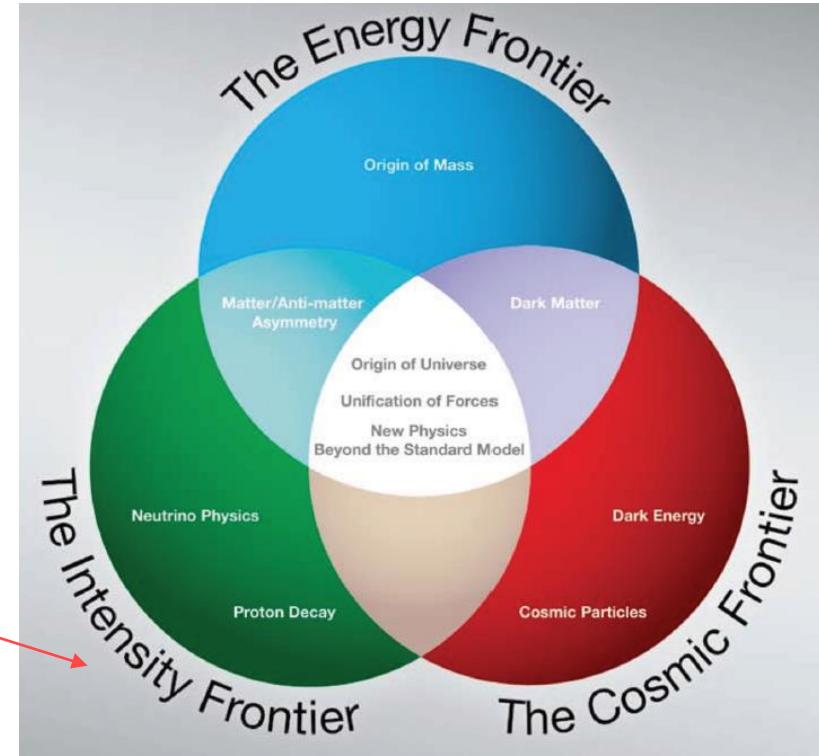
IV. CalcLoop

V. Summary and outlook

Precision: gateway to discovery

➤ Discovery via precision

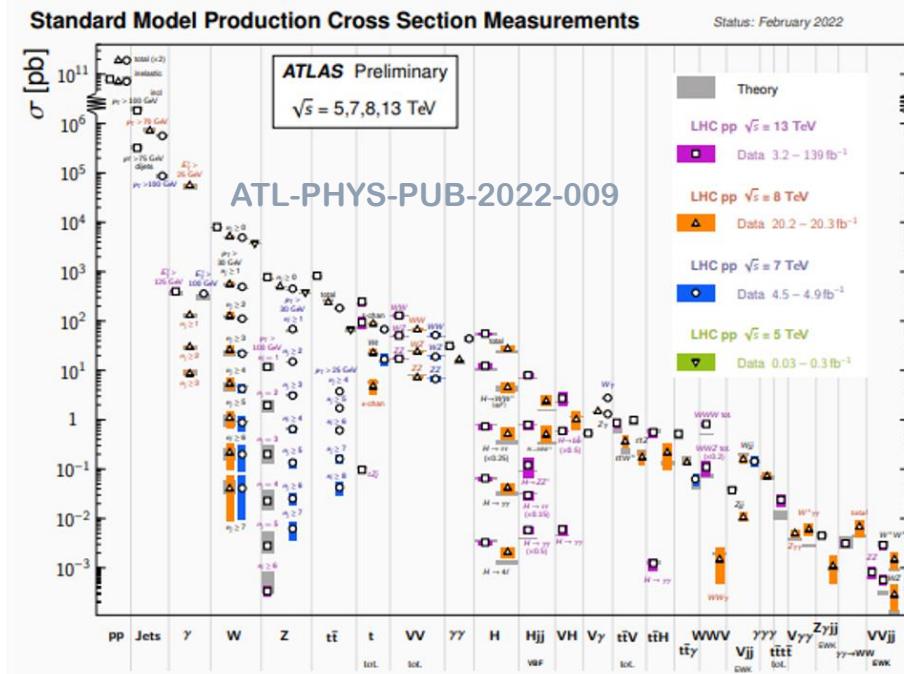
- (HL-)LHC, BELLII, EIC, CEPC/ILC/FCC-ee
- Search anomalous deviations from theory
- Interplay between exp. and th.



Era of precision physics at the LHC

➤ High-precision data

- Many observables probed at **present level precision**
- At least **NNLO QCD and NLO EW** corrections generally required (plus parton shower, resummation, etc.)



