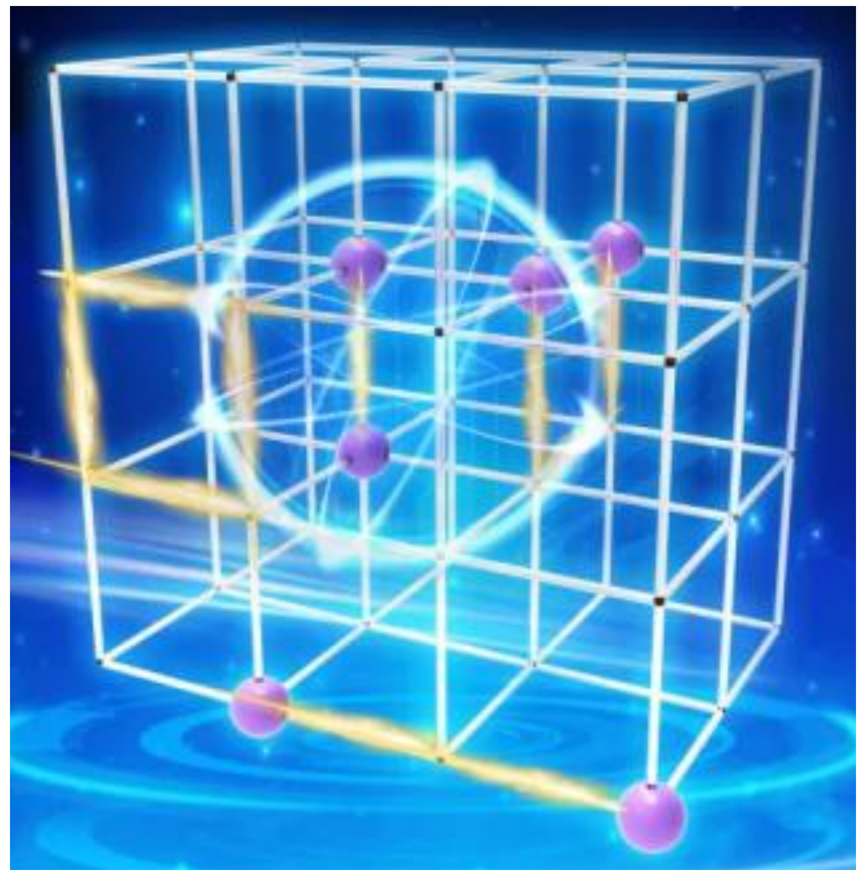
Inverse Problem Approach

——A novel non-perturbative QCD method



Fu-Sheng Yu

Lanzhou University

Based on [Ao-Sheng Xiong(熊傲昇), Ting Wei(魏婷), FSY, arXiv:2211.13753]

2023.10.08，第三届中国格点QCD研讨会

1. Motivation: non-perturbative approaches

- **Lattice QCD**: the only recognized first-principle method

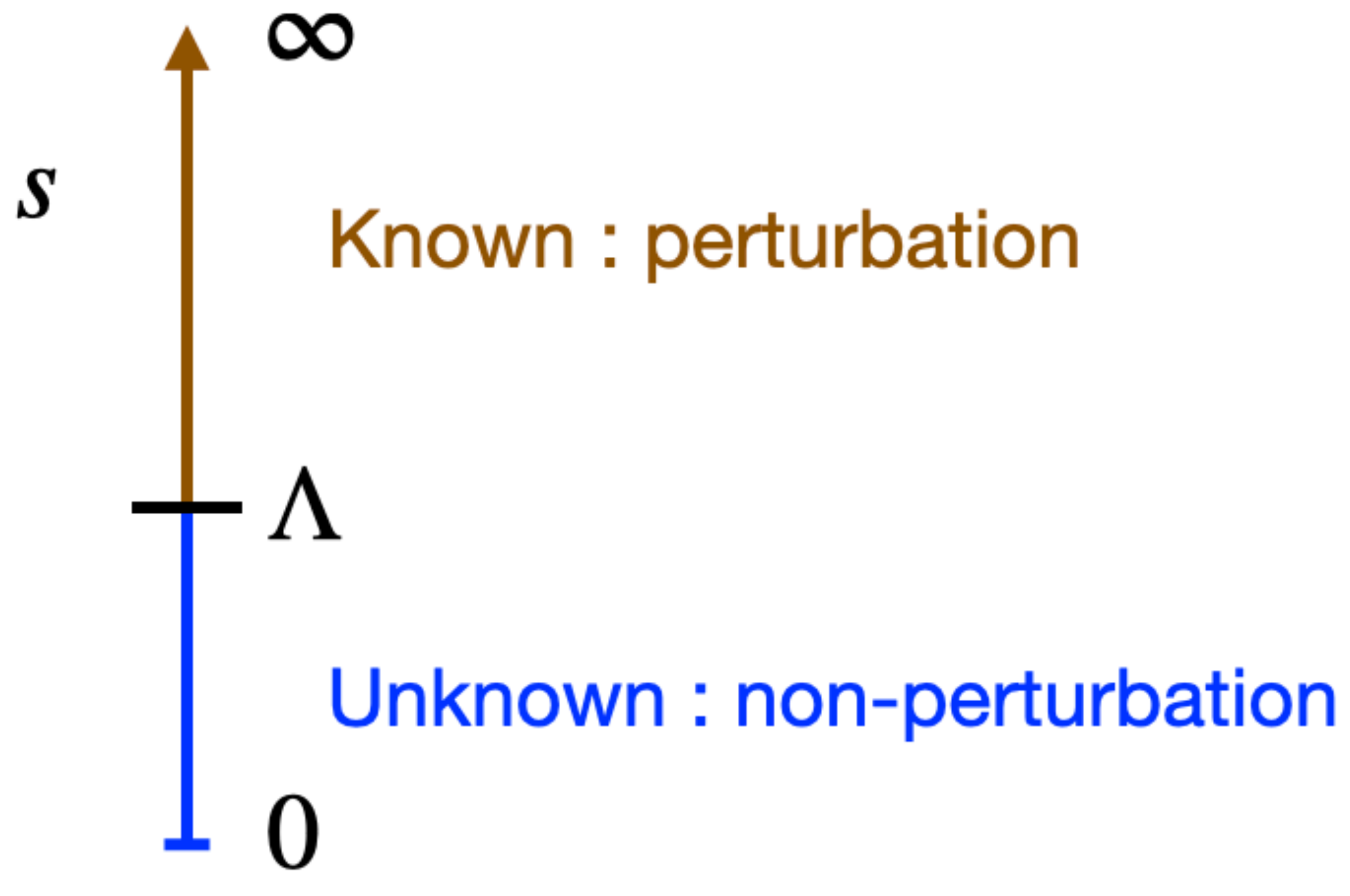
- QCD sum rules, Dyson-Schwinger Equation, Chiral perturbation theory, Holographic QCD, Light-front quantization, Other EFTs and phenomenological models

- Each of them has its advantages and shortcomings.

- It is always welcome to develop a new theoretical method for non-perturbation, to make complimentary predictions what are difficult by the above methods.

- Inverse problem is such a new method. Its development is learning Lattice QCD. It might collaborate with Lattice QCD.

The main idea of the inverse problem approach



$$\Pi(q^2) = i\int d^4x e^{iq\cdot x}\langle \mathcal{O}(x)\mathcal{O}(0)\rangle$$

Dispersion Relation:

$$\mathcal{R}e[\Pi(s)] = \frac{1}{\pi}\mathcal{P}\int_0^\infty \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

If $s > \Lambda$,

$$\mathcal{P}\int_0^\Lambda \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_\Lambda^\infty \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

2.ill-posedness of the inverse problem

- Dispersion relation: first-class Fredholm integration equation

If $s > \Lambda$, $$\mathcal{P}\int_0^{\Lambda}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_{\Lambda}^{\infty}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

Existence of solution ? Uniqueness of the solution ?? Stability of the solution ???

2.ill-posedness of the inverse problem

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y)$$

- The operator $K : X \to Y$, $\quad Kx = y, \quad x \in X, \quad y \in Y$
- Inverse problem: solve x by known of K and y, $\quad x = K^{-1}y$
- Definition of well-posedness:

Define: *The operator equation (3.1) is called well-posed if the following holds [8]:*

1.Existence: For every $g \in G$ there is (at least one) $f \in F$ such that $Kf = g$;

2.Uniqueness: For every $g \in G$ there is at most one $f \in F$ with $Kf = g$;

3.Stability: The solution f depends continuously on g; that is, for every sequence $(f_n) \subset F$ with $Kf_n \to Kf (n \to \infty)$, it follows that $f_n \to f (n \to \infty)$

- Ill-posedness: At least one of the above conditions is not satisfied

- If well-posed, K^{-1} must be a bounded or continuous operator, otherwise ill-posed.

2. Proof of the ill-posedness

Proof of uniqueness:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

Proof. Since K is a linear operator, we know that $Kf_1 - Kf_2 = K(f_1 - f_2) = 0$. Setting $f = f_1 - f_2$, we just need to prove that $Kf = 0$ implies $f(x) = 0$, a. e. $x \in [a,b]$.

It is easy to obtain that $Kf = \int_a^b \frac{1}{y-x} f(x)dx = \int_a^b \left(\frac{1}{y}\sum_{k=0}^{\infty}(\frac{x}{y})^k\right) f(x)dx$. Since $x \in [a,b]$, $y \in [c,d]$, $c > b$, we know $|\frac{x}{y}| \le |\frac{b}{c}| < 1$, which implies that $\left|\sum_{k=0}^{\infty}(\frac{x}{y})^k f(x)\right| \le \sum_{k=0}^{\infty}(\frac{b}{c})^k |f(x)|$ for all $x \in [a,b]$. Combined with $\int_a^b |f(x)|dx < +\infty$ and the control convergence theorem, we have

$$y\int_a^b \frac{1}{y-x} f(x)dx = \sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx = 0, \quad y \in [c,d]. \tag{3.4}$$

If $d = +\infty$, by using (3.4), we have

$$\int_a^b f(x)dx + \frac{1}{y}\int_a^b xf(x)dx + \cdots + \frac{1}{y^k}\int_a^b x^k f(x)dx + \cdots = 0, \quad y \in (c, +\infty). \tag{3.5}$$

Letting $y \to +\infty$ in (3.5), we have $\int_a^b f(x)dx = 0$. Then multiplying y on both sides of (3.5) and letting $y \to +\infty$, we also have $\int_a^b xf(x)dx = 0$. Repeating above process, we can obtain that

$$\int_a^b x^k f(x)dx = 0, \quad k = 0, 1, 2, \cdots. \tag{3.6}$$

If $d < +\infty$, taking $z \in D := \{z \in \mathbb{C} : |z| \ge c\}$, we have

$$\left|\sum_{k=0}^{\infty}\frac{1}{z^k}\int_a^b x^k f(x)dx\right| \le \sum_{k=0}^{\infty}\frac{1}{c^k}\left|\int_a^b x^k f(x)dx\right| \le \sum_{k=0}^{\infty}\frac{b^k}{c^k}\int_a^b |f(x)|dx < +\infty,$$

which implies that the series $\sum_{k=0}^{\infty}\frac{1}{z^k}\int_a^b x^k f(x)dx$ is convergent uniformly on D. Since $\frac{1}{z^k}\int_a^b x^k f(x)dx$ is analytic on D for each k and use the Weierstrass theorem, we conclude that the series $\sum_{k=0}^{\infty}\frac{1}{z^k}\int_a^b x^k f(x)dx$ is analytic on D. Further, we know $\sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx$ is real analytic on $y \in (c, +\infty)$. Combined with the analytic continuation, we know that (3.4) holds for $y > c$, i. e.

$$\sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx = 0, \quad y \in (c, +\infty).$$

Similar to the proof process of the case $d = +\infty$, we also conclude that $\int_a^b x^k f(x)dx = 0$, $k = 0, 1, 2, \cdots$ for $d < +\infty$.

2. Proof of the ill-posedness

Proof of uniqueness:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

Proof. Since K is a linear operator, we know that $Kf_1 - Kf_2 = K(f_1 - f_2) = 0$. Setting $f = f_1 - f_2$, we just need to prove that $Kf = 0$ implies $f(x) = 0$, a. e. $x \in [a,b]$.

Since $C[a,b]$ is dense in $L^2(a,b)$, then for $f(x) \in L^2(a,b)$ and any $\epsilon > 0$, there exists $\tilde{f}(x) \in C[a,b]$, such that $\|f - \tilde{f}\|_{L^2(a,b)} < \epsilon$. On the other hand, for $\tilde{f}(x) \in C[a,b]$, there exists a polynomial $Q_n(x)$ of degree $n \in \mathbb{N}$, such that $\|\tilde{f} - Q_n\|_{C[a,b]} < \epsilon$ by the Weierstrass theorem. Therefore, we have

$$\begin{aligned}\|f - Q_n\|_{L^2(a,b)} &\le \|f - \tilde{f}\|_{L^2(a,b)} + \|\tilde{f} - Q_n\|_{L^2(a,b)} \\ &\le \epsilon + \sqrt{b-a}\|\tilde{f} - Q_n\|_{C[a,b]} \\ &< \epsilon + \epsilon\sqrt{b-a},\end{aligned}$$

By using (3.6), we know that $\int_a^b f(x)Q_n(x)dx = 0$. Combined with the Cauchy inequality, we have

$$\begin{aligned}\|f\|^2_{L^2(a,b)} &= \int_a^b f^2(x)dx = \int_a^b \left(f^2(x) - f(x)Q_n(x)\right)dx \\ &\le \int_a^b |f(x)| \cdot |f(x) - Q_n(x)|dx \\ &\le \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}}\left(\int_a^b |f(x) - Q_n(x)|^2 dx\right)^{\frac{1}{2}} \\ &= \|f\|_{L^2(a,b)}\|f - Q_n\|_{L^2(a,b)} \\ &\le (\epsilon + \epsilon\sqrt{b-a})\|f\|_{L^2(a,b)},\end{aligned}$$

which implies that $\|f\|_{L^2(a,b)} \le \epsilon + \epsilon\sqrt{b-a}$.

Letting $\epsilon \to 0$, we have $\|f\|_{L^2(a,b)} = 0$, i. e. $f(x) = 0$, a. e. $x \in [a,b]$. The proof is completed. □

2. Proof of the ill-posedness

Proof of instability:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

We show the instability of the inverse problem of dispersion relation by the special case. Taking $a = 0$, $b = 1$, $c = 2$, $d = 3$, $f_2(x) = f_1(x) + \sqrt{n}\cos(n\pi x)$, and $f_{1,2}$ are the solutions of $g_{1,2}$ with $g_i(y) = \int_0^1 \frac{1}{y-x} f_i(x)dx$. As $n \to \infty$, it is obvious that

$$\|f_2 - f_1\|_{L^2(0,1)} = \left(\int_0^1 (\sqrt{n}\cos(n\pi x))^2 dx\right)^{1/2} = \frac{\sqrt{n}}{\sqrt{2}} \to \infty, \tag{3.7}$$

and

$$\|g_2 - g_1\|_{L^2(2,3)} = \frac{1}{\sqrt{n\pi}}\left(\int_2^3 \left(\int_0^1 \left(\frac{1}{y-x}\right)^2 \sin(n\pi x)dx\right)^2 dy\right)^{1/2} \leq \frac{1}{\sqrt{n\pi}} \to 0. \tag{3.8}$$

That means the solutions could be changed infinitely even though the noise of the input data is approaching to vanish. So the inverse problem is unstable.

2. Proof of the ill-posedness

The inverse problem of dispersion relation is ill-posed   See 2211.13753

1) Existence ✓

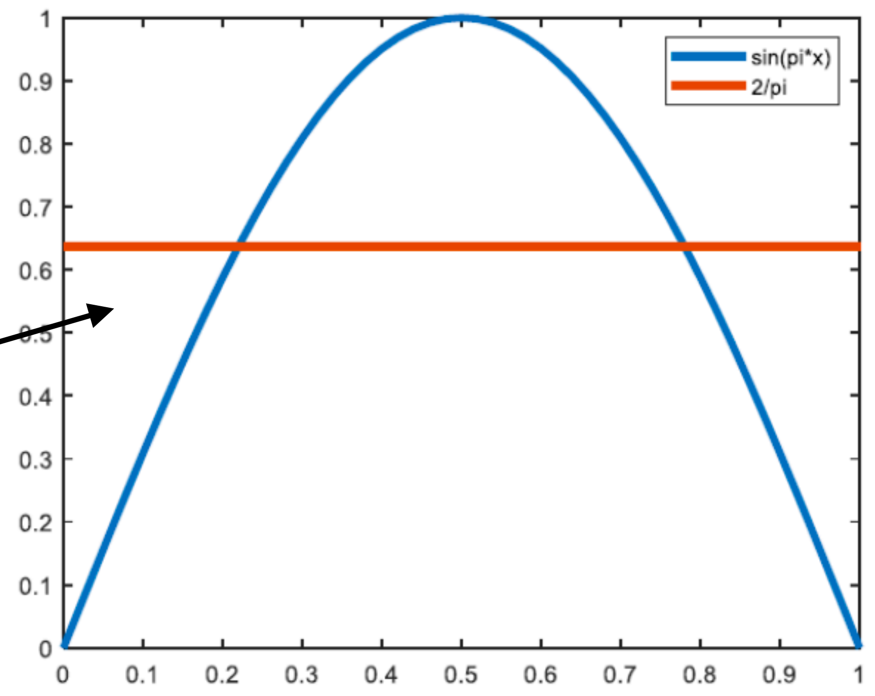
2) Uniqueness ✓

3) Stability ✗

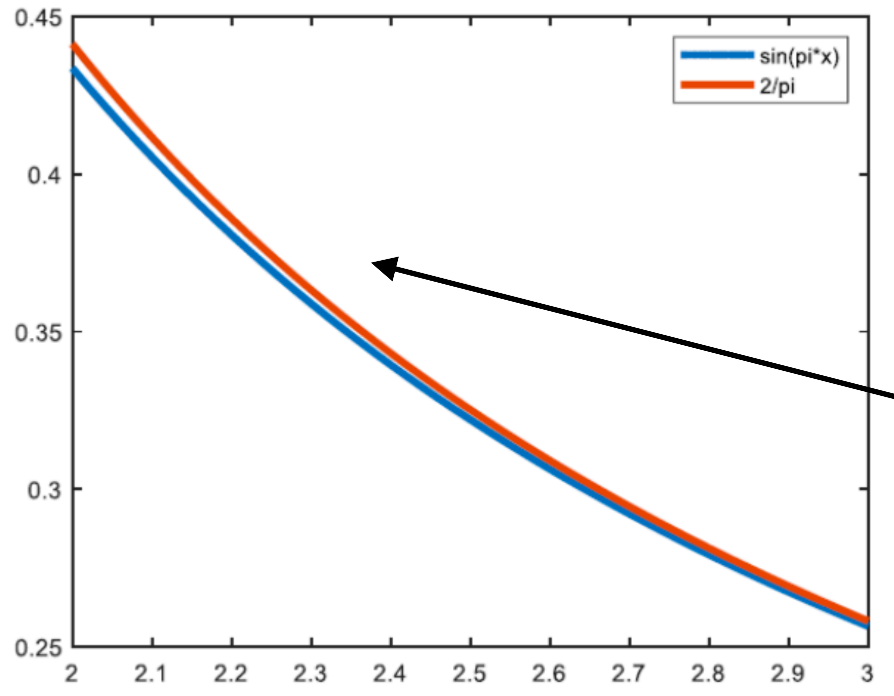
$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

$f(x)$

Instability



$g(y)$



uniqueness

Can we find a good solution? And how?

3. Regularization method

Define: *A regularization strategy is a family of linear and bounded operators $R_\alpha : G \to F, \alpha > 0$, such that $\lim_{\alpha\to 0} R_\alpha K f = f$ for all $f \in F$, where the α is the regularization parameter [8].*

- Construct a bounded operator which is approximate to K^{-1},
- Ill-posed problem => well-posed approximate problem, so that $f_\alpha^\delta = R_\alpha g^\delta$
- f_α^δ is the approximate solution related to both α and δ.
- An effective regularization strategy is to satisfy $f_\alpha^\delta \to f$, as $\|g^\delta - g\| \leq \delta \to 0$

$$\begin{aligned}\left\|f_\alpha^\delta - f\right\|_F &\leq \left\|R_\alpha g^\delta - R_\alpha g\right\|_F + \|R_\alpha g - f\|_F \\ &\leq \|R_\alpha\| \left\|g^\delta - g\right\|_G + \|R_\alpha K f - f\|_F \\ &\leq \delta \|R_\alpha\| + \|R_\alpha K f - f\|_F\end{aligned}$$

$$Kf = g, f \in F, g \in G$$

$$\lim_{\alpha\to 0} R_\alpha K f = f$$

3. Tikhonov Regularization

$$f_\alpha^\delta = R_\alpha g^\delta \qquad R_\alpha := (\alpha I + K^* K)^{-1} K^* : G \to F \qquad \alpha f_\alpha^\delta + K^* K f_\alpha^\delta = K^* g^\delta$$

$$f_\alpha^\delta = \arg\min_{f\in L^2(a,b)} J(f), \quad J(f) = \frac{1}{2}\|Kf - g^\delta\|^2_{L^2(c,d)} + \frac{\alpha}{2}\|f\|^2_{L^2(a,b)}$$

A priori condition: $f = K^* v,\ v \in G,\ \|v\|_G \le E$ $\qquad \|f_\alpha^\delta - f\|_F \le \dfrac{\delta}{2\sqrt{\alpha}} + \dfrac{\sqrt{\alpha}E}{2}$

Take $\alpha = \delta/E$

$$\boxed{\|f_\alpha^\delta - f\|_F \le \sqrt{\delta E} \to 0, \quad \delta \to 0}$$

- **The most important: the uncertainty converges to vanishing as $\delta \to 0$.**
- **It exists an upper limit !**
  The uncertainty must be controllable.

3. Selection rules of the Regularization parameter

A-priori methods are always difficult to use in practice.

A-posterior methods can be tried.

