Study the singularity property of single loop diagram by a geometric method

Jia-Jun Wu (UCAS)

Collaborator: Ming-yang Duan

Paper is processing…..

The 7th Symposium on "Symmetries and the emergence of Structure in QCD"

山东日照 ShanDong RiZhao 2023.7.21

中国科学院大学
University of Chinese Academy of Sciences

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- Landau Formulas => Geometric method
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- Box Singularity
- Pentagon => Hexagon => N Polygons Singularity
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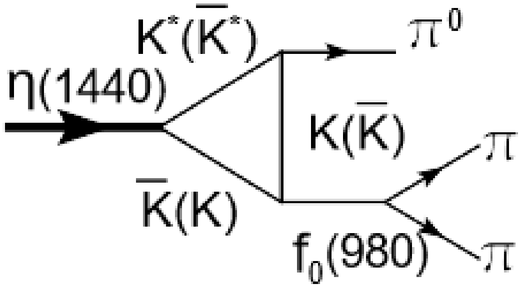
Motivation：From Triangle Singularity

L. D. Landau, Nucl. Phys. 13, no.1, 181-192 (**1960**)
S. Coleman, R.E. Norton, Nuovo Cim. **1965**, 38, 438–442,
R. Karplus, C.M. Sommerfield, E.H. Wichmann, PR **1958**, 111, 1187–1190.
J.D. Bjorken, Ph.D. Thesis, Stanford University, Stanford, CA, USA, **1959**.
C. Schmid, Phys. Rev. **1967**, 154, 1363,

F.K. Guo, X. H. Liu, S. Sakai
PPNP 2020, 112, 103757

Structures	Processes	Loops	I/F	Refs.
2.1 GeV [141]	$\gamma p^+ \to N^*(2030) \to K^+\Lambda(1405)$	$K^*\Sigma\pi$	I	[142]
2.1 GeV	$\pi^- p^+ \to K^0\Lambda(1405),\ pp \to pK^+\Lambda(1405)$	$K^*\Sigma\pi$	I	[143]
1.88 GeV	$\Lambda_c^+ \to \pi^+\pi^0\pi\Sigma$	$\bar{K}^*N\bar{K}$	I	[144, 145][a]
$N(1700)$ [10]	$N(1700) \to \pi\Delta$	$\rho N\pi$	I	[146]
$N(1875)$ [10]	$N(1875) \to \pi N(1535)$	$\Sigma^*K\Lambda$	I	[147]
$\Delta(1700)$ [148–150]	$\gamma p \to \Delta(1700) \to \pi N(1535) \to p\pi^0\eta$	$\Delta\eta p$	I	[151]
2.2 GeV [152]	$\Lambda_c^+ \to \pi^0\phi p$	$\Sigma^*K^*\Lambda$	F	[153]
1.66 GeV [154, 155]	$\Lambda_c^+ \to \pi^+K^-p$	$a_0\Lambda\eta, \Sigma^*\eta\Lambda$	F	[156]
$P_c(4450)$ [35]	$\Lambda_b^0 \to K^-J/\psi p$	$\Lambda(1890)\chi_{c1}p$	F	[157–160][b]
		$N(1900)\chi_{c1}p$	F	[159]
peaks relevant for P_c	$\Lambda_b^0 \to K^-J/\psi p$	$\bar{D}_{sJ}\Lambda_c^{(*)}\bar{D}^{(*)}$	F	[36, 158]

Structures	Processes	Loops	I/F	Refs.
$\rho(1480)$ [78, 79]	$\pi^-p \to \phi\pi^0 n$	$K^*\bar{K}K$	I	[80, 81]
$\eta(1405/1475)$ [82–86]	$\eta(1405/1475) \to \pi f_0$	$K^*\bar{K}K$	I	[87–91][a,b]
$f_1(1420)$ [92]	$f_1(1285) \to \pi a_0/\pi f_0$	$K^*\bar{K}K$	I	[89, 93–95][b]
$a_1(1420)$ [96, 97]	$a_1(1260) \to f_0\pi \to 3\pi$	$K^*\bar{K}K$	I	[97–99]
1.4 GeV [100]	$J/\psi \to \phi\pi^0\eta/\phi\pi^0\pi^0$	$K^*\bar{K}K$	I	[101][b]
1.42 GeV	$B^- \to D^{*0}\pi^- f_0(a_0), \tau \to \nu_\tau\pi^- f_0(a_0)$	$K^*\bar{K}K$	I	[102, 103]
	$D_s^+ \to \pi^+\pi^0 f_0(a_0), \bar{B}_s^0 \to J/\psi\pi^0 f_0(a_0)$	$K^*\bar{K}K$	I	[104, 105]
$f_2(1810)$ [10]	$f_2(1640) \to \pi\pi\rho$	$K^*\bar{K}^*K$	I	[106]
1.65 GeV	$\tau \to \nu_\tau\pi^- f_1(1285)$	$K^*\bar{K}^*K$	I	[107]
1515 MeV	$J/\psi \to K^+K^- f_0\,(a_0)$	$\phi\bar{K}K$	I	[108]
2.85 GeV, 3.0 GeV	$B^- \to K^-\pi^-D_{s0}^*/K^-\pi^-D_{s1}$	$K^{*0}D^{(*)0}K^+$	I	[109, 110]
5.78 GeV	$B_c^+ \to \pi^0\pi^+B_s^0$	$\bar{K}^{*0}B^+\bar{K}$	F	[111]
[4.01, 4.02] GeV	$[\bar{D}^{*0}D^{*0}] \to \gamma X$	$D^{*0}\bar{D}^{*0}D^0$	I	[112]
4015 MeV	$e^+e^- \to \gamma X$	$D^{*0}\bar{D}^{*0}D^0$	I	[113, 114]
4015 MeV	$B \to KX\pi$, $pp/p\bar{p} \to X\pi$+anything	$D^{*0}\bar{D}^{*0}D^0$	I	[115, 116]
$\Upsilon(11020)$ [117, 118]	$e^+e^- \to Z_b\pi$	$B_1(5721)\bar{B}B^*$	I	[119, 120]
3.73 GeV	$X \to \pi^0\pi^+\pi^-$	$D^{*0}\bar{D}^0D^0$	F	[121]
[4.22, 4.24] GeV	$e^+e^- \to \gamma J/\psi\phi/\pi^0J/\psi\eta$	$D_{s0(s1)}^*\bar{D}_s^{(*)}D_s^{(*)}$	F	[122]
[4.08, 4.09] GeV	$e^+e^- \to \pi^0J/\psi\eta$	$D_{s0(s1)}^*\bar{D}_s^{(*)}D_s^{(*)}$	F	[122]
$Z_c(3900)$ [31, 32]	$e^+e^- \to J/\psi\pi^+\pi^-$	$D_1\bar{D}D^*$	F	[119, 123–127][c]
		$D_0^*(2400)\bar{D}^*D$	F	[128, 129]
$Z_c(4020, 4030)$ [33, 130]	$e^+e^- \to \pi^+\pi^-h_c(\psi')$	$D_{1(2)}\bar{D}^{(*)}D^{(*)}$	F	[125]
$X(4700)$ [131, 132]	$B^+ \to K^+J/\psi\phi$	$K_1(1650)\psi'\phi$	F	[133]
$Z_c(4430)$ [30, 134]	$\bar{B}^0 \to K^-\pi^+J/\psi$	$\bar{K}^{*0}\psi(4260)\pi^+$	F	[135]
$Z_c(4200)$ [136, 137]	$\bar{B}^0 \to K^-\pi^+\psi(2S)$	$\bar{K}_2^*\psi(3770)\pi^+$	F	[135]
	$\Lambda_b^0 \to p\pi^-J/\psi$	$N^*\psi(3770)\pi^-$	F	[135]
$X(4050)^\pm$ [138]	$\bar{B}^0 \to K^-\pi^+\chi_{c1}$	$\bar{K}^{*0}X\pi^+$	F	[139]
$X(4250)^\pm$ [138]	$\bar{B}^0 \to K^-\pi^+\chi_{c1}$	$\bar{K}_2^*\psi(3770)\pi^+$	F	[139]
$Z_b(10610)$ [34]	$e^+e^- \to \Upsilon(1S)\pi^+\pi^-$	$B_J^*\bar{B}^*B$	F	[128]



BESIII collaboration,
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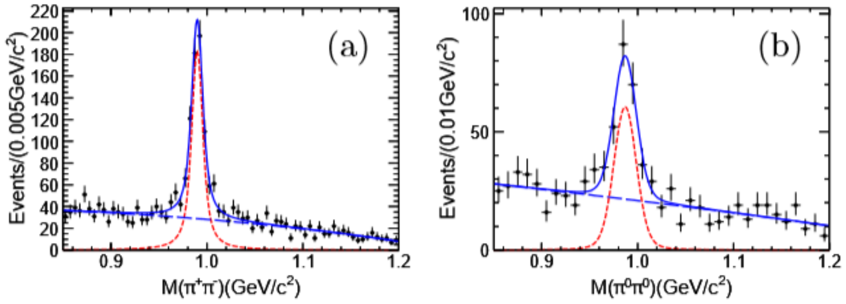
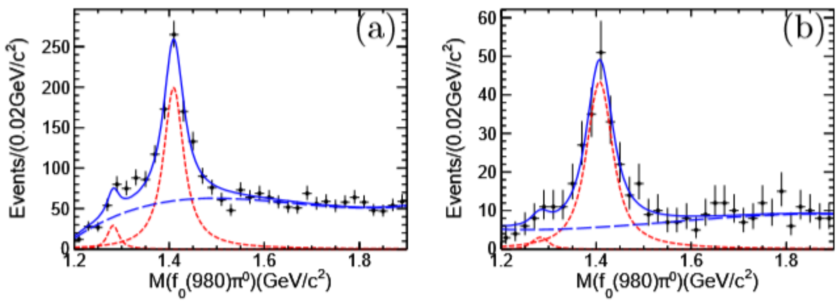




Motivation： From Triangle Singularity

1.It happens at a pure kinematic point

-> Model independent

2.The effect of Loop

-> Understand hadronic loop properties

3.Provide a peak structure

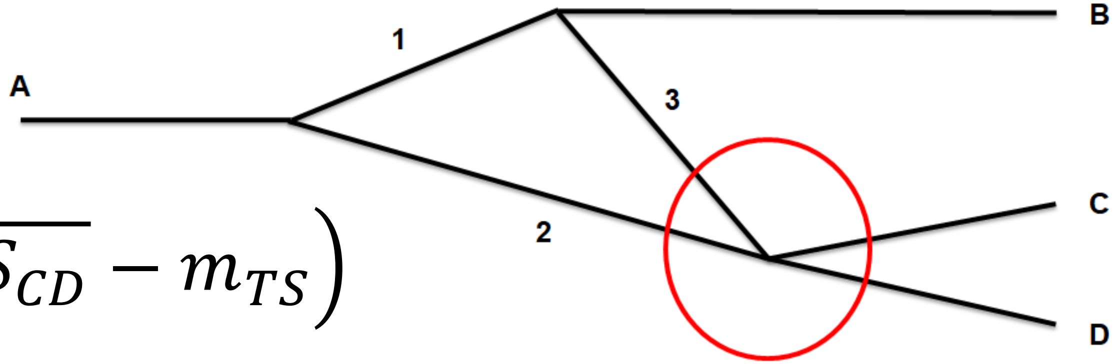
-> May mixing with resonance

4.To extract the nature of hadron

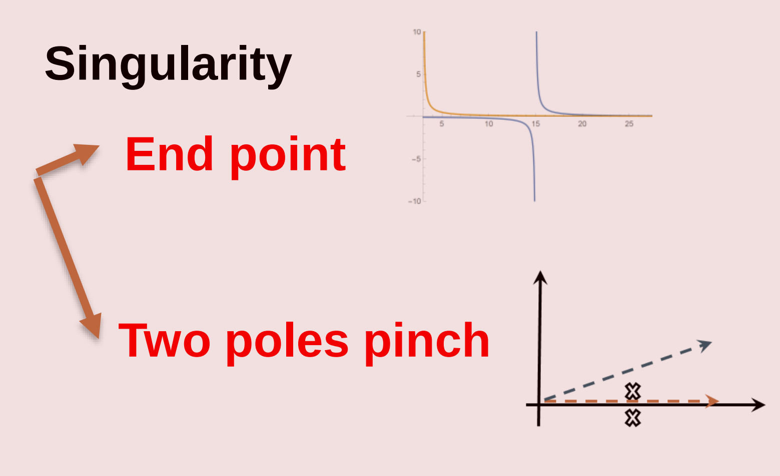
-> Study the coupling in the special point