Automatic NNLO perturbative calculation is highly demanded

➤ A “billion-dollar project”

- Halving total uncertainty \approx building another LHC
- Note: LHC cost about 10 billion

Perturbative QFT

1. Generate Feynman amplitudes

- Feynman diagrams and Feynman rules (New developments: unitarity, recurrence relation, CHY, ...)
- Express amplitudes as linear combinations of FIs with rational coefficients

2. Calculate Feynman loop integrals (FIs)

- Integral reduction + Master integrals calculation

3. Perform phase-space integrations

- Monte Carlo simulation with IR subtractions
- Relating to loop integrals via reverse unitarity (if no jet)

$$\int \frac{d^D p}{(2\pi)^D} (2\pi) \delta_+(p^2) = \int \frac{d^D p}{(2\pi)^D} \left(\frac{i}{p^2 + i0^+} + \frac{-i}{p^2 - i0^+} \right)$$

Fully automatic calculation: packages ABC

	Generate amplitudes	Manipulate amplitudes	Integral reduction	Master integrals calculation
Package used	FeynArts or qgraf	CalcLoop	Blade	AMFlow
Notes	https://feynarts.de/ http://cfif.ist.utl.pt/~paulo/qgraf.html	https://gitlab.com/yqma/CalcLoop	https://gitlab.com/multiloop-pku/blade	https://gitlab.com/multiloop-pku/amflow

- Fully automatic, valid to any-loop order
- The key: AMFlow
- Main challenge: integral reduction is time/resource consuming

Implementing block-triangular form,
usually improves efficiency by $O(10^2)$

The dawn of automatic multi-loop calculation!

Automatic NLO correction obtained more than 10 years ago: MadGraph, Helac, FDC, etc

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Current status: master integrals calculation

➤ Main methods

- Sector decomposition Hepp, (1966)
Binoth, Heinrich, 0004013
F. Feng, Z. Li, ...
- Mellin-Barnes representation Usyukina (1975)
Smirnov, 9905323
J. Wang, ...
- Difference equations Laporta, 0102033
Lee, 0911.0252
- Differential equations Kotikov, PLB(1991)



Analytical : if ϵ -form exists

Henn, 1304.1806
L.B Chen, L.L. Yang, G. Yang, Y. Zhang, ...

See also Prof. Yang's talk

Numerical: general and efficient

X. Liu, YQM, C. Y. Wang, 1711.09572
Hidding, 2006.05510
X. Liu, YQM, 2201.11669

Powered by auxiliary mass flow

Auxiliary mass terms

➤ Auxiliary FIs

$$I_{\vec{\nu}}^{\text{aux}}(D, \vec{s}, \eta) = \int \prod_{i=1}^L \frac{d^D \ell_i}{i\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 - \lambda_1 \eta + i0^+)^{\nu_1} \cdots (\mathcal{D}_K - \lambda_K \eta + i0^+)^{\nu_K}}$$

- $\lambda_i \geq 0$ (typically 0 or 1), an auxiliary mass if $\lambda_i > 0$
- Analytical function of η
- Physical result obtained by (causality)

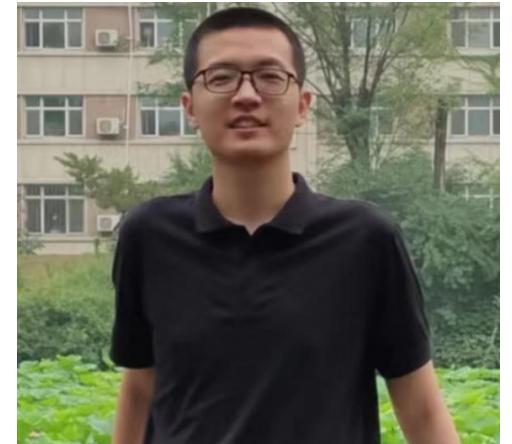
$$I_{\vec{\nu}}(D, \vec{s}) \equiv \lim_{\eta \rightarrow i0^-} I_{\vec{\nu}}^{\text{aux}}(D, \vec{s}, \eta)$$

- 1) Setup η -DEs; 2) Calculate boundary conditions; 3) Solve η -DEs

➤ η -DEs for MIs in auxiliary family using IBP

$$\frac{\partial}{\partial \eta} \vec{I}^{\text{aux}}(D, \vec{s}, \eta) = A(D, \vec{s}, \eta) \vec{I}^{\text{aux}}(D, \vec{s}, \eta)$$