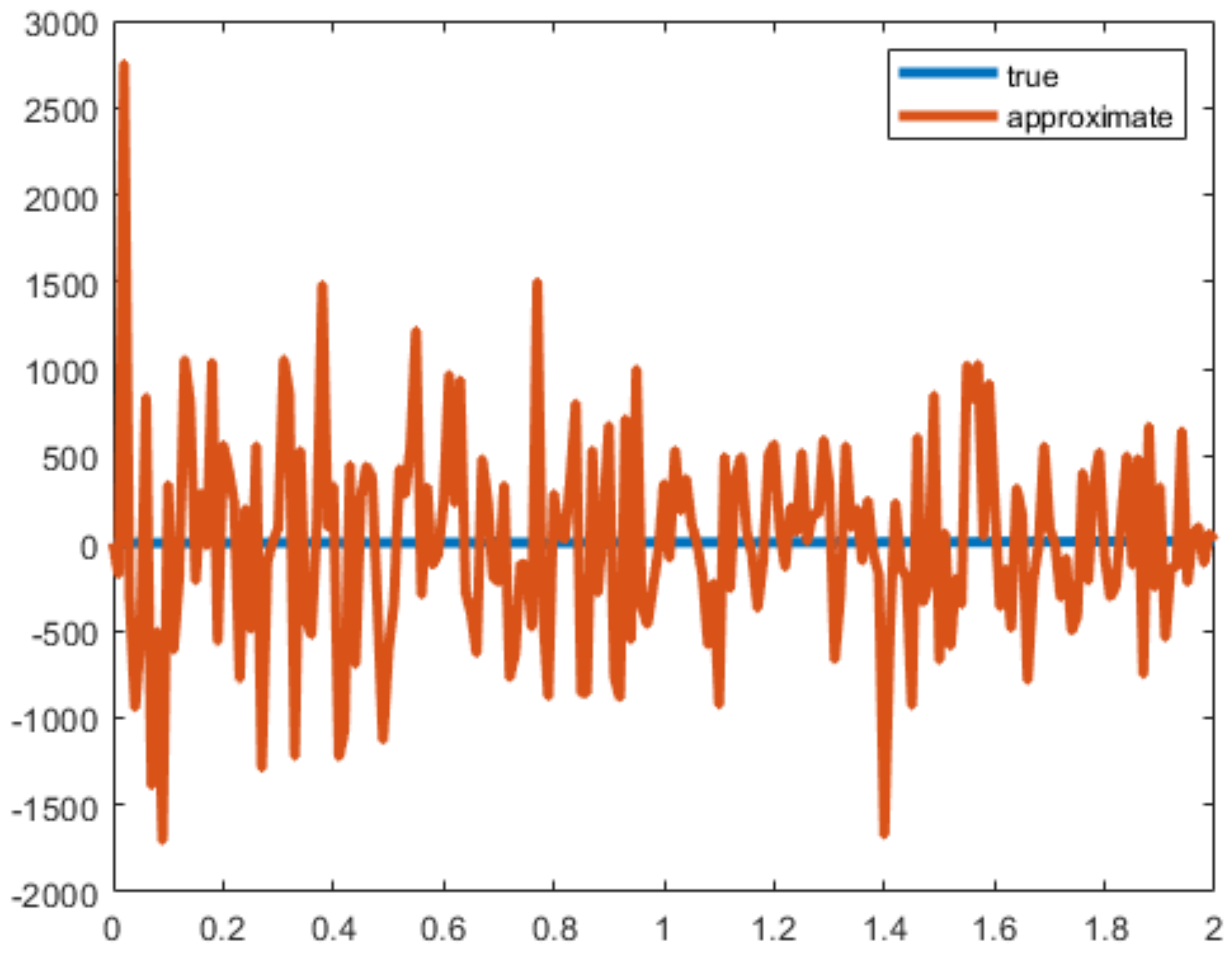
L-curve method: $$\alpha = \underset{f_\alpha^\delta \in L^2(a,b)}{\arg\min} \left(\left\| f_\alpha^\delta \right\|_F \left\| g^\delta - K f_\alpha^\delta \right\|_G \right)$$

Both of $\|f_\alpha^\delta\|$ and $\|g^\delta - Kf_\alpha^\delta\|$ should be minimized together,

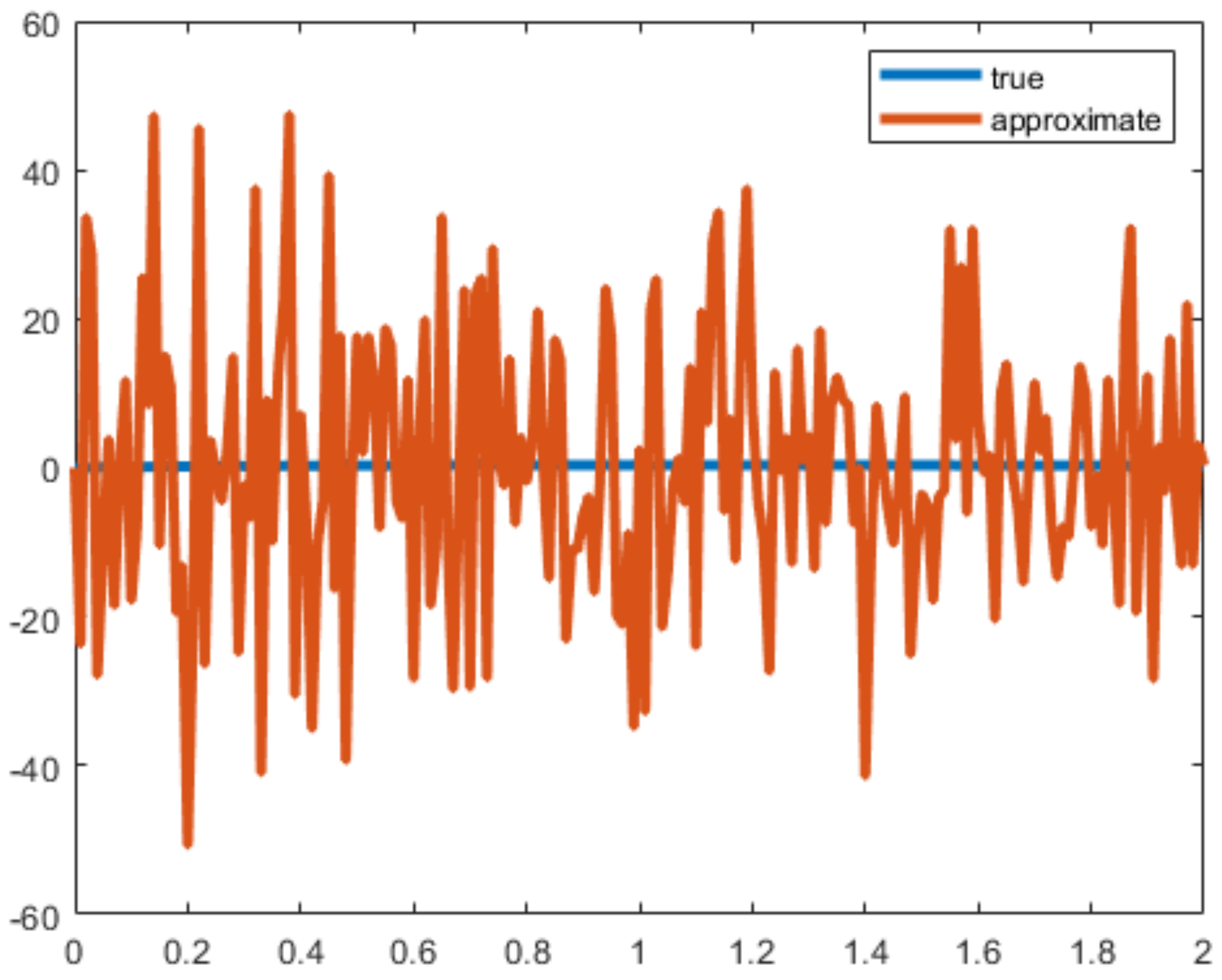
considering $$f_\alpha^\delta = \underset{f \in L^2(a,b)}{\arg\min} J(f), \quad J(f) = \frac{1}{2}\|Kf - g^\delta\|^2_{L^2(c,d)} + \frac{\alpha}{2}\|f\|^2_{L^2(a,b)}$$

4. Test: Importance of regularization

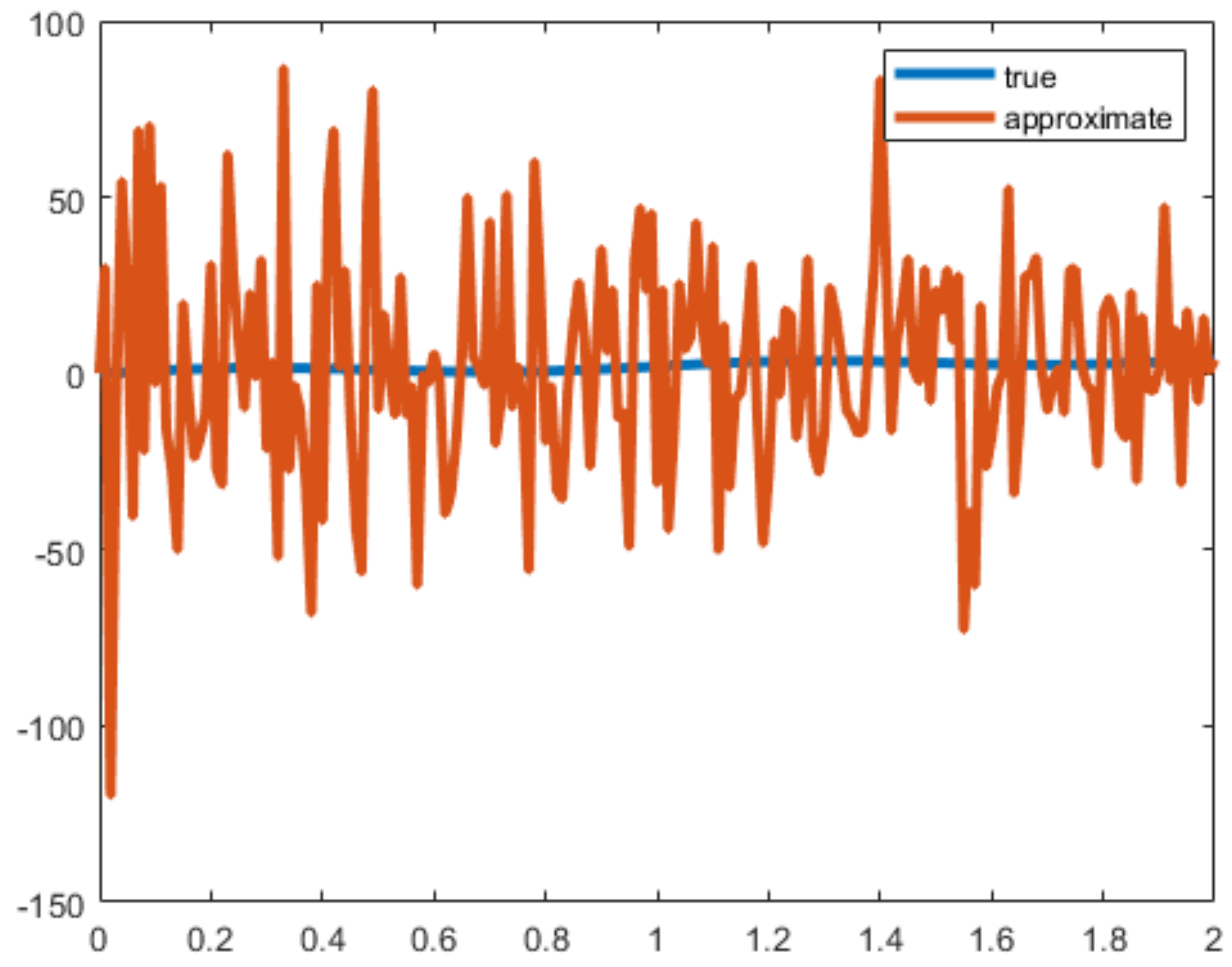
The solutions without any regularization:



model 1



model 2



model 3

- It can be clearly seen that the solutions are unstable and far from the true values.
- The ill-posed inverse problems can not be solved without any regularization.

4. Test: Impact of improved regularization method

- **The regularization method works well for the three models**

- Non-stationary Tikhonov regularization for model 3

(1) $\quad Compute \quad r_k^\delta = g^\delta - K f_k$

(2) $\quad Solve \quad h_k = min\{\frac{1}{2}||Kh - r_k^\delta||_{L^2} + \frac{\alpha_k}{2}||h||_{H^1}\}$

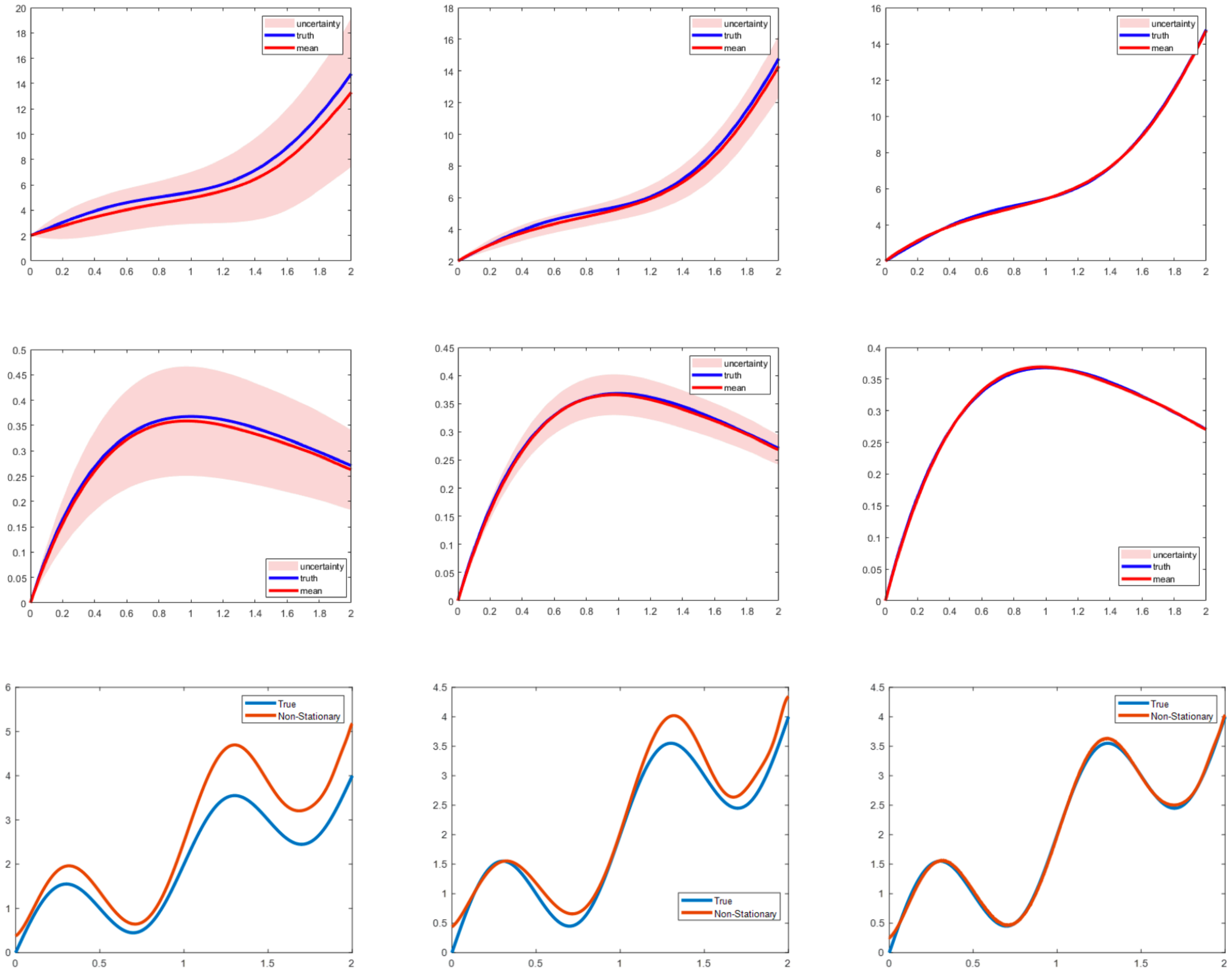
(3) $\quad Update \quad f_{k+1} = f_k + h_k$

