An introduction to
high energy nuclear collisions

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QCD与中高能暑期学校, 山东大学-青岛, 2023/07/10

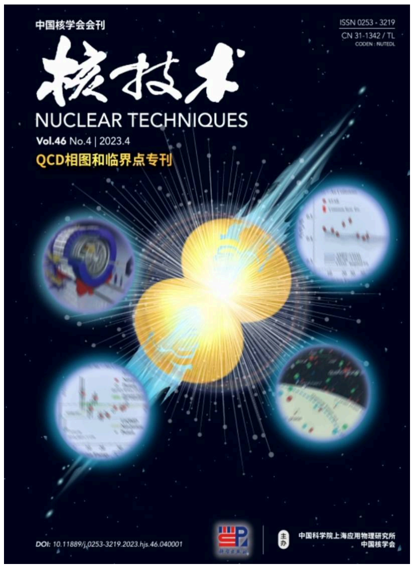
Outline

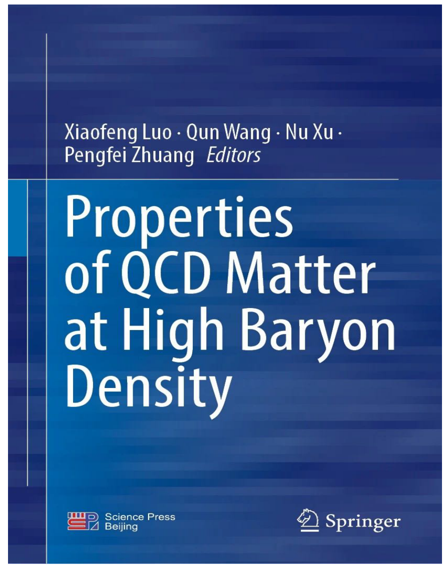
- **Natural Unit system**
- **Kinematics and bulk properties of nuclear collisions**
- **Some aspects of high energy nuclear collisions**
- **Main reference:**

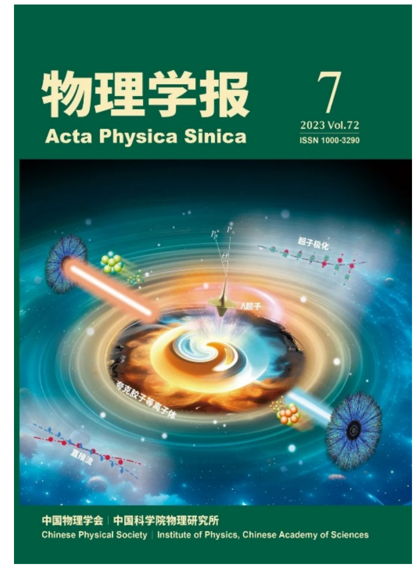
夸克胶子等离子体(从大爆炸到小爆炸),
八木浩辅、初田哲男、三明康郎 著
王群、马余刚、庄鹏飞 译
中国科学技术大学出版社 (2022年第2次印刷)

第3部分(第10-16章)

Outline







Other references:

(1) Properties of QCD Matter at High Baryon Density, X.F. Luo, Q. Wang, N. Xu, P.F. Zhuang (ed.), Science Press 2022

(2) QCD相图和临界点(专刊), 陈列文, 黄梅, 刘玉鑫, 罗晓峰, 马余刚 (编辑), Nucl.Tech. 46, 4 (2023)

(3) 高能重离子碰撞过程的自旋与手征效应(专刊), 梁作堂, 王群, 马余刚 (编辑), 物理学报, 72卷第7期 (2023)

Natural unit system

- SI unit in daily life: [length, mass, time] → [m, kg, s]
- To describe small particle with high speed conveniently we need the natural unit. For example, a nucleus, its mass about 10^{-27}-10^{-25} kg, size about 10^{-15} m, nucleon speed inside the nucleus: in the magnitude of $c = 3\times10^{8}$ m/s. In these cases, it is not convenient to use SI unit.
- cgs unit: [length, mass, time] → [cm, g, s] → $\mathrm{cm}^{a}\mathrm{g}^{b}\mathrm{s}^{c}\mathrm{K}^{d}$
- The dynamics of microscopic particles is controlled by quantum mechanics: (reduced) Planck constant $\hbar$
- Natural unit system:

  [angular momentum, velocity, energy] → $\hbar^{\alpha}c^{\beta}\mathrm{eV}^{\gamma}k_B^{\delta}$

From cgs-Gaussian to natural unit

- **natural unit → cgs unit**

$$
\begin{aligned}
1\,c &= 3 \times 10^{10}\,\mathrm{cm} \cdot \mathrm{s}^{-1} \\
1\,\hbar &= 1.05 \times 10^{-27}\,\mathrm{g} \cdot \mathrm{cm}^2 \cdot \mathrm{s}^{-1} \\
1\,\mathrm{eV} &= 1.6 \times 10^{-12}\,\mathrm{g} \cdot \mathrm{cm}^2 \cdot \mathrm{s}^{-2} \\
1\,k_\mathrm{B} &= 1.3806488 \times 10^{-16}\mathrm{g} \cdot \mathrm{cm}^2 \cdot \mathrm{s}^{-2} \cdot \mathrm{K}^{-1}
\end{aligned}
$$

- **cgs unit → natural unit**

$$
\begin{aligned}
1\,\mathrm{s} &= 1.52 \times 10^{15}\,\hbar \cdot \mathrm{eV}^{-1} \\
1\,\mathrm{cm} &= 5.06 \times 10^{4}\,\hbar \cdot \mathrm{eV}^{-1} \cdot c \\
1\,\mathrm{g} &= 5.6 \times 10^{32}\,\mathrm{eV} \cdot c^{-2} \\
1\,\mathrm{K} &= 8.617 \times 10^{-5}\,\mathrm{eV} \cdot k_\mathrm{B}^{-1}
\end{aligned}
$$

Natural unit convention

- **natural unit convention:** $\hbar = c = k_B = 1$**, so any quantity has unit** $\mathbf{eV}^{\gamma}$

$$
\begin{aligned}
[\mathrm{time}] &= \mathrm{eV}^{-1} \\
[\mathrm{length}] &= \mathrm{eV}^{-1} \\
[\mathrm{mass}] &= \mathrm{eV} \\
[\mathrm{temperature}] &= \mathrm{eV}
\end{aligned}
$$

- **Another example, if we say a particle move at speed 0.5, it actually means its speed is 0.5c.**

Maxwell's equations in cgs-Gaussian

- Unrationalized Gaussian units in electromagnetism: the factor 4π appears in the Maxwell's equation and it is absent in the Coulomb's force law. Maxwell's equations in cgs unrationalized Gaussian unit read

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \underline{4\pi}\rho \\
\nabla \times \mathbf{B} &= \frac{\underline{4\pi}}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}
\end{aligned}
$$

E and B have the same unit: Gauss (Gs)

Maxwell's equations in cgs-Gaussian

- **The inverse-square force laws (Coulomb's law and Biot-Savart's Law)**

$$\mathbf{F} = \frac{q_1 q_2}{r^3}\mathbf{r}$$

$$\mathbf{F} = \frac{1}{c^2}\int\int \frac{I_1 d\mathbf{l}_1 \times (I_2 d\mathbf{l}_2 \times \mathbf{r})}{r^3}$$

There is no 4π in force laws

- **Rationalized Gaussian (Lorentz-Heaviside) unit is related to unrationalized Gaussian by**

$$\mathbf{E}_{\mathrm{LH}} = \frac{1}{\sqrt{4\pi}}\mathbf{E}_{\mathrm{unrat-Gauss}}$$

$$q_{\mathrm{LH}} = \sqrt{4\pi}\, q_{\mathrm{unrat-Gauss}}$$

Electric charge in cgs-Gaussian units

- In the cgs-Gaussian units, the charge is in the electrostatic unit (esu) which can be determined from the Coulomb's law

$$F = \frac{q^2}{r^2} \to \mathrm{esu}^2 = \mathrm{g}\cdot\mathrm{cm}\cdot\mathrm{s}^{-2}\times\mathrm{cm}^2 = \mathrm{g}\cdot\mathrm{cm}^3\cdot\mathrm{s}^{-2}$$
$$\to \quad \mathrm{esu} = \mathrm{g}^{1/2}\cdot\mathrm{cm}^{3/2}\cdot\mathrm{s}^{-1}$$