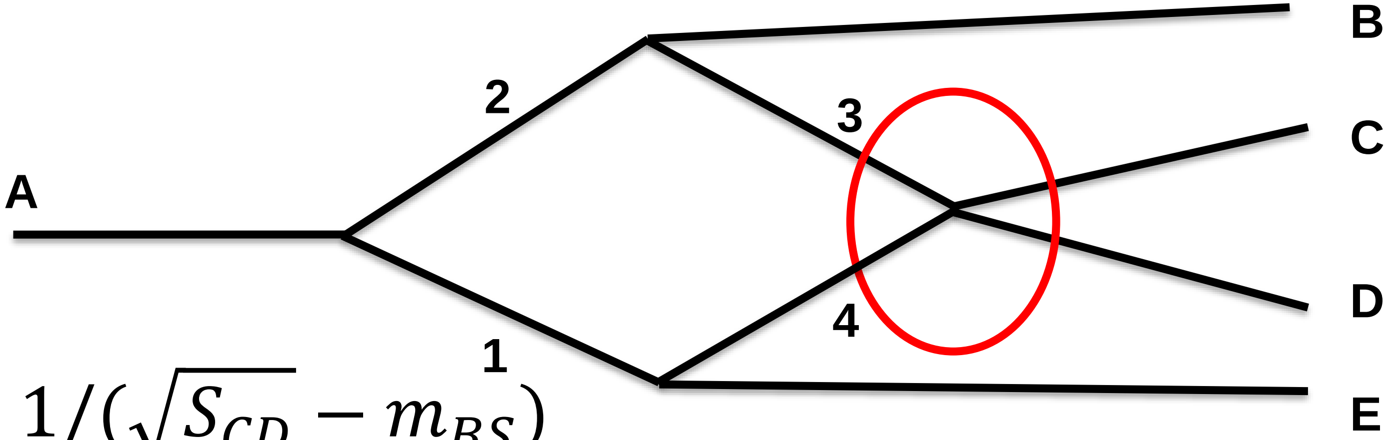
5. ……

Motivation： From Triangle Singularity

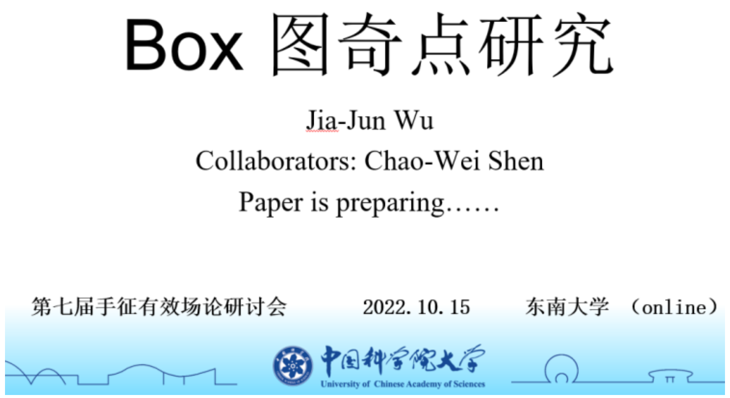


$$\sim \ln\left(\sqrt{S_{CD}} - m_{TS}\right)$$





$$\sim 1/(\sqrt{S_{CD}} - m_{BS})$$


Box 图奇点研究

Jia-Jun Wu

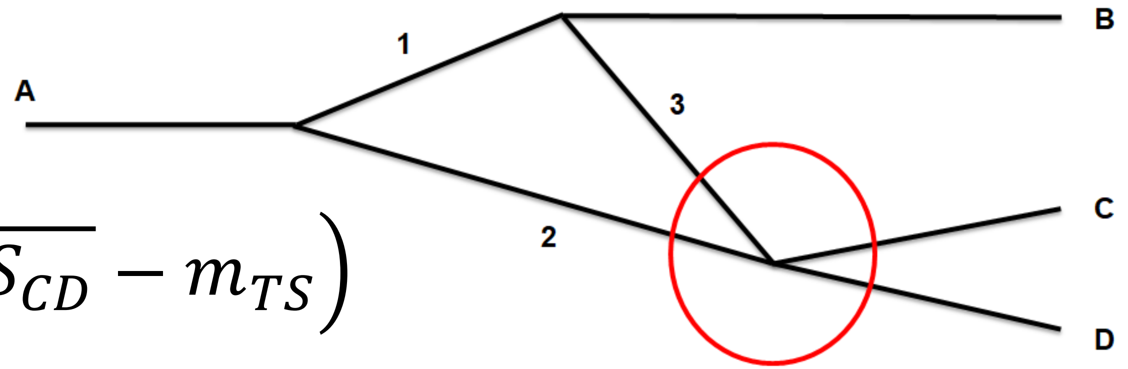
Collaborators: Chao-Wei Shen

Paper is preparing……

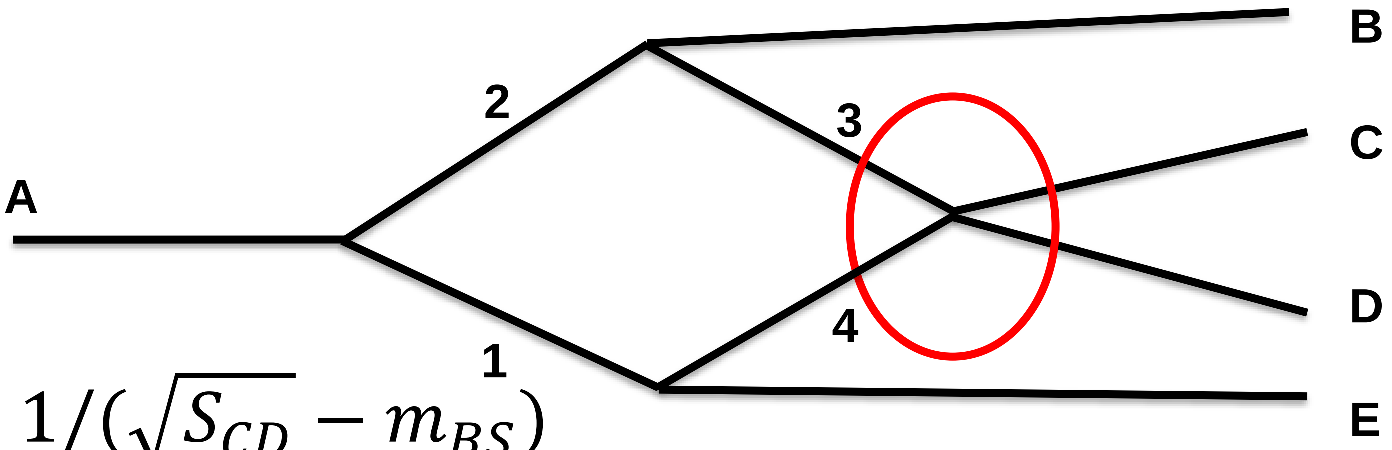
第七届手征有效场论研讨会 2022. 10. 15 东南大学 （online）

中国科学院大学 University of Chinese Academy of Sciences

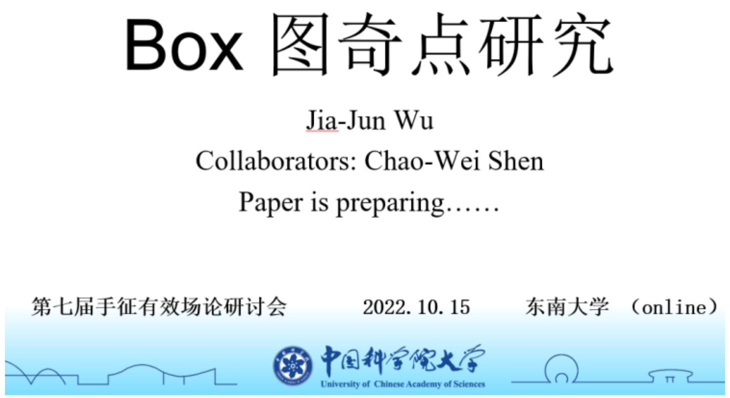
Motivation： From Triangle Singularity



$$\sim \ln\left(\sqrt{S_{CD}} - m_{TS}\right)$$

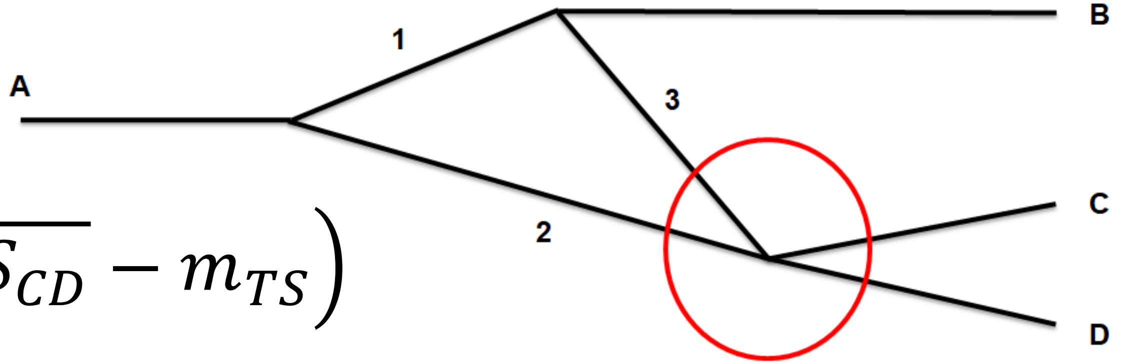


$$\sim 1/(\sqrt{S_{CD}} - m_{BS})$$



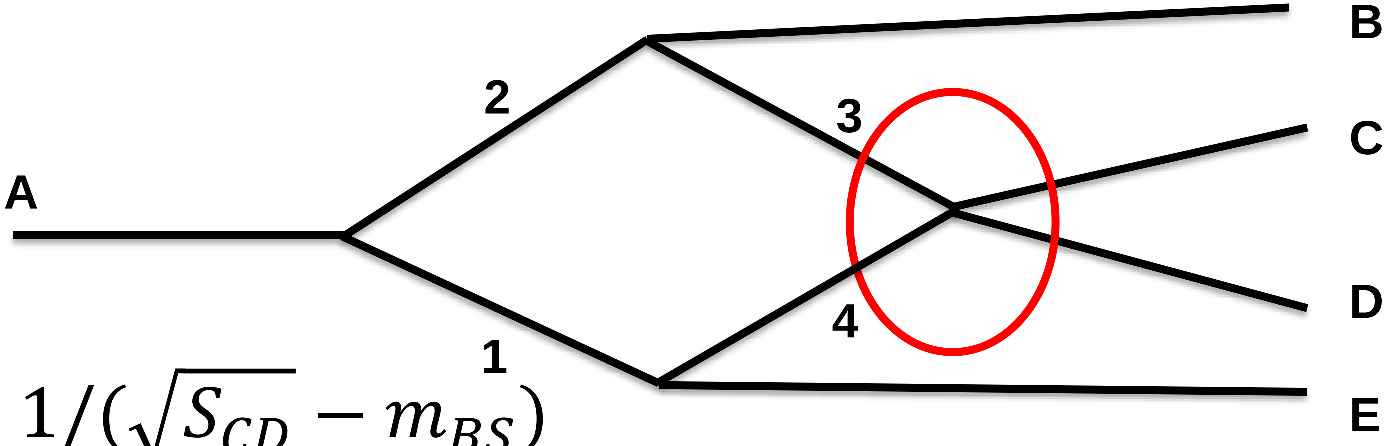
More and More vertexes

Motivation： From Triangle Singularity



$$\sim \ln\left(\sqrt{S_{CD}} - m_{TS}\right)$$

Question:
How to determine the singularity condition of a single loop with $n \geq 3$ intermediate states ?



$$\sim 1/(\sqrt{S_{CD}} - m_{BS})$$

Box 图奇点研究

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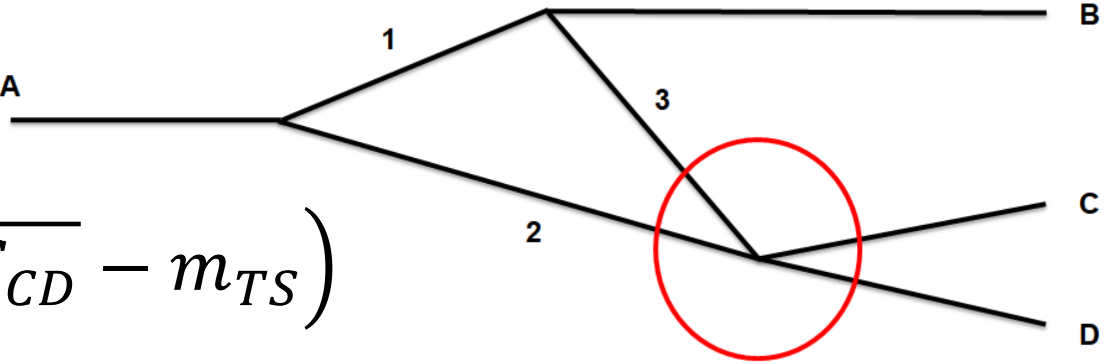
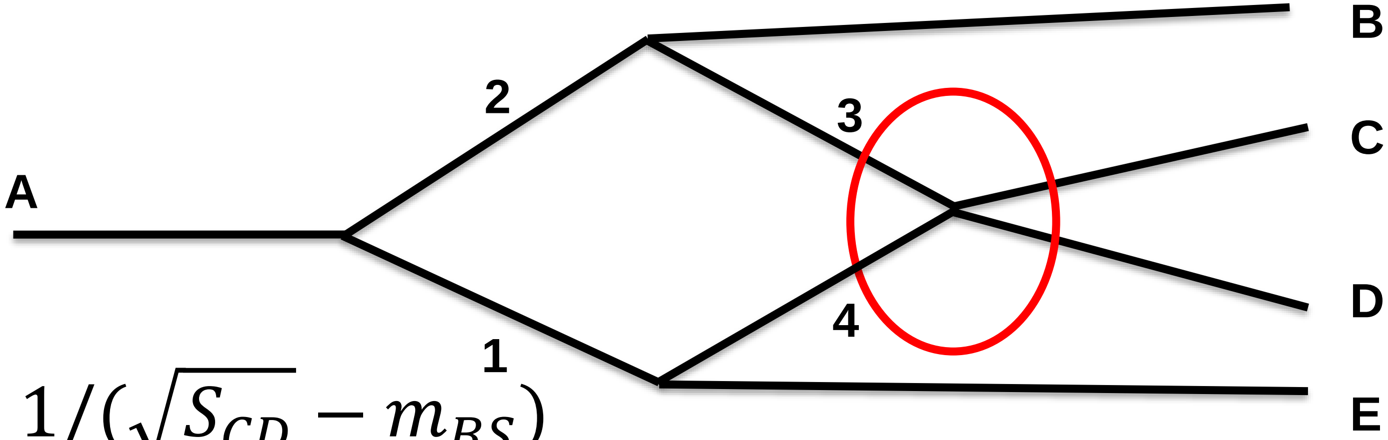


$$\sim \ln\left(\sqrt{s_{CD}} - m_{TS}\right)$$



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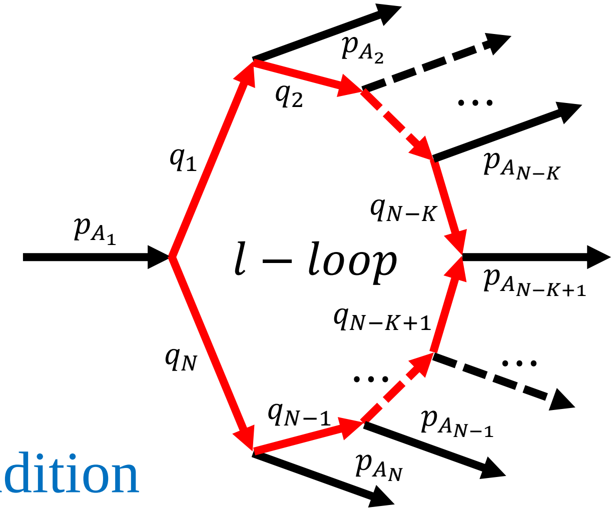
Landau Formulas => Geometric method

