

# Lattice QCD calculation of the invisible decay $J/\psi \rightarrow \gamma\nu\bar{\nu}$

孟雨

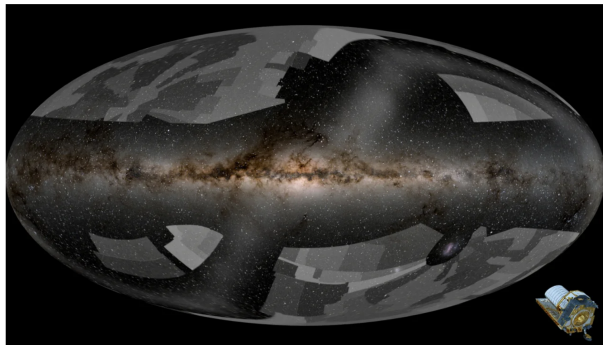
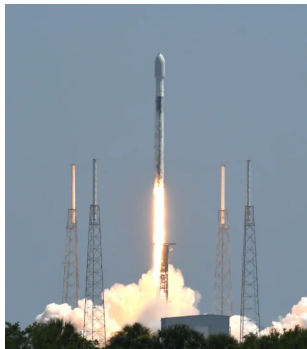
郑州大学

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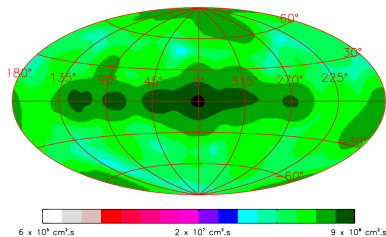
- Motivation
- Methodology:  $\eta_c \rightarrow \gamma\gamma$
- Lattice studies on  $J/\psi \rightarrow \gamma\nu\bar{\nu}$
- Summary

# Motivation: search for the new physics

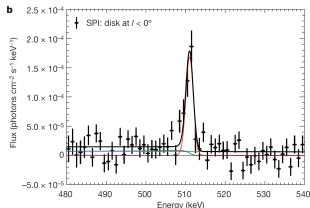


- Euclid space telescope: launching at 11:12 a.m, Sat July 1, 2023
- Mission: 3D map of the universe, with billions of galaxies that stretch 10 billion light-years away

# Motivation: 511keV $\gamma$ -Ray in Galactic Center



**Figure 1.** 508.25–513.75 keV *INTEGRAL* SPI exposure map. Units are in  $\text{cm}^2 \times \text{s}$ . This map takes into account the differential sensitivity of SPI across its field of view.



Nature, **451**,159(2008)

- First observation, *Astrophys.J.* 172, L1(1972).
- One of the views: **light dark matter** (1-100MeV) annihilating into  $e^+e^-$  pairs  
*PRL*,92,101301(2004)

# Current status of the dark matter search on the collider

- Remarkable candidates
  - **Dark photon  $A'$** : coupling to the photon through the kinetic mixing  $\frac{\epsilon}{2}F^{\mu\nu}F'_{\mu\nu}$ , free parameters are  $m_{A'}$  and  $\epsilon$
  - **CP-odd Higgs Boson  $A^0$** :  $m_{A^0}$  and  $g_{a\ell\bar{\ell}}$
  - **Axion-like particle  $a$** :  $m_a$  and  $g_{a\gamma\gamma}$
- Dark matter can be produced on the collider
  - **Direct production:  $e^+e^- \rightarrow \gamma X$**   
BABAR,PRL119,131804(2017)      BESIII,PLB839,137785(2023)
  - **Resonant production:  $e^+e^- \rightarrow J/\psi/\Upsilon(1S) \rightarrow \gamma + X$**   
CLEO,PRD81,091101(2010)      BABAR,PRL107,021804(2011)  
Belle,PRL122,011801(2019)      Belle,PRL128,081804 (2022)  
BESIII,PRD101,112005(2020)      BESIII,PRD105,012008(2022)  
BESIII,PLB838,137698(2023)

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BESIII,PLB838,137698(2023)

## CLEO 2010

PHYSICAL REVIEW D 81, 091101(R) (2010)

### Search for the decay $J/\psi \rightarrow \gamma + \text{invisible}$

(CLEO Collaboration)

$$J/\psi \rightarrow \gamma + X$$

invisible to detector

light dark matter ?