- We know that the Coulomb's force law in the SI unit system has the following form

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2}$$

Vacuum electric permittivity is

$$\epsilon_0 = 8.8542\times 10^{-12}\ \mathrm{C}^2\mathrm{N}^{-1}\mathrm{m}^{-2}$$
$$\frac{1}{4\pi\epsilon_0} = 8.99\times 10^{9}\ \mathrm{C}^{-2}\mathrm{N}^{1}\mathrm{m}^{2}$$
$$1\,\mathrm{C} = 3\times 10^{9}\ \mathrm{esu}$$
$$1\,\mathrm{e} = 1.602\times 10^{-19}\ \mathrm{C} = 4.8\times 10^{-10}\ \mathrm{esu}$$

EM field in cgs-Gaussian

- **In the SI system, the unit of the electric field is Volt/m=N/C, while in the unrationalized Gaussian units, the electric and magnetic fields have the same unit: Gauss (G). So we have**

$$
\begin{aligned}
1\,\mathrm{G} &= \frac{\mathrm{dyn}}{\mathrm{esu}} = \mathrm{g}^{1/2}\cdot\mathrm{cm}^{-1/2}\cdot\mathrm{s}^{-1} \\
&= 6.92\times 10^{-2}\,(\hbar c)^{-3/2}\cdot\mathrm{eV}^2 \\
1\,\mathrm{Volt} &= 1\,\mathrm{N}\cdot\mathrm{m/C} = \frac{10^7\,\boxed{\mathrm{dyn}\cdot\mathrm{cm}}}{3\times 10^9\,\boxed{\mathrm{esu}}} \\
&= \frac{1}{300}\mathrm{g}^{1/2}\cdot\mathrm{cm}^{1/2}\cdot\mathrm{s}^{-1} = \frac{1}{300}\boxed{\mathrm{statVolt}} \\
\underline{1\,\mathrm{erg}} &\underline{= 1\,\mathrm{statVolt}\cdot\mathrm{esu}}
\end{aligned}
$$

$$1\,\mathrm{eV} = 1.6\times 10^{-12}\,\mathrm{g}\cdot\mathrm{cm}^2\cdot\mathrm{s}^{-2}$$

How to recover exact unit from natural unit

- **In natural unit, a physical quantity has the unit: $\mathbf{eV}^{\gamma}$, under the convention $\hbar = c = k_B = 1$, how to find its exact form? We assume its exact form is $\hbar^{\alpha} c^{\beta} \mathrm{eV}^{\gamma} k_B^{\delta}$ where α, β, δ are to be determined, we can write its cgs form**

$$\begin{aligned} \hbar^{\alpha} c^{\beta} \mathrm{eV}^{\gamma} k_{\mathrm{B}}^{\delta} &= \mathrm{cm}^{a} \mathrm{g}^{b} \mathrm{s}^{c} \mathrm{K}^{d} \\ &= \mathrm{cm}^{2\alpha+\beta+2\gamma+2\delta} \mathrm{g}^{\alpha+\gamma+\delta} \mathrm{s}^{-\alpha-\beta-2\gamma-2\delta} \mathrm{K}^{-\delta} \end{aligned}$$

- **Once a, b, c, d in cgs unit are determined from physical relations involving this quantity, we can determine**

$$\begin{aligned} \alpha &= a + c \\ \beta &= a - 2b \\ \delta &= -d \end{aligned}$$

From cgs-Gaussian to natural unit

- **So the quantity with the dimension**

$$[\boldsymbol{D}] = \mathbf{cm}^{a}\mathbf{g}^{b}\mathbf{s}^{c}\mathbf{K}^{d} = \hbar^{\alpha}c^{\beta}\mathbf{eV}^{\gamma}k_B^{\delta}$$