- Loop integral $\int \frac{Bd^4kd^4l\cdots}{A_1A_2A_3\cdots}$, Feynman Parameterization

$$\frac{1}{A_1A_2A_3\cdots} = (n-1)! \int_0^1 \cdots \int_0^1 \frac{d\alpha_1 d\alpha_2 \cdots d\alpha_n \delta(\alpha_1+\alpha_2+\cdots+\alpha_n-1)}{(\alpha_1A_1+\alpha_2A_2+\cdots+\alpha_nA_n)^n}$$

- Denomintor function $f = \alpha_1A_1 + \alpha_2A_2 + \cdots$
- The divergence condition:
- (1) $A_i \equiv q_i^2 - m_i^2 = 0 \rightarrow q_i^2 = m_i^2$ （or $\alpha_i = 0$ ） => on shell condition
- (2) $\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \cdots = 0, \rightarrow -\sum_{i=1}^{N-k} \alpha_i q_i + \sum_{j=N-k}^{N} \alpha_j q_j = 0$



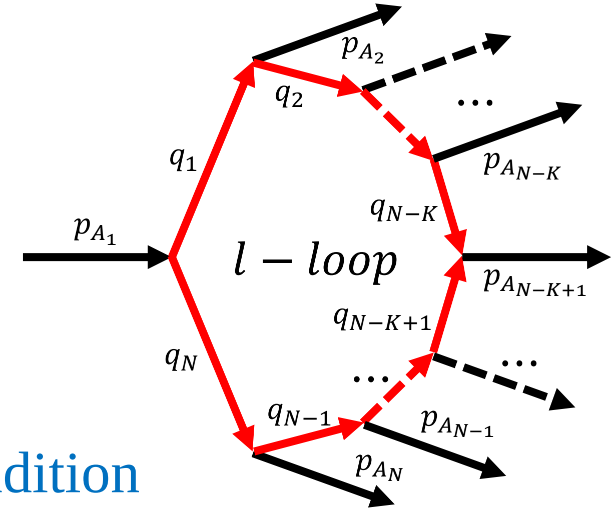
=> Complicated Algebra Expansion

Landau Formulas => Geometric method

- Loop integral $\int \frac{B d^4k d^4l \cdots}{A_1 A_2 A_3 \cdots}$, Feynman Parameterization

$$\frac{1}{A_1 A_2 A_3 \cdots} = (n-1)! \int_0^1 \cdots \int_0^1 \frac{d\alpha_1 d\alpha_2 \cdots d\alpha_n \delta(\alpha_1 + \alpha_2 + \cdots + \alpha_n - 1)}{(\alpha_1 A_1 + \alpha_2 A_2 + \cdots + \alpha_n A_n)^n}$$

- Denomintor function $f = \alpha_1 A_1 + \alpha_2 A_2 + \cdots$
- The divergence condition:
- (1) $A_i \equiv q_i^2 - m_i^2 = 0 \rightarrow q_i^2 = m_i^2$ (or $\alpha_i = 0$) => on shell condition
- (2) $\sum_i \alpha_i \frac{\partial A_i}{\partial k} = \sum_i \alpha_i \frac{\partial A_i}{\partial l} = \cdots = 0, \rightarrow -\sum_{i=1}^{N-k} \alpha_i q_i + \sum_{j=N-k}^{N} \alpha_j q_j = 0$



=> **Complicated Algebra Expansion**

Translate this condition to a geometric condition:

One point should be in a Hypercube which is constructed by outgoing four momenta.

Landau Formulas => Geometric method

- Loop integral $\int \frac{B d^4k d^4l \cdots}{A_1 A_2 A_3 \cdots}$, Feynman Parameterization

$$\frac{1}{A_1 A_2 A_3 \cdots} = (n-1)! \int_0^1 \cdots \int_0^1 \frac{d\alpha_1 d\alpha_2 \cdots d\alpha_n \delta(\alpha_1+\alpha_2+\cdots+\alpha_n-1)}{(\alpha_1 A_1+\alpha_2 A_2+\cdots+\alpha_n A_n)^n}$$

- Denomintor function $f = \alpha_1 A_1 + \alpha_2 A_2 + \cdots = \varphi + K(k,l,\cdots)$
- The divergence condition:
- (1) $A_i \equiv q_i^2 - m_i^2 = 0 \rightarrow q_i^2 = m_i^2$ （or $\alpha_i = 0$ ） => on shell condition
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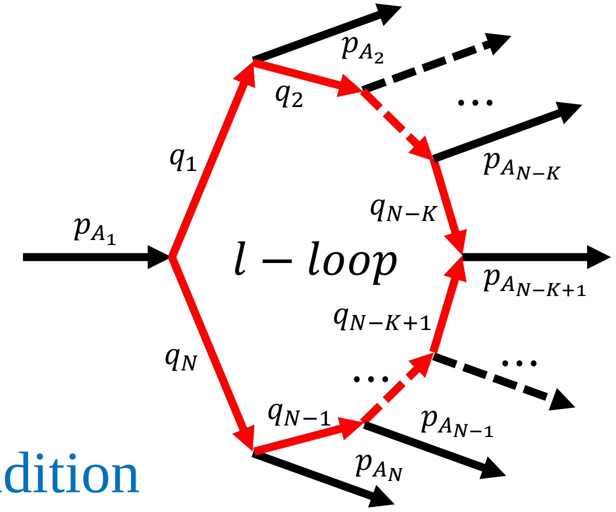


=> Complicated Algebra Expansion

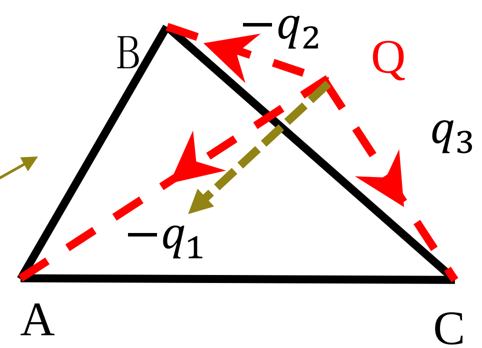
Translate this condition to a geometric condition:

One point should be in a Hypercube which is constructed by outgoing four momenta.

Then it is easy to extract physical condition, even by eyes!

Triangle Singularity as an example

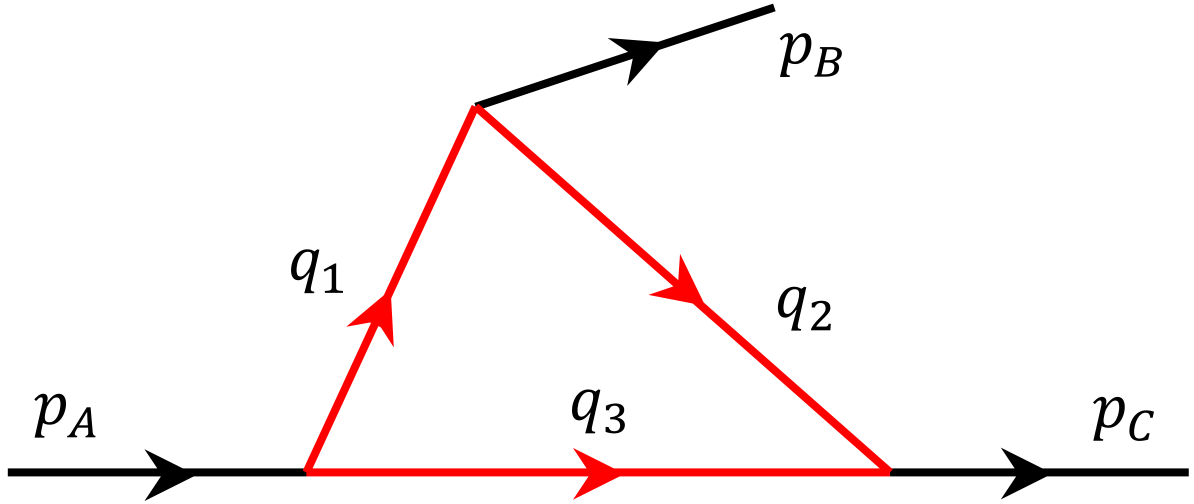
Triangle Singularity

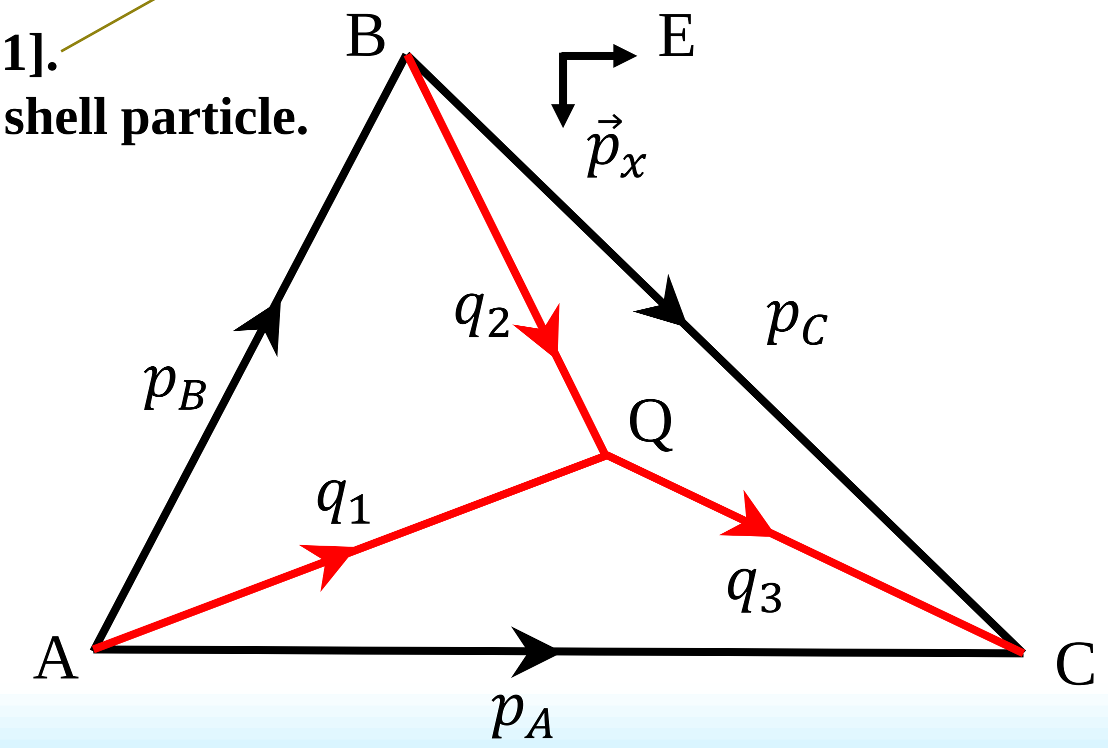


- **Triangle Loop integral** $\int \frac{d^4 q_3}{A_1 A_2 A_3}$**, conditions (1)** $q_i^2 = m_i^2$**, (2)** $\sum \alpha_i q_i = 0$

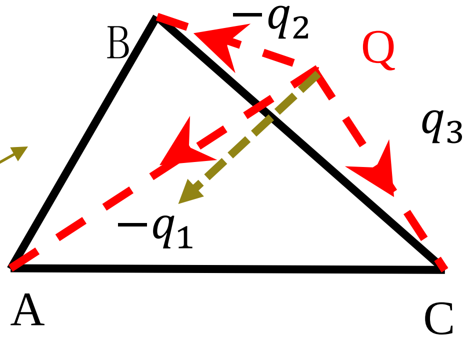
=> $-\alpha_1 q_1 - \alpha_2 q_2 + \alpha_3 q_3 = 0$

Firstly, Q should be on the plain of ABC.
Secondly, Q should be in the triangle since α_i are all [0,1].
Thirdly, Q should be the left of B since E > 0 for the on shell particle.