X. Liu, YQM, C. Y. Wang, 1711.09572



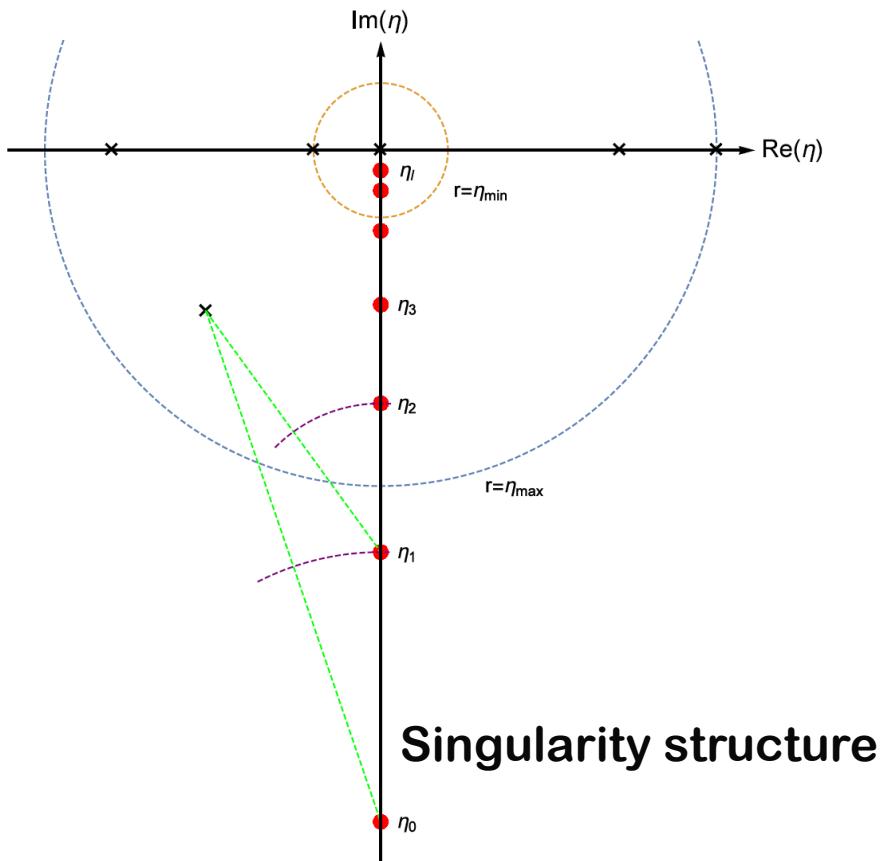
刘霄, 牛津大学



王辰宇, 慕尼黑工业大学

Flow of auxiliary mass

➤ Solve ODEs of MIs



$$\frac{\partial}{\partial \eta} \vec{I}^{\text{aux}}(D, \vec{s}, \eta) = A(D, \vec{s}, \eta) \vec{I}^{\text{aux}}(D, \vec{s}, \eta)$$

- If $\vec{I}^{\text{aux}}(D, \vec{s}, \infty)$ is known, solving ODEs numerically to obtain $\vec{I}^{\text{aux}}(D, \vec{s}, i0^-)$
- A well-studied mathematical problem

Step1: Asymptotic expansion at $\eta = \infty$

Step2: Taylor expansion at analytical points

Step3: Asymptotic expansion at $\eta = 0$

- Efficient to get high precision :
ODEs, known singularity structure

Boundary values at $\eta \rightarrow \infty$

➤ Nonzero integration regions as $\eta \rightarrow \infty$

- Linear combinations of loop momenta: $\mathcal{O}(\sqrt{|\eta|})$ or $\mathcal{O}(1)$

Beneke, Smirnov, 9711391
Smirnov, 9907471

➤ Simplify propagators at $\eta \rightarrow \infty$

- ℓ_L is the $\mathcal{O}(\sqrt{|\eta|})$ part of loop momenta
- ℓ_S is the $\mathcal{O}(1)$ part of loop momenta
- p is linear combination of external momenta

$$\frac{1}{(\ell_L + \ell_S + p)^2 - m^2 - \kappa \eta} \sim \frac{1}{\ell_L^2 - \kappa \eta}$$

- Unchange if $\ell_L = 0$ and $\kappa = 0$

➤ Boundary FIs after simplification

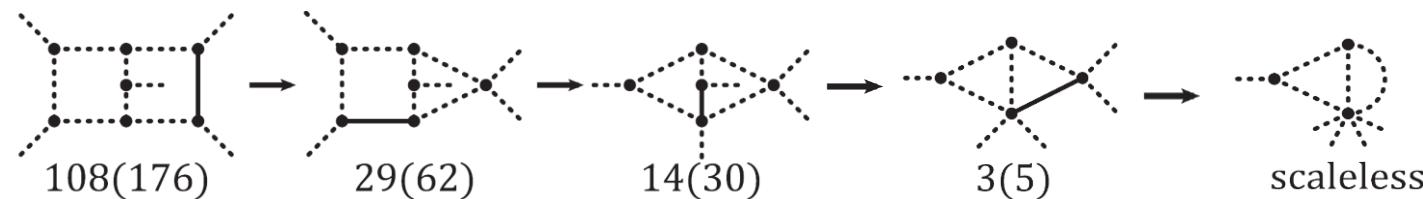
1. Vacuum integrals
2. Simpler FIs with less denominators, if all loop momenta are $\mathcal{O}(1)$