(4) $\quad$ Stop by the L-curve method

model 1

model 2

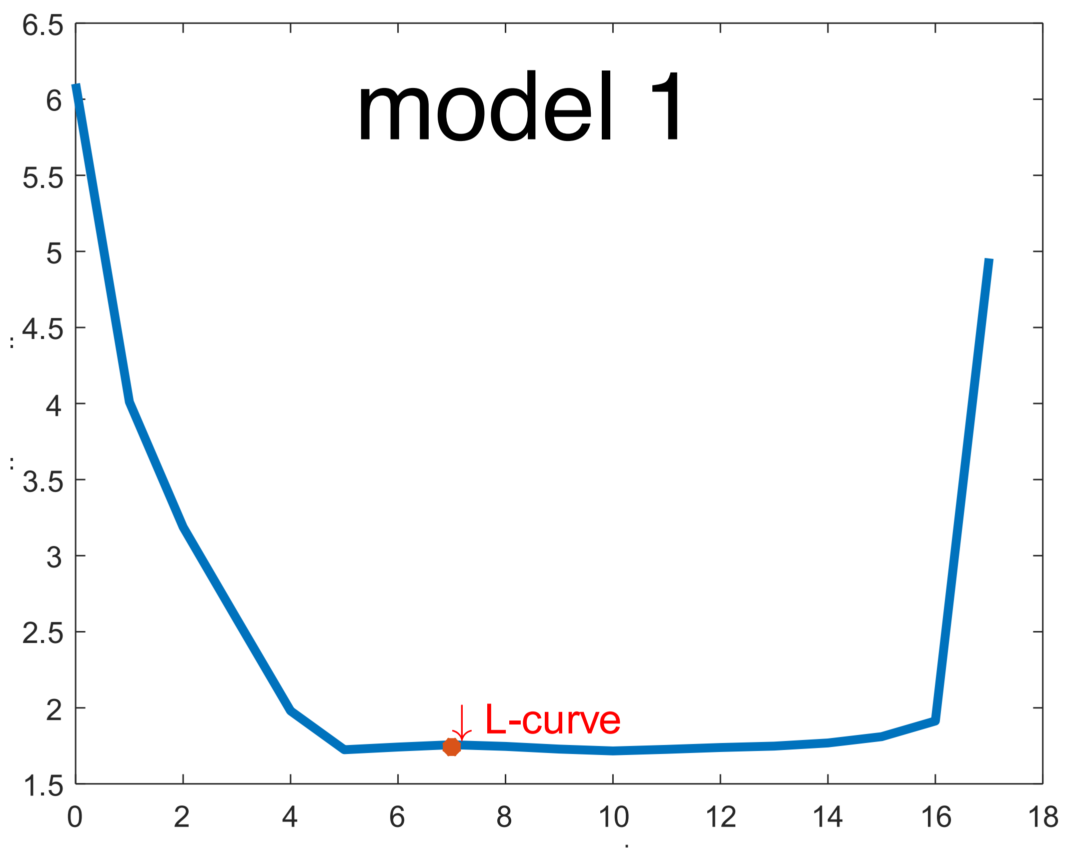
model 3

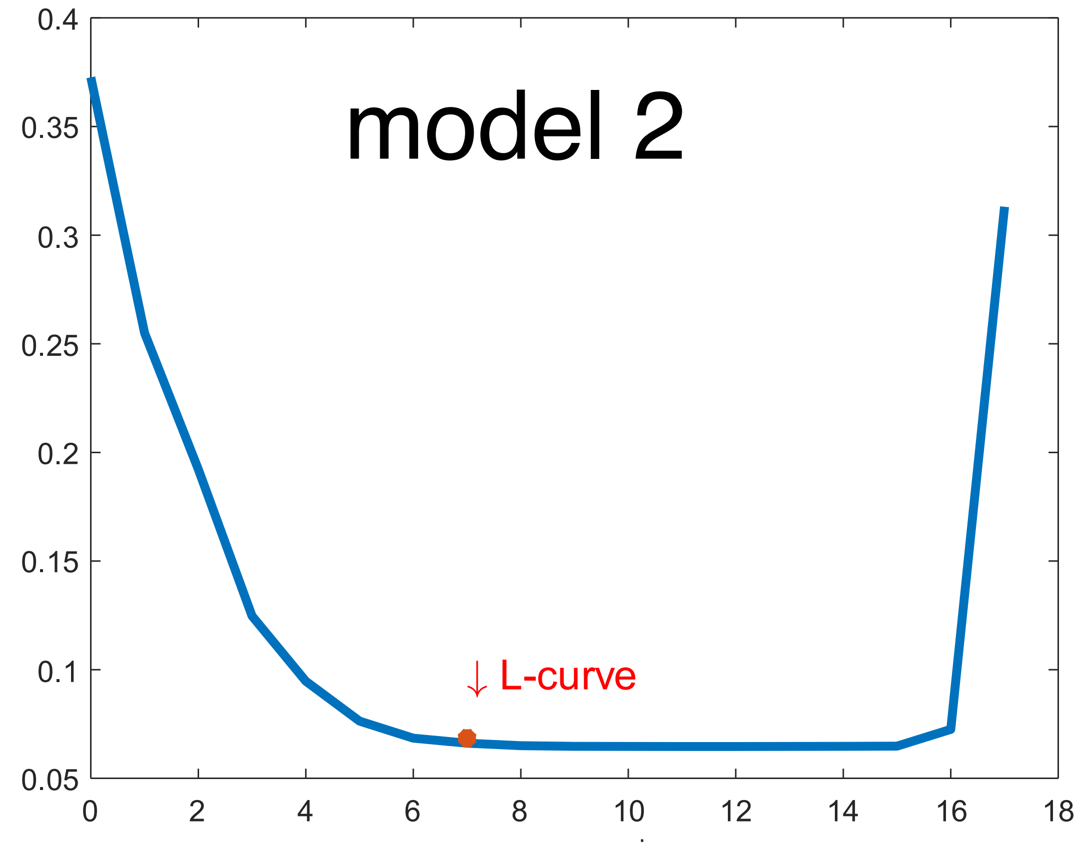


Input errors: 30% 10% 1%

4. Test: Insensitivity to α and Λ

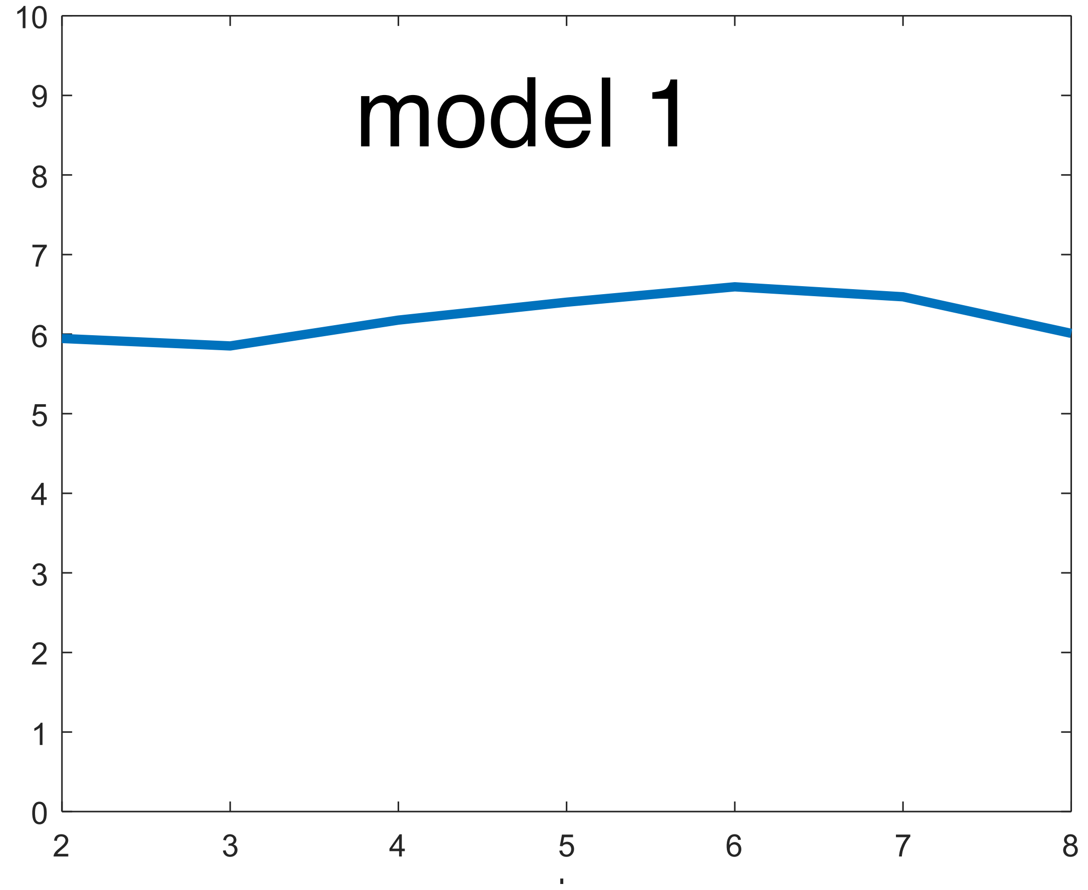
$\|f_\alpha^\delta - f\|_{H^1}$



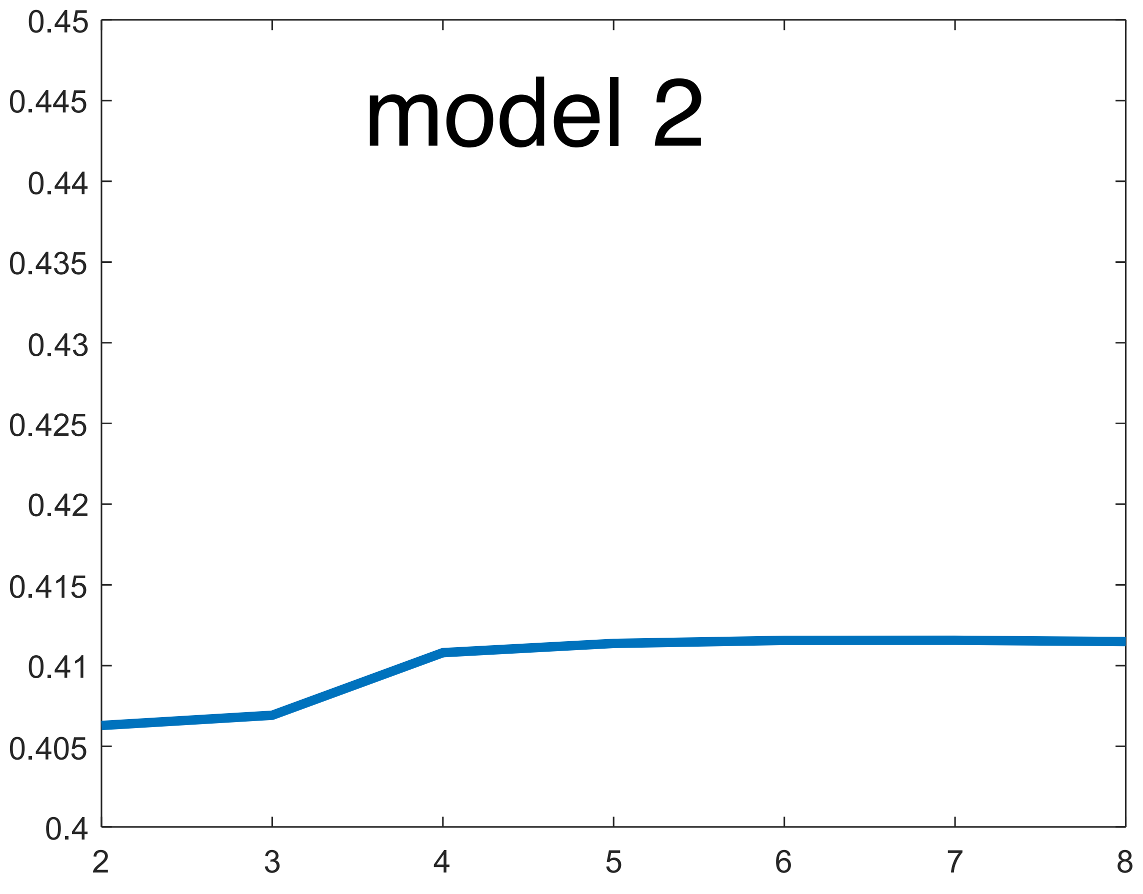


$-\log_{10}\alpha$

$f_\alpha^\delta(1)$



$f_\alpha^\delta(1)$

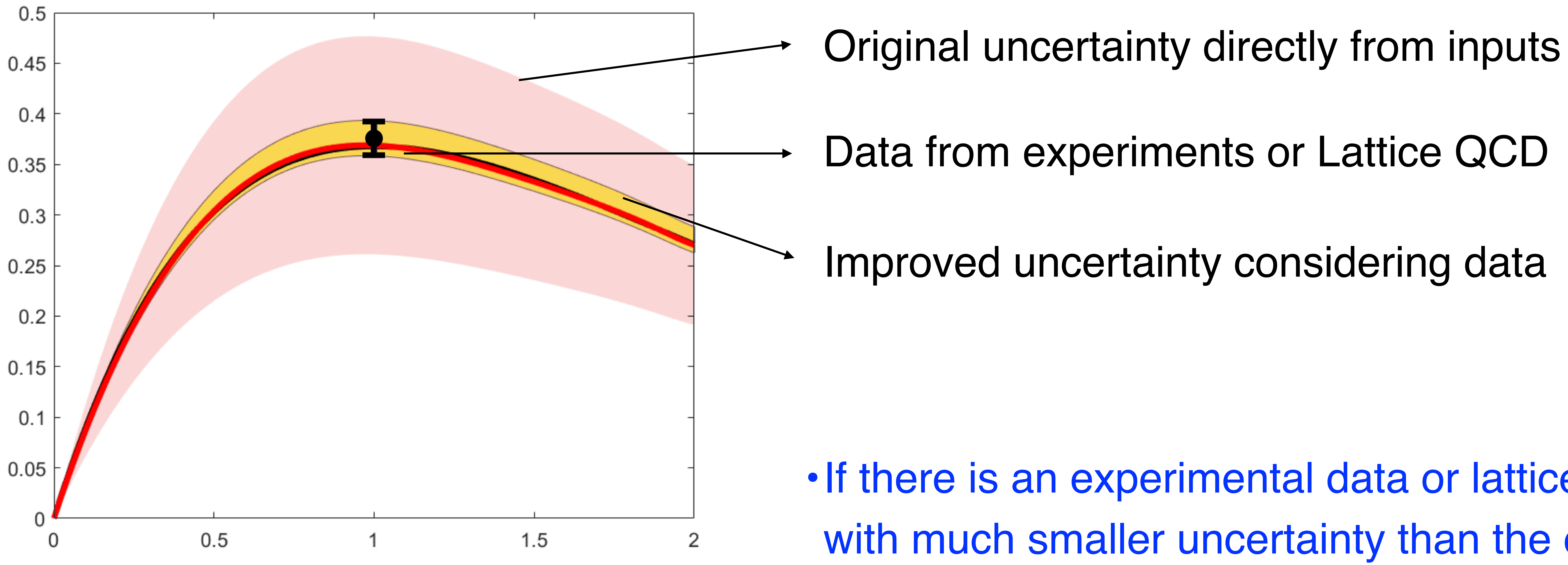


Λ

- Solutions are insensitive to the regularization parameter and the separation scale.
- The uncertainties of the inverse problem can be well controlled.

4. Test: Constrained data

- This method can combine with experiments and Lattice QCD to improve the precision of predictions



- If there is an experimental data or lattice data with much smaller uncertainty than the original solutions, we can use it to constrain the solution to be more precise in the whole range.

Criteria of a good theoretical approach

(1) Well defined in mathematics $\longrightarrow$ Dispersion relation + proof of ill-posedness

(2) Realization in numerical calculations $\longrightarrow$ Regularization methods

(3) Can be systematically improved $\longrightarrow$ Errors converge to vanishing as $\delta \to 0$

(4) Simple at the beginning $\longrightarrow$ Tikhonov regularization

Inverse problem approach has a potential to be a first-principle approach

5. Physical perspectives

(1) The whole non-perturbative region is solved simultaneously.

- **Advantage for the excited states**. Even higher precision by combination with experiments or Lattice QCD for the ground states.

(2) Modifying the QCD sum rules, with excited states and density function solved directly, without the assumption from global quark-hadron duality.

- Can calculate whatever QCD sum rules did, but with reasonable uncertainties.
- Advantage for the low q^2 region of transition form factors of $B \to K^{(*)}$ by light-cone QCD sum rules.

(3) Might solve the inverse problem in the Lattice QCD. And many others…

Summary

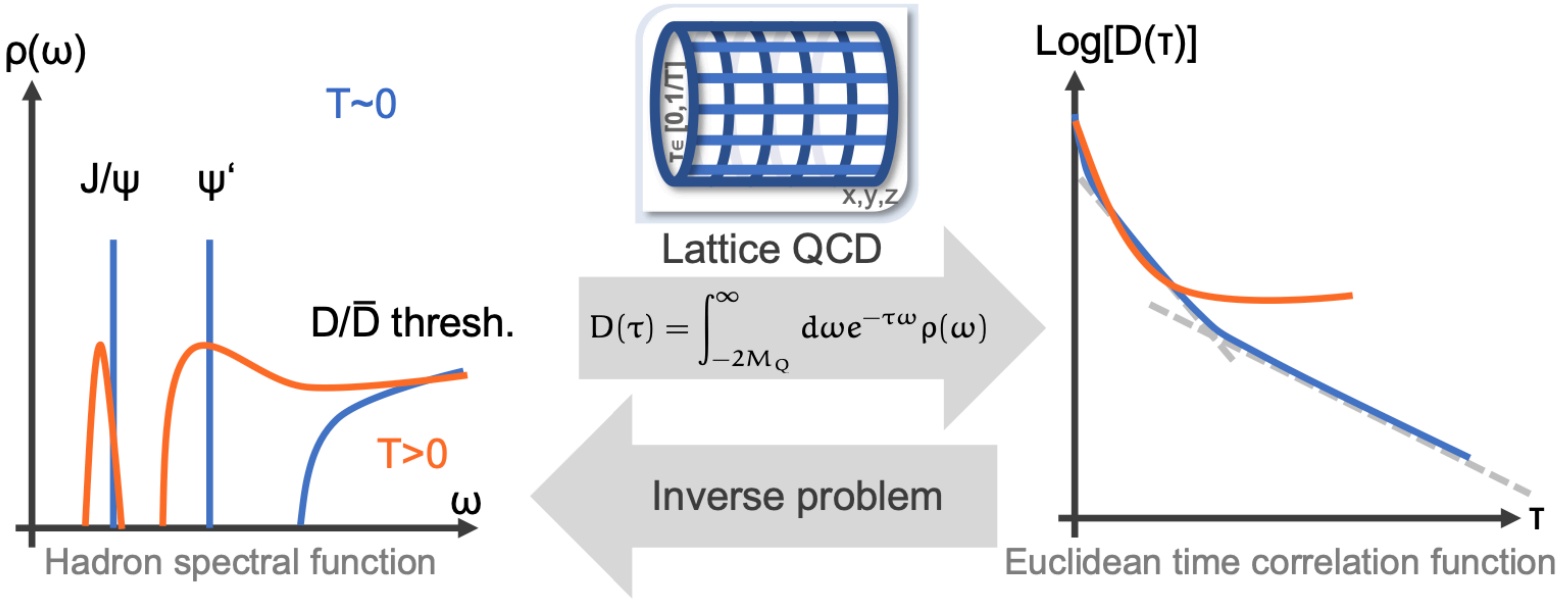
- We propose a novel method to calculate the non-perturbative quantities.
- With the **dispersion relation** of QFT, the non-perturbative quantities are obtained by solving the **inverse problem** with the perturbative calculations as inputs.
- **The precision of the predictions can be systematically improved, without any artificial assumptions.**
- **The mathematical basis has been provided.**
- Physical applications are expected.

Thank you!

Backups

- Firstly proposed to solve the problem of understanding of $D^0 - \overline{D}^0$ mixing [H.n.Li, H.Umeeda, **FSY**, F.Xu, 2001.04079]
- **Physical applications:**
  - muon g-2 [H.n.Li, H.Umeeda, 2004.06451]
  - modifying the QCD sum rules [H.n.Li, H.Umeeda, 2006.16593]
  - glueballs [H.n.Li, 2109.04956]
  - pion distribution amplitudes [H.n.Li, 2205.06746]
  - neutral meson mixings [H.n.Li, 2208.14798]
  - understandings of fermion masses and EW masses [H.n.Li, 2302.01761, 2304.05921, 2306.03463]
- Its **mathematical basis** should be provided [A.S.Xiong, T.Wei, **FSY**, 2211.13753].

Inverse problem in Lattice QCD



Rothkopf, 2211.10680

Spectral function reconstruction from Euclidean lattices

Inverse problem in Lattice QCD

Hadronic on the Lattice

Lattice QCD: **Euclidean** field theory using the path-integral formalism: **time-dependent matrix elements** are problematic.

$$W_{\mu\nu} = \frac{1}{4\pi}\int d^4z e^{iq\cdot z}\left\langle p,s\left|\left[\underline{J_\mu^\dagger(z)\ J_\nu(0)}\right]\right|p,s\right\rangle$$

Euclidean hadronic tensor:

$$\tilde{W}_{\mu\nu}(\vec{p},\vec{q},\tau=t_2-t_1)=\sum_{\vec{x}_2\vec{x}_1} e^{-i\vec{q}\cdot(\vec{x}_2-\vec{x}_1)}\langle p,s\,|\,J_\mu^\dagger(\vec{x}_2,t_2)J_\nu(\vec{x}_1,t_1)\,|\,p,s\rangle$$

Back to Minkowski space by solving the inverse problem:

$$\tilde{W}_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\tau)=\int d\nu W_{\mu\nu}(\boldsymbol{p},\boldsymbol{q},\nu)e^{-\nu\tau}$$

K.F. Liu and S. J. Dong, PRL 72, 1790 (1994)

K.-F. Liu, PRD62, 074501 (2000)

***J. Liang** et. al.,PRD101, 114503 (2020)*

***J. Liang** et. al., PRD102, 034514 (2020)*

Jian Liang's talk @ 2nd EicC CDR workshop

Maximum entropy method (MEM)

$$D(\tau) = \int_0^{\infty} K(\tau,w)A(w)\,dw$$

- **MEM is a method to circumvent these difficulties** by making a statistical inference of the most probable SPF (or sometimes called the image in the following) as well as its reliability on the basis of a limited number of noisy data.

- Its **basis is Bayes' Theorem**:

$$P[X|Y] = \frac{P[Y|X]P[X]}{P[Y]}$$

From Bayes' Theorem, we can get :

$$\boxed{P[A|DH]} = \frac{P[D|AH]P[A|H]}{P[D|H]}.$$

The most probable image is $A(w)$ that satisfies the condition: $\frac{\delta P[A|DH]}{\delta A} = 0.$

(1) Firstly, they make:

$$P[D|AH] = \frac{1}{Z_L} e^{-L},$$

$$L = \frac{1}{2}\sum_{i,j}(D(\tau_i) - D_A(\tau_i))C_{ij}^{-1}(D(\tau_j) - D_A(\tau_j)),$$

In the case where $P[A|H] = 0$, **maximizing $P[A|DH]$ is equivalent to standard $\chi^2 - \mathrm{fitting}$**. However, the $\chi^2 - fitting$ does not work.

hep-lat/0011040

The precision can be systematically improved

Without any beyond-control assumptions, the precision can be systematically improved:

(1) Suitable regularization method and selection rule of the regulators

(2) Higher precision of input data

(3) Combination with higher precise data of experiments or Lattice QCD.

1. The main idea of the inverse problem approach

If $s > \Lambda$, $$\mathcal{P}\int_0^{\Lambda} \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_{\Lambda}^{\infty}\frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

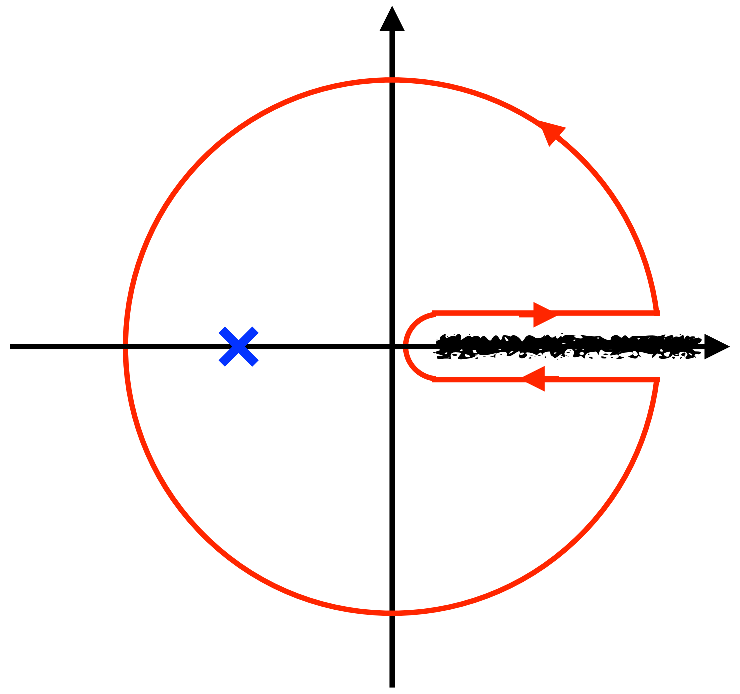
- With the dispersion relation of QFT, the non-perturbative quantities are obtained by solving the inverse problem with the perturbative calculations as inputs.
- Using the regularization method, the solutions are stable, and can be converged to the true value as the input errors approaching zero.
- The precision of the predictions can be systematically improved, without any artificially assumptions.

1. Dispersion relations and inverse problems

Dispersion relation:

- Based on Quantum Field Theory and correlation functions
- Analyticity of QFT, relation between a physical point and the curves, or relation between the real and imaginary parts

$$\Pi(q^2) = i \int d^4x e^{iq\cdot x} \langle \mathcal{O}(x)\mathcal{O}(0)\rangle$$



$$Re[\Pi(s)] = \frac{1}{\pi}\int_0^\infty \frac{Im[\Pi(s')]}{s-s'}ds'$$

- The above formula is just an example. Any dispersion relation would be studied similarly.

1. Dispersion relations and inverse problems

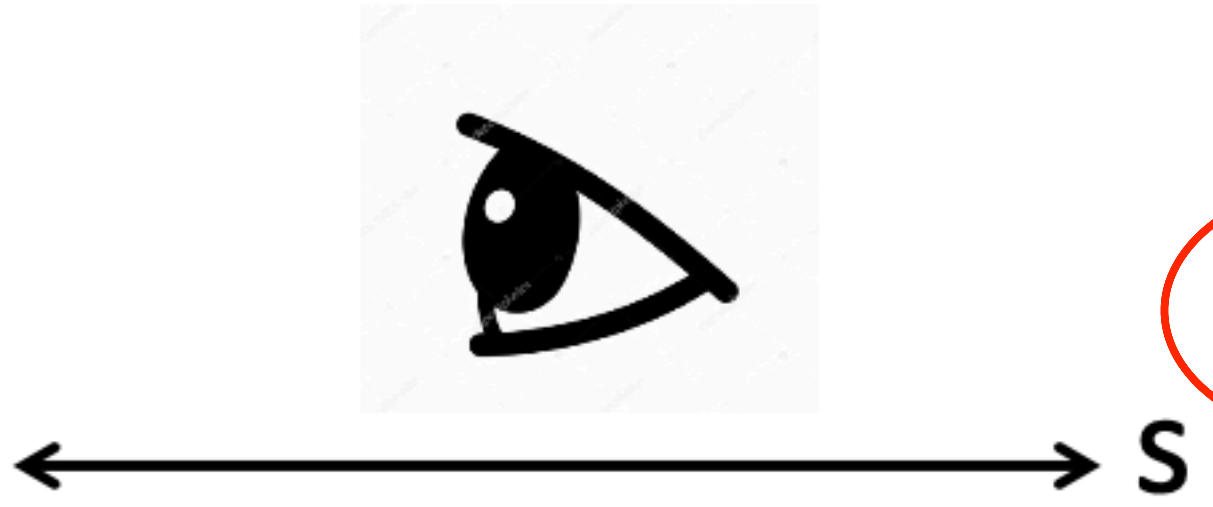
If $s > \Lambda$, $$\mathcal{P}\int_0^{\Lambda} \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds' = \pi\mathcal{R}e[\Pi(s)] - \mathcal{P}\int_{\Lambda}^{\infty} \frac{\mathcal{I}m[\Pi(s')]}{s-s'}ds'$$

To be solved

calculable

"charge distribution" at low s

"potential" at high s



$$\varphi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}dV'$$

2. ill-posedness of the inverse problem

- $Kx = y$ ==> $x = K^{-1}y$. Discretization?

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5 \end{cases} \quad \Longrightarrow \quad x_1 = 1, \;\; x_2 = 1$$

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5.01 \end{cases} \quad \Longrightarrow \quad x_1 = -59, \;\; x_2 = 41$$

- A very small noise might cause a large change of solutions

2. ill-posedness of the inverse problem

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5 \end{cases} \quad \Rightarrow \quad x_1 = 1, \;\; x_2 = 1$$

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ 1.9999x_1 + 3.0001x_2 = 5.01 \end{cases} \quad \Rightarrow \quad x_1 = -59, \;\; x_2 = 41$$

- A very small noise might cause a large change of solutions

$$K = \begin{pmatrix} 2 & 3 \\ 1.9999 & 3.0001 \end{pmatrix}, \qquad |K| = 0.0005, \qquad K^{-1} = \frac{K^*}{|K|} = \begin{pmatrix} 6000.2 & -6000 \\ -3999.8 & 4000 \end{pmatrix}$$

K^{-1} enhances the errors

- In the continuum limit, K^{-1} is unbounded. The problem is ill-posed.

2. ill-posedness of the inverse problem

- The operator $K : X \to Y$, $\quad Kx = y, \quad x \in X, \quad y \in Y$
- The inverse problem of dispersion relation must be ill-posed.
- K is a linear bounded compact operator. It doesn't have a bounded inverse operator in the infinite dimensional space.

Proof. It is easily to check that $Kf_1 + Kf_2 = K(f_1 + f_2)$ and $\alpha Kf = K(\alpha f)$ so the $K : F \to G$ operator is a linear operator. For any $f \in L^2(a, b)$, by the Cauchy inequality, we have

$$\begin{aligned} \|Kf\|^2_{L^2(c,d)} &= \int_c^d (Kf)^2 dy = \int_c^d \left(\int_a^b \frac{1}{y-x} f(x)dx\right)^2 dy \qquad (3.2) \\ &\leq \int_c^d \int_a^b \left(\frac{1}{y-x}\right)^2 dx \int_a^b f^2(x)dxdy \leq \left(\frac{1}{c-b}\right)^2 (b-a)(d-c)\|f\|^2_{L^2(a,b)} = M\|f\|^2_{L^2(a,b)} < +\infty, \end{aligned}$$

where $M > 0$ is a constant. Thus, from the form of the equation (3.2), we easily know $K : F \to G$ is a bounded operator.

Since $c > b$, the mth order derivative of Kf exists for any $m \in \mathbb{N}$ and by the Cauchy inequality, we have

$$\left\|\frac{\partial^m (Kf)}{\partial y^m}\right\|^2_{L^2(c,d)} = \int_c^d \left(\int_a^b \frac{(-1)^m m!}{(y-x)^{m+1}} f(x)dx\right)^2 dy \leq C\|f\|^2_{L^2(a,b)}, \qquad (3.3)$$

where $C > 0$ is a constant depending on a, b, c, d only. Therefore, $Kf \in H^m(c, d)$ for any $m \in \mathbb{N}$. Since m is arbitrary, by the embedding theorem, we know $Kf \in C^\infty[c, d]$. And since $H^1(c, d)$ is embedded into $L^2(c, d)$ compactly, we know the operator K is a compact operator. The proof is completed $\square$

4. Test of Toy Models

- Questions on the inverse problem approach:

  (1) **Regularization:** How important are the regularization methods?
  Can the solutions be systematically improved by the regularization method and the method of selecting the regularization parameter?

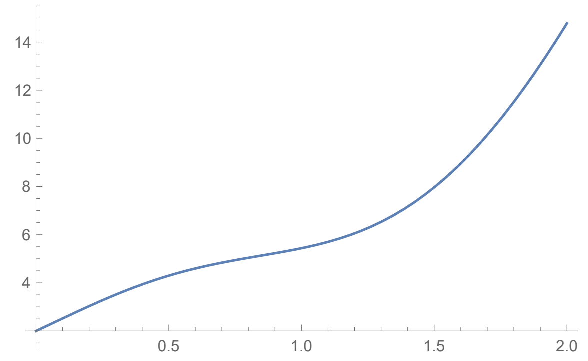
  (2) **Impact of input uncertainties:** What is the dependence of the errors of solutions on the uncertainties of inputs? Larger, smaller or similar?

  (3) **Impact of α and Λ:** How sensitive are the solutions to the parameters α and Λ?
  Does it exist a plateau?

