R-RATIO FROM LATTICE QCD

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Introduction

We present a calculation of the R-ratio on the lattice. The R-ratio is defined as the hadronic cross section ratio $\sigma(e^+e^- \to \mathrm{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. Fig. 1 shows R in the light, charm, and beauty regions [1]. The extraction of spectral densities from Euclidean lattice correlation function is one kind of inverse problem. By using the method of Bayesian Reconstruction (BR) [2], we compute the R-ratio with energy up to 2.5 GeV. Since the lattice spectrum is discrete, we need to smear the reconstructed spectral functions in order to compare them with the experimental results. Our results are consistent with the PDG and KNT19 compilation of R-ratio experimental measurements smeared with the same Gaussian kernels. This study provides a new possibility that lattice calculation can contribute to the muon $g-2$ studies. And, more generally, it demonstrates a systematic approach to tackle the inverse problem with sophisticated error control.

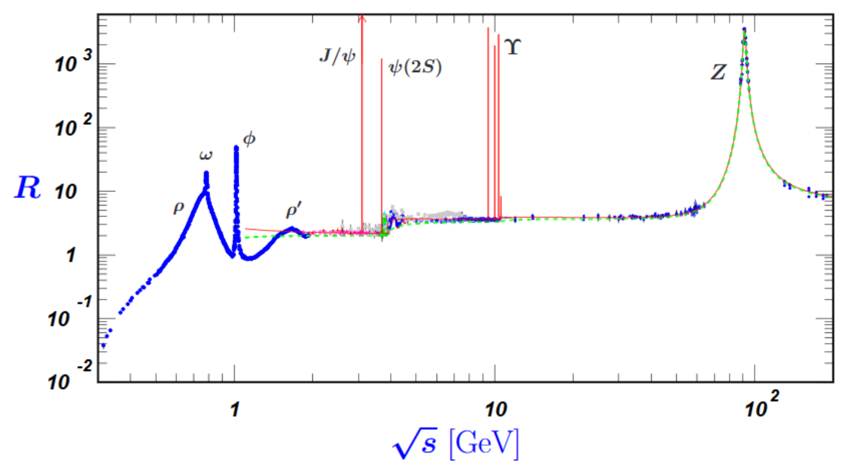

Figure 1: Experimental results of the R-ratio.

Numerical Calculation

The electromagnetic current and current-current correlation function are expressed as

$$J^{em}_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s, \tag{1}$$

$$C(t) = \langle J^{em}_\mu(t) J^{em}_\nu(0)\rangle = \int d\omega \rho(\omega) e^{-\omega t}, \tag{2}$$

so numerically the spectral function $\rho(\omega)$ can be extracted by solving the corresponding inverse problem. One can calculate the R-ratio with the following expression:

$$R(\omega) = \frac{12\pi^2}{\omega^2}\rho(\omega). \tag{3}$$

In the practical calculation on the ensembles with two degenerate light flavors, $C(t)$ can be decomposed into two pieces

$$C(t) = \frac{5}{9}C^{con}(t; m_l) + \frac{1}{9}C^{con}(t; m_s), \tag{4}$$

where m_l is the isosymmetric light quark mass and m_s is the strange quark mass. We use the high-precision current-current correlation functions for both light and strange quarks, but no charm and no disconnected insertions for now.

Symbol	$L^3\times T$	m_π(MeV)	a(fm)	L(fm)	$N_{\rm cfg}$
48I	$48^3\times 96$	139	0.11406(26)	5.475	100
64I	$64^3\times 128$	139	0.08365(25)	5.354	92
24D	$24^3\times 64$	141	0.1940(19)	4.656	232
32D	$32^3\times 64$	141	0.1940(19)	6.208	134
48D	$48^3\times 96$	141	0.1940(19)	9.312	47

Table 1: The gauge ensembles used.