Iterative strategy

➤ For boundary FIs with less denominators:

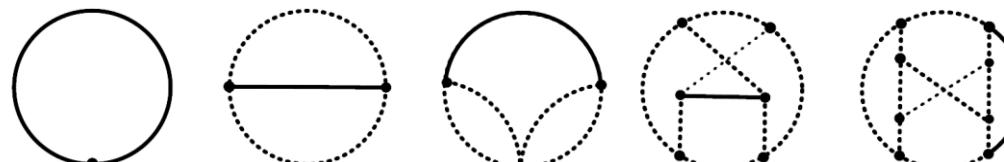
- Calculate them again use AMF method, even simpler boundary FIs as input (besides vacuum integrals)

X. Liu, YQM, 2107.01864



- Eventually, leaving only (single-mass) vacuum integrals as input
- Kinematic information can be recovered by linear algebra!

➤ Typical single-mass vacuum MIs



Baikov, Chetyrkin, 1004.1153
Lee, Smirnov, Smirnov, 1108.0732
Georgoudis, et. al., 2104.08272

- Much simpler to be calculated
- The same number of loops and spacetime dimensions

From vacuum integrals to p-integrals

➤ A family of single-mass vacuum integrals

$$I_{\vec{\nu}}(D, m^2) = \int \prod_{i=1}^L \frac{d^D \ell_i}{i\pi^{D/2}} \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{(\mathcal{D}_1 + i0^+)^{\nu_1} \cdots (\mathcal{D}_K + i0^+)^{\nu_K}}$$

$$\mathcal{D}_1 = \ell_1^2 - m^2 + i0^+$$

- m^2 : the only scale. Can choose $m^2 = 1$

➤ Propagator (p-)integrals

$$\hat{I}_{\vec{\nu}'}(\ell_1^2) = \int \left(\prod_{i=2}^L \frac{d^D \ell_i}{i\pi^{D/2}} \right) \frac{\mathcal{D}_{K+1}^{-\nu_{K+1}} \cdots \mathcal{D}_N^{-\nu_N}}{\mathcal{D}_2^{\nu_2} \cdots \mathcal{D}_K^{\nu_K}}$$

$$\begin{aligned}\vec{\nu}' &= (\nu_2, \dots, \nu_N) \\ \nu &= \sum_{i=1}^N \nu_i\end{aligned}$$

- As ℓ_1^2 is the only scale: $\hat{I}_{\vec{\nu}'}(\ell_1^2) = (-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1} \hat{I}_{\vec{\nu}'}(-1)$
- L -loop single-mass vacuum integral expressed by $(L-1)$ -loop p-integral

$$I_{\vec{\nu}} = \int \frac{d^D \ell_1}{i\pi^{D/2}} \frac{(-\ell_1^2)^{\frac{(L-1)D}{2} - \nu + \nu_1}}{(\ell_1^2 - 1 + i0^+)^{\nu_1}} \hat{I}_{\vec{\nu}'}(-1) = \frac{\Gamma(\nu - LD/2) \Gamma(LD/2 - \nu + \nu_1)}{(-1)^{\nu_1} \Gamma(\nu_1) \Gamma(D/2)} \hat{I}_{\vec{\nu}'}(-1)$$

From p-integrals to vacuum integrals

➤ Apply AMF method on $(L - 1)$ -loop p-integral

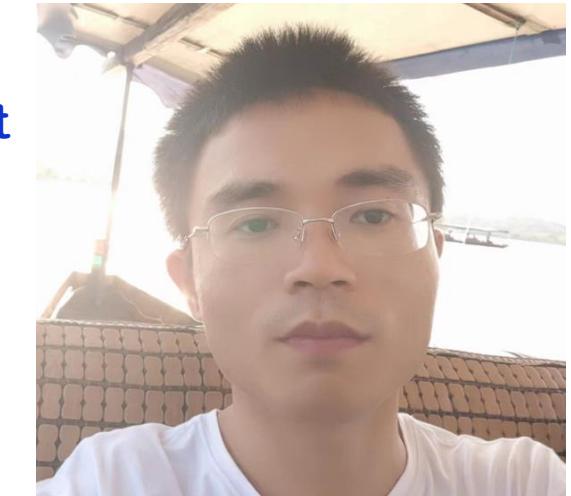
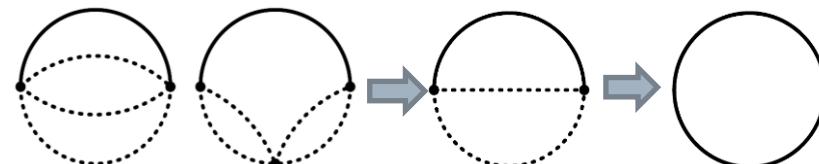
Z. F. Liu, YQM, 2201.11637