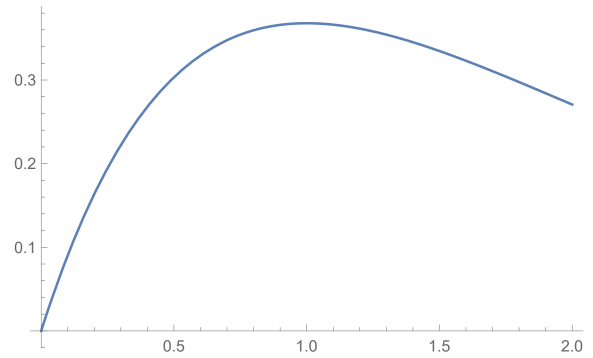
  (4) **Impact of more conditions:** Can the solutions be improved if we known more conditions?

4. Test of Toy Models

- Simple at the beginning: Tikhonov regularization + L-curve method for the regulator
  - They are simple in mathematics and in practice and thus are very helpful to develop the new approach in the future.
- Uncertainties are the most important issue. $b_i = \mu_i \pm \sigma_i$

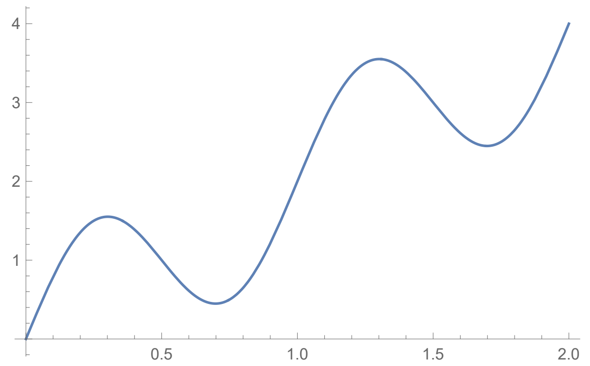


$$f(x) = a_1 f_1(x) + a_2 f_2(x) \qquad g^\delta(y) = b_1 g_1(y) + b_2 g_2(y) \qquad g_i(y) = \int_a^b \frac{f_i(x)}{y-x} dx.$$



Model 1: a monotonic function as $f_1(x) = \sin(\pi x)$, $f_2(x) = e^x$;

Model 2: a simple non-monotonic function as $f_1(x) = xe^{-x}$, $f_2(x) = 0$;

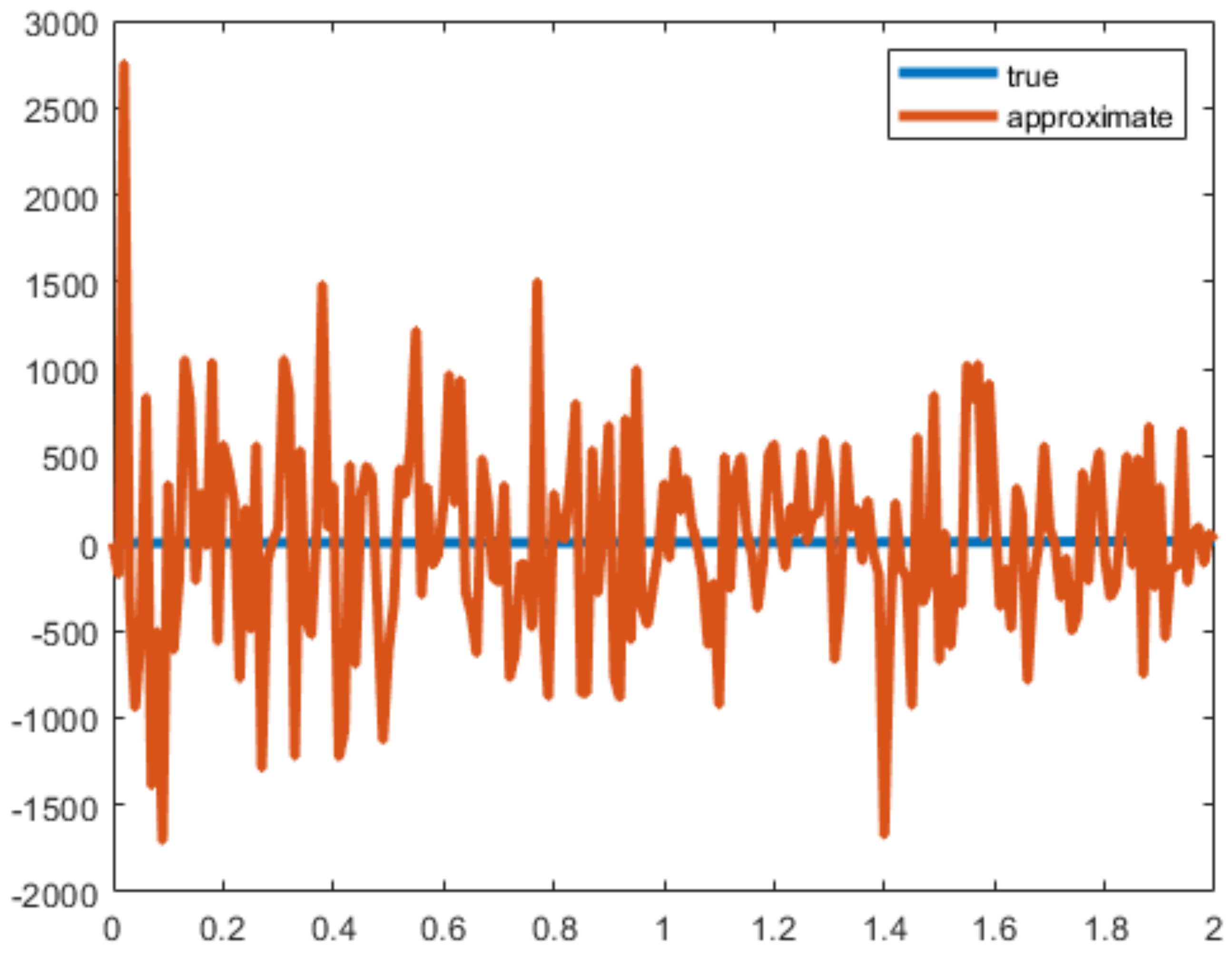


Model 3: an oscillating function as $f_1(x) = \sin(2\pi x)$, $f_2(x) = x$.

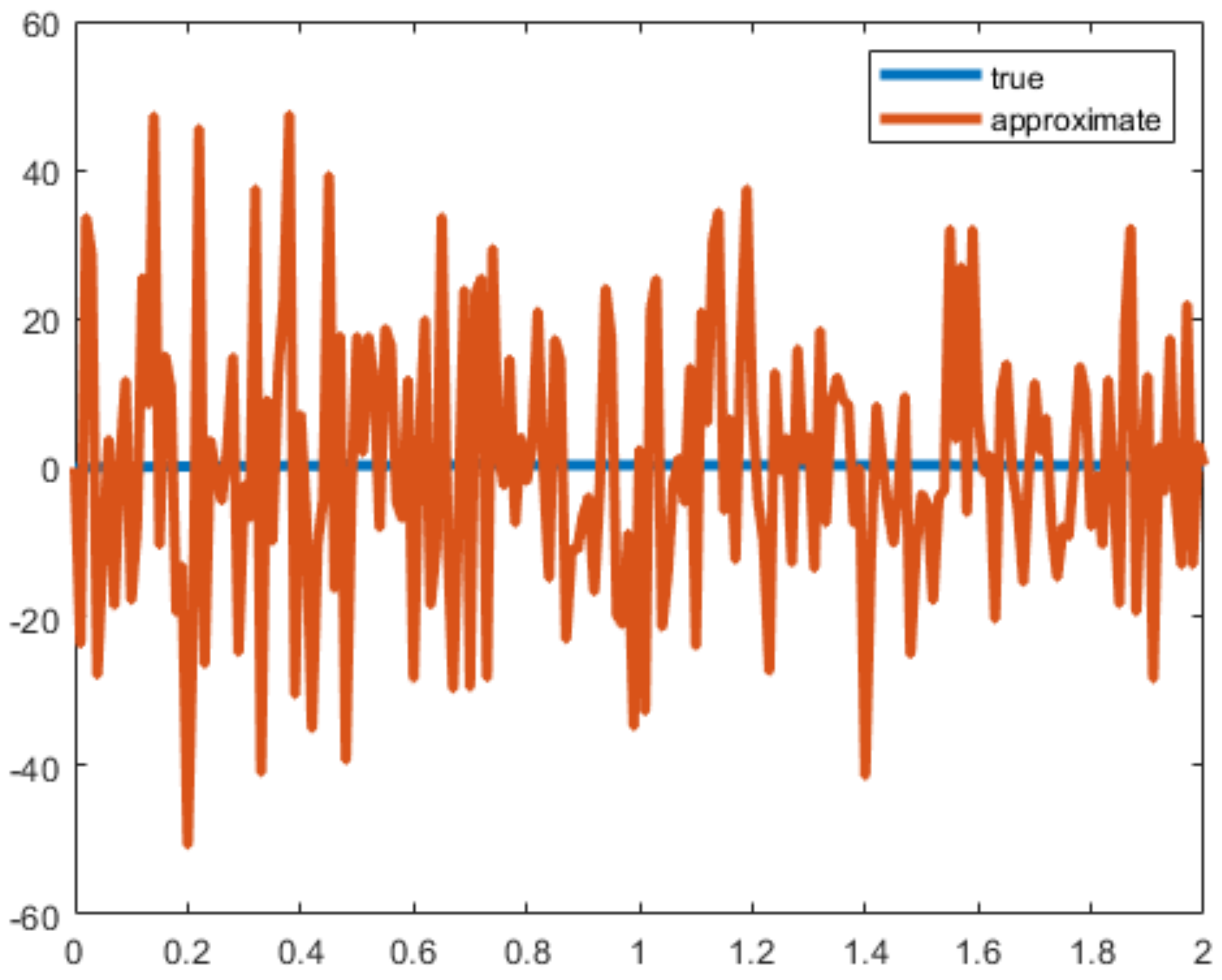
They are either helpful to clarify the properties of inverse problems or close to the real physical problem

4. Test: Importance of regularization

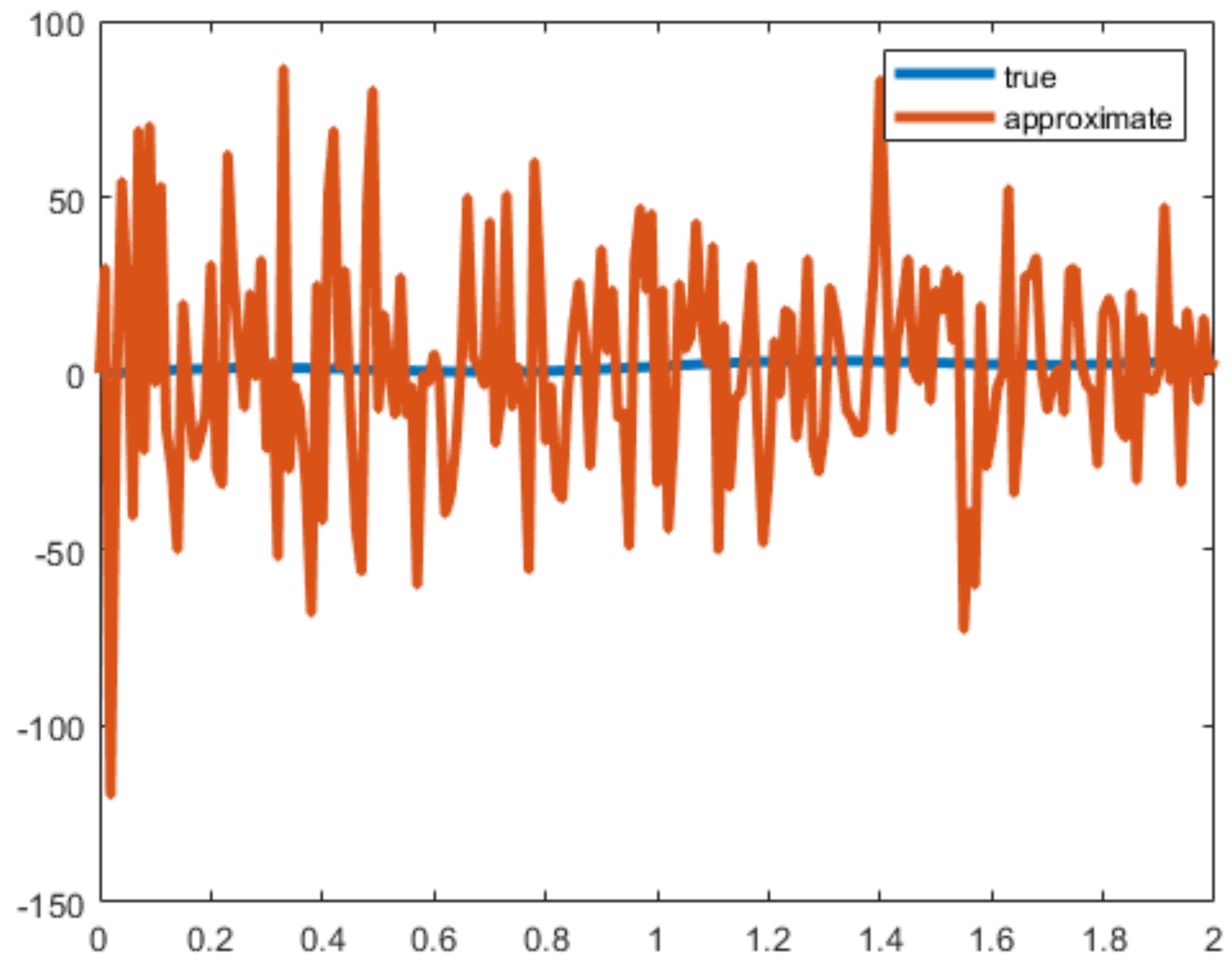
The solutions without any regularization:



model 1



model 2



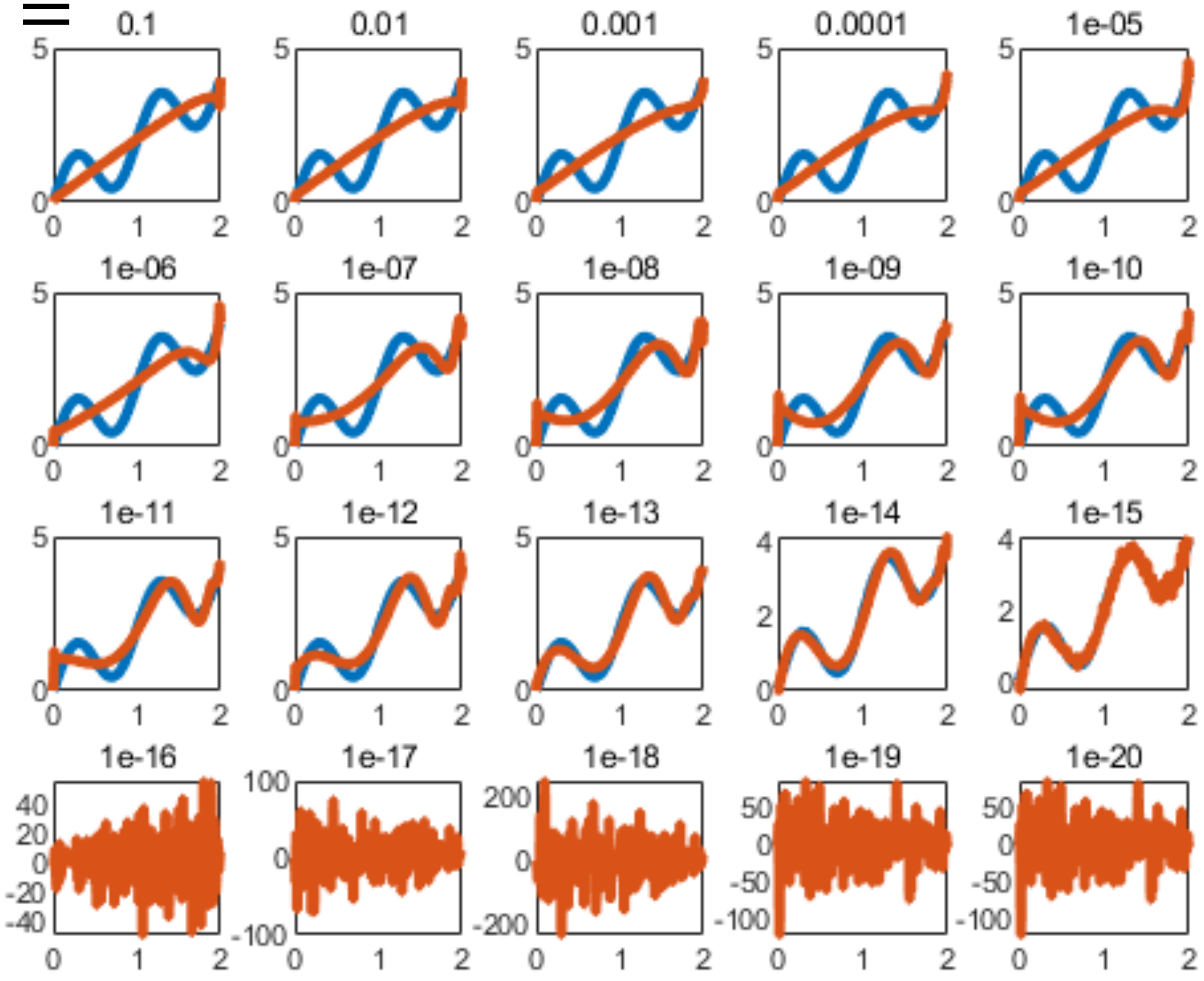
model 3

- It can be clearly seen that the solutions are unstable and far from the true values.
- The ill-posed inverse problems can not be solved without any regularization.

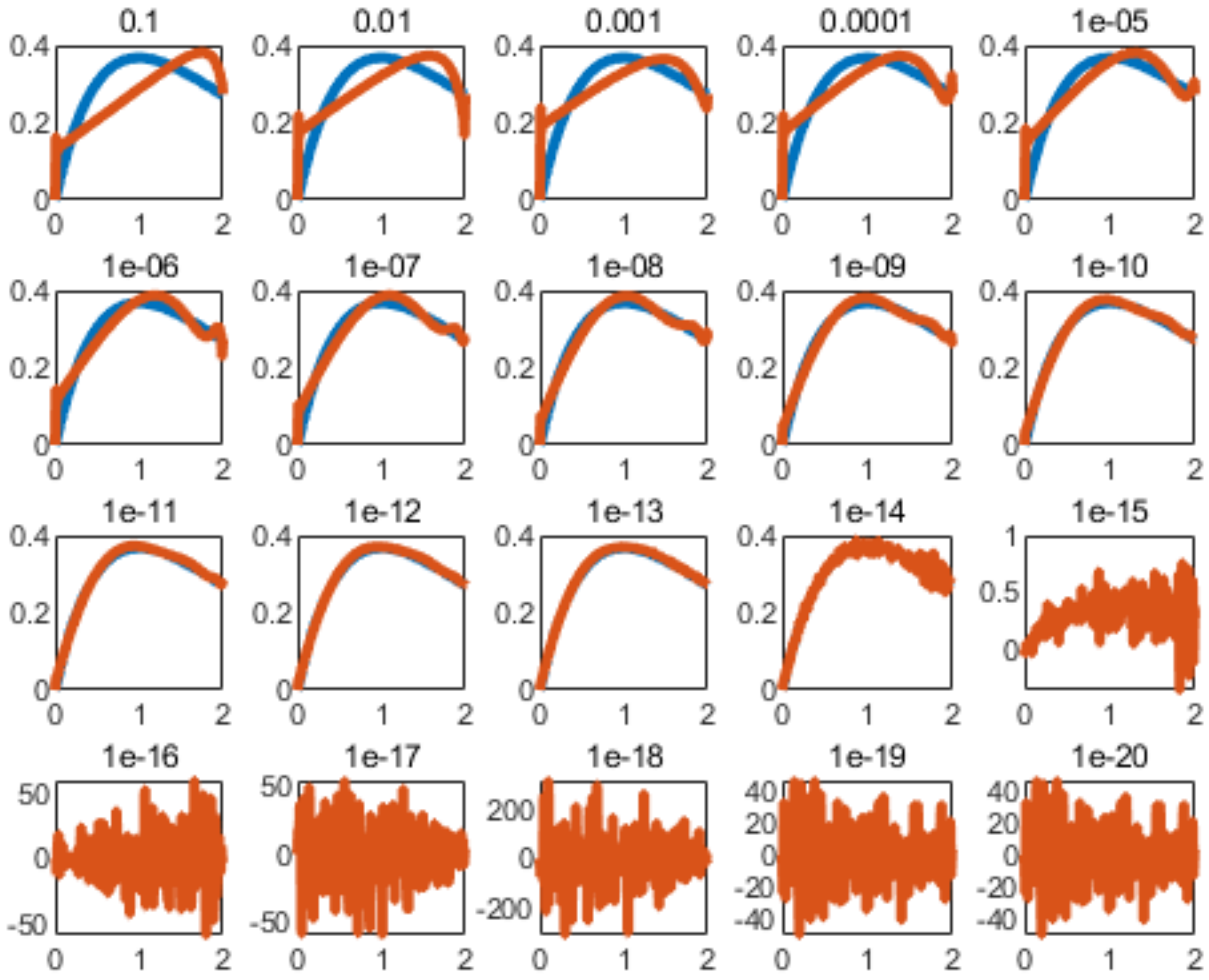
4. Test: Importance of regularization

The solutions with Tikhonov regularization:

$\alpha =$



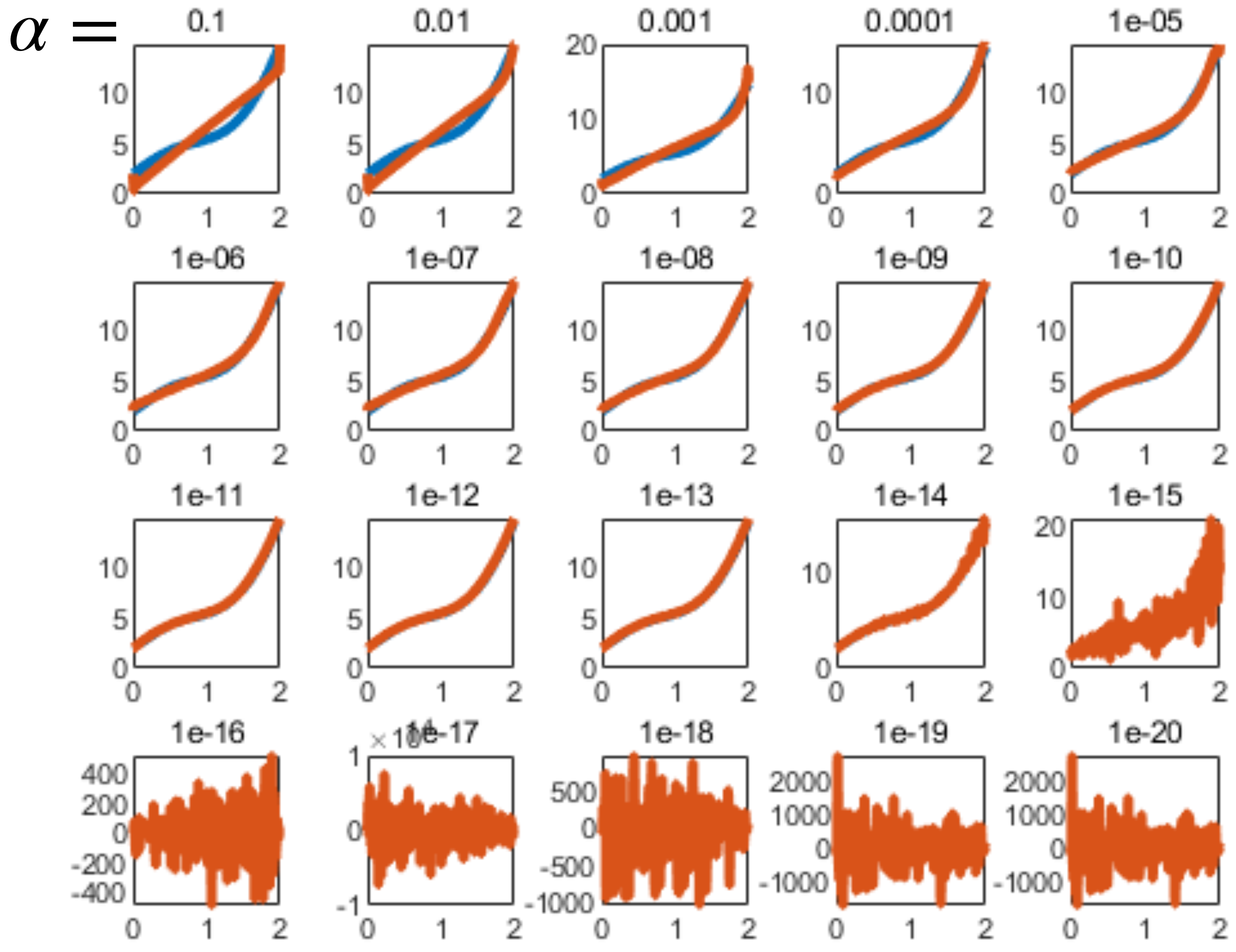
model 3



model 2

4. Test: Importance of regularization

The solutions with Tikhonov regularization:

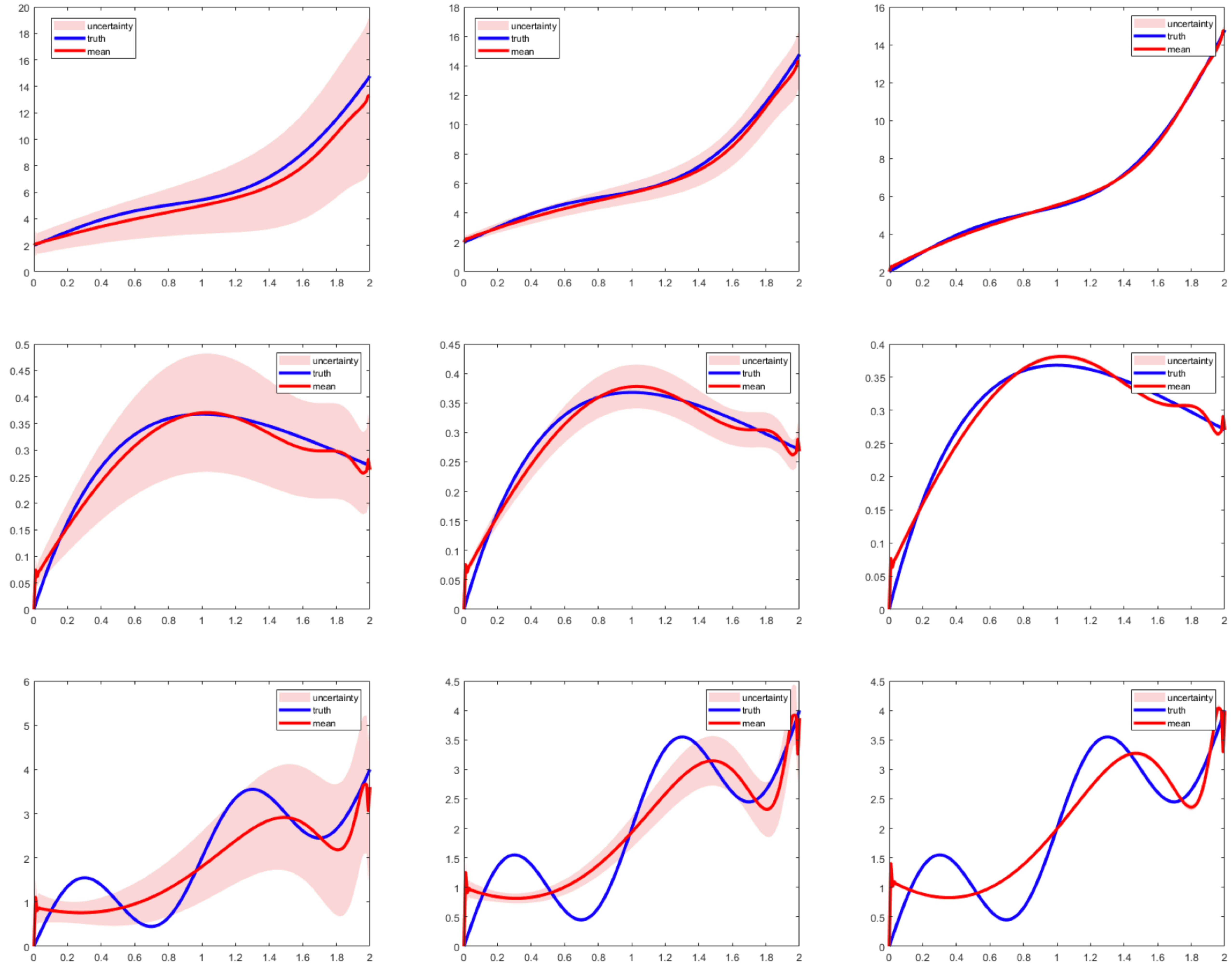


model 1

- It can be seen clearly that some values of regularization parameters can give good results.
- The ill-posed inverse problems can be solved by regularization.
- The regularization parameter can be neither too small (not enough for regularization), nor too large (dominate over the original problem)
- But α still works by ranging several orders of magnitude.
- The regularization methods are very important in solving the inverse problems.

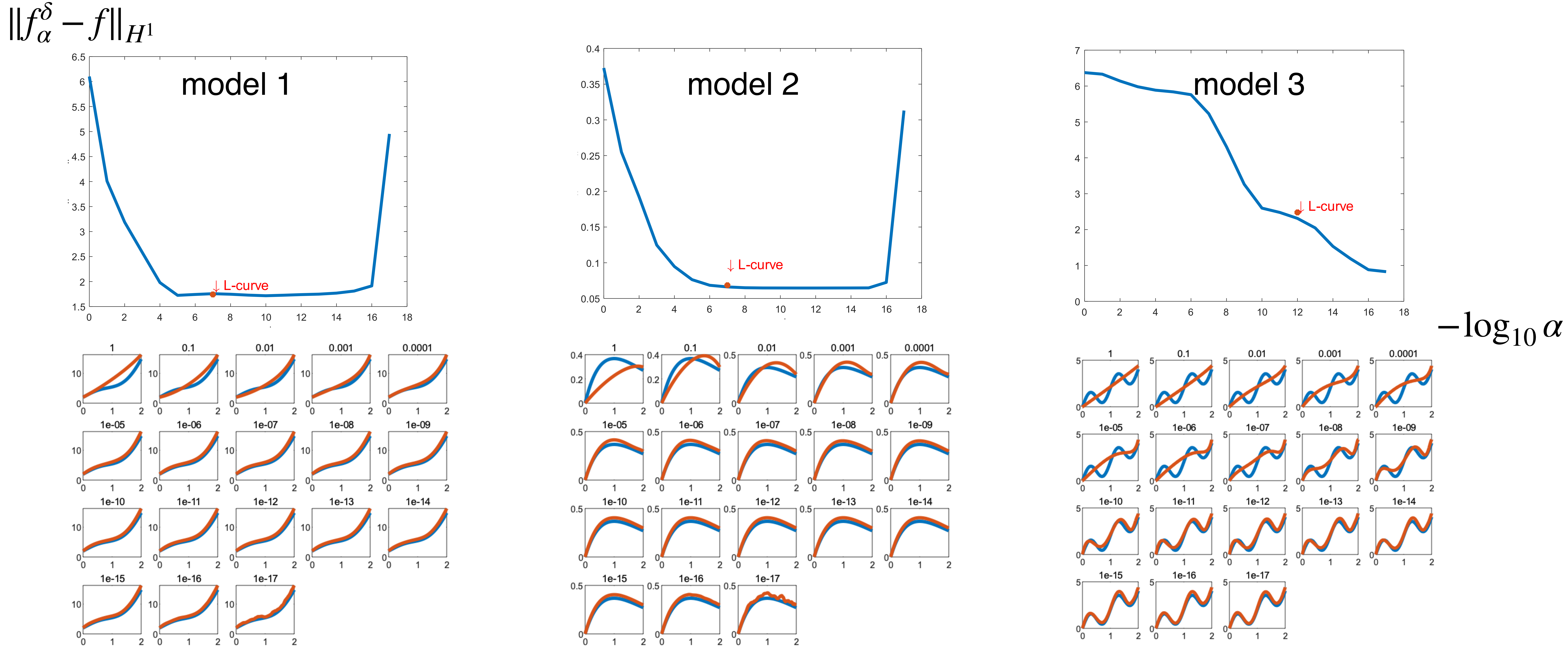
4. Test: Impact of input uncertainties

- **The most important issue is to control the uncertainties!**
  - The uncertainties of the solutions are almost at the same level of the input errors.
  - The smaller the input errors are, the more precise the solutions are.
  - **The precision of the predictions can be systematically improved by lowering down the input errors.**



Input errors: 30% 10% 1%

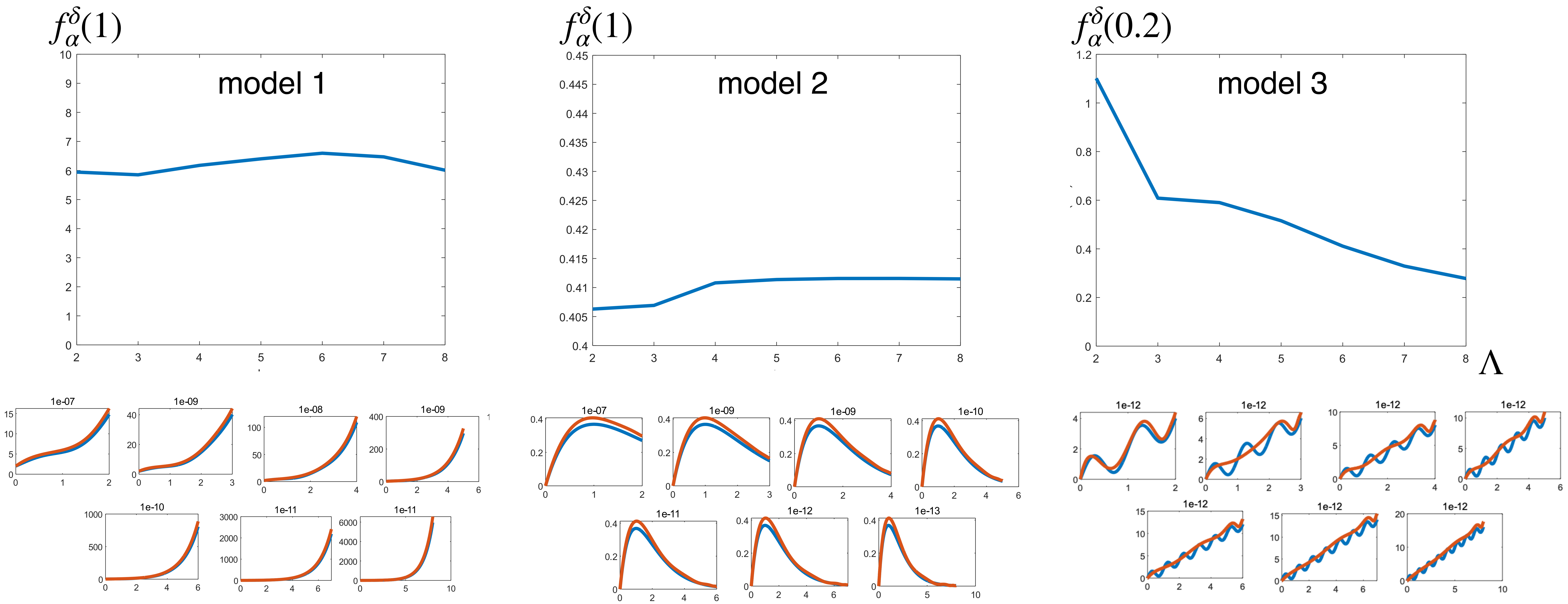
4. Test: Plateaus of the regularization parameter α



There exist plateaus. Solutions are insensitive to regularization parameter. L-curve method is suitable.

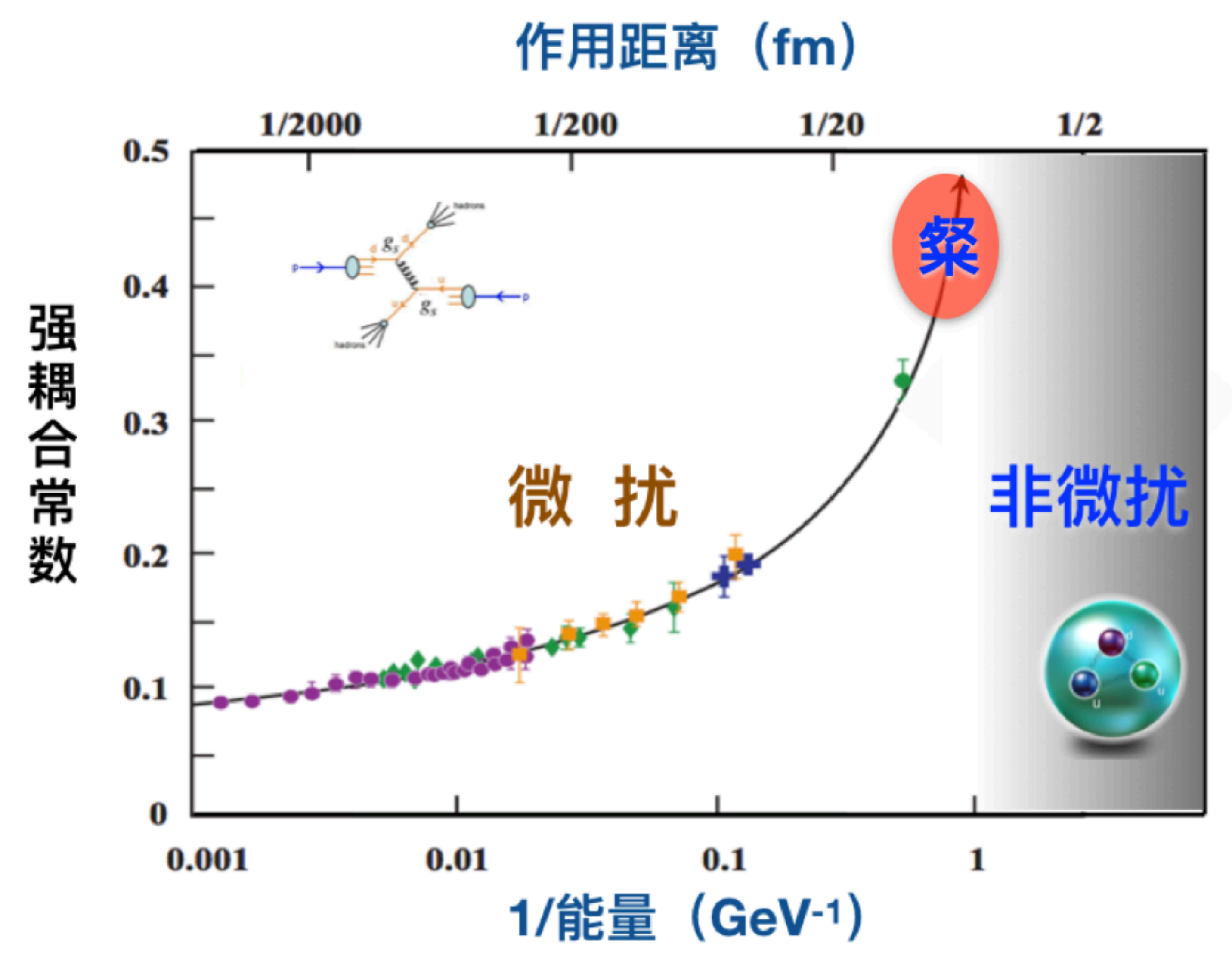
The inverse problem approach works for the non-perturbative calculations.

4. Test: Plateaus of the separation scale Λ



- There exist plateaus.
- Solutions are insensitive to the separation scale for monotonic and simple non-monotonic functions.
- The continuous condition at Λ might be even more helpful.

$D^0 - \overline{D}^0$ Mixing

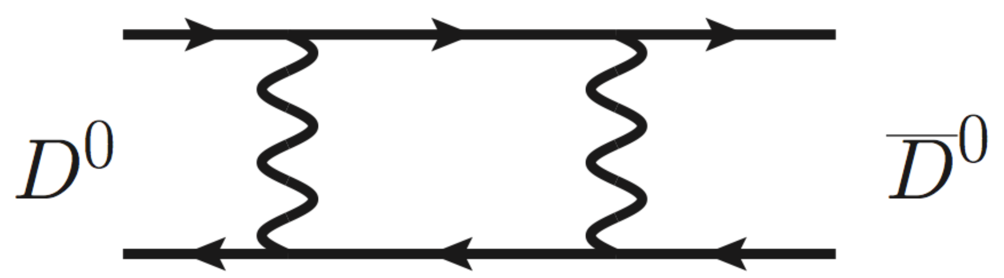


- The time evolution

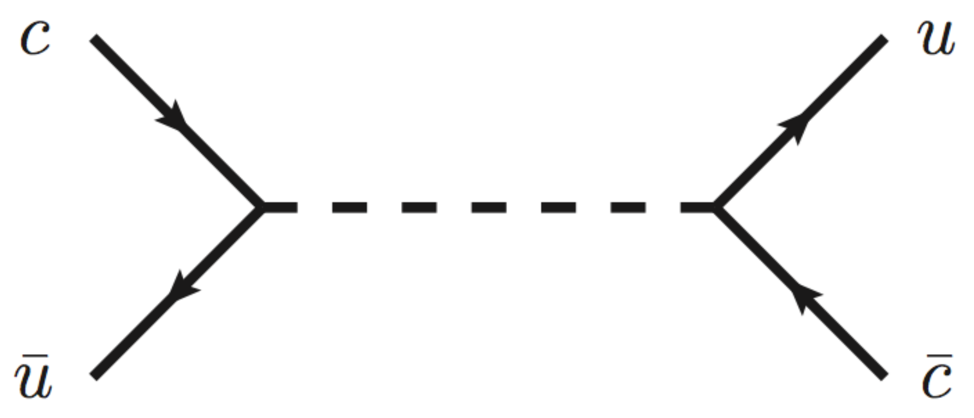
$$i\frac{\partial}{\partial t}\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix} = \left(\mathbf{M} - \frac{i}{2}\mathbf{\Gamma}\right)\begin{pmatrix} D^0(t) \\ \overline{D}^0(t) \end{pmatrix}$$

- Mixing parameters: Mass and Width differences

$$\boxed{x \equiv \frac{\Delta m}{\Gamma} = \frac{m_1 - m_2}{\Gamma}} \qquad \boxed{y \equiv \frac{\Delta\Gamma}{2\Gamma} = \frac{\Gamma_1 - \Gamma_2}{2\Gamma}}$$



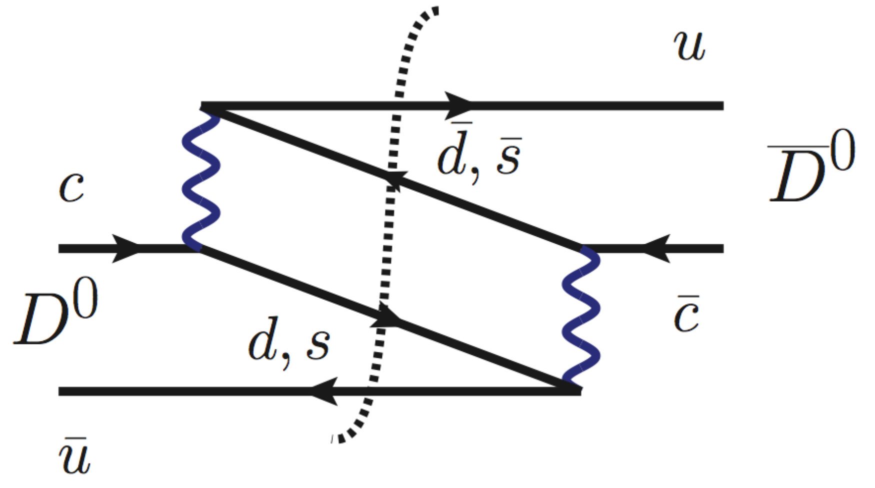
- Useful to search for new physics,
- but less understood in the Standard Model

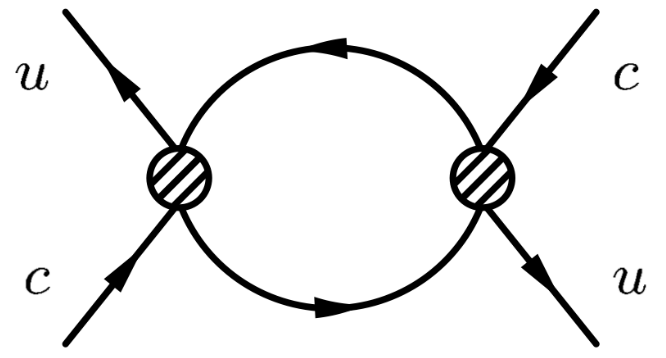


Falk, et al, ’02; Cheng, Chiang, ’10

$$y_{\mathrm{PP+PV}} = (3.6 \pm 2.6) \times 10^{-3}$$

- **Before 2017, exclusive approach is hopeful**

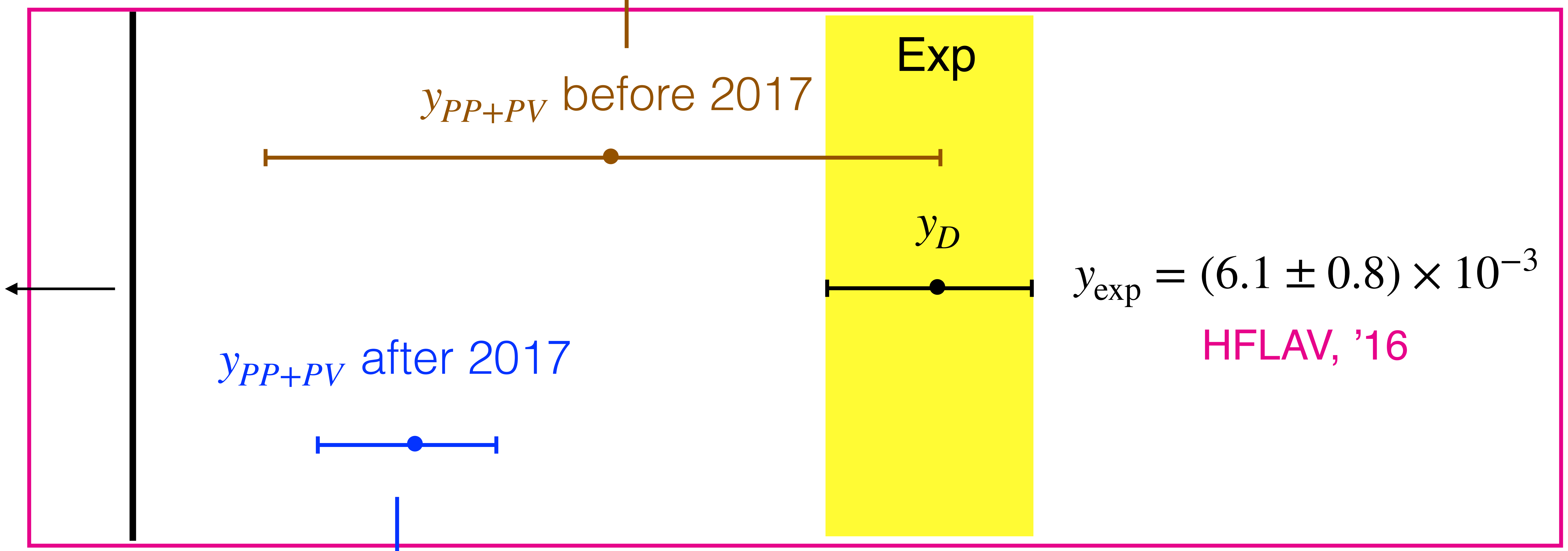




Inclusive approach doesn’t work

$$y_{\mathrm{incl}} \sim 10^{-7}$$

Lenz, *et al*, ’12



$$y_{\mathrm{exp}} = (6.1 \pm 0.8) \times 10^{-3}$$

HFLAV, ’16

- **After 2017, exclusive approach is dying**

$$y_{\mathrm{PP+PV}} = (2.1 \pm 0.7) \times 10^{-3}$$

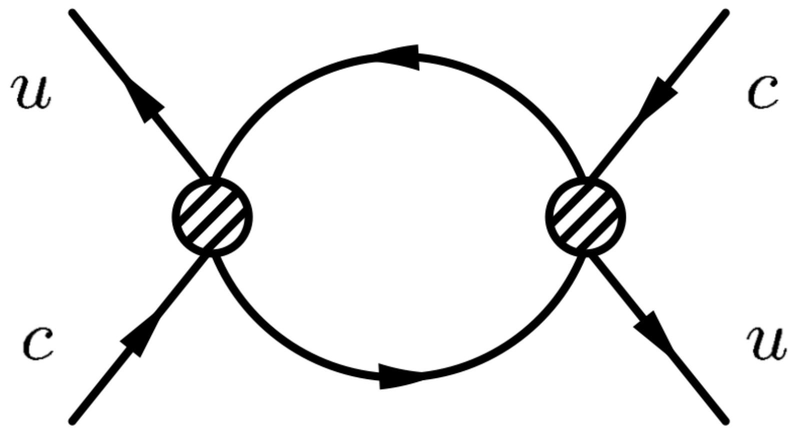
Jiang, **FSY**, Qin, Li, Lü, ’17

No theoretical methods work for D0 mixing

No theoretical predictions for indirect CP violation

Inclusive Approach

Theory / Exp. comparison (for inclusive)