The upper limit corresponding to  $m_X = 0$

is  $4.3 \times 10^{-6}$  at the 90% confidence level

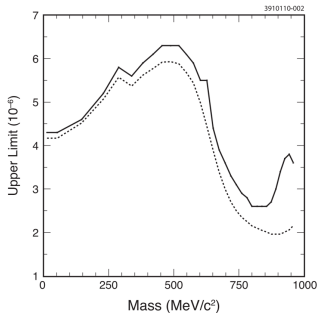


FIG. 4. The 90% confidence-level upper limits for  $J/\psi \rightarrow \gamma X$ , where  $X$  is invisible to the detector. The dashed line shows the results for statistical uncertainties alone, and the solid line includes systematic and statistical uncertainties.

## BESIII

PHYSICAL REVIEW D 101, 112005 (2020)

### Search for the decay $J/\psi \rightarrow \gamma + \text{invisible}$

Upper bound  $\mathcal{B} \sim 7.0 \times 10^{-7}$

PHYSICAL REVIEW D 105, 012008 (2022)

### Search for a CP-odd light Higgs boson in $J/\psi \rightarrow \gamma A^0$

Upper bound  $\mathcal{B}(J/\psi \rightarrow \gamma A^0) \times \mathcal{B}(A^0 \rightarrow \mu^+ \mu^-) \sim (1.2 - 778.0) \times 10^{-9}$

PHYSICS LETTERS B 838, 137698 (2023)

### Search for an axion-like particle in radiative $J/\psi$ decays

We search for an axion-like particle (ALP)  $a$  through the process  $\psi(3686) \rightarrow \pi^+ \pi^- J/\psi$ ,  $J/\psi \rightarrow \gamma a$ ,  $a \rightarrow \gamma\gamma$  in a data sample of  $(2.71 \pm 0.01) \times 10^9$   $\psi(3686)$  events collected by the BESIII detector. No significant ALP signal is observed over the expected background, and the upper limits on the branching fraction of the decay  $J/\psi \rightarrow \gamma a$  and the ALP-photon coupling constant  $g_{a\gamma\gamma}$  are set at 95% confidence level in the mass range of  $0.165 \leq m_a \leq 2.84$  GeV/ $c^2$ . The limits on  $\mathcal{B}(J/\psi \rightarrow \gamma a)$  range from  $8.3 \times 10^{-8}$  to  $1.8 \times 10^{-6}$  over the search region, and the constraints on the ALP-photon coupling are the most stringent to date for  $0.165 \leq m_a \leq 1.468$  GeV/ $c^2$ .



# Future experiments in the search for dark matter

- **Super  $\tau$ - Charm Facility (STCF)**

arXiv:2303.15790

Samples about 100 larger than the present largest  $\tau$ -charm factory - BESIII

2.6	New light particles beyond the SM . . . . .	52
2.6.1	Particles in the dark sector . . . . .	52
2.6.2	Millicharged particles . . . . .	54

- **Belle II: The Belle II Physics Book**

arXiv:1808.10567

16.3	Experiment: Quarkonium Decay	569
16.3.1	Searches for BSM physics in invisible $\Upsilon(1S)$ decays	569
16.3.2	Probe of new light $CP$ even Higgs bosons from bottomonium $\chi_{b0}$ decay	571
16.3.3	Search for a $CP$ -odd Higgs boson in radiative $\Upsilon(3S)$ decays	572

- **LHCb II: Physics case for an LHCb Upgrade II**

arXiv:1808.08865

8.6	Searches for prompt and detached dark photons . . . . .	97
8.7	Searches for semileptonic and hadronic decays of long-lived particles . . . . .	98

## Standard model background in $J/\psi$ invisible decay

Only after we exactly determine the standard model background can the precise experimental investigations of the  $J/\psi \rightarrow \gamma + \text{invisible}$  decay possibly provide us rigorous constraints on new physics

In the standard model, the background is

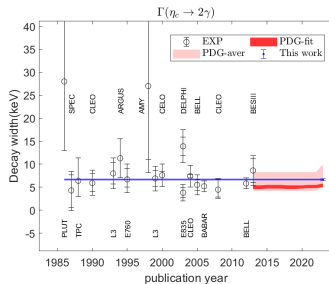
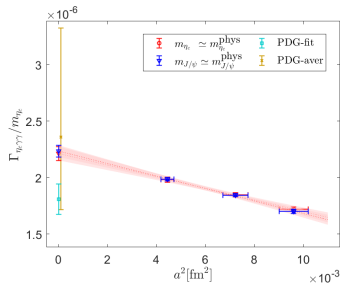
$$J/\psi \rightarrow \gamma + \nu + \bar{\nu}$$

Since the neutrinos are only invisible particles

We target on an first-principle calculation on this invisible decay.