Triangle Singularity



- **Triangle Loop integral** $\int \frac{d^4 q_3}{A_1 A_2 A_3}$, **conditions (1)** $q_i^2 = m_i^2$, **(2)** $\sum \alpha_i q_i = 0$

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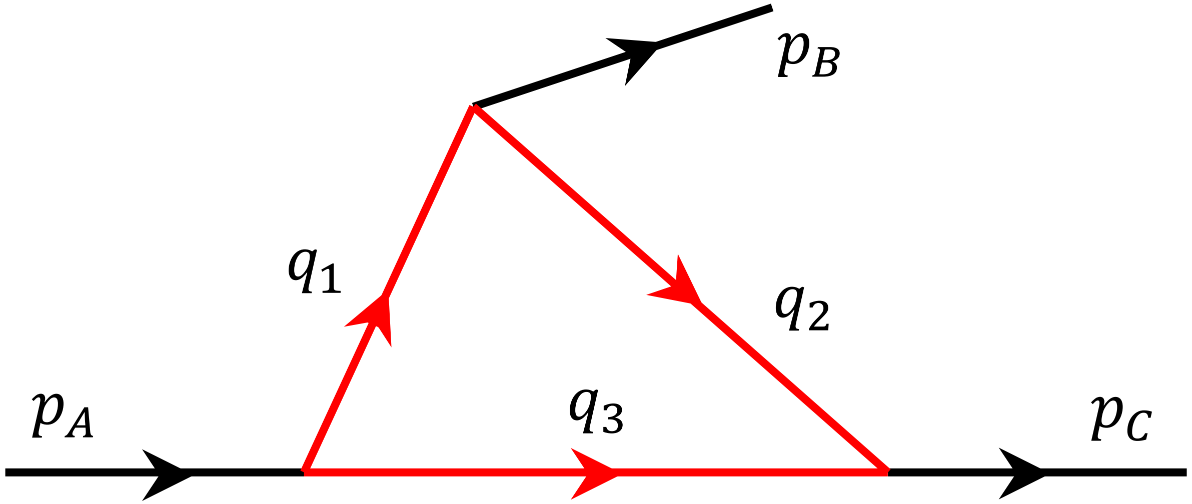
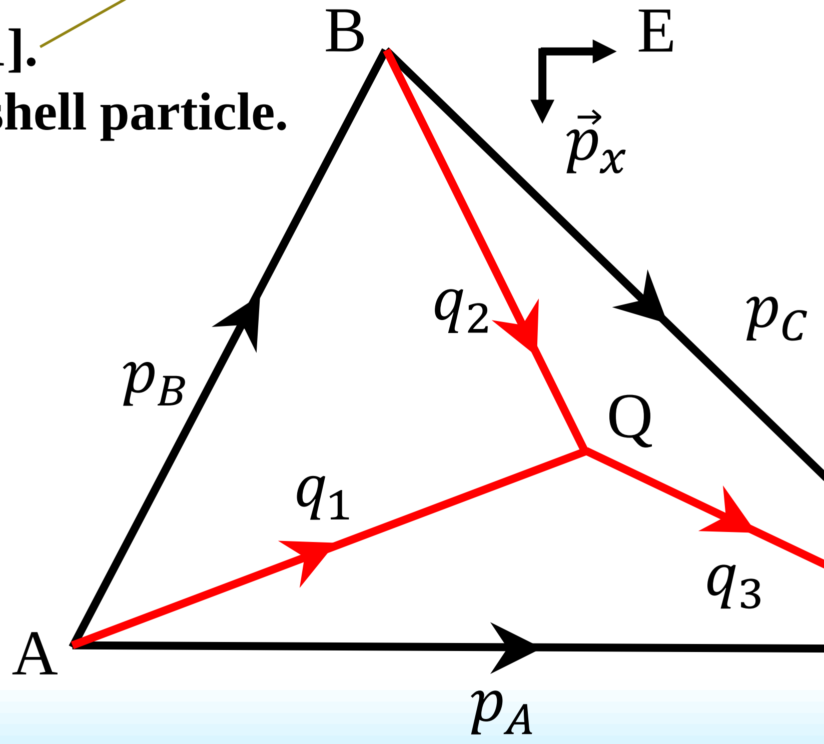
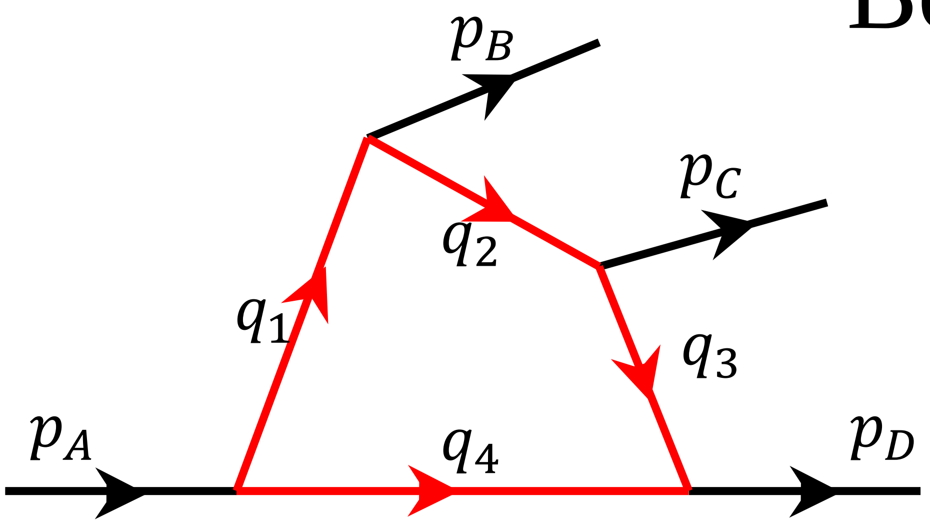
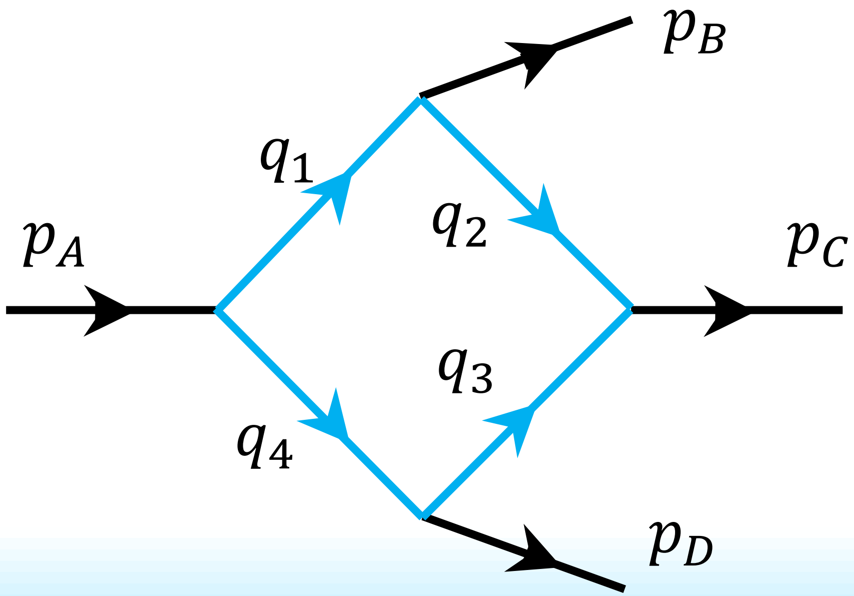


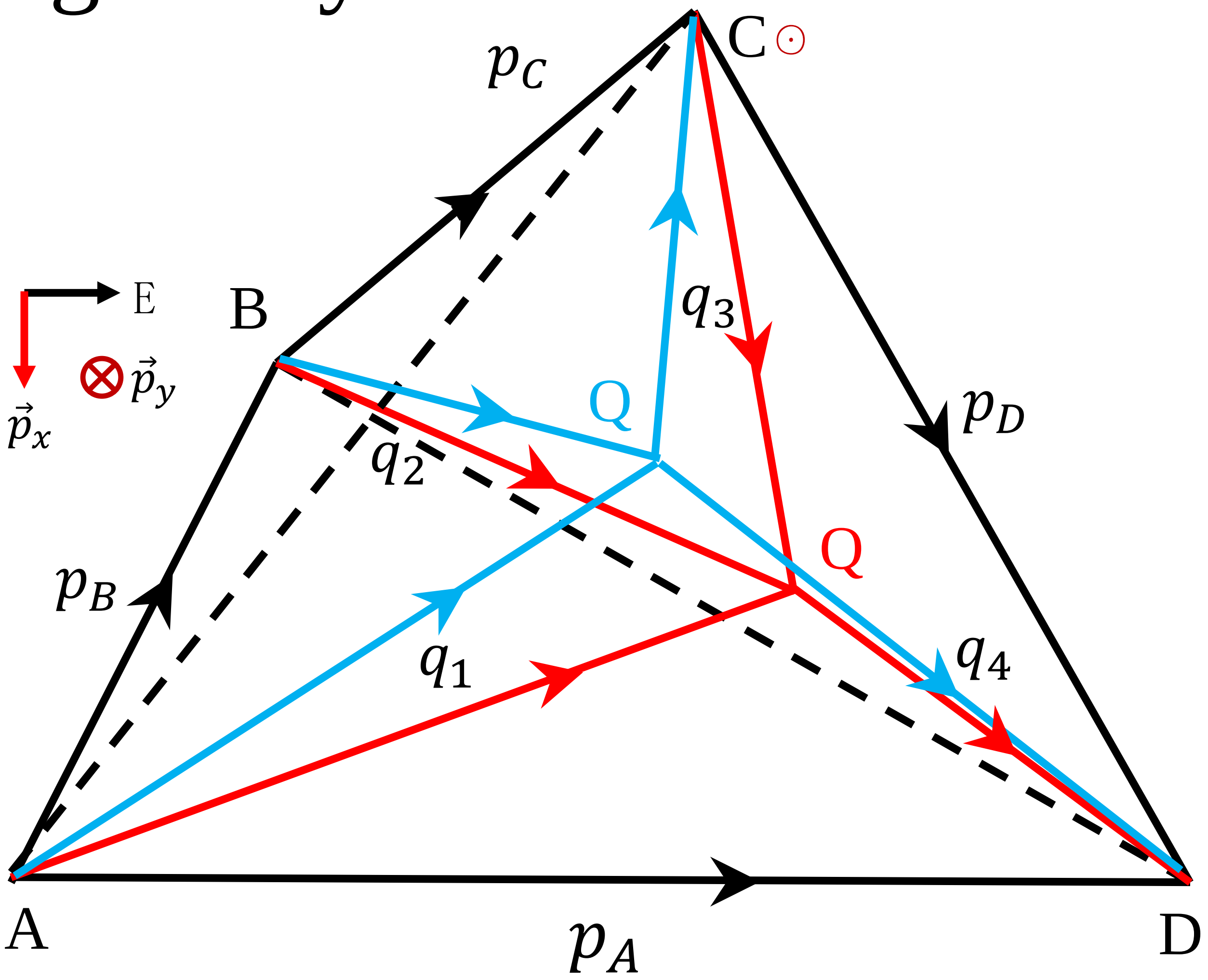


By eyes, the slope of BQ is larger than that of QC, corresponding to $v_2 > v_3$.