quark level

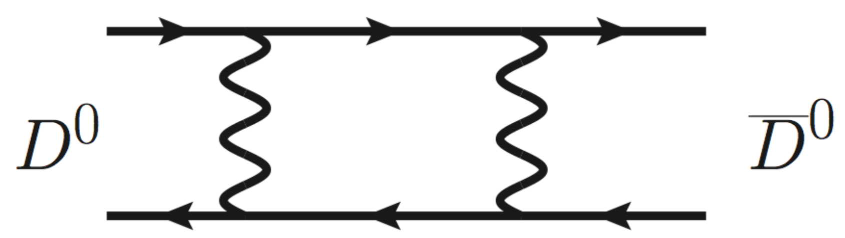
Short-distance

D meson	B_s meson	B_d meson
Hagelin 1981, Cheng 1982 Buras, Slominski and Steger 1984 **NLO QCD** Golowich and Petrov 2005	Artuso, Borissov and Lenz, 2016	Artuso, Borissov and Lenz, 2016
SM $\begin{cases} x \simeq 6 \times 10^{-7} \\ y \simeq 6 \times 10^{-7} \end{cases}$ Suppressed by GIM	SM $\begin{cases} \Delta M_s = (18.3 \pm 2.7)\mathrm{ps}^{-1} \\ \Delta\Gamma_s = (0.088 \pm 0.020)\mathrm{ps}^{-1} \end{cases}$	SM $\begin{cases} \Delta M_d = (0.528 \pm 0.078)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_d = (2.61 \pm 0.59) \cdot 10^{-3}\ \mathrm{ps}^{-1} \end{cases}$
HFLAV at Moriond2019 Exp. $\begin{cases} x = (3.9^{+1.1}_{-1.2}) \times 10^{-3} \\ y = (6.51^{+0.63}_{-0.69}) \times 10^{-3} \end{cases}$	HFLAV Exp. $\begin{cases} \Delta M_s = (17.757 \pm 0.021)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_s = (0.082 \pm 0.006)\ \mathrm{ps}^{-1} \end{cases}$	HFLAV Exp. $\begin{cases} \Delta M_d = (0.5055 \pm 0.0020)\ \mathrm{ps}^{-1} \\ \Delta\Gamma_d = 0.66(1 \pm 10) \cdot 10^{-3}\ \mathrm{ps}^{-1} \end{cases}$

- For B_s, B_d mesons, the data are reproduced within 1σ.
- For D meson, the order of magnitude is not reproduced within leading-power.

Inverse Problem

$D^0 - \overline{D}^0$ mixing

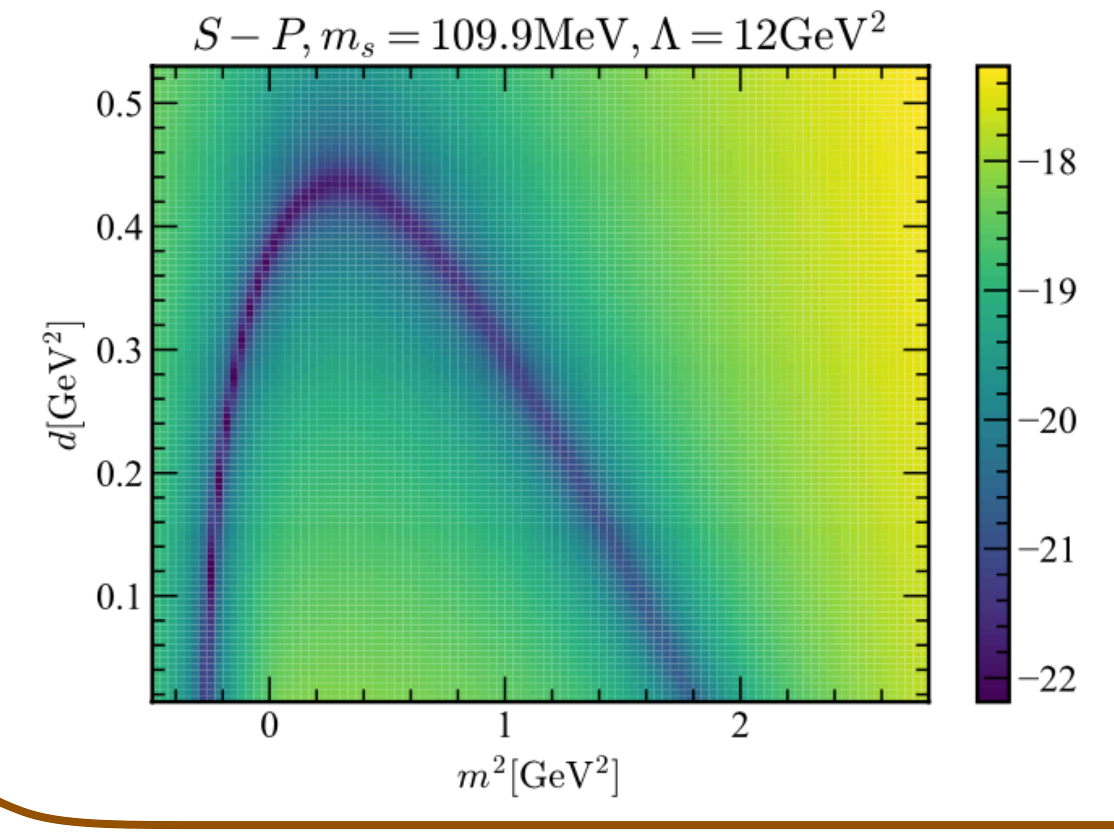


$$\int_0^\Lambda ds' \frac{y(s')}{s-s'} = \pi x(s) - \int_\Lambda^\infty ds' \frac{y(s')}{s-s'} \equiv \omega(s)$$

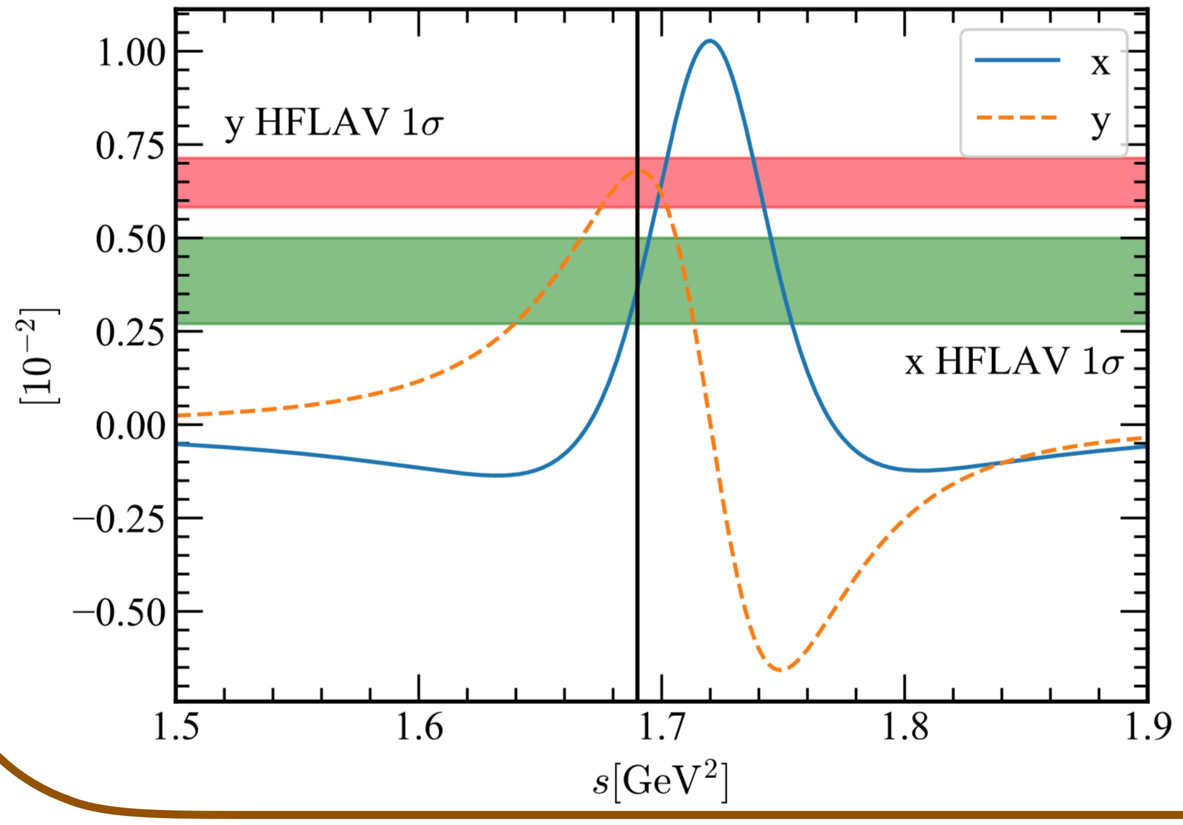
parametrization:

$$y(s) = \frac{Ns[b_0 + b_1(s-m^2) + b_2(s-m^2)^2]}{[(s-m^2)^2+d^2]^2}$$

Li, Umeeda, Xu, **FSY**, PLB(2020)



multiple solutions



Additional conditions:
data of x and y as inputs

Predict indirect CPV

$q/p = 1.0002e^{i0.006^\circ}$

consistent with data

$q/p = (0.969^{+0.050}_{-0.045})e^{i(-3.9^{+4.5}_{-4.6})^\circ}$

A real prediction

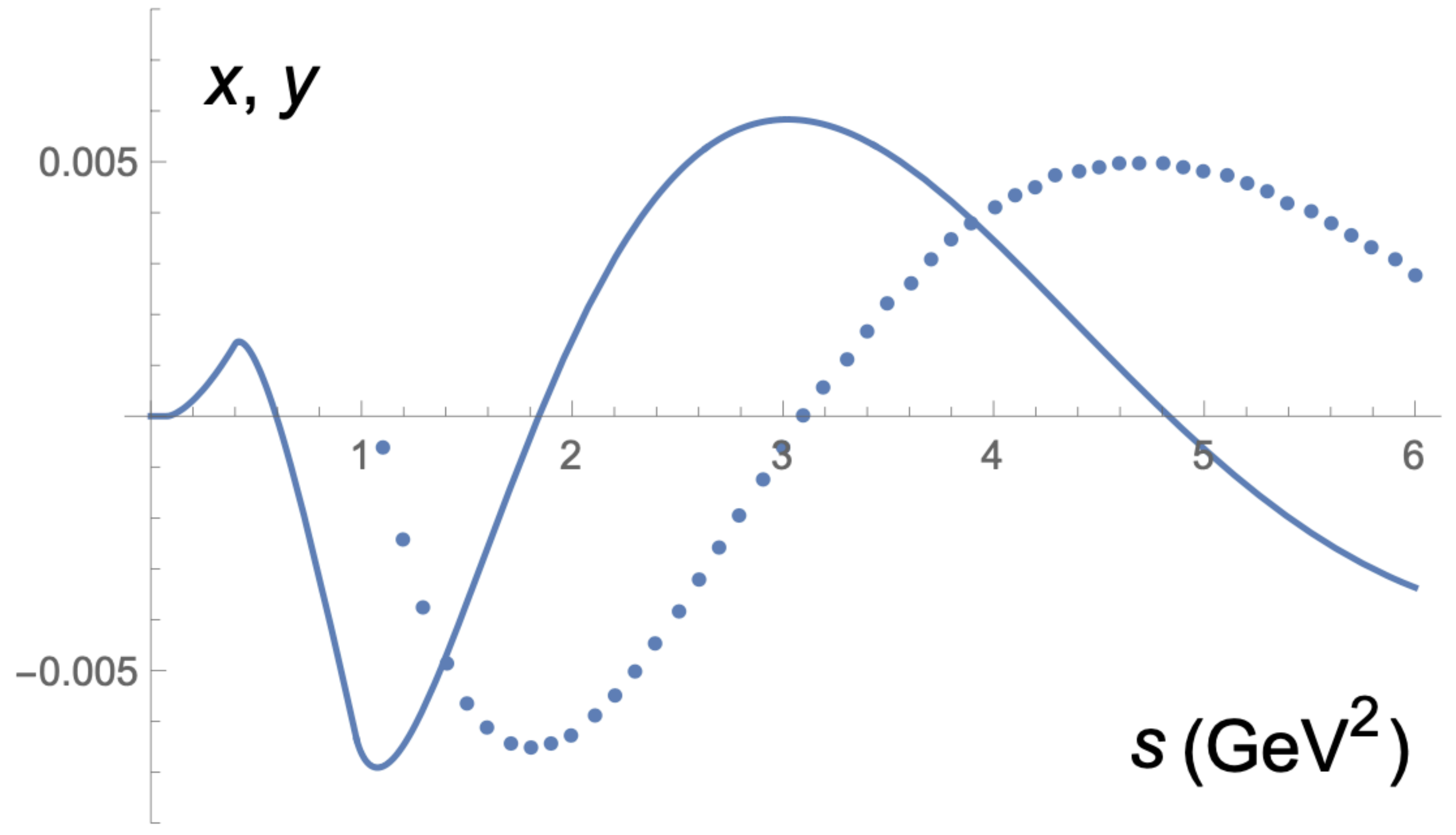


FIG. 7: Behaviors of $x(s)$ (dotted line) and $y(s)$ (solid line) for $\Lambda = 4.3$ GeV^2.

Inverse problem: $x(m_D^2) = (0.21^{+0.04}_{-0.07})\%, \quad y(m_D^2) = (0.52 \pm 0.03)\%.$

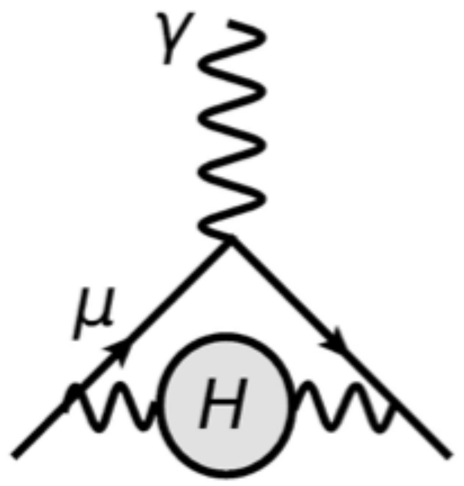
Experiment: $x = (0.44^{+0.13}_{-0.15})\%, \quad y = (0.63 \pm 0.07)\%,$

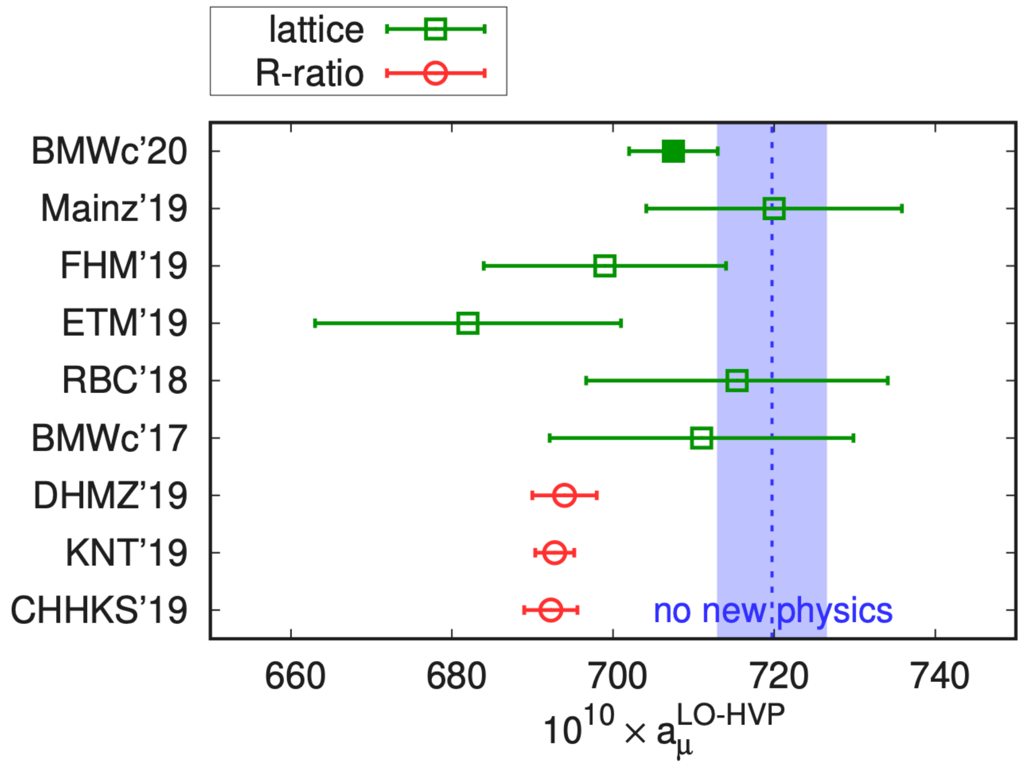
H.n.Li, 2208.14798

- Perspective: Using the Tikhonov regularization could provide more reasonable uncertainties.

muon g-2

- Muon g-2: 4.2σ deviation from the SM Muon g-2, PRL(2021)
- Dominate uncertainty of the SM prediction: hadronic vacuum polarization (HVP) Aoyama, et al, Phys.Rept(2020)





- Inverse Problem:

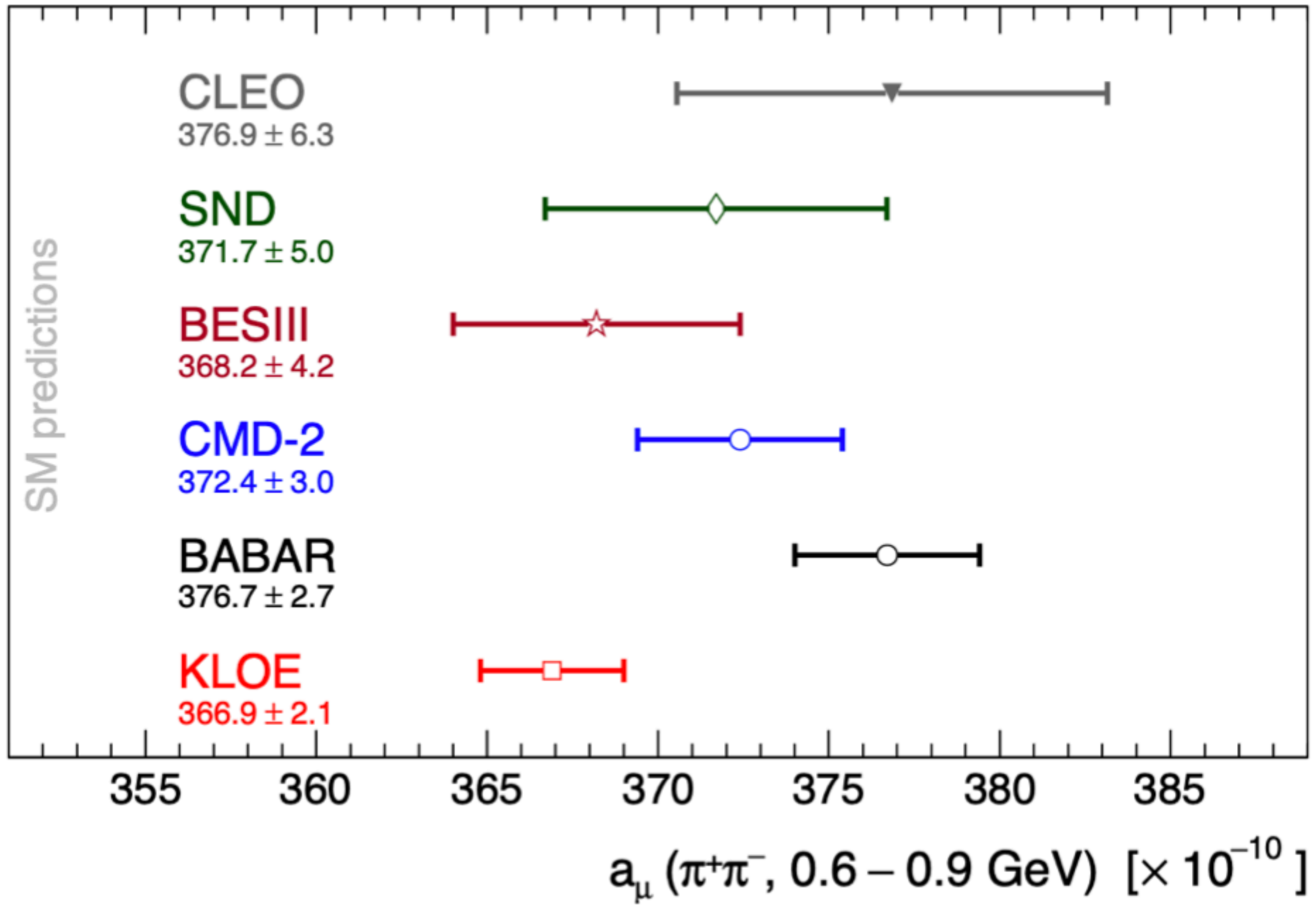
$$\int_{\lambda_r}^{\Lambda_r} ds' \frac{\mathrm{Im}\Pi_r(s')}{s'(s'+s)} - \pi \frac{\Pi_r(0)}{s} = -\pi \frac{\Pi_r(-s)}{s} - \int_{\Lambda_r}^{\infty} ds' \frac{\mathrm{Im}\Pi_r(s')}{s'(s'+s)} \qquad r = \rho,\ \omega,\ \phi$$

- Result: Inverse problem: $a_\mu^{\mathrm{HVP}} = (641^{+65}_{-63}) \times 10^{-10}$ H.n.Li, Umeeda, 2004.06451