1) IBP to setup η -DEs

2) Single-mass vacuum integrals no more than $(L - 1)$ loops as input

Single-mass vacuum integrals with L loops are determined by
that with no more than $(L - 1)$ loops (besides IBP)

- Boundary: 0-loop p-integrals equal 1



刘志峰, 浙江大学

➤ Only IBPs are needed to determine Fls!

Fls \triangleq Linear algebra

Package: AMFlow

➤ Download

X. Liu, YQM, 2201.11669

Link: <https://gitlab.com/multiloop-pku/amflow>

Name	Last commit	Last update
📁 diffeq_solver	update	5 months ago
📁 examples	update	3 months ago
📁 ibp_interface	fix_a_bug_for_mpi_version	1 week ago
📄 AMFlow.m	fix mass mode	2 months ago
📄 CHANGELOG.md	update changelog	1 week ago
📄 FAQ.md	update	6 months ago
📄 LICENSE.md	test	7 months ago
📄 README.md	update	3 months ago
📄 options_summary	update	3 months ago

Sang	2202.11615
Tao	2204.06385
Chen	2204.13500
Armadillo	2205.03345
Chaubey	2205.06339
Zhang	2205.06124
Abreu	2206.03848
Bonciani	2206.10490
Feng	2207.14259
Feng	2208.04302
Chaubey	2208.05837
Sang	2208.10118
Tao	2209.15521
Sang	2210.02979
Henn	2210.13505
Badger	2210.17477
Jakubčík	2211.08446
Abreu	2211.08838
Wang	2211.13713

➤ Description

- The first (method and) package that can calculate any FI (with any number of loops, any D and \vec{s} , or linear propagators) to arbitrary precision, *given sufficient resource*
- Integral reduction is the bottleneck

《Science》 report

<https://doi.org/10.1126/science.adg0720> (2022-11-30)

几十年来，理论粒子物理学家一直在努力解决费曼积分这个棘手的难题。在他们要计算的各个问题中，费曼积分都处于核心地位。

现在把它简化为线性代数问题，从而给出了解决方案。方法将为寻找新物理提供必不可少的理论预言。

For decades, theoretical particle physicists have struggled with vexing calculus problems called Feynman integrals. They are central to every calculation they make—from predicting how magnetic a particle called the muon should be, to estimating the rate at which Higgs bosons should emerge at the Large Hadron Collider (LHC). Now, theorists have found a way to solve the integrals numerically by reducing them to linear algebra. The method promises faster and more precise theoretical calculations, which are essential for searching for hints of new particles and forces.

Stefan Weinzierl教授指出：“该方法原则上普适于任何场景，可用以处理任意费曼积分，运用效果好得令人惊讶”。

“It’s surprising that [the method] works so well,” says Stefan Weinzierl, a theoretical physicist at Johannes Gutenberg University of Mainz who has written an 800-page book on the integrals. “In principle, it’s absolutely general, so you can treat any Feynman integral with it.”

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Current status: integral reduction

➤ IBP reduction is all you need (having AMFlow)

- Laporta algorithm

Laporta, 0102033

➤ Difficulties of IBP method

- Complicated intermediate expression

- Resource-consuming due to large scale of linear equations:

E.g. Laporta 1910.01248

Hundreds GB RAM

E.g. J. Klappert et al., 2008.06494

Months of runtime using super computer

E.g. Davies, Herren, Steinhauser, 1911.10214
(wall time 860 days)

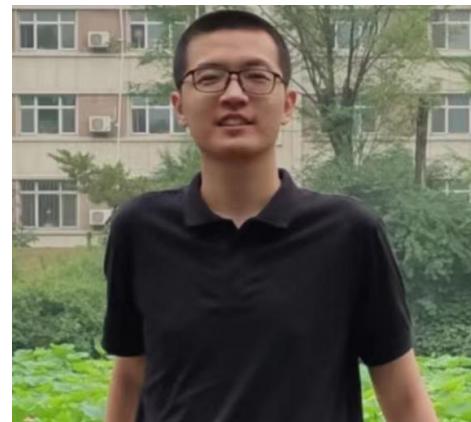
Current status: integral reduction

➤ Selected developments

See also Prof. Zhang's talk

- Finite field: solving intermediate express swell Manteuffel, Schabinger, 1406.4513
- Syzygy equations: trimming IBP system Gluza, Kajda, Kosower, 1009.0472
Larsen, Zhang, et. al., 1511.01071, 1805.01873, 2104.06866
- Block-triangular form: minimize IBP system (needs input) Liu, YQM, 1801.10523, Guan, Liu, YQM, 1912.09294
- A better choice of basis: UT basis/ D-factorized Usovitsch, 2002.08173
S. Abreu, et al., PRL (2019) A. V. Smirnov, V. A. Smirnov , 2002.08042