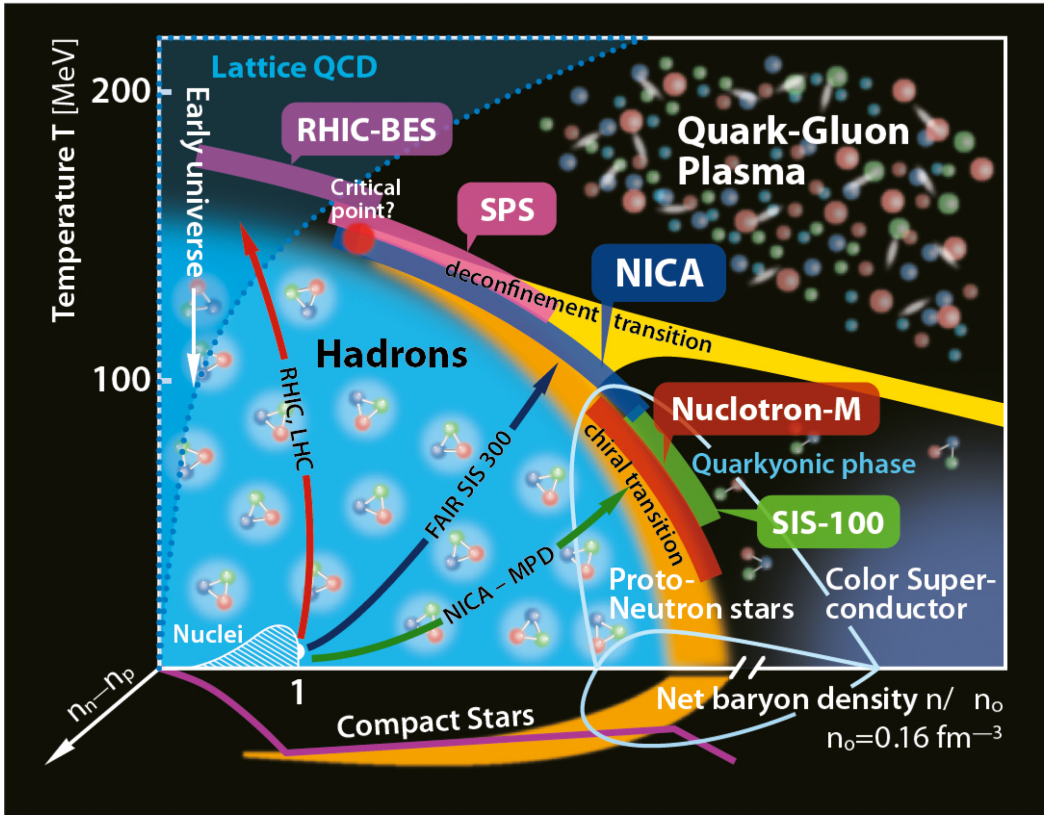
- **For non-thermal quantity, we have $d = 0$ and $\delta = 0$.**

$$\begin{aligned} \alpha &= a + c \\ \beta &= a - 2b \\ \gamma &= b + d - a - c \\ \delta &= -d \end{aligned} \qquad \begin{aligned} a &= 2\alpha + \beta + 2\gamma + 2\delta \\ b &= \alpha + \gamma + \delta \\ c &= -\alpha - \beta - 2\gamma - 2\delta \\ d &= -\delta \end{aligned}$$

Problem: derive these relations

Why high energy nuclear collisions (heavy ion collisions)

- QCD phases as properties of strong interaction matter: quark-gluon plasma (QGP, new state of matter)
- Two forms of QGP: (a) QGP at high T in the early universe; (b) QGP at high baryon density ρ_B in cores of compact (neutron) stars
- In 1974, T.D. Lee and W. Greiner proposed high energy nuclear collisions to form QGP in the laboratory