BR and Smearing Method

BR is an improved Bayesian method. The Bayesian probability is

$$P[\rho|D,\alpha,m] \propto e^{Q(\rho)}, \tag{5}$$

where $P[\rho|D,\alpha,m]$ denotes the conditional probability that ρ is the solution given lattice data D, prior information m, and a hyper parameter α. Specifically, $Q = \alpha S - L - \gamma(L-N_\tau)^2$, where

$$S = \sum_j \left[1 - \frac{\rho(\omega_j)}{m(\omega_j)} + \log\left(\frac{\rho(\omega_j)}{m(\omega_j)}\right)\right]\Delta\omega_j, \tag{6}$$

which is an alternative way to encode the constraint from the prior. L is the likelihood function. γ is a numerically large number such that the term $\gamma(L-N_\tau)^2$ helps to prevent over-fitting. The hyper parameter α here is integrated over as

$$P[\rho|D,m] = \frac{P[D|\rho,I]}{P[D|m]}\int d\alpha P[\alpha|D,m], \tag{7}$$

where $P[\alpha|D,m]$ is the probability of α in the present of D and m.

The search space of the maximum of $P[\rho|D,\alpha,m]$ is the whole parameter space and one needs high-precision architecture [3] (e.g., 512-bit floating point numbers).

In principle, the obtained lattice spectral function consists of discrete delta peaks, while the experimental one is continuous. For a fair comparison with the experiments, we need to smear both the lattice and experimental results. We use the normalized Gaussian kernels for this purpose:

$$G_\sigma(\omega) = \exp(-\omega^2/2\sigma^2)/\sqrt{2\pi\sigma^2}, \tag{8}$$

$$R_\sigma(E) = \int d\omega G_\sigma(E-\omega)R(\omega). \tag{9}$$

Errors and Results

We carefully checked the finite volume and finite lattice spacing effects. As shown in the left panel of Fig. 2, the volume dependence of the R-ratio is negligible. While a clear difference at small ω can be seen in the right panel which manifests the lattice spacing effects and a continuum extrapolation is carried out.

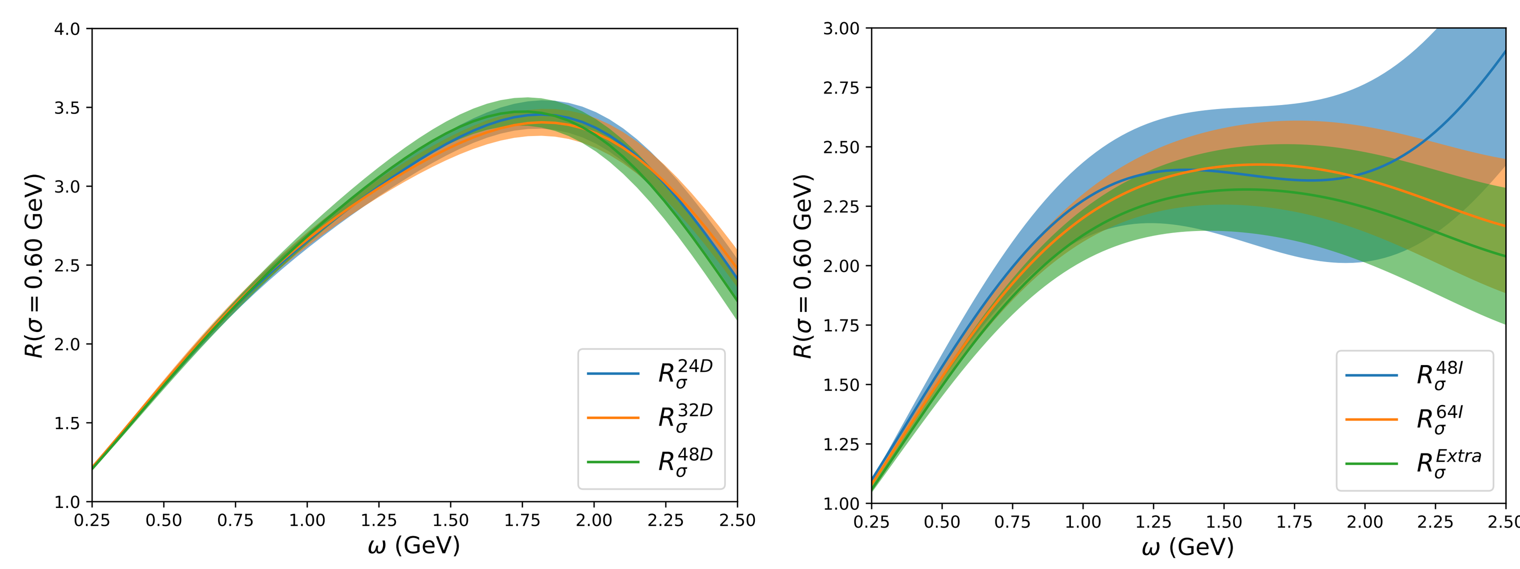

Figure 2: The volume and lattice spacing dependence.