# Methodology: $\eta_c \rightarrow 2\gamma$

Scalar function method: YM et al, Sci.Bull 68,1880(2023),2109.09381



$$\Gamma(\eta_c \rightarrow 2\gamma) = \begin{cases} 6.67(16)(6) \text{ keV} \\ 5.4(4) \text{ keV} & \text{PDG-fit} \\ 7.04^{+2.9}_{-1.9} \text{ keV} & \text{PDG-aver} \end{cases}$$

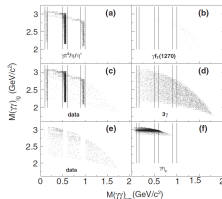
- HPQCD,  $\Gamma_{\eta_c \gamma} = 6.788(45)_{\text{fit}}(41)_{\text{sys}}$  keV, PRD108,014513(2023),2305.06231

# Discussion

- CLEO(08) and BESIII(13) extract the branching fraction of  $\eta_c \rightarrow 2\gamma$  by

$$J/\psi \rightarrow \gamma \eta_c \rightarrow 3\gamma$$

- Using  $1.06 \times 10^8 \psi(3686) \rightarrow \pi^+ \pi^- J/\psi$



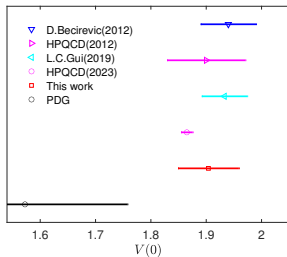
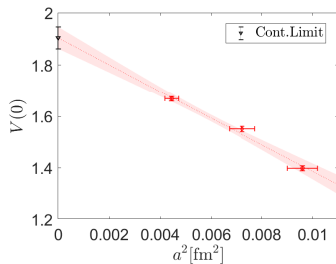
VALUE ( $10^{-4}$ )	CL%	EVTS	DOCUMENT ID	TECN	COMMENT
$1.68 \pm 0.12$					OUR FIT
$2.2^{+0.9}_{-0.8}$					OUR AVERAGE
$2.7 \pm 0.8 \pm 0.6$			<sup>1</sup> ABUKIM 2013I	BES3	
$0.7^{+1.6}_{-0.7} \pm 0.2$		$1.2^{+2.8}_{-1.1}$	<sup>2</sup> ADAMS 2008	CLEO	$\psi(2S) \rightarrow \pi^+ \pi^- J/\psi$
			● ● We do not use the following data for averages, fits, limits, etc. ● ●		
$2.0^{+0.9}_{-0.7} \pm 0.2$		13	<sup>3</sup> WICHT 2008	BELL	$B^\pm \rightarrow K^\pm \gamma \gamma$
$2.80^{+0.67}_{-0.58} \pm 1.0$			<sup>4</sup> ARMSTRONG 1995F	E760	$\bar{p} p \rightarrow \gamma \gamma$
< 9	90		<sup>5</sup> BISELLO 1991	DM2	$J/\psi \rightarrow \gamma \gamma \gamma$
$6^{+4}_{-3} \pm 4$			<sup>4</sup> BAGLIN 1987B	SPEC	$\bar{p} p \rightarrow \gamma \gamma$
< 18	90		<sup>6</sup> BLOOM 1983	CBAL	$J/\psi \rightarrow \eta_c \gamma$

<sup>1</sup> ABUKIM 2013I reports  $[\Gamma(\eta_c(1S) \rightarrow \gamma \gamma) / \Gamma_{\text{total}}] \times [B(J/\psi(1S) \rightarrow \gamma \eta_c(1S))] = (4.5 \pm 1.2 \pm 0.6) \times 10^{-6}$  which we divide by our best value  $B(J/\psi(1S) \rightarrow \gamma \eta_c(1S)) = 0.017 \pm 0.004$ . Our first error is their experiment's error and our second error is the systematic error from using our best value.

<sup>2</sup> ADAMS 2008 reports  $[\Gamma(\eta_c(1S) \rightarrow \gamma \gamma) / \Gamma_{\text{total}}] \times [B(J/\psi(1S) \rightarrow \gamma \eta_c(1S))] = (1.2^{+2.7}_{-1.1} \pm 0.3) \times 10^{-6}$  which we divide by our best value  $B(J/\psi(1S) \rightarrow \gamma \eta_c(1S)) = 0.017 \pm 0.004$ . Our first error is their experiment's error and our second error is the systematic error from using our best value.