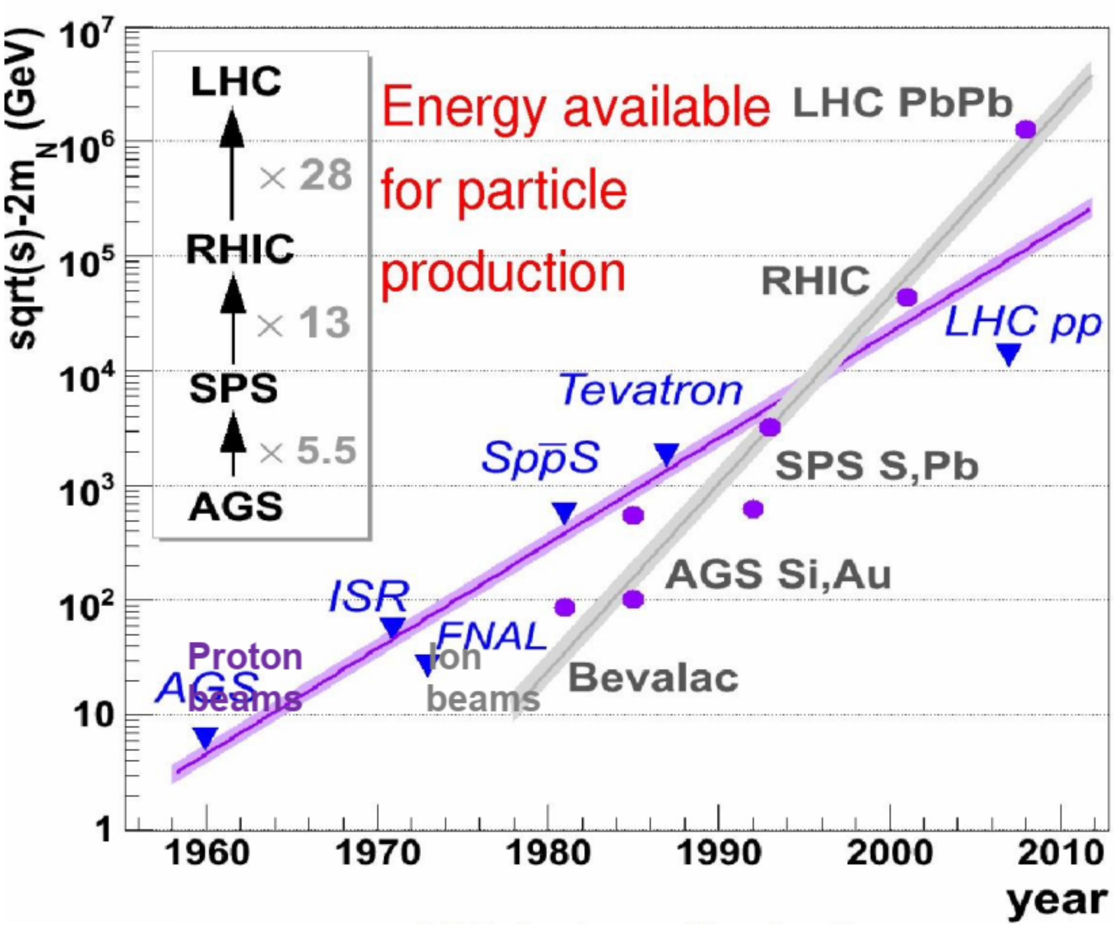
Why high energy nuclear collisions (heavy ion collisions)