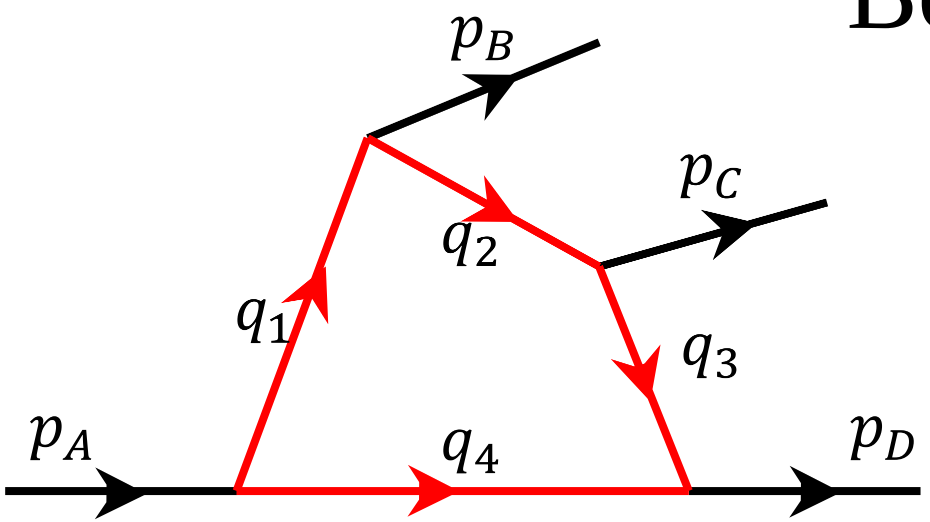
Box Singularity

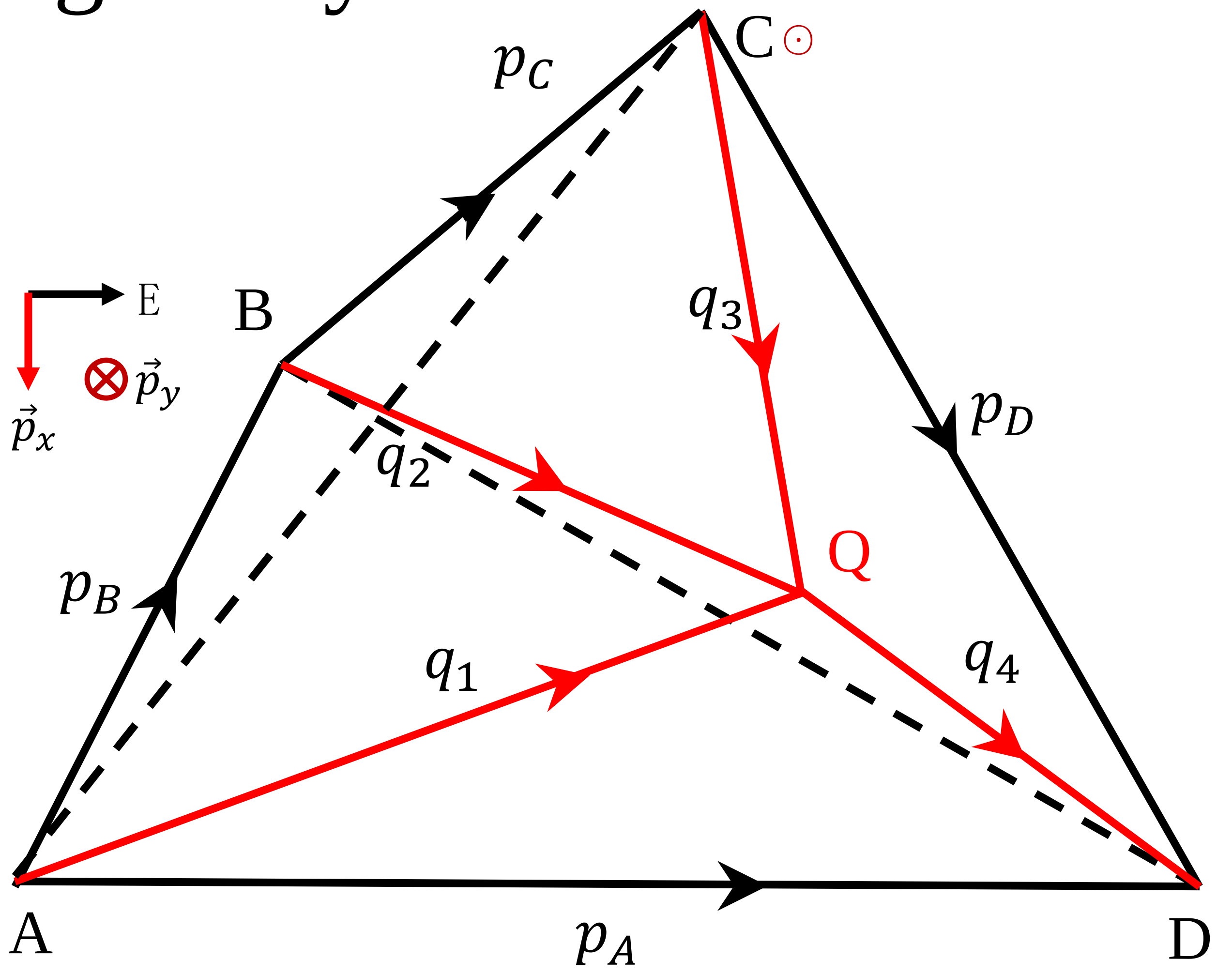






Box Singularity





1. C is just on the plain of ABC
all momenta on the same line.

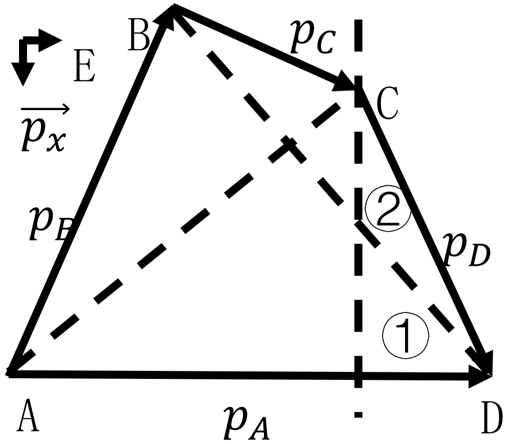
2. C is in the front of the plain of ABC
{E, $\vec{p}_x$, $\vec{p}_y$} three dimensions system

Box Singularity

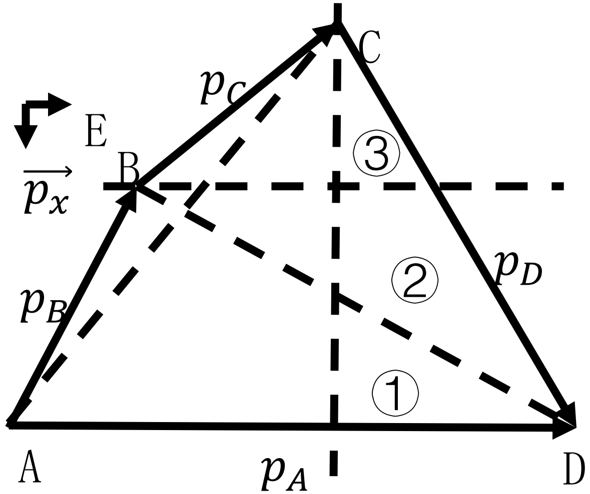
Condition :
$\boldsymbol{max\{v_2, v_3\} > v_4}$

1. C is just on the plain of ABC
all momenta on the same line.

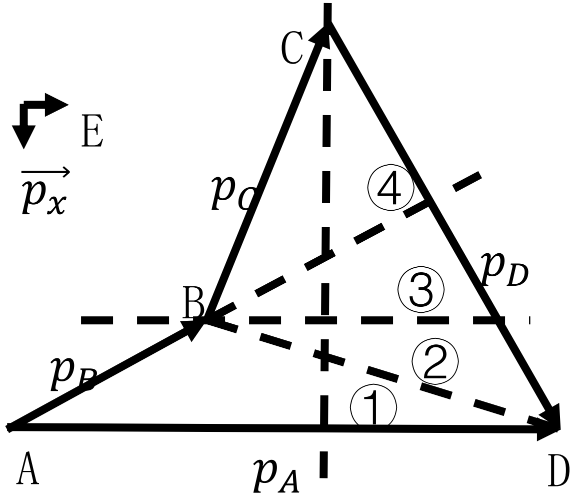
2. C is in the front of the plain of ABC
$\{E, \vec{p}_x, \vec{p}_y\}$ three dimensions system



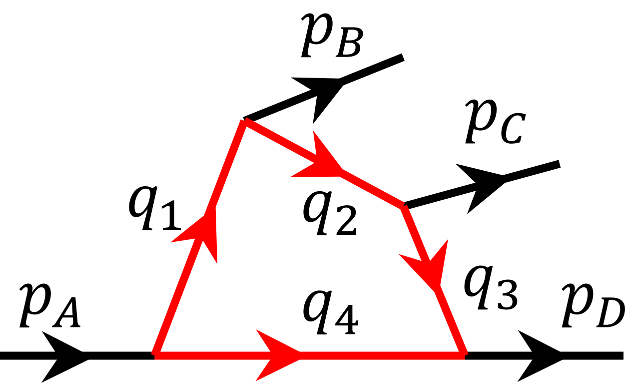
(1) $v_D > v_C > 0 > v_B$

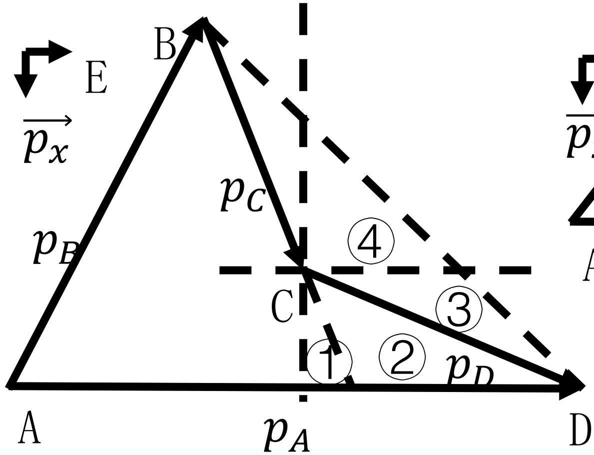


(2) $v_D > 0 > v_C > v_B$

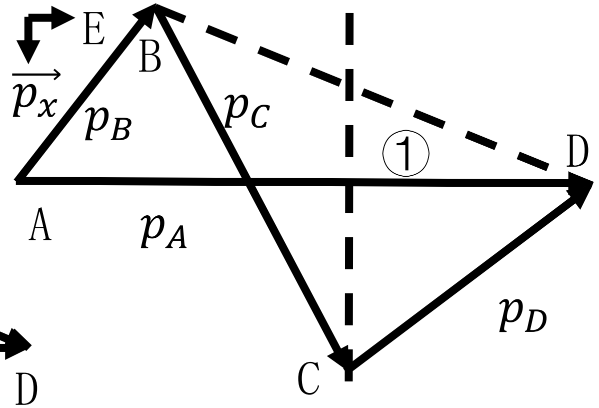


(3) $v_D > 0 > v_B > v_C$

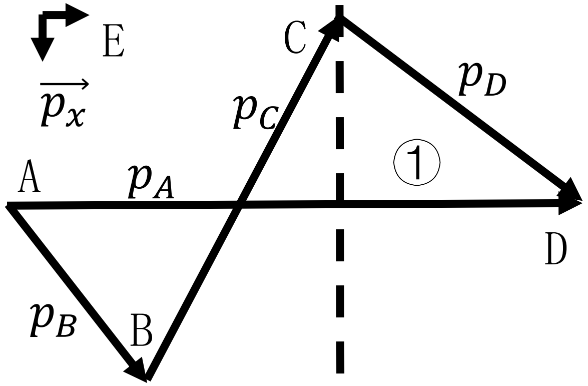




(4) $v_C > v_D > 0 > v_B$



(5) $v_C > 0 > v_D$

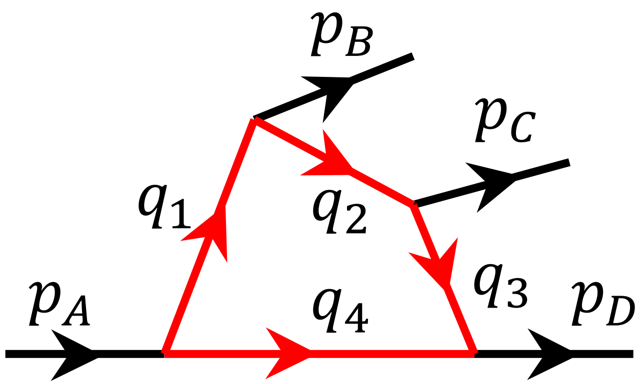


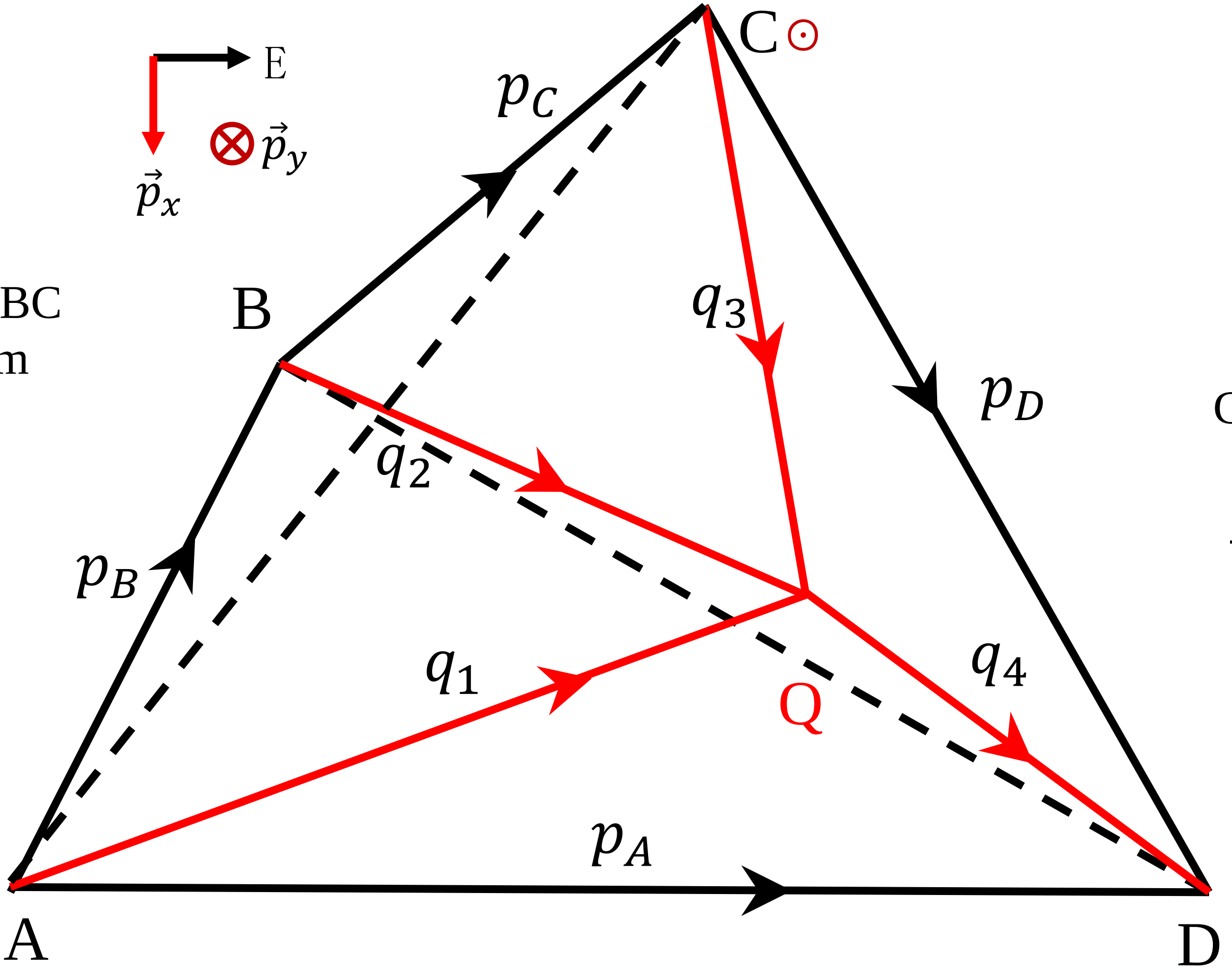
(6) $v_B > 0 > v_C$

Box Singularity

1. C is just on the plain of ABC
all momenta on the same line.

2. C is in the front of the plain of ABC
$\{E, \vec{p}_x, \vec{p}_y\}$ three dimensions system





Condition:

$$v_{3\perp 4} > 0 > v_{2\perp 4}$$

$$\frac{v_4 - v_{2\parallel 4}}{-v_{2\perp 4}} + \frac{v_4 - v_{3\parallel 4}}{v_{3\perp 4}} < 0$$

Box Singularity

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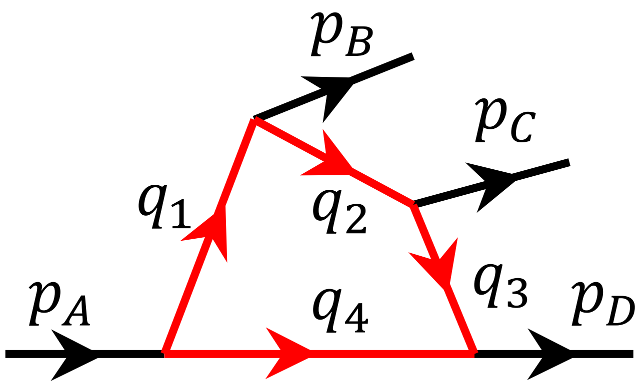