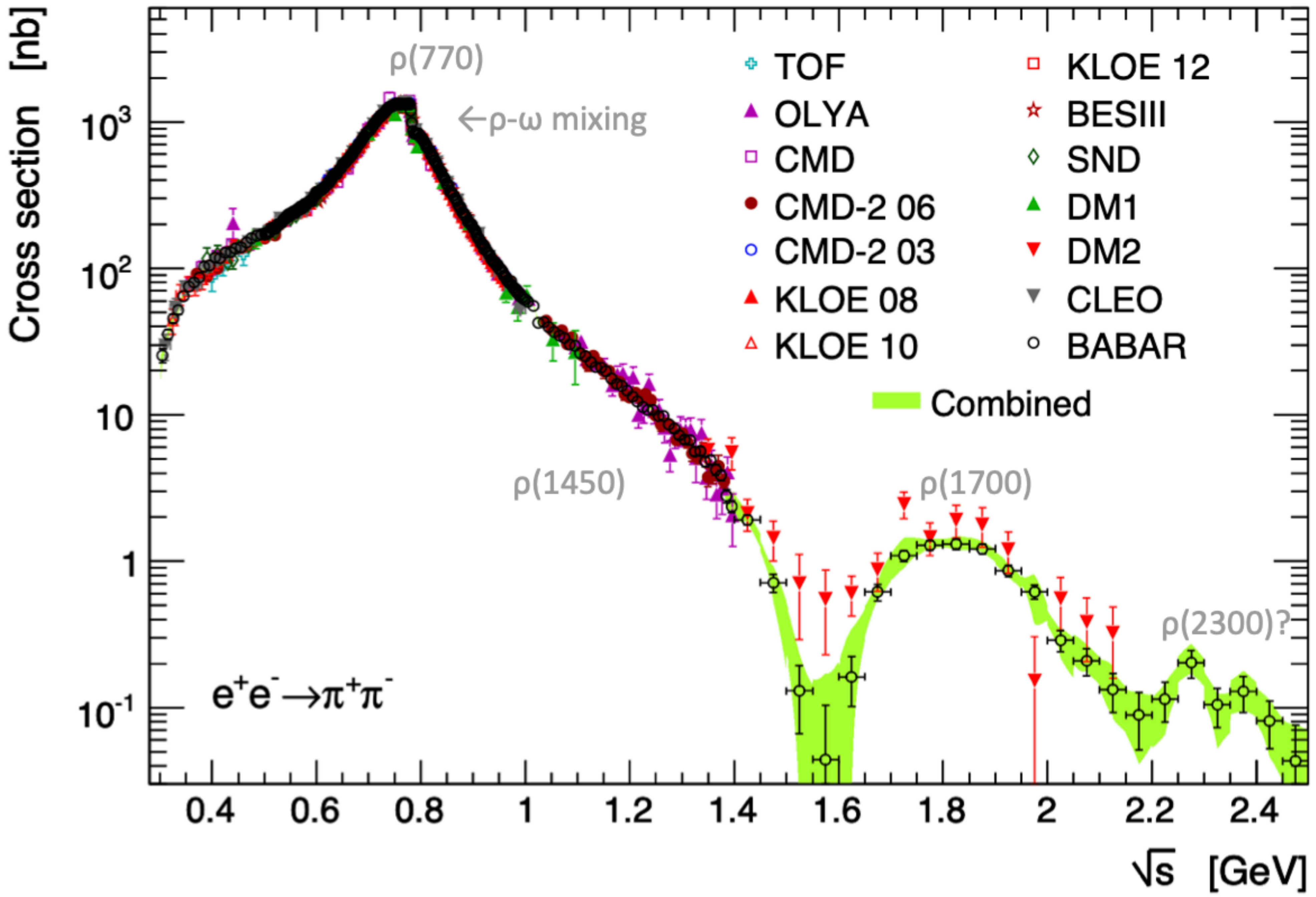
Non-perturbative properties can be revealed from asymptotic QCD by solving an inverse problem.

muon g-2

Figure from DHMZ, EPJC80 (2020) 241



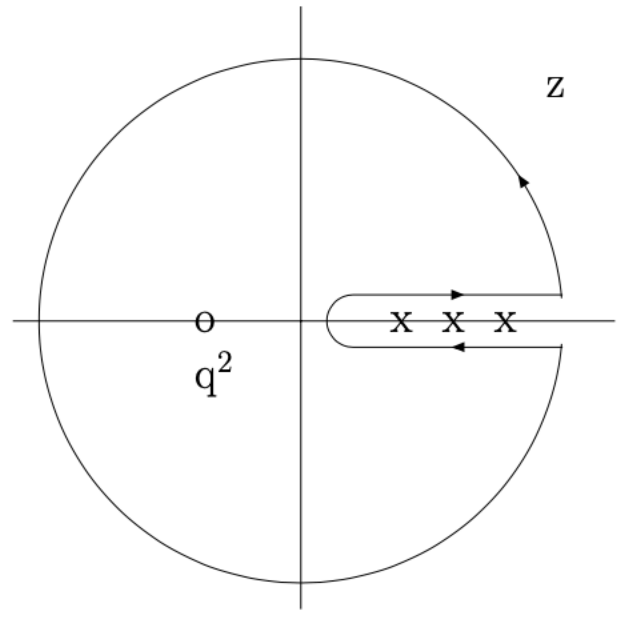
Figures from DHMZ, EPJC80 (2020) 241



Zhiqing Zhang's slides

- Perspective: 4-loop pQCD combined with experimental data at reliable regions might solve the BABAR-KLOE problem and lower down the uncertainty of predictions.

QCD sum rules



- Conventional QCD sum rules $\Pi_{\mu\nu}(q^2) = i\int d^4x e^{iq\cdot x}\langle 0|T[J_\mu(x)J_\nu(0)]|0\rangle$

Dispersion relation: $$\Pi(q^2) = \frac{1}{2\pi i}\oint ds\frac{\Pi(s)}{s-q^2} = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds\frac{\text{Im }\Pi(s)}{s-q^2-i\epsilon}$$

$$\text{Im}\Pi(q^2) = \pi f_V^2\delta(q^2-m_V^2) + \pi\rho^h(q^2)\theta(q^2-s_h)$$

Quark-hadron duality: $$\rho^h(s) = \frac{1}{\pi}\text{Im}\Pi^{\text{pert}}(s)\theta(s-s_0)$$

$$\int_{s_h}^{\infty} ds\frac{\rho^h(s)}{s-q^2} = \frac{1}{\pi}\int_{s_0}^{\infty} ds\frac{\text{Im}\Pi^{\text{pert}}(s)}{s-q^2}$$

- Uncertainty sources: quark-hadron duality. Results are very sensitive to the effective threshold s_0

QCD sum rules

- Excited states and continuum spectrum can be directly solved by the inverse problem.
- Avoid the quark-hadron duality

$$\begin{aligned} \mathrm{Im}\Pi(q^2) &= \pi f_\rho^2 \delta(q^2 - m_\rho^2) + \pi f_{\rho(1450)}^2 \delta(q^2 - m_{\rho(1450)}^2) + \pi f_{\rho(1700)}^2 \delta(q^2 - m_{\rho(1700)}^2) \\ &\quad + \pi f_V^2 \delta(q^2 - m_V^2) + \pi \rho^h(q^2), \end{aligned}$$

H.n.Li, Umeeda, 2006.16593

$$m_{\rho(770)}(m_{\rho(1450)}, m_{\rho(1700)}, m_{\rho(1900)}) \approx 0.78\ (1.46,\ 1.70,\ 1.90)\ \mathrm{GeV}$$

$$f_{\rho(770)}(f_{\rho(1450)}, f_{\rho(1700)}, f_{\rho(1900)}) \approx 0.22\ \ (0.19,\ 0.14,\ 0.14)\ \ \mathrm{GeV}$$

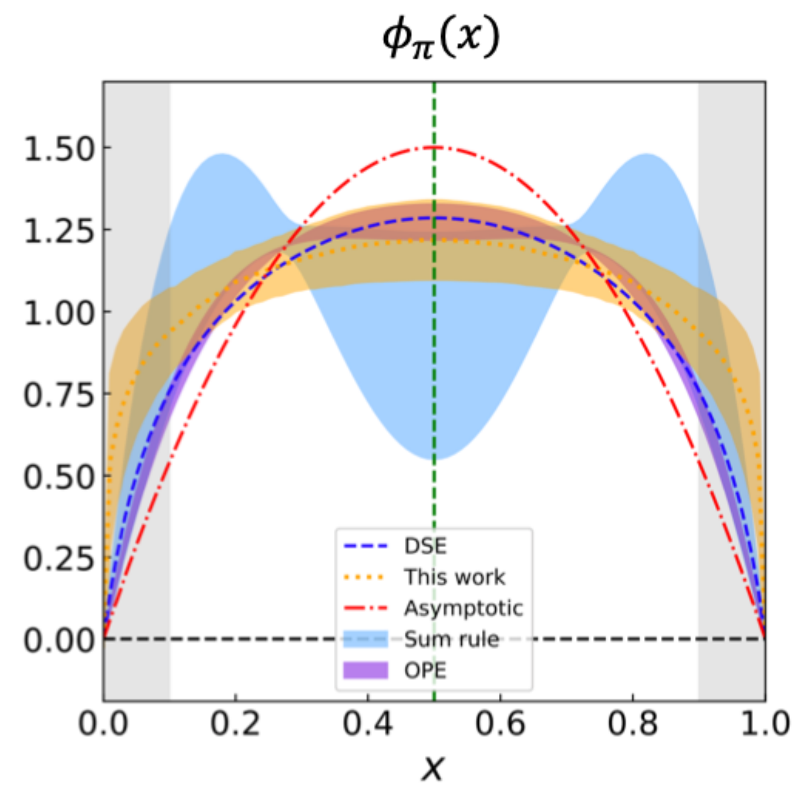
- Perspective: Inverse problem modifies the QCD sum rules.
  - Provide under-controlled uncertainties.
  - Calculate whatever calculated.
  - Advantage to excited states, no matter how much the pole contributes.

Light-cone distribution amplitudes

- Theoretical uncertainties on baryon CPV are dominated by the baryon LCDAs.
- Limited knowledge for nucleons. VERY very limited for all the others, especially for HIGH TWISTs.
- LaMET and Lattice QCD

  Z.F.Deng, C.Han, W.Wang, J.Zeng, J.L.Zhang, 2304.09004

  Hua, et al, 2021



- Inverse Problem can give very high moments.

H.n.Li, 2205.06746

$$(a_2^\pi, a_4^\pi, a_6^\pi, a_8^\pi, a_{10}^\pi, a_{12}^\pi, \cdots, a_{32}^\pi, a_{34}^\pi)|_{\mu=2\,\mathrm{GeV}}$$
$$= (0.1775^{+0.0036}_{-0.0040}, 0.0957^{+0.0011}_{-0.0012}, 0.0762^{+0.0006}_{-0.0003}, 0.0688^{+0.0016}_{-0.0012}, 0.0643^{+0.0021}_{-0.0017}, 0.0603^{+0.0024}_{-0.0019},$$
$$\cdots, 0.0089^{+0.0004}_{-0.0006}, 0.0028^{+0.0001}_{-0.0003}),$$

- Perspective: Tikhonov regularization could provide more reasonable uncertainties.

反问题是什么？

● 反问题：

$$x? \rightarrow \boxed{K} \rightarrow y$$

原因或输入　　过程或模型　　结果或输出

例：

		正问题	反问题
小学期间	x和K是整数	求$y=Kx$	$x=\frac{1}{K}*y$
中学期间	x是实数 K是映射或函数	求$y=K(x)$	隐函数定理
大学期间	x是向量 K是矩阵	求$y=K*x$	$x=K^{-1}y$
	$x=x(t)$是函数 K是积分运算	求$y(t)=\int\frac{x(s)}{t-s}ds$	?
泛函：算子方程	x是函数空间 K是算子	求$y=Kx$	$x=K^{-1}y$

反问题是什么?

● 反问题：

$x?$	$\rightarrow$	$\boxed{K}$	$\rightarrow$	y
原因或输入		过程或模型		结果或输出

例：

常规一维函数	若 $y = ax + b(a \neq 0)$，已知y_1，求x_1 =?		(存在且唯一)
	$y = sin(x)$，	已知y_2，求x_2 =?	(存在但不唯一)
矩阵问题	$y = Kx$,	已知y_3，求x_3 =?	依赖具体情况而定
	矩阵K非奇异		(存在且唯一)
	矩阵K奇异：需要额外验证其他条件		(存在但不唯一) (不存在)

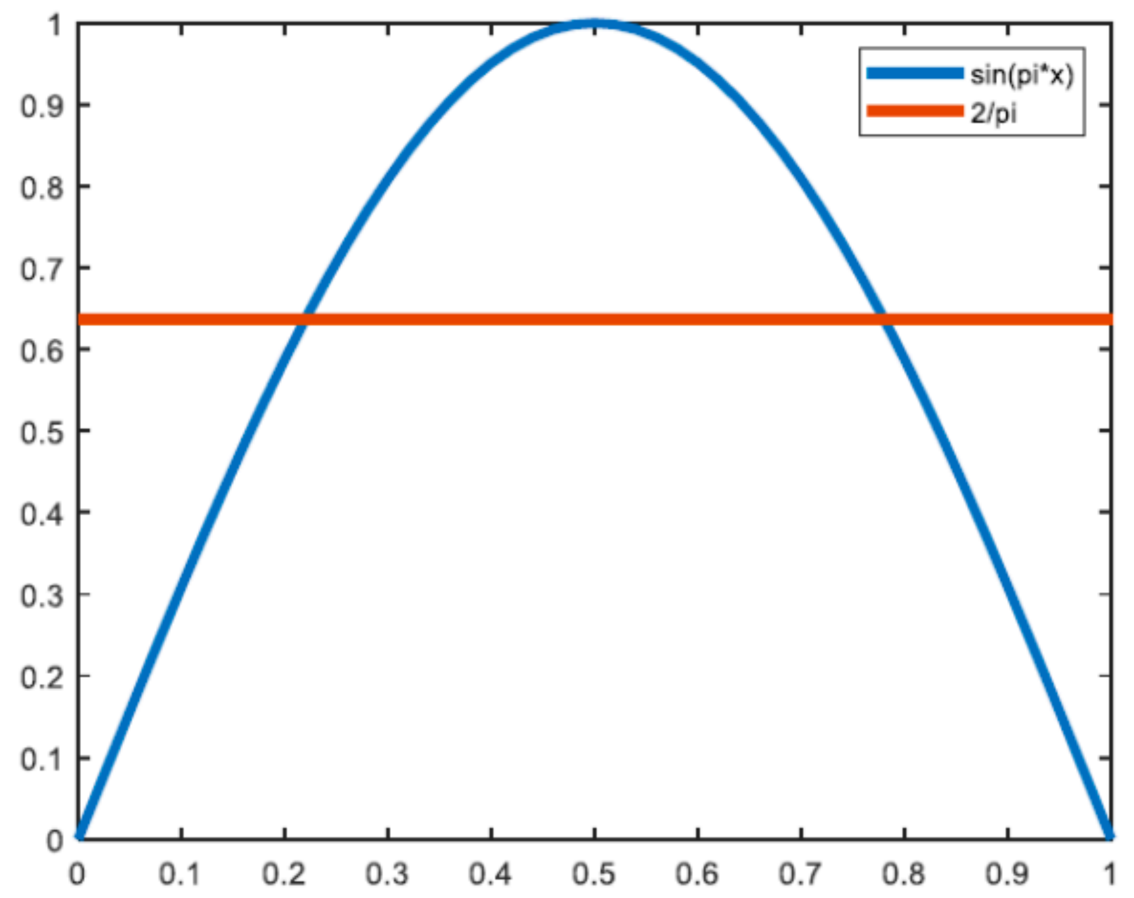
反问题是什么?

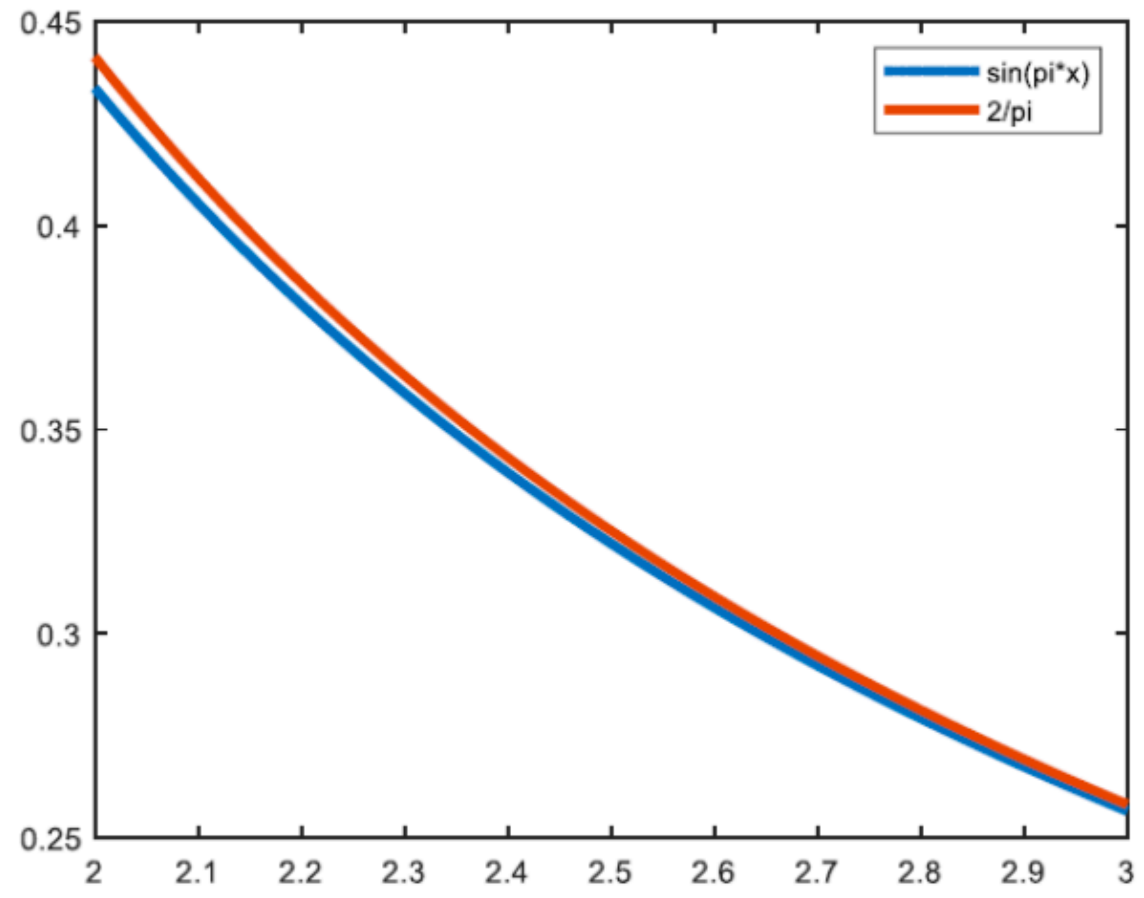
● **反问题:**

$$\underset{\text{原因或输入}}{x?} \rightarrow \boxed{\underset{\text{过程或模型}}{K}} \rightarrow \underset{\text{结果或输出}}{y}$$

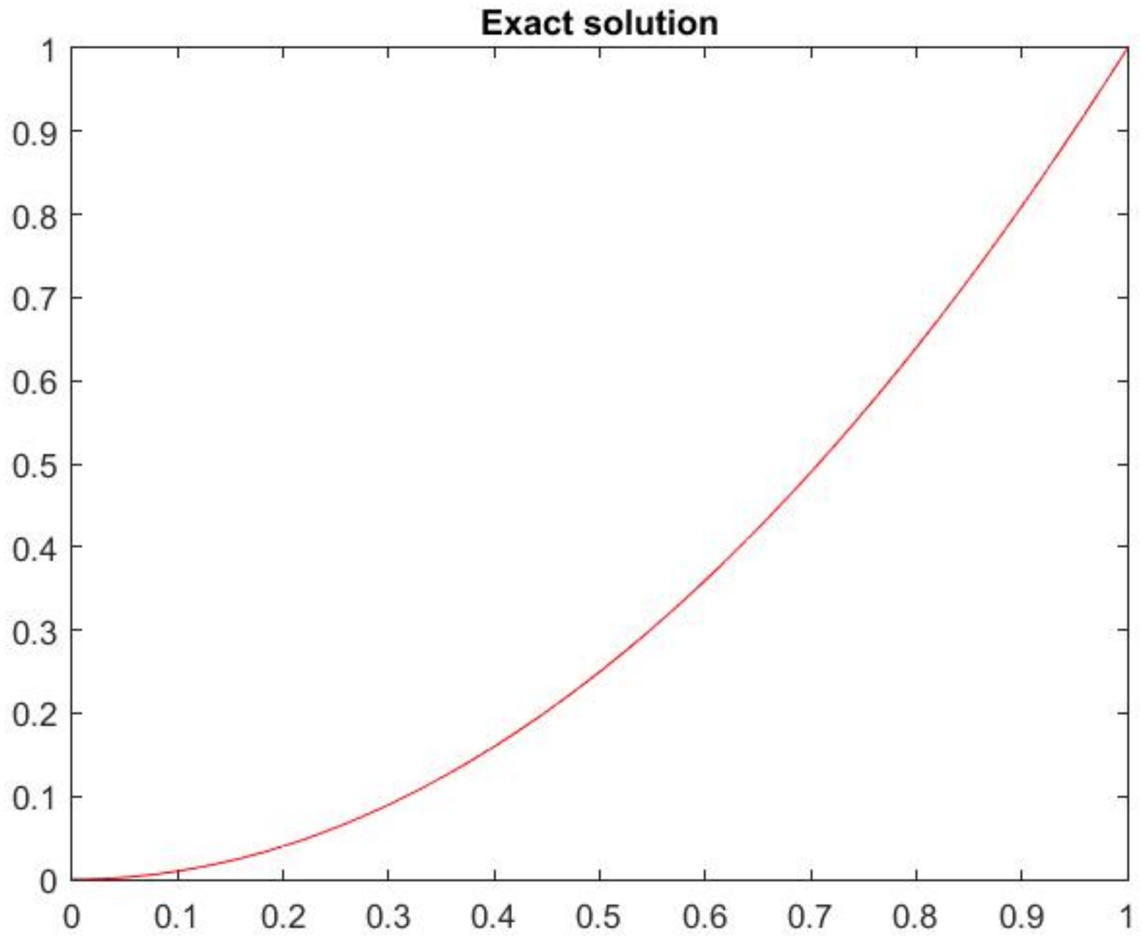
例:

积分方程 $\int_0^1 \frac{f(x)}{y-x} dx = g(y), y \in [2,3]$ **知道$g(y)$，求$f(x)$** $=?$