The contribution of  $J/\psi \rightarrow \gamma \eta_c$  is crucial

$$\eta_c \rightarrow 2\gamma \quad \& \quad J/\psi \rightarrow \gamma\eta_c$$



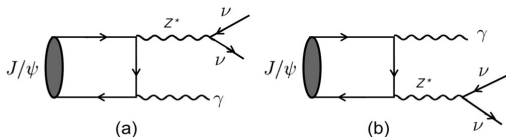
- Our result  $\text{Br}(J/\psi \rightarrow \gamma\eta_c) = 2.31(13)\%$  [To appear]  $\Leftarrow 1.7(4)\%$  PDG
- Combining BESIII(13),  $\Gamma(\eta_c \rightarrow 2\gamma) = 6.2(1.7)(0.9)\text{keV} \Leftarrow 8.64(2.56)(1.92)\text{keV}$
- **Remark:** direct measurement with no-dependence on  $J/\psi \rightarrow \gamma\eta_c$  is urgent, or much improved  $J/\psi \rightarrow 3\gamma$ , both statistically and systematically.
  - Model-independent input for  $J/\psi \rightarrow 3\gamma$  amplitude with **full Dalitz space**.  
YM et al, PRD 102,054506(2020)

$$J/\psi \rightarrow \gamma \nu \bar{\nu}$$

$J/\psi \rightarrow \gamma \nu \bar{\nu}$  decay from lattice QCD

arXiv:2309.15436

# $J/\psi \rightarrow \gamma\nu\bar{\nu}$ : Formalism



- Amplitude

$$i\mathcal{M} = -i \frac{(qce)G_F}{\sqrt{2}} [H_{\mu\nu\alpha}(q, p) \epsilon_{J/\psi}^\alpha(p) \epsilon^{\nu*}(q) \bar{u}(q_1) \gamma^\mu (1 - \gamma_5) v(q_2)]$$

- Hadronic function

$$\begin{aligned}
 H_{\mu\nu\alpha}(q, p) &= \int d^4x e^{-iqx} \langle 0 | T \{ J_\mu^{\text{em}}(x) J_\nu^Z(0) \} | J/\psi(p)_\alpha \rangle \\
 &\equiv \epsilon_{\mu\nu\alpha\beta} q_\beta F_{\gamma\nu\bar{\nu}} \quad \text{PRD } \mathbf{90}, 077501(2014)
 \end{aligned}$$

- Decay width

$$\Gamma(J/\psi \rightarrow \gamma\nu\bar{\nu}) = \frac{\alpha G_F^2}{3\pi^2} \int_0^{\frac{m_{J/\psi}}{2}} |\vec{q}|^3 (m_{J/\psi} - |\vec{q}|) |F_{\gamma\nu\bar{\nu}}|^2 d|\vec{q}|$$

- Form factor

$$\begin{aligned} F_{\gamma\nu\bar{\nu}}(E_\gamma, \Delta t) &= \frac{1}{6p \cdot q} \epsilon_{\mu\nu\alpha\beta} p_\beta H_{\mu\nu\alpha}(q, p) \\ &= -\frac{i}{6E_\gamma} \int e^{E_\gamma t} dt \int d^3\vec{x} j_0(E_\gamma|\vec{x}|) \epsilon_{\mu\nu\alpha 0} \mathcal{H}_{\mu\nu\alpha}(x, \Delta t) \end{aligned}$$