刘霄, 牛津大学



关鑫, 北京大学 ->斯坦福直线加速器中心(SLAC)



Block-triangular form

➤ Improved linear system

$$Q_{11} I_1 + Q_{12} I_2 + Q_{13} I_3 + Q_{14} I_4 + \dots + Q_{1N} I_N = 0$$

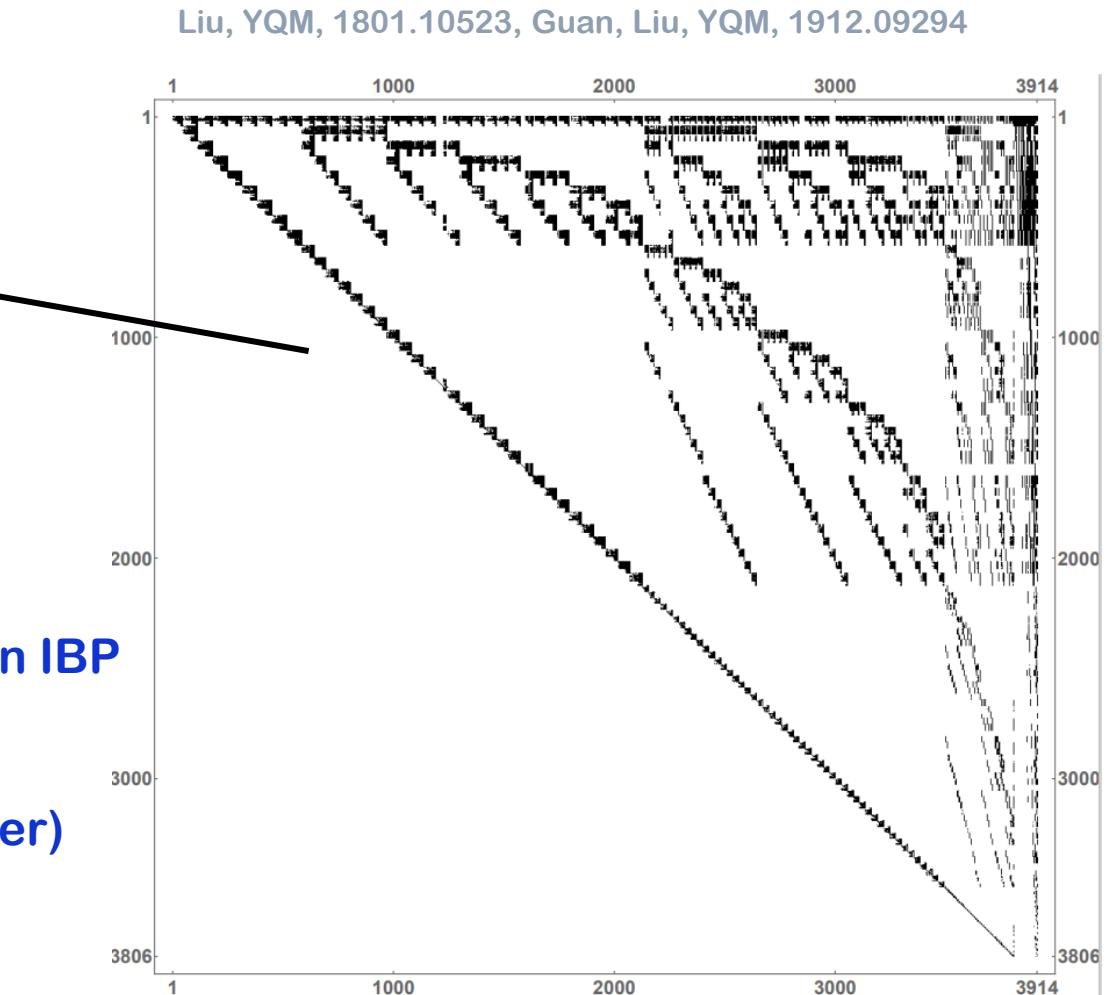
$$Q_{21} I_1 + Q_{22} I_2 + Q_{23} I_3 + Q_{24} I_4 + \dots + Q_{2N} I_N = 0$$

$$Q_{33} I_3 + Q_{34} I_4 + \dots + Q_{3N} I_N = 0$$

$$Q_{43} I_3 + Q_{44} I_4 + \dots + Q_{4N} I_N = 0$$

...

- Simple relations among Feynman integrals
- Several orders of magnitude equations less than IBP
- Nice block-triangular structure, efficient for numerical sampling (finite field / floating number)



Search algorithm

➤ Decomposition of $Q_i(\vec{s}, \epsilon)$ $\sum Q_i(\vec{s}, \epsilon) I_i(\vec{s}, \epsilon) = 0$

$$Q_i(\vec{s}, \epsilon) = \sum_{\mu_0=0}^{\epsilon_{max}} \sum_{\mu} \tilde{Q}_i^{\mu_0 \mu_1 \dots \mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \dots s_r^{\mu_r}$$

- $\tilde{Q}_i^{\mu_0 \mu_1 \dots \mu_r}$ are unknowns
- $\mu_1 + \dots + \mu_r = d_i$

➤ Input from numerical IBP $I_i(\vec{s}, \epsilon) = \sum_{j=1}^n C_{ij}(\vec{s}, \epsilon) M_j(\vec{s}, \epsilon)$

$$\Rightarrow \sum_{\mu_0, \mu} \sum_{j=1}^n \tilde{Q}_i^{\mu_0 \dots \mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \dots s_r^{\mu_r} C_{ij}(\vec{s}, \epsilon) M_j(\vec{s}, \epsilon) = 0$$

➤ Linear equations: $\sum_{\mu_0, \mu} \tilde{Q}_i^{\mu_0 \dots \mu_r} \epsilon^{\mu_0} s_1^{\mu_1} \dots s_r^{\mu_r} C_{ij}(\vec{s}, \epsilon) = 0$

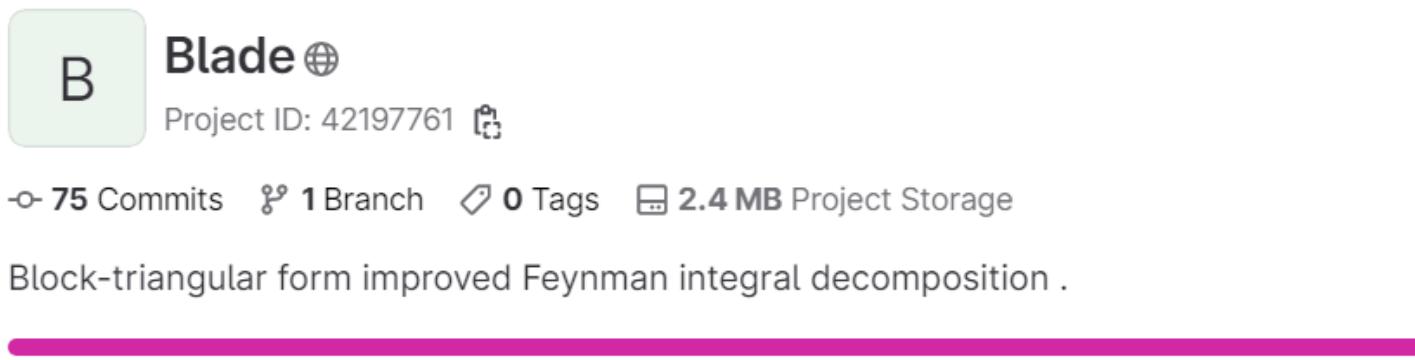
- With enough constraints $\Rightarrow \tilde{Q}_i^{\mu_0 \dots \mu_r}$
- With finite field technique, equations can be efficiently solved
- Relations among $G \equiv \{I_1, I_2, \dots, I_N\}$ can be determined

Package: Blade

➤ Download

Guan, Liu, YQM, Wu, To appear

Link: <https://gitlab.com/multiloop-pku/blade>

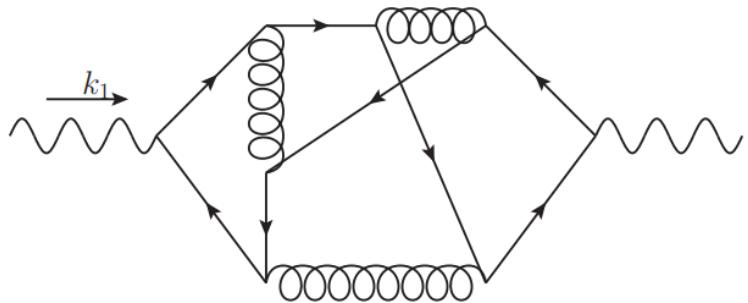


➤ Description

- Automatic IBP reduction
- Usually improve the efficiency by 1-2 orders

A four-loop example

➤ $e^+e^- \rightarrow \gamma^* \rightarrow t\bar{t}$ at N³LO_{QCD} (**forward scattering**)