李可染:核子重如牛碰撞生新态

北京五道口清华科技园

Nuclear stopping power and nuclear transparency

Experiments in HIC: past and future



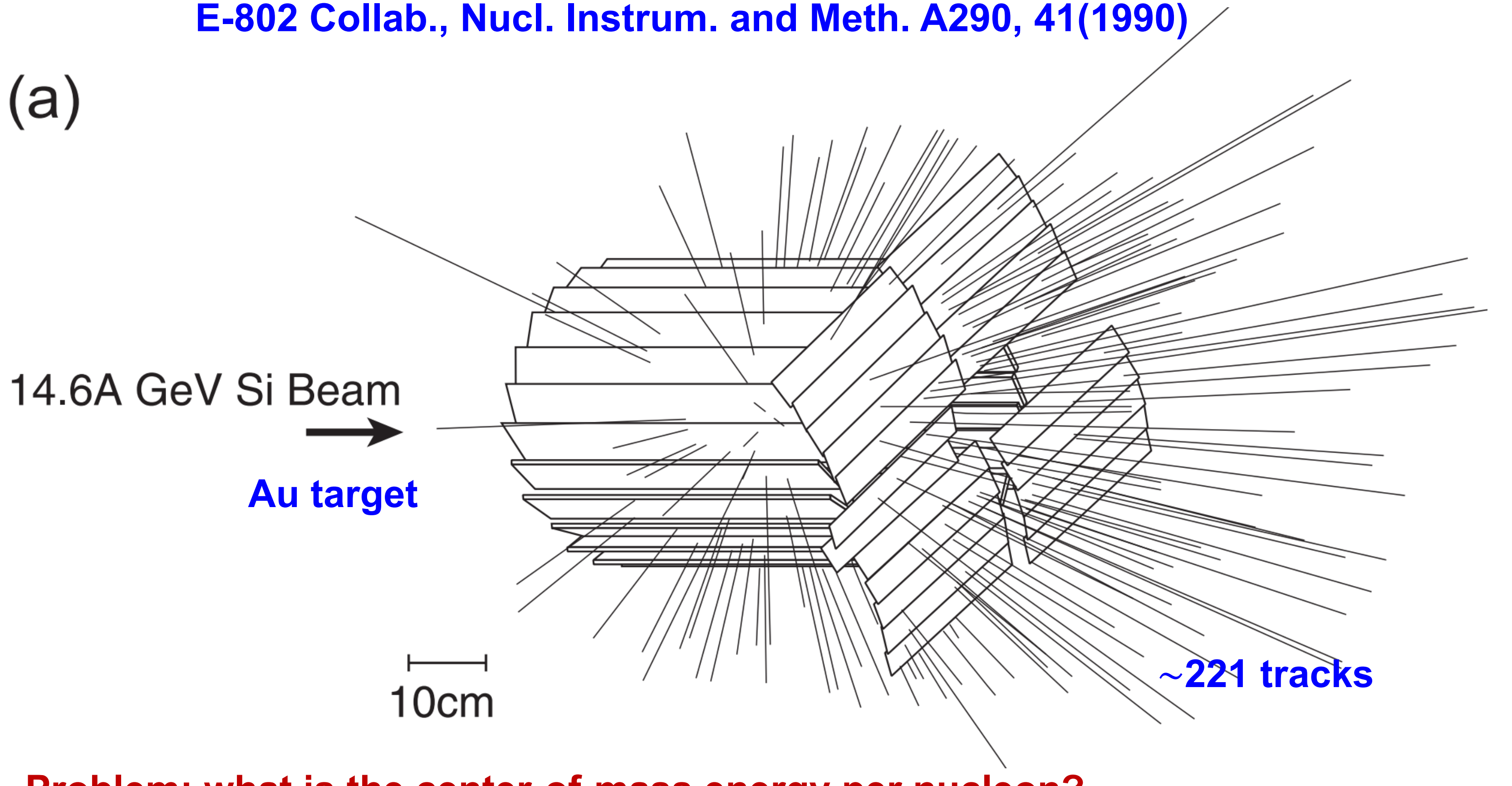
RHIC@BNL, 2000--,
$\sqrt{s_{NN}} \leq 200$ **GeV**
[beam energy scan $\sqrt{s_{NN}}$ **= 7.7, 11.5, 19.6, 27, 39, and 62.4 GeV]**

LHC @ CERN
Run I, 2009-13: $\sqrt{s_{NN}}$ **= 2.76TeV**
Run II, 2015-18: $\sqrt{s_{NN}}$ **= 5.02TeV**
Run III (HL-LHC): $\sqrt{s_{NN}}$ **= 5.5TeV**