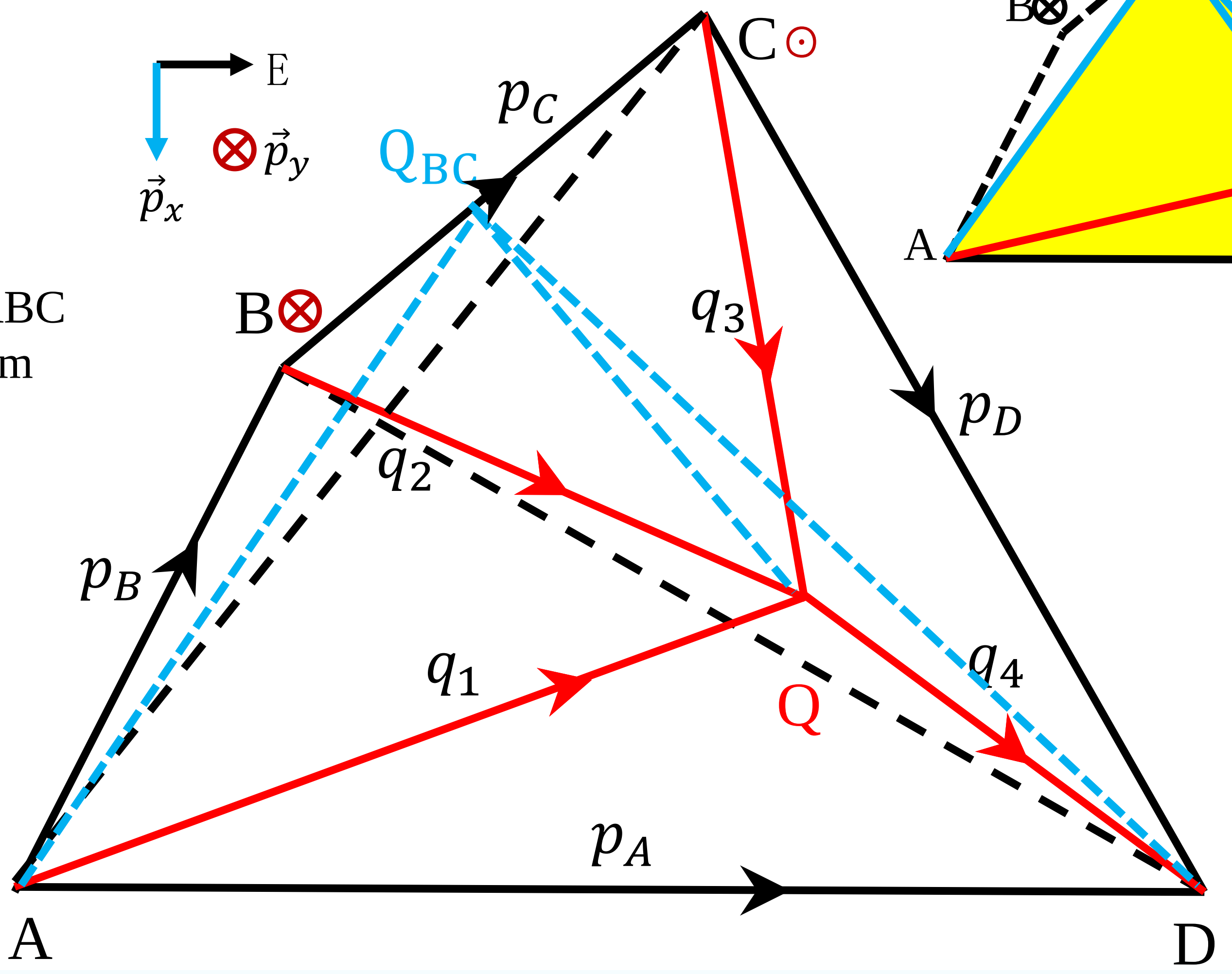

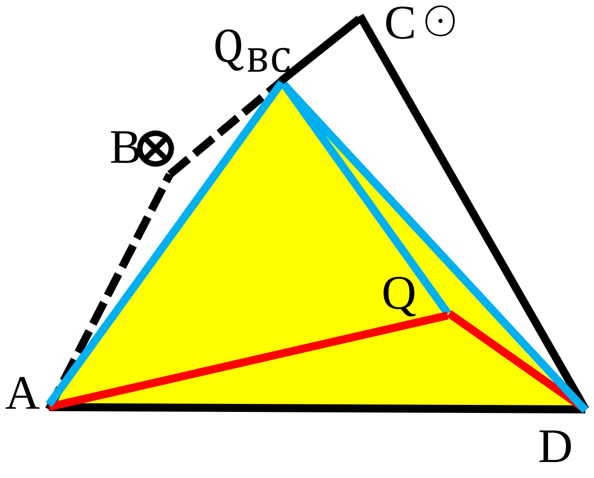

Box Singularity

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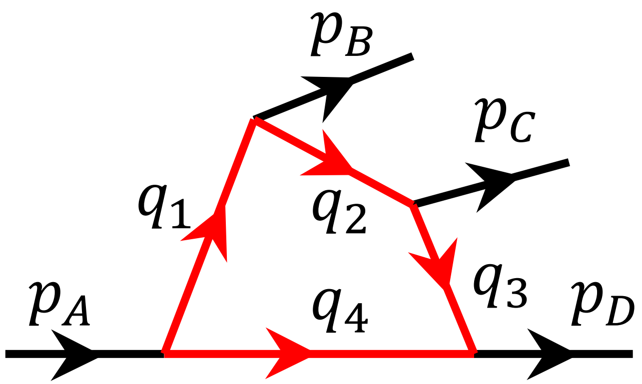
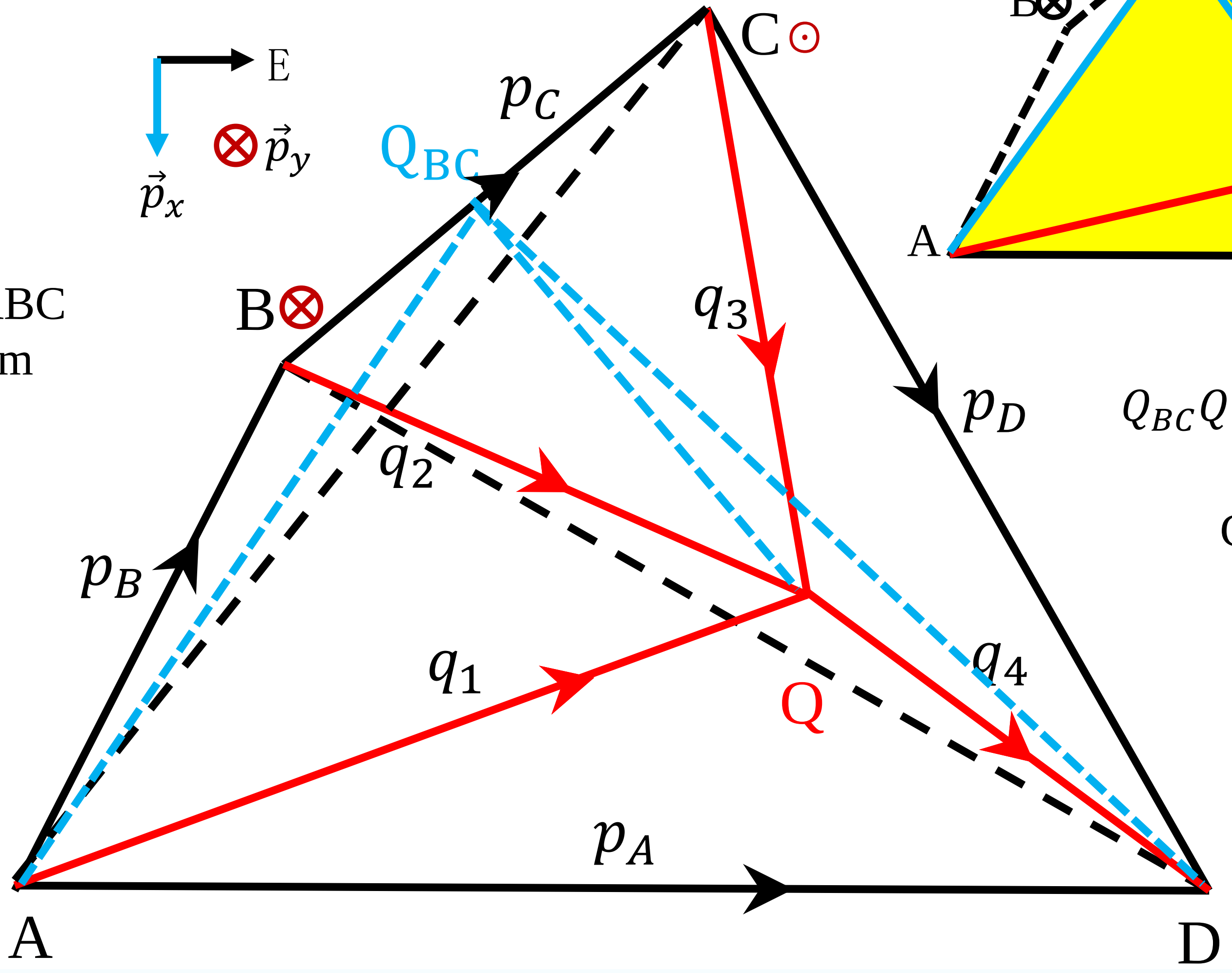
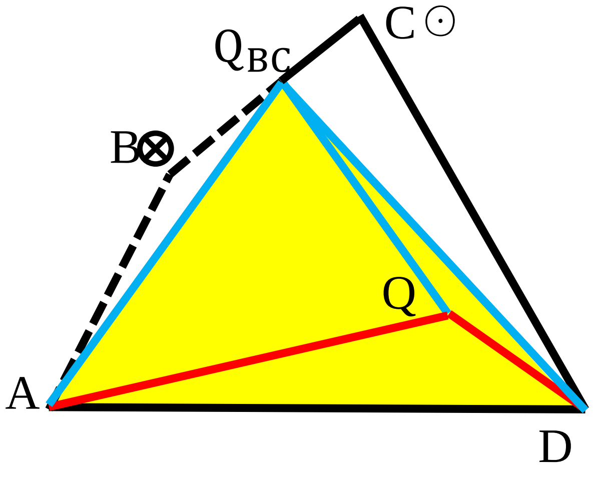






$$Q_{BC}Q:\ \ v_{23} = \frac{v_{2x}v_{3y} - v_{3x}v_{2y}}{v_{3y} - v_{2y}}$$

Condition:

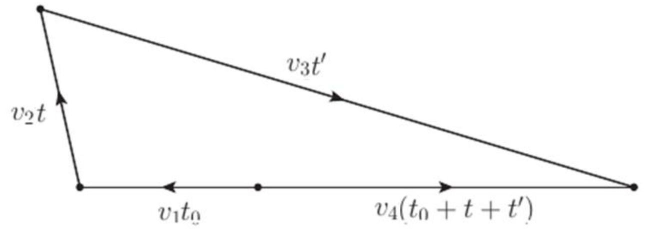
$$v_{3y} > 0 \geq v_{2y}$$

$$v_{23} > v_4$$

$$\frac{v_4 - v_{2x}}{-v_{2y}} + \frac{v_4 - v_{3x}}{v_{3y}} < 0$$

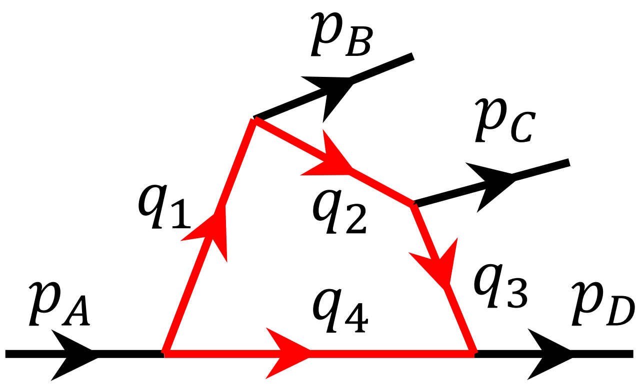
Real Collision between 3 and 4

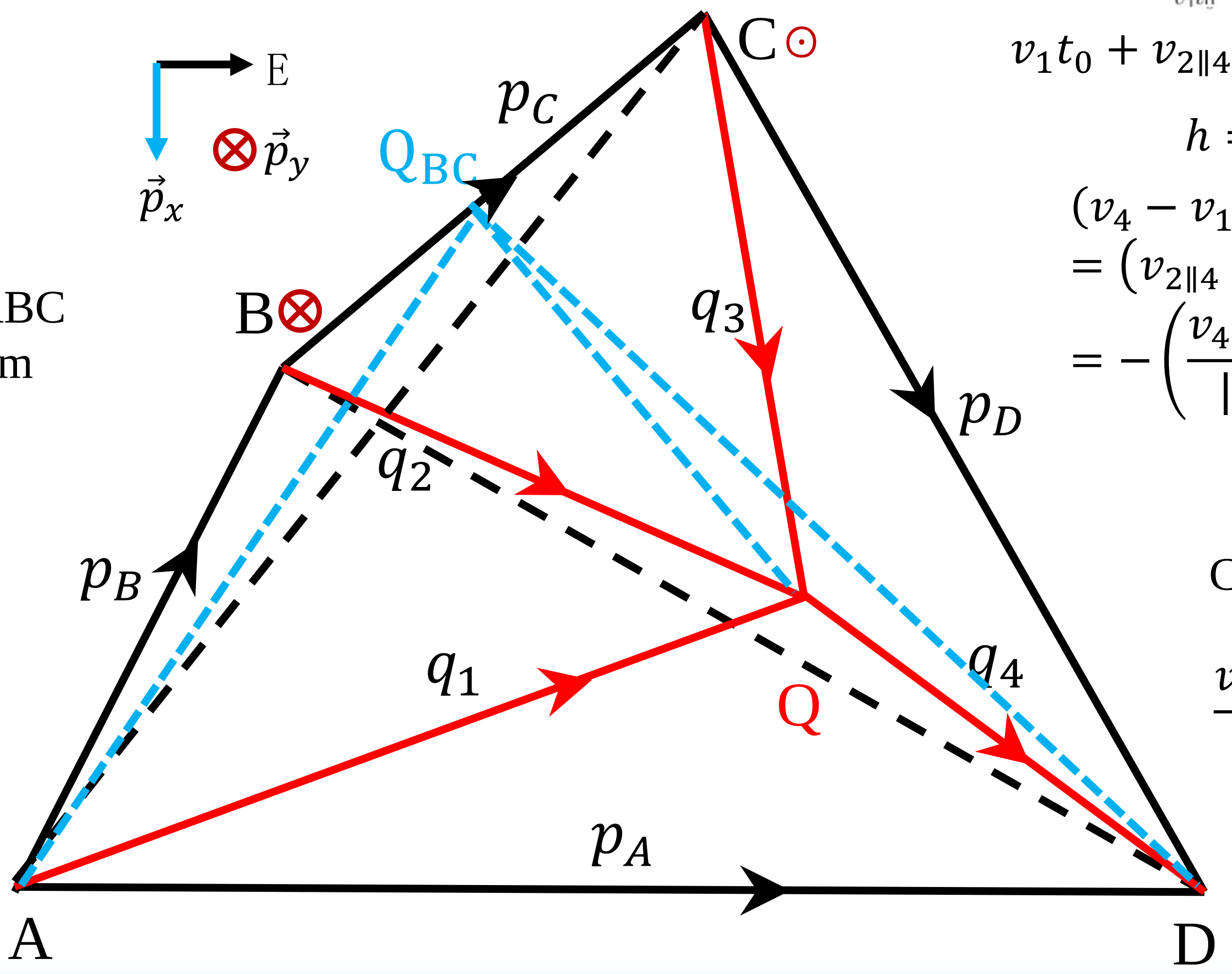
Box Singularity



1. C is just on the plain of ABC
all momenta on the same line.

2. C is in the front of the plain of ABC
$\{E, \vec{p}_x, \vec{p}_y\}$ three dimensions system





$$v_1 t_0 + v_{2\parallel 4} t + v_{3\parallel 4} t' = v_4(t_0 + t + t')$$

$$h = |v_{2\perp 4}| t = |v_{3\perp 4}| t'$$

$$(v_4 - v_1)t_0 = (v_{2\parallel 4} - v_4)t + (v_{3\parallel 4} - v_4)t' = -\left(\frac{v_4 - v_{3\parallel 4}}{|v_{3\perp 4}|} + \frac{v_4 - v_{2\parallel 4}}{|v_{2\perp 4}|}\right) h > 0$$

Condition:

$$v_{3y} > 0 \geq v_{2y}$$

$$\frac{v_4 - v_{2x}}{-v_{2y}} + \frac{v_4 - v_{3x}}{v_{3y}} < 0$$

Real Collision between 3 and 4

Important Step:
3-dimentions → 2-dimentions

Box Singularity

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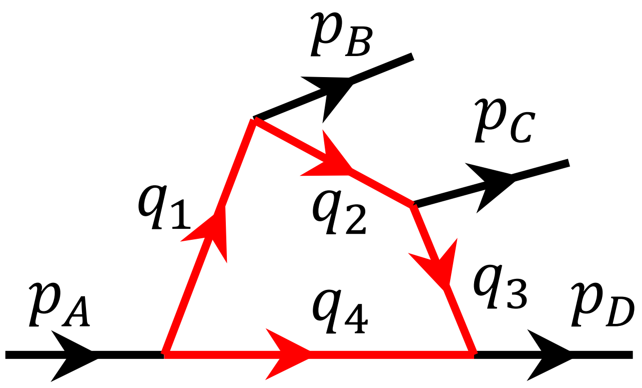
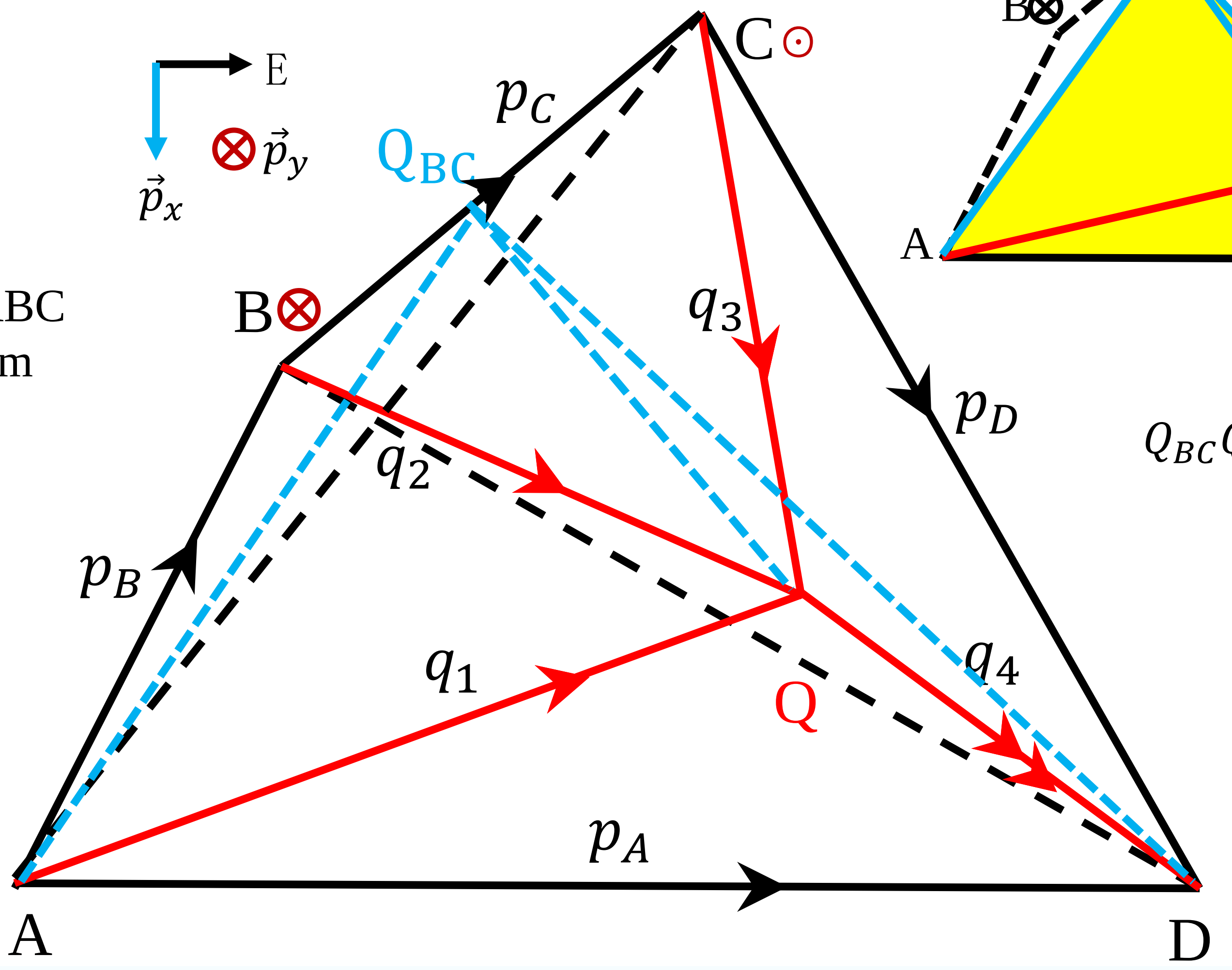
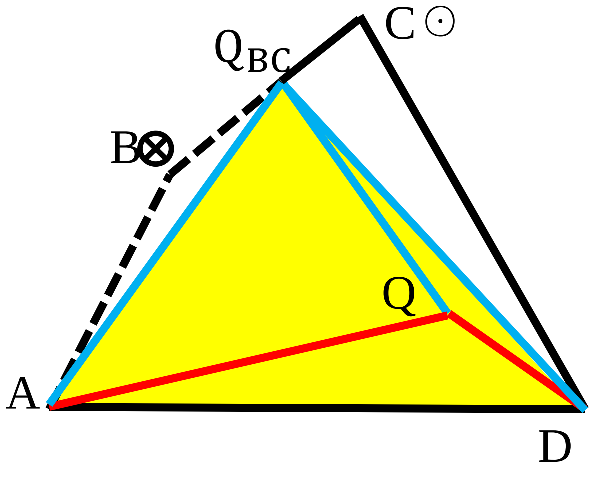






$$Q_{BC}Q:\ v_{23} = \frac{v_{2x}v_{3y} - v_{3x}v_{2y}}{v_{3y} - v_{2y}}$$

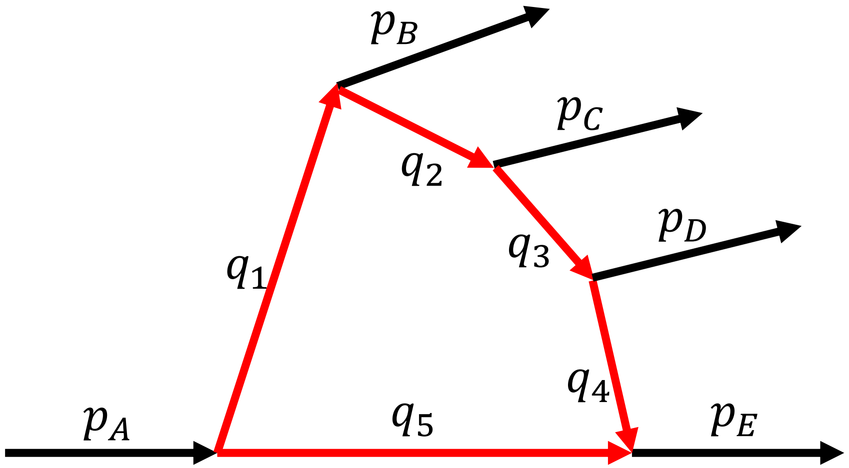
Condition:

$$v_{3y} > 0 \geq v_{2y}$$

$$v_{23} > v_4$$

$$\frac{v_4 - v_{2x}}{-v_{2y}} + \frac{v_4 - v_{3x}}{v_{3y}} < 0$$

Pentagon => Hexagon => N Polygons Singularity



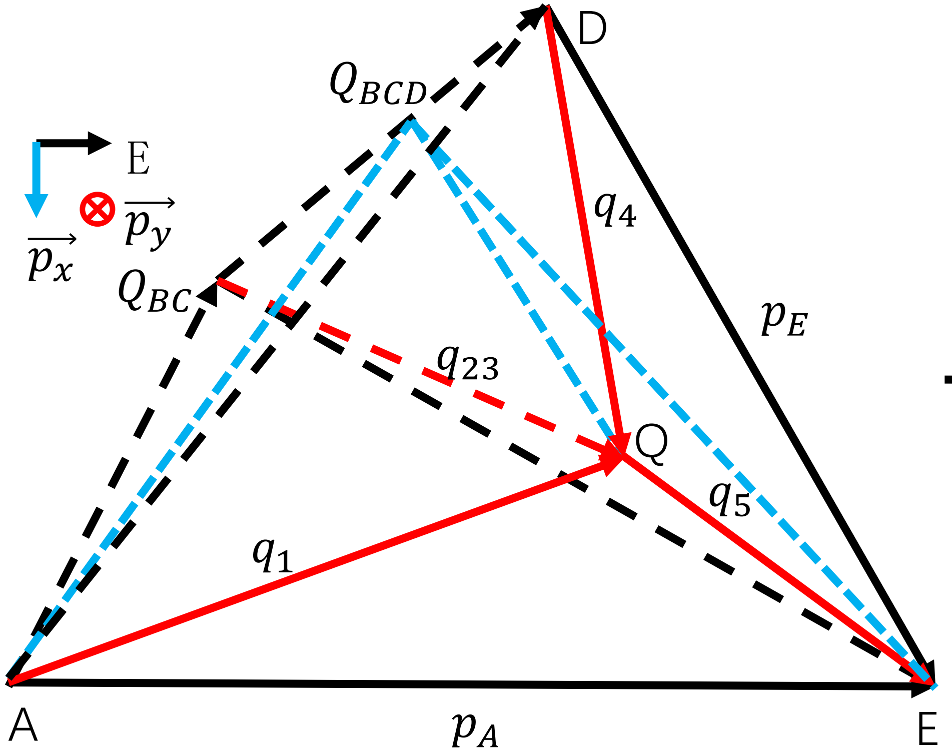
(1) Plain (2) 3 Dimension (3) 4 Dimension

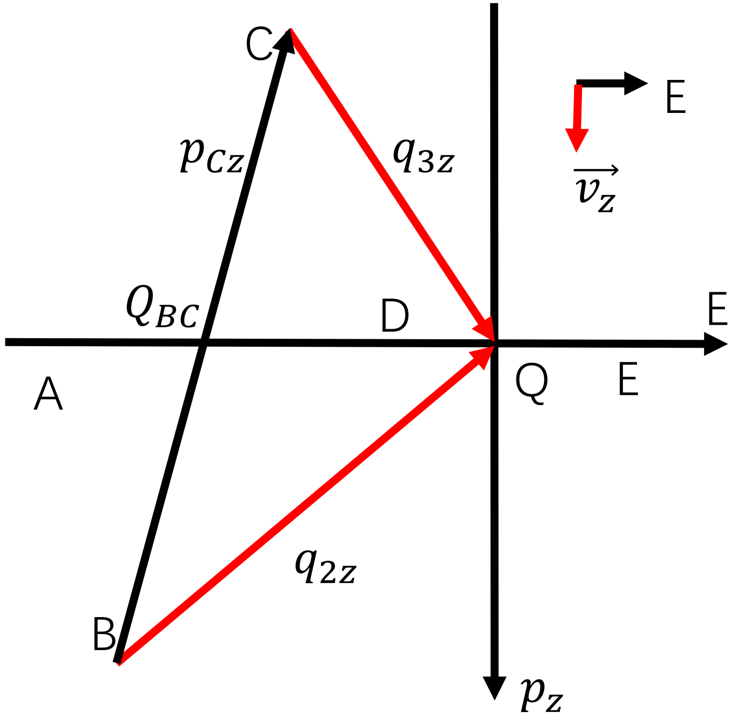
Same as before but more freedom

4 Dimension => How to draw ???

Important Step:

4-dimentions → 3-dimentions → 2-dimentions





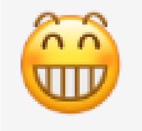
Conditions: $v_{3z} > 0 > v_{2z}$,

$v_{4y} > 0 > v_{23y}$, $\frac{v_{23x}v_{4y} - v_{4x}v_{23y}}{v_{4y} - v_{23y}} > v_5$,

Where $v_{23x} = \frac{v_{2x}v_{3z} - v_{3x}v_{2z}}{v_{3z} - v_{2z}}$, $v_{23y} = \frac{v_{2y}v_{3z} - v_{3y}v_{2z}}{v_{3z} - v_{2z}}$

Pentagon => Hexagon => N Polygons Singularity

5-dimentions → 4-dimentions → 3-dimentions → 2-dimentions

But our world is just in 4-dimentions time space. 