- Decay width evaluated by a Monte-Carlo method

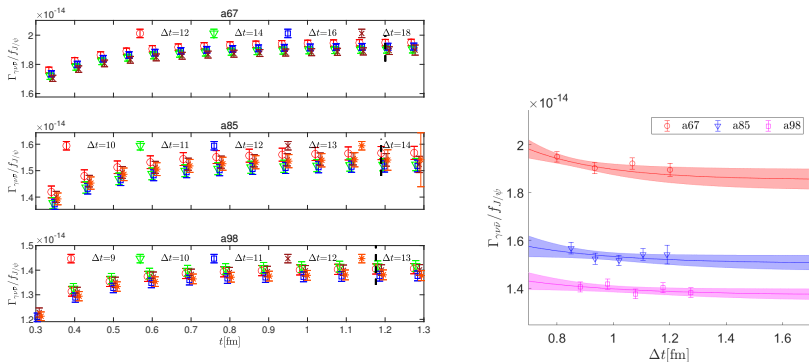
$$\Gamma_{\gamma\nu\bar{\nu}}(\Delta t) = \frac{\alpha G_F^2}{3\pi^2} \frac{m_{J/\psi}}{2N_{MC}} \sum_{i=1}^{N_{MC}} \left( E_\gamma^3 (m_{J/\psi} - E_\gamma) |F_{\gamma\nu\bar{\nu}}(E_\gamma, \Delta t)|^2 \right)_i$$

- Dimensionless quantity  $R_f \equiv \Gamma_{\gamma\nu\bar{\nu}}/f_{J/\psi}$  and  $\Delta t$  dependence

$$R_f(\Delta t) = R_f + \zeta \cdot e^{-(m_{J/\psi}^{(1)} - m_{J/\psi})\Delta t}$$

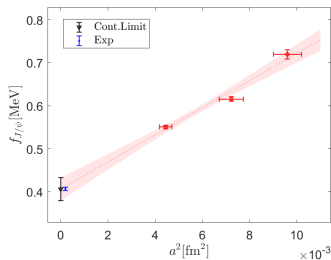
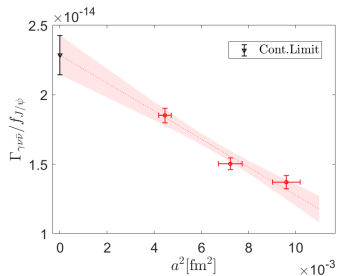


# Excited-state contamination



- A slight dependence on the excited-state of  $J/\psi$  ( $\Delta t$ -dependence).
- Dashed black line: a suitable time truncation  $t_{\text{cut}} \sim 1.2$  fm.
- The right: an extrapolation for  $\Delta t$  at  $t_{\text{cut}}$ .

# Continuous limit



- The first lattice QCD calculation [arXiv:2309.15436](https://arxiv.org/abs/2309.15436)

$$\text{Br}[J/\psi \rightarrow \gamma\nu\bar{\nu}] = 1.00(9)(7) \times 10^{-10}$$

- Phenomenological estimation  $\sim 0.7 \times 10^{-10}$   
Dao-Neng Gao, PRD **90**,077501(2014)

## Discussion: next generation experiments

- Dark matter search by  $J/\psi \rightarrow \gamma + \text{invisible}$  [STCF]
  - STCF has the potential to improve the upper limit to  $10^{-8}$ , even possibly  $10^{-9}$  level.
  - Standard model background  $J/\psi \rightarrow \gamma\nu\bar{\nu}$  is calculated on the lattice with a branching fraction of  $1.00(9)(7) \times 10^{-10}$ .
- Dark matter search by  $\Upsilon \rightarrow \gamma + \text{invisible}$  [Belle II]
  - The design luminosity of Belle II is 80 times larger than Belle, which has an upper limit of  $10^{-6}$ .
  - Naive phenomenological estimation  $\mathcal{B}(\Upsilon(1S) \rightarrow \gamma\nu\bar{\nu}) \sim 10^{-9}$  considering  $\mathcal{B} \sim M^2$ .
  - No lattice calculation of  $\mathcal{B}(\Upsilon(1S) \rightarrow \gamma\nu\bar{\nu})$ .

- We present first lattice calculation on the invisible decay  $J/\psi \rightarrow \gamma\nu\bar{\nu}$  using a scalar function method.
- Various systematics are examined, including finite-volume effects, excited-state contamination, and discretization effects.
- An exact branching fraction  $\text{Br}[J/\psi \rightarrow \gamma\nu\bar{\nu}] = 1.00(9)(7) \times 10^{-10}$  is determined, providing theoretical support for the dark matter search in future experiments.

Thank you for attention!