NICA@JINR, 2021,
$3 < \sqrt{s_{NN}} < 11$ **GeV**

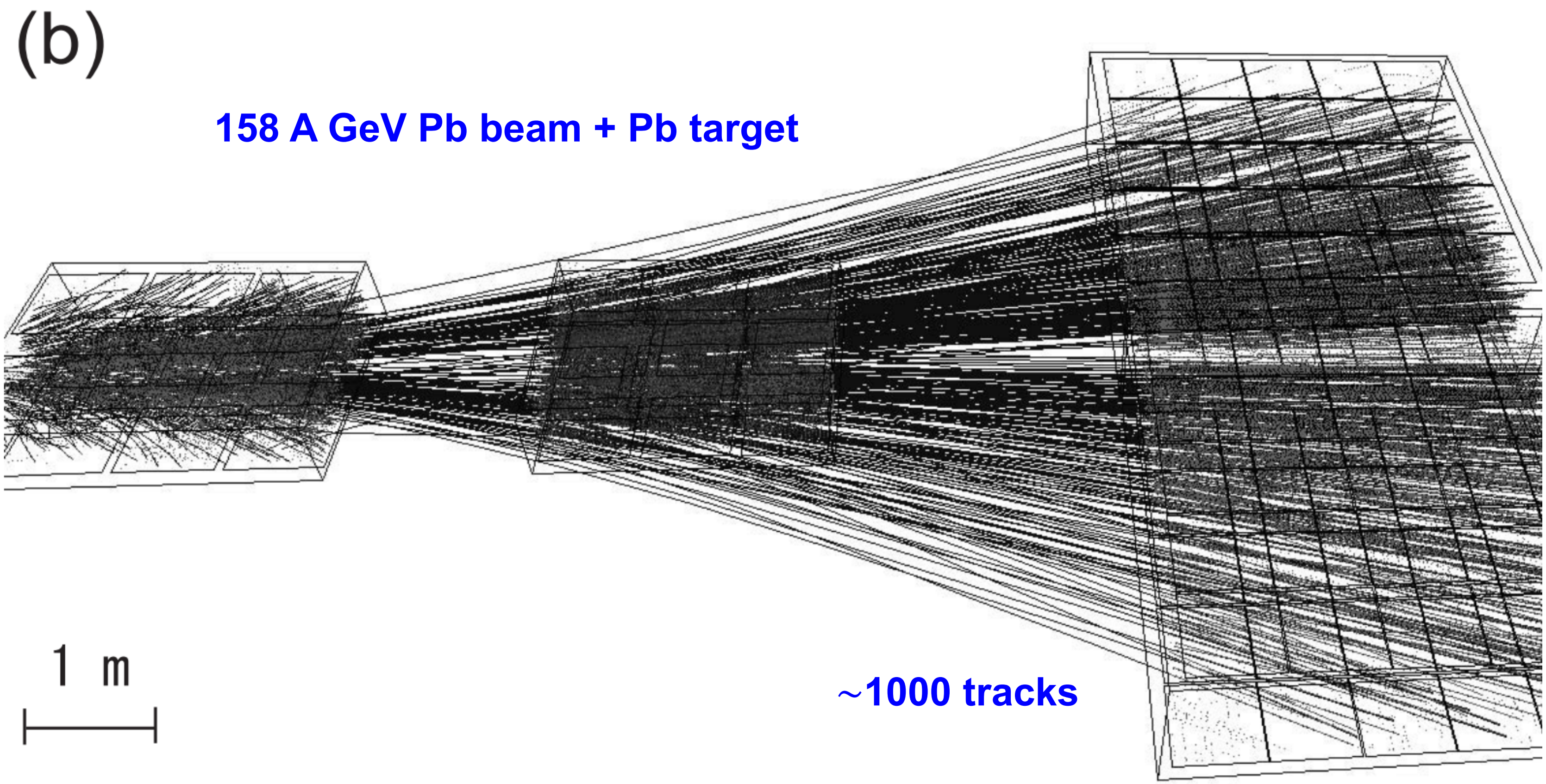
AGS@ BNL

E-802 Collab., Nucl. Instrum. and Meth. A290, 41(1990)