Thus, N>5 is similar N=5, but several free choices for which 5 points to construct a Hypercube to hide Q point as defined before.

$2 \leq a < b < c \leq N-1$,

Satisfy: $v_{b;z} > 0 > v_{a;z}$, $v_{c;y} > 0 > v_{a,b;y}$, $\frac{v_{a,b;x}v_{c;y} - v_{c;x}v_{a,b;y}}{v_{c;y} - v_{a,b;y}} > v_N$,

Where $v_{a,b;x} = \frac{v_{a;x}v_{b;z} - v_{b;x}v_{a;z}}{v_{b;z} - v_{a;z}}$, $v_{a,b;y} = \frac{v_{a;y}v_{b;z} - v_{b;y}v_{a;z}}{v_{b;z} - v_{a;z}}$

Summary

We introduce a geometric method to find the singularity condition of single loop with any intermediate particles.

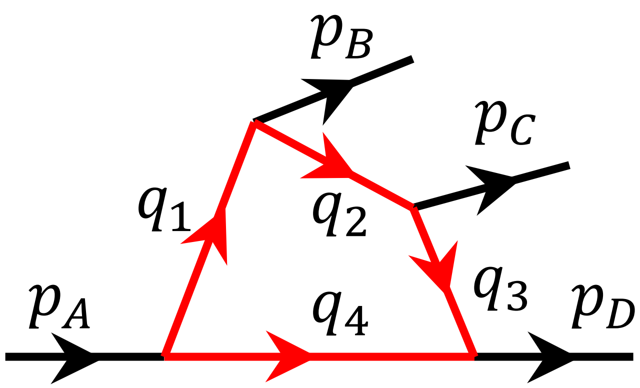
1. One point should be in the a Hypercube which is organized by outgoing four momenta.

2. A geometric method of the dimension reduction.

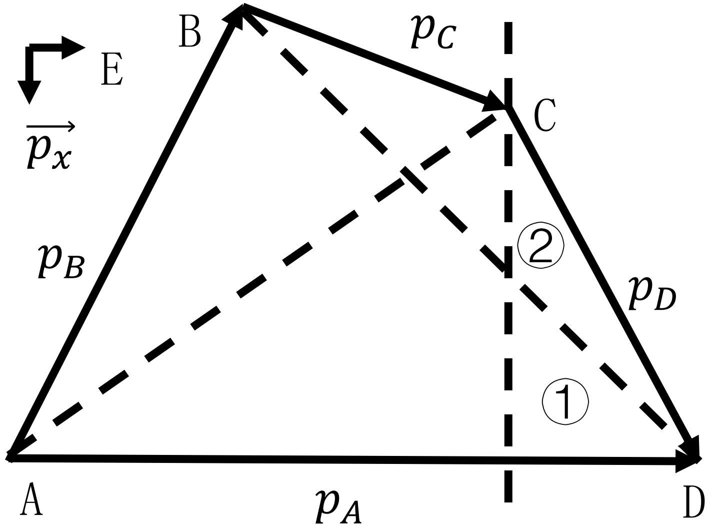
Thanks for attention!

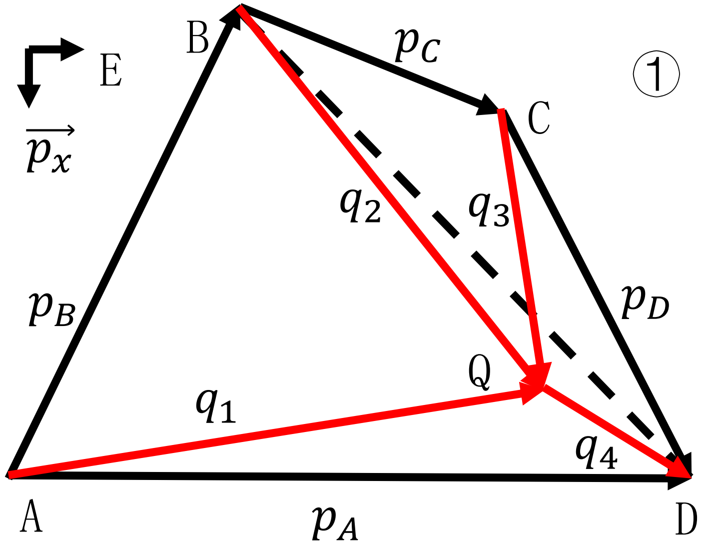
Backup

Box Singularity

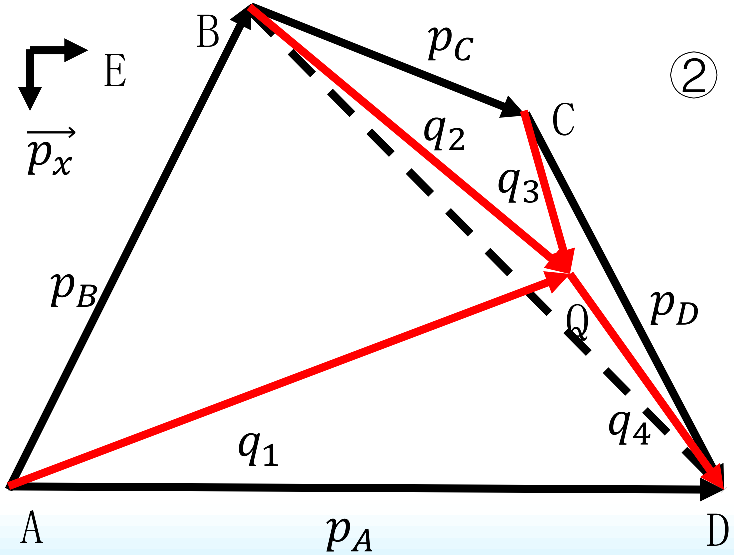


（1） $v_D > v_C > 0 > v_B$， two cases





① $v_3 > v_2 > v_4$



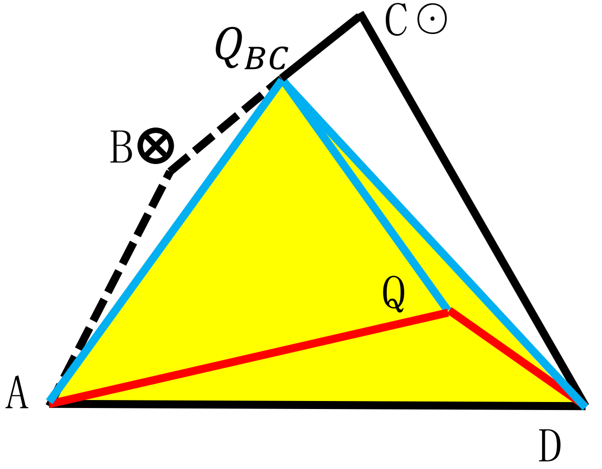
② $v_3 > v_4 > v_2$

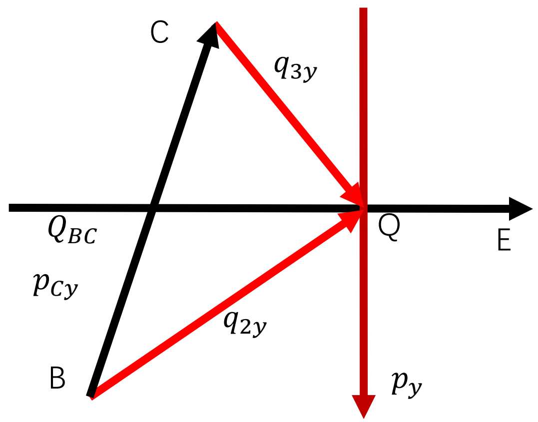
1. C is just on the plain of ABC all momenta on the same line.

2. C is in the front of the plain of ABC $\{E, \vec{p}_x, \vec{p}_y\}$ three dimensions system

Important Step:
3-dimentions drop to
2-dimentions

Box Singularity





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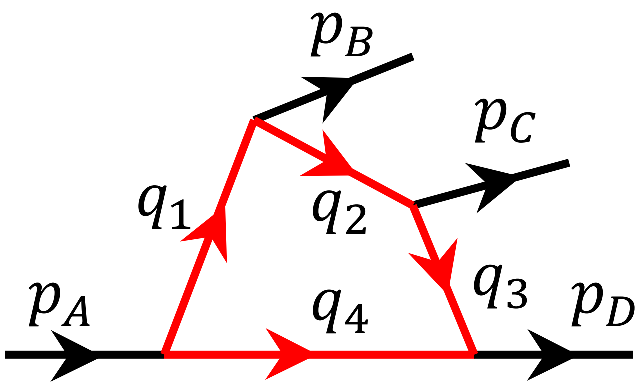


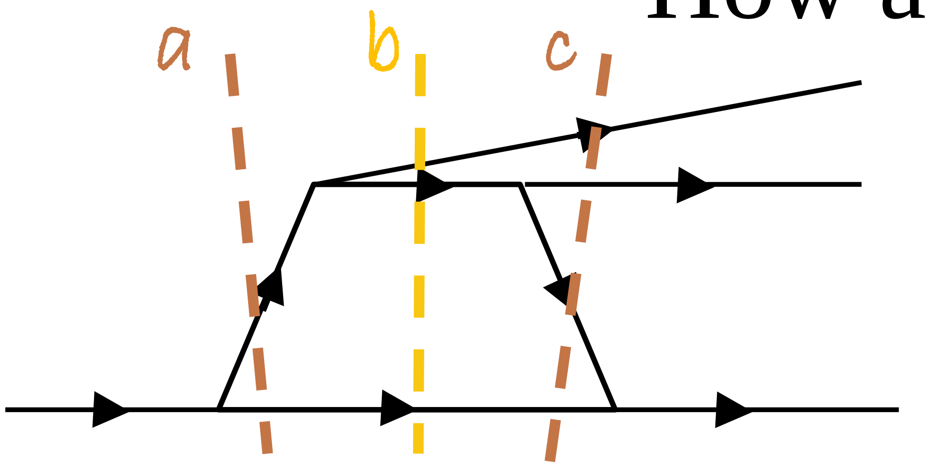
由于 $\frac{Q_{BC}C}{BC} = \frac{q_{3y}}{q_{3y}-q_{2y}} = \frac{v_{3y}E_3}{v_{3y}E_3-v_{2y}E_2}$，则

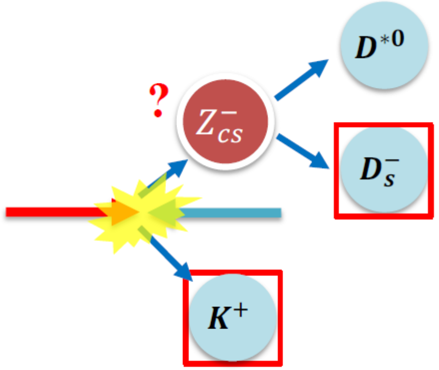
$$\begin{cases} E_{23} = E_3 + \frac{v_{3y}E_3}{v_{3y}E_3-v_{2y}E_2}(E_2-E_3) = \frac{v_{3y}-v_{2y}}{v_{3y}E_3-v_{2y}E_2}E_2E_3 \\ q_{23} = v_{3x}E_3 + \frac{v_{3y}E_3}{v_{3y}E_3-v_{2y}E_2}(v_{2x}E_2-v_{3x}E_3) = \frac{v_{2x}v_{3y}-v_{3x}v_{2y}}{v_{3y}E_3-v_{2y}E_2}E_2E_3 \end{cases}$$

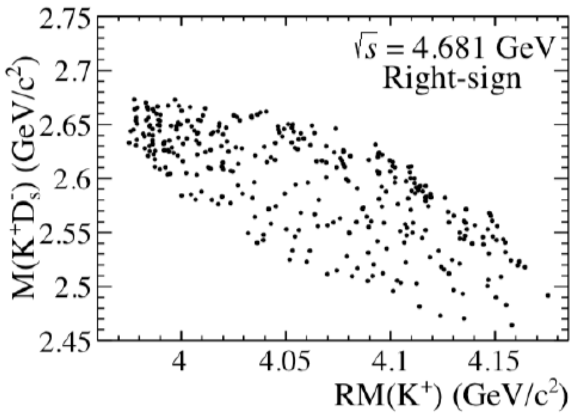
于是

$$v_{23} = \frac{q_{23}}{E_{23}} = \frac{v_{2x}v_{3y}-v_{3x}v_{2y}}{v_{3y}-v_{2y}}$$

How about Box Singularity







From
Peirong’ s Talk