# Back-up

# Hadronic function in Minkowski and Euclidean space

- Minkowski

$$H_{\mu\nu\alpha}(q, p) = i \sum_{n, \vec{q}} \frac{1}{E_\gamma - E_n + i\epsilon} \langle 0 | J_\mu^{\text{em}}(0) | n(\vec{q}) \rangle \langle n(\vec{q}) | J_\nu^Z(0) | J/\psi(p) \rangle_\alpha$$
$$- i \sum_{n', \vec{q}} \frac{1}{E_\gamma + E_{n'} - m_{J/\psi} - i\epsilon} \langle 0 | J_\nu^Z(0) | n'(-\vec{q}) \rangle \langle n'(-\vec{q}) | J_\mu^{\text{em}}(0) | J/\psi(p) \rangle_\alpha$$

- Euclidean

$$H_{\mu\nu\alpha}^E(q, p) = i \sum_{n, \vec{q}} \frac{1 - e^{-(E_n - E_\gamma)T/2}}{E_\gamma - E_n + i\epsilon} \langle 0 | J_\mu^{\text{em}}(0) | n(\vec{q}) \rangle \langle n(\vec{q}) | J_\nu^Z(0) | J/\psi(p) \rangle_\alpha$$
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- A naive relation requires

- $\delta E_n \equiv E_n - E_\gamma > 0$
- $\delta E_{n'} \equiv E_\gamma + E_{n'} - m_{J/\psi} > 0$

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- $\delta E_n \equiv E_n - E_\gamma > 0$    $|n\rangle \xrightarrow{\text{lowest}} |J/\psi\rangle$
- $\delta E_{n'} \equiv E_\gamma + E_{n'} - m_{J/\psi} > 0$    $|n'\rangle \xrightarrow{\text{lowest}} |\eta_c\rangle$

Note: a general approach to exponentially growing terms

Norman H. Christ et al, PRD 103,014507(2021)



- $N_f = 2$  twisted mass ensembles

Ens	$a$ (fm)	$L^3 \times T$	$N_{\text{conf}} \times T$	$m_\pi$ (MeV)	$t$
a67	0.0667(20)	$32^3 \times 64$	$197 \times 64$	300	12-18
a85	0.085(2)	$24^3 \times 48$	$200 \times 48$	315	10-14
a98	0.098(3)	$24^3 \times 48$	$236 \times 48$	365	9-13

- Numerical results:  $\delta E_{\eta_c}(\vec{n} \neq 0) > 0$

Ensemble	a67	a85	a98
$a\delta E_{\eta_c}( \vec{n} ^2 = 0)$	-0.0343(2)	-0.0372(3)	-0.0387(3)
$a\delta E_{\eta_c}( \vec{n} ^2 = 1)$	0.1781(2)	0.2446(3)	0.2380(4)
$a\delta E_{\eta_c}( \vec{n} ^2 = 2)$	0.2758(3)	0.3737(3)	0.3611(4)
$a\delta E_{\eta_c}( \vec{n} ^2 = 3)$	0.3544(3)	0.4751(4)	0.4587(4)
$a\delta E_{\eta_c}( \vec{n} ^2 = 4)$	0.4223(4)	0.5636(5)	0.5426(5)

Note that  $\langle 0 | J_\nu^Z(0) | \eta_c(\vec{0}) \rangle \langle \eta_c(\vec{0}) | J_\mu^{\text{em}}(0) | J/\psi(\vec{0})_\alpha \rangle = 0$

# Hadronic function in Minkowski and Euclidean space

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$$- i \sum_{n', \vec{q}} \frac{1 - e^{-(E_\gamma + E_{n'} - m_{J/\psi})T/2}}{E_\gamma + E_{n'} - m_{J/\psi} - i\epsilon} \langle 0 | J_\nu^Z(0) | n'(-\vec{q}) \rangle \langle n'(-\vec{q}) | J_\mu^{\text{em}}(0) | J/\psi(p) \rangle_\alpha$$

- **A naive relation** requires

- $\delta E_n \equiv E_n - E_\gamma > 0$    $|n\rangle \underline{\underline{\text{lowest}}}$   $|J/\psi\rangle$
- $\delta E_{n'} \equiv E_\gamma + E_{n'} - m_{J/\psi} > 0$    $|n'\rangle \underline{\underline{\text{lowest}}}$   $|\eta_c\rangle$

# Finite-volume effects

- Examining the finite-volume effects ( $R \equiv |\vec{x}|$ )

$$F_{\gamma\nu\bar{\nu}}(E_\gamma, \Delta t) = -\frac{i}{6E_\gamma} \int e^{E_\gamma t} dt \int_0^R d^3\vec{x} j_0(E_\gamma|\vec{x}|) \epsilon_{\mu\nu\alpha 0} \mathcal{H}_{\mu\nu\alpha}(x, \Delta t)$$