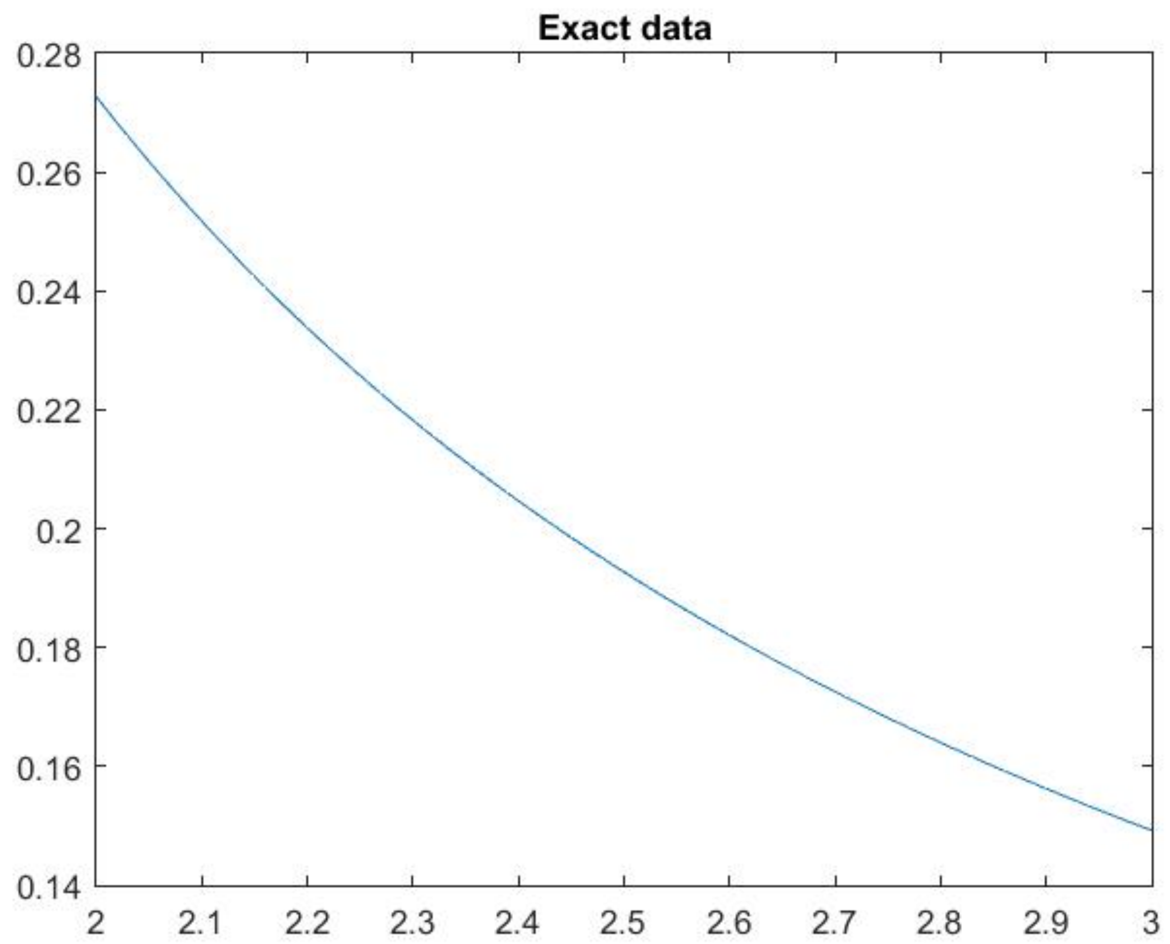


存在性未知;
唯一性未知;
但不稳定

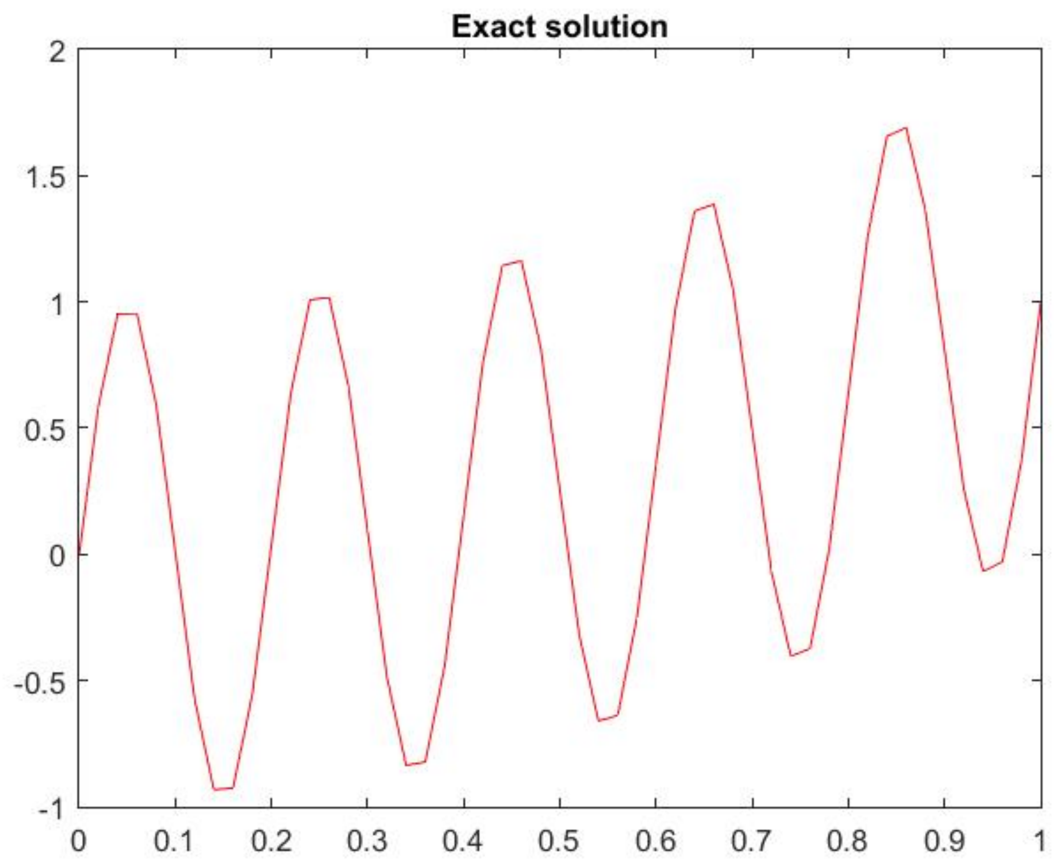


$x(t) = t^2$

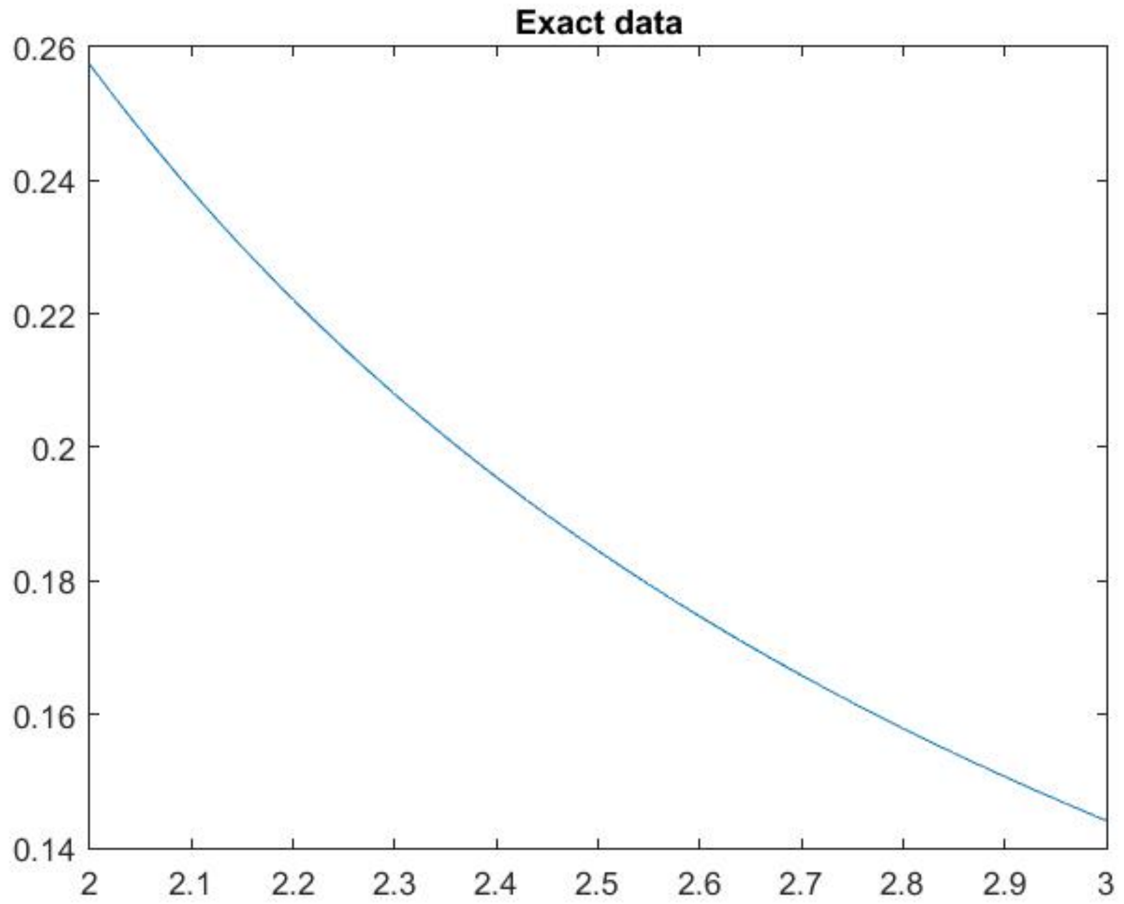
正问题



$y(s) = Kx$



$x(t) = t^2 + \sin(10\pi t)$



$y = Kx$

Proof of uniqueness:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

Proof. Since K is a linear operator, we know that $Kf_1 - Kf_2 = K(f_1 - f_2) = 0$. Setting $f = f_1 - f_2$, we just need to prove that $Kf = 0$ implies $f(x) = 0$, a. e. $x \in [a,b]$.

It is easy to obtain that $Kf = \int_a^b \frac{1}{y-x} f(x)dx = \int_a^b \left(\frac{1}{y}\sum_{k=0}^{\infty}(\frac{x}{y})^k\right) f(x)dx$. Since $x \in [a,b]$, $y \in [c,d]$, $c > b$, we know $|\frac{x}{y}| \leq |\frac{b}{c}| < 1$, which implies that $\left|\sum_{k=0}^{\infty}(\frac{x}{y})^k f(x)\right| \leq \sum_{k=0}^{\infty}(\frac{b}{c})^k |f(x)|$ for all $x \in [a,b]$. Combined with $\int_a^b |f(x)|dx < +\infty$ and the control convergence theorem, we have

$$y\int_a^b \frac{1}{y-x} f(x)dx = \sum_{k=0}^{\infty}\frac{1}{y^k}\int_a^b x^k f(x)dx = 0, \quad y \in [c,d]. \tag{3.4}$$

If $d = +\infty$, by using (3.4), we have

$$\int_a^b f(x)dx + \frac{1}{y}\int_a^b xf(x)dx + \cdots + \frac{1}{y^k}\int_a^b x^k f(x)dx + \cdots = 0, \quad y \in (c, +\infty). \tag{3.5}$$

Letting $y \to +\infty$ in (3.5), we have $\int_a^b f(x)dx = 0$. Then multiplying y on both sides of (3.5) and letting $y \to +\infty$, we also have $\int_a^b xf(x)dx = 0$. Repeating above process, we can obtain that

$$\int_a^b x^k f(x)dx = 0, \quad k = 0, 1, 2, \cdots. \tag{3.6}$$

Proof of uniqueness:

If $d < +\infty$, taking $z \in D := \{z \in \mathbb{C} : |z| \geq c\}$, we have

$$\Big|\sum_{k=0}^{\infty} \frac{1}{z^k} \int_a^b x^k f(x)dx\Big| \leq \sum_{k=0}^{\infty} \frac{1}{c^k} \Big|\int_a^b x^k f(x)dx\Big| \leq \sum_{k=0}^{\infty} \frac{b^k}{c^k} \int_a^b |f(x)|dx < +\infty,$$

which implies that the series $\sum_{k=0}^{\infty} \frac{1}{z^k} \int_a^b x^k f(x)dx$ is convergent uniformly on D. Since $\frac{1}{z^k} \int_a^b x^k f(x)dx$ is analytic on D for each k and use the Weierstrass theorem, we conclude that the series $\sum_{k=0}^{\infty} \frac{1}{z^k} \int_a^b x^k f(x)dx$ is analytic on D. Further, we know $\sum_{k=0}^{\infty} \frac{1}{y^k} \int_a^b x^k f(x)dx$ is real analytic on $y \in (c, +\infty)$. Combined with the analytic continuation, we know that (3.4) holds for $y > c$, i. e.

$$\sum_{k=0}^{\infty} \frac{1}{y^k} \int_a^b x^k f(x)dx = 0, \quad y \in (c, +\infty).$$

Similar to the proof process of the case $d = +\infty$, we also conclude that $\int_a^b x^k f(x)dx = 0$, $k = 0, 1, 2, \cdots$ for $d < +\infty$.

Proof of uniqueness:

Since $C[a,b]$ is dense in $L^2(a,b)$, then for $f(x) \in L^2(a,b)$ and any $\epsilon > 0$, there exists $\tilde{f}(x) \in C[a,b]$, such that $\|f - \tilde{f}\|_{L^2(a,b)} < \epsilon$. On the other hand, for $\tilde{f}(x) \in C[a,b]$, there exists a polynomial $Q_n(x)$ of degree $n \in \mathbb{N}$, such that $\|\tilde{f} - Q_n\|_{C[a,b]} < \epsilon$ by the Weierstrass theorem. Therefore, we have

$$\begin{aligned}
\|f - Q_n\|_{L^2(a,b)} &\le \|f - \tilde{f}\|_{L^2(a,b)} + \|\tilde{f} - Q_n\|_{L^2(a,b)} \\
&\le \epsilon + \sqrt{b-a}\|\tilde{f} - Q_n\|_{C[a,b]} \\
&< \epsilon + \epsilon\sqrt{b-a},
\end{aligned}$$

By using (3.6), we know that $\int_a^b f(x)Q_n(x)dx = 0$. Combined with the Cauchy inequality, we have

$$\begin{aligned}
\|f\|^2_{L^2(a,b)} &= \int_a^b f^2(x)dx = \int_a^b \left(f^2(x) - f(x)Q_n(x)\right)dx \\
&\le \int_a^b |f(x)| \cdot |f(x) - Q_n(x)|dx \\
&\le \left(\int_a^b f^2(x)dx\right)^{\frac{1}{2}} \left(\int_a^b |f(x) - Q_n(x)|^2 dx\right)^{\frac{1}{2}} \\
&= \|f\|_{L^2(a,b)}\|f - Q_n\|_{L^2(a,b)} \\
&\le (\epsilon + \epsilon\sqrt{b-a})\|f\|_{L^2(a,b)},
\end{aligned}$$

which implies that $\|f\|_{L^2(a,b)} \le \epsilon + \epsilon\sqrt{b-a}$.

Letting $\epsilon \to 0$, we have $\|f\|_{L^2(a,b)} = 0$, i. e. $f(x) = 0$, a. e. $x \in [a,b]$. The proof is completed. □

Proof of instability:

$$\int_a^b \frac{f(x)}{y-x}\mathrm{d}x = g(y), \quad y \in [c,d], \quad c > b, \quad a > 0$$

We show the instability of the inverse problem of dispersion relation by the special case. Taking $a = 0$, $b = 1$, $c = 2$, $d = 3$, $f_2(x) = f_1(x) + \sqrt{n}\cos(n\pi x)$, and $f_{1,2}$ are the solutions of $g_{1,2}$ with $g_i(y) = \int_0^1 \frac{1}{y-x} f_i(x) dx$. As $n \to \infty$, it is obvious that

$$\|f_2 - f_1\|_{L^2(0,1)} = \left(\int_0^1 (\sqrt{n}\cos(n\pi x))^2 dx \right)^{1/2} = \frac{\sqrt{n}}{\sqrt{2}} \to \infty, \tag{3.7}$$

and

$$\|g_2 - g_1\|_{L^2(2,3)} = \frac{1}{\sqrt{n}\pi} \left(\int_2^3 \left(\int_0^1 \left(\frac{1}{y-x} \right)^2 \sin(n\pi x) dx \right)^2 dy \right)^{1/2} \le \frac{1}{\sqrt{n}\pi} \to 0, \tag{3.8}$$

That means the solutions could be changed infinitely even though the noise of the input data is approaching to vanish. So the inverse problem is unstable.

Numerical Method of Tikhonov Regularization

$$\varphi_i(x) = \begin{cases} \frac{x-x_{i-1}}{h}, x \in [x_{i-1}, x_i], \\ -\frac{x-x_{i+1}}{h}, x \in [x_i, x_{i+1}], \\ 0, otherwise, \end{cases}$$

$$\boxed{f_\alpha^\delta = \arg\min_{f \in L^2(a,b)} J(f) = \arg\min_{f \in L^2(a,b)} \left(\frac{1}{2} \|Kf - g^\delta\|^2_{L^2(c,d)} + \frac{\alpha}{2} \|f\|^2_{L^2(a,b)} \right)}$$

$$\varphi_0(x) = \begin{cases} -\frac{x-x_1}{h}, x \in [x_0, x_1], \\ 0, otherwise, \end{cases}$$

$$f^\delta_{\alpha,n}(x) = \sum_{i=0}^{n} c_i \varphi_i(x)$$

$$\varphi_n(x) = \begin{cases} \frac{x-x_{n-1}}{h}, x \in [x_{n-1}, x_n], \\ 0, otherwise. \end{cases}$$

$$J(f^\delta_{\alpha,n}) = \frac{1}{2} \left\| \sum_{i=0}^{n} c_i K\varphi_i - g^\delta \right\|^2_{L^2(c,d)} + \frac{\alpha}{2} \left\| \sum_{i=0}^{n} c_i \varphi_i \right\|^2_{L^2(a,b)}$$

$$= \frac{1}{2} \sum_{i,j=0}^{n} c_i c_j (K\varphi_i, K\varphi_j)_{L^2(c,d)} - \sum_{i=0}^{n} c_i (K\varphi_i, g^\delta)_{L^2(c,d)} + \frac{1}{2}(g^\delta, g^\delta)_{L^2(c,d)} + \frac{\alpha}{2} \sum_{i,j=0}^{n} c_i c_j (\varphi_i, \varphi_j)_{L^2(a,b)}$$

$$X_n = \text{span}\{\varphi_0, \varphi_1, \cdots, \varphi_n\}$$

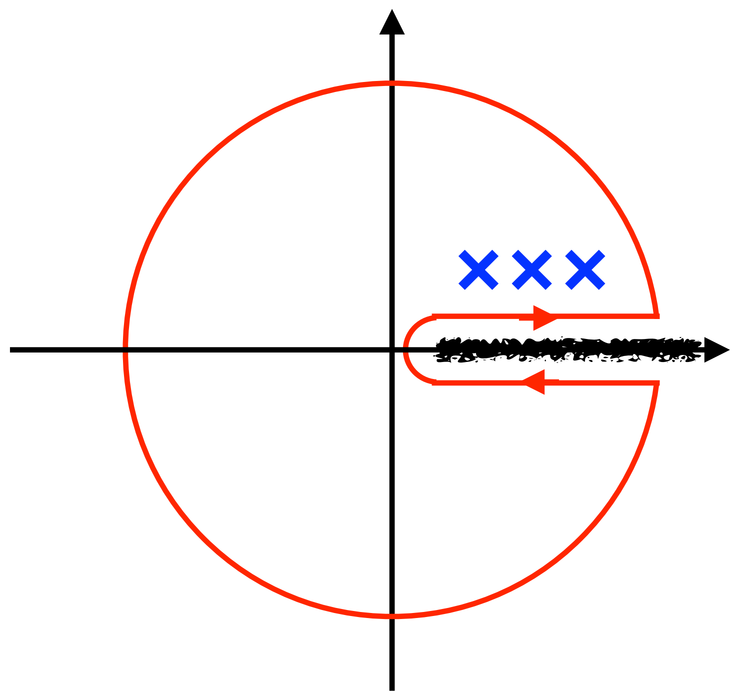
$$A_{ij} = (K\varphi_i, K\varphi_j)_{L^2(c,d)} \qquad B_{ij} = (\varphi_i, \varphi_j)_{L^2(a,b)} \qquad C = (c_0, c_1, \cdots, c_n)^T$$

$$f^\delta_{\alpha,n}(x) = \sum_{i=0}^{n} c_i \varphi_i(x)$$

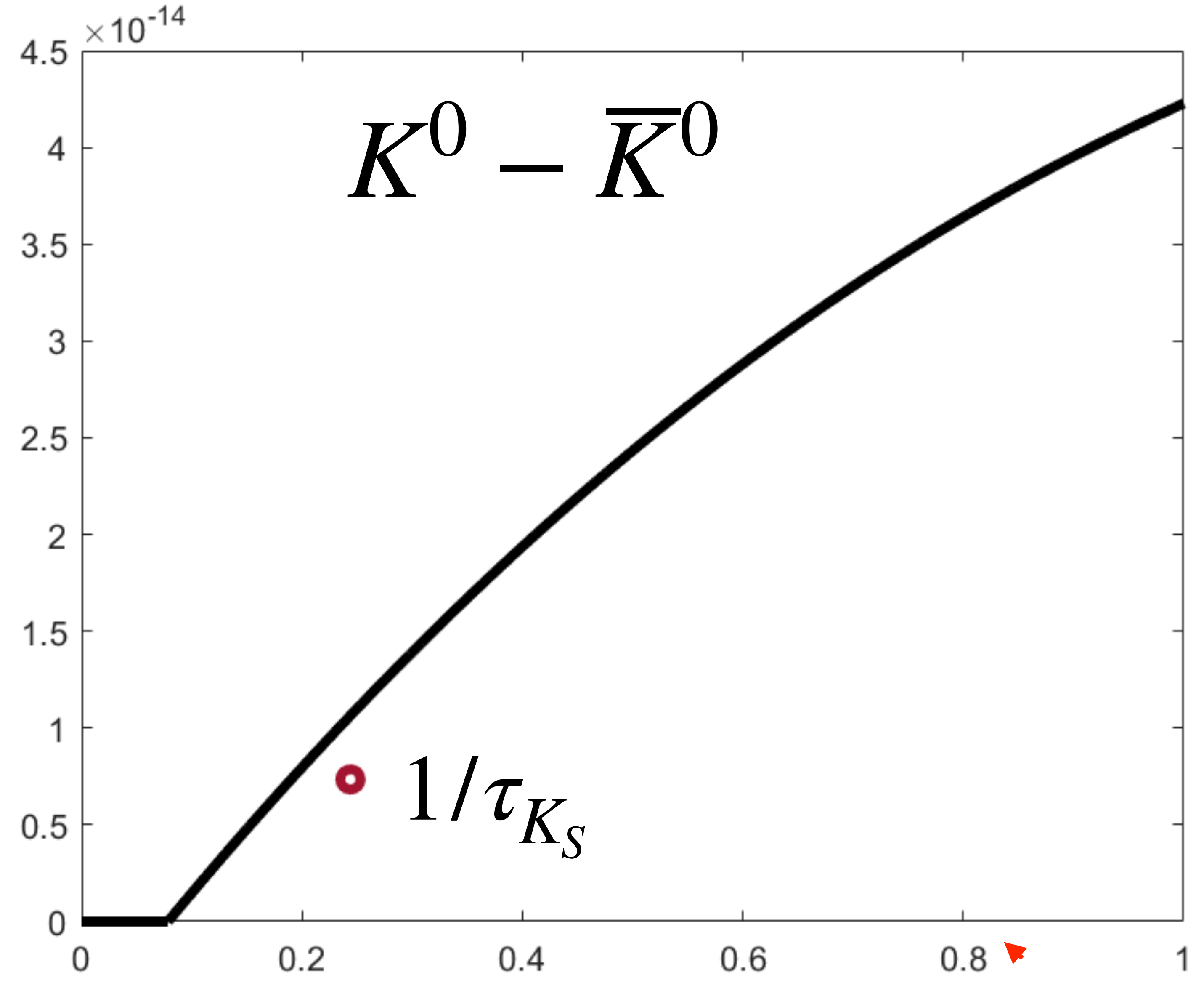
$$(A + \alpha B)C = D \qquad D_i = (K\varphi_i, g^\delta)_{L^2(c,d)}$$

Theorem 4.3. *If the noise δ and the regularization parameter α are fixed, we have $\|f^\delta_{\alpha,n} - f^\delta_\alpha\|_{L^2(a,b)} \to 0$, as $n \to \infty$.*

5. Physical applications: neutral meson mixing



$$\Delta\Gamma_K = 2|\Gamma_{12}|$$



$$\frac{(s-s_1)(s_1-s_2)(s_2-s)}{2\pi}\int_{s_{th}}^{\Lambda}\frac{\Gamma_{12}(s')}{(s'-s)(s'-s_1)(s'-s_2)}ds'$$

$$= (s_1-s_2)M_{12}(s) + (s_2-s)M_{12}(s_1) + (s-s_1)M_{12}(s_2)$$

$$-\frac{(s-s_1)(s_1-s_2)(s_2-s)}{2\pi}\int_{\Lambda}^{\infty}\frac{\Gamma_{12}(s')}{(s'-s)(s'-s_1)(s'-s_2)}ds'$$

$$\Gamma_{21}^q = \frac{1}{2M_{B_q}}\mathrm{Disc}\langle\overline{B}_q|i\int d^4x\, T\left(\mathcal{H}_{eff}^{\Delta B=1}(x)\mathcal{H}_{eff}^{\Delta B=1}(0)\right)|B_q\rangle$$