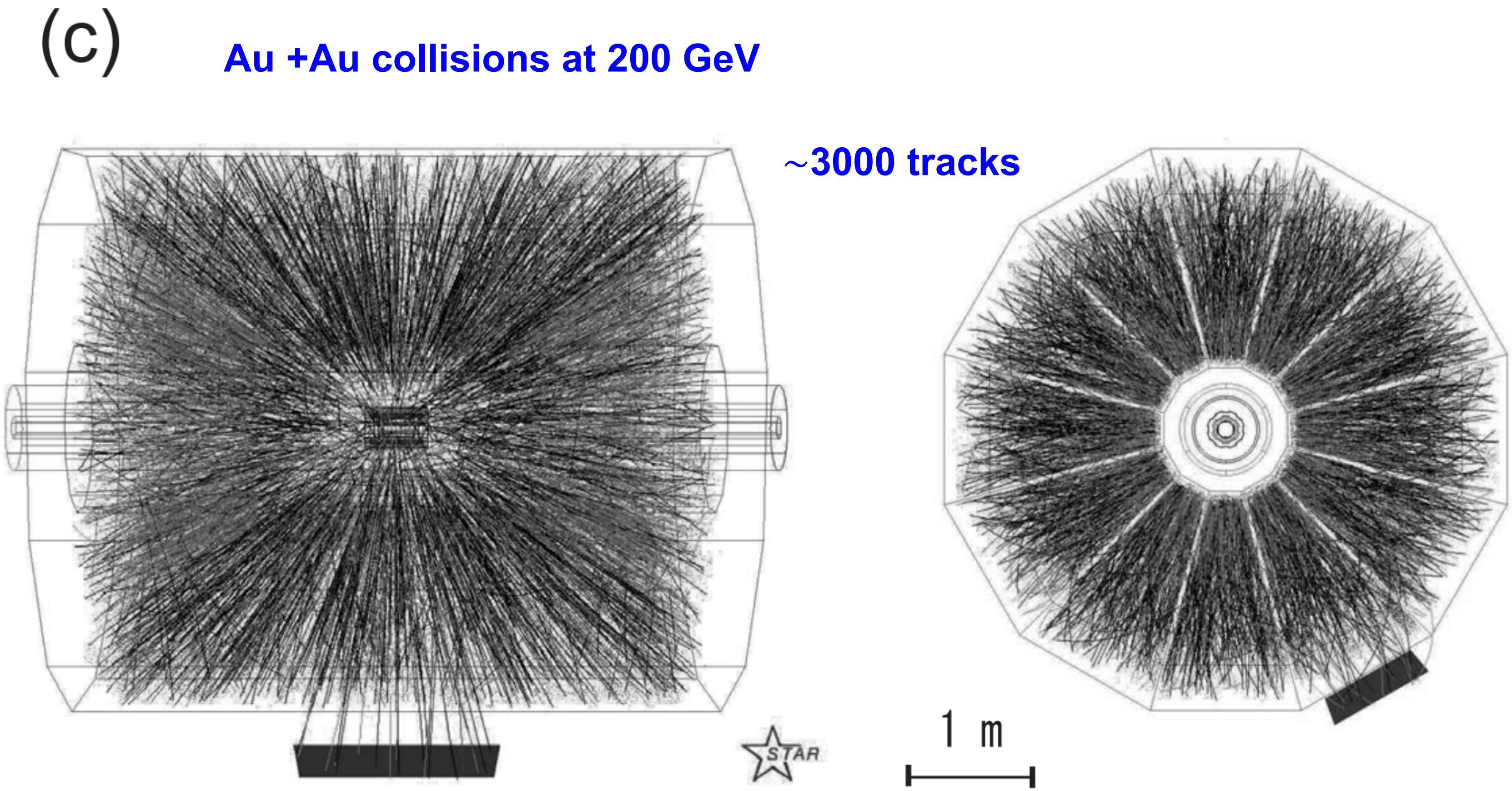
Problem: what is the center-of-mass energy per nucleon?

SPS@ CERN