Chen, Guan, He, Liu, YQM 2209.14259

- Functions of ϵ and m_t^2/s
- Feynman integrals up to degree 4

# int.	# MIs	t_{search}/h	t_{IBP}/s	t_{solve}/s	# IBP	# sample	# primes
43788	369	8	432	4.5	64	4555	7

- About two orders of magnitude faster than plain IBP

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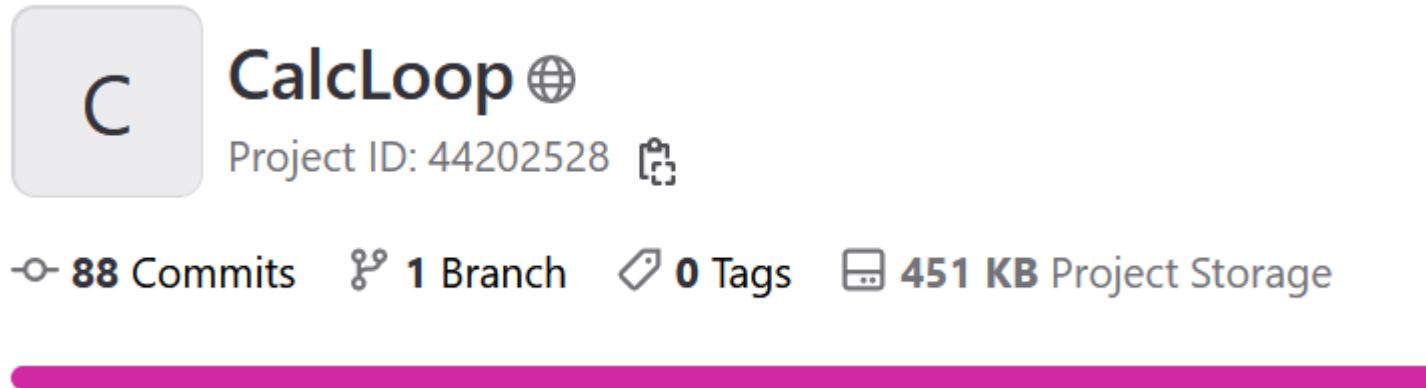
V. Summary and outlook

Package: CalcLoop

➤ Download (to be finished)

YQM, To appear

Link: <https://gitlab.com/yqma/CalcLoop>



➤ Description

- Automatic high order perturbative calculation

Example

- E.g. $e^+e^- \rightarrow Q\bar{Q}$ at NNLO QCD

```
files={"twoloop","tree"};
dir=FileNameJoin[{filename,"integrals",StringJoin@@files}];

ampAssoc=AmplitudeSquared[dir,amps@files[[1]],amps@files[[2]]]//CLTiming;

fiAssoc=FamilyDecomposition[FileNameJoin[{dir,dirXsection}],ampAssoc@"Amplitude",
  "CutInformation"->ampAssoc@"PhaseSpace"]//CLTiming;

resLoop2=RunAMFlow[FileNameJoin[{dir,dirXsection}],kinematics]//Expand//CLTiming;
```

Summary

- AMFlow: in principle any FI can be calculated
- Blade: improve the efficiency of IBP reduction significantly
- CalcLoop: towards fully automatic high-order perturbative calculation
- Integral reduction: bottleneck of most current cutting-edge problems, stay tuned

Thank you!

Adaptive search strategy

➤ Semi-analytic

- The number of unknowns of full-analytic block-triangular form may be too large
- Keep a subset of variables analytic \Rightarrow easy to search
- The integral set is the same \Rightarrow still very efficient
- More than one block-triangular form is needed

$$Q_i(\vec{z}) = \sum_{\mu} \tilde{Q}_i^{\mu_1 \dots \mu_r} z_1^{\mu_1} \dots z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-1,0}, z_r) = \sum_{\mu_r} \tilde{Q}_i^{\mu_0} z_r^{\mu_r}$$

$$Q_i(z_{1,0}, \dots z_{r-2,0}, z_{r-1}, z_r) = \sum_{\mu_{r-1}, \mu_r} \tilde{Q}_i^{\mu_{r-1} \mu_r} z_{r-1}^{\mu_{r-1}} z_r^{\mu_r}$$

....

➤ Adaptive search

1. $n = 1$
2. Search n -variable block-triangular form within time limit T
3. If search succeed, $n++$ and go to step 2, otherwise go to step 4
4. Perform reduction by solving the most efficient linear system(i -variable)

➤ Exploit full potential of block-triangular form