Our results with both statistical error and systematic errors caused by the prior function and continuum extrapolation with two smearing parameters σ = 0.6 GeV and 0.7 GeV are shown in Fig. 3. We found that the main error comes from the prior function of BR, and our results are well consistent with the experimental values.

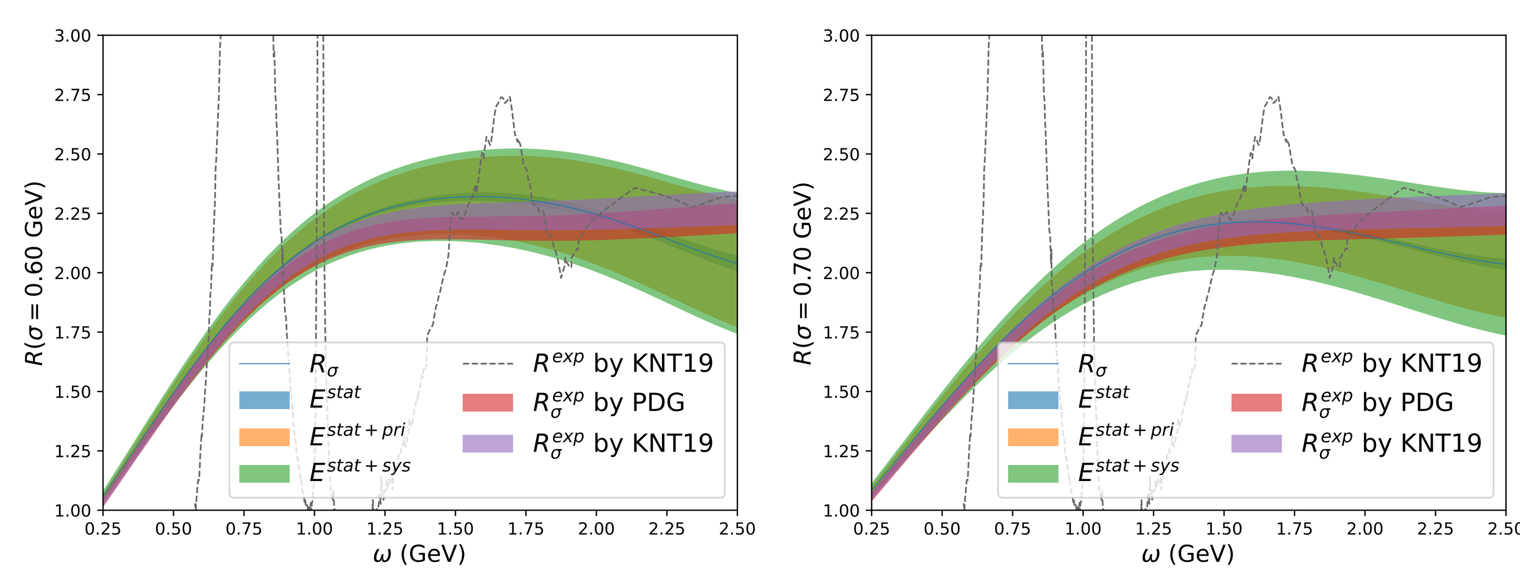

Figure 3: The R-ratio with both statistical and systematic errors.

Summary and Outlook

R-ratio as a function of the center-of-mass energy of the electrons, allows to predict the leading hadronic contribution (HVP) to the muon anomalous magnetic moment (a_μ). ETMC recently has a study of the R-ratio using an improved Backus-Gilbert method [4], while in this work, we present a new method based on BR to solve the inverse problems and calculate the R-ratio through spectral functions. We find that the finite volume effects are negligible. We demonstrate a full consideration of both the statistical error of lattice data and the systematic error caused by the prior function and the lattice spacing extrapolation. We are trying to optimize our results to provide a new possibility for calculating the muon $g-2$. More generally, this new approach can be applied to address the inverse problems arising in many aspects of lattice QCD.

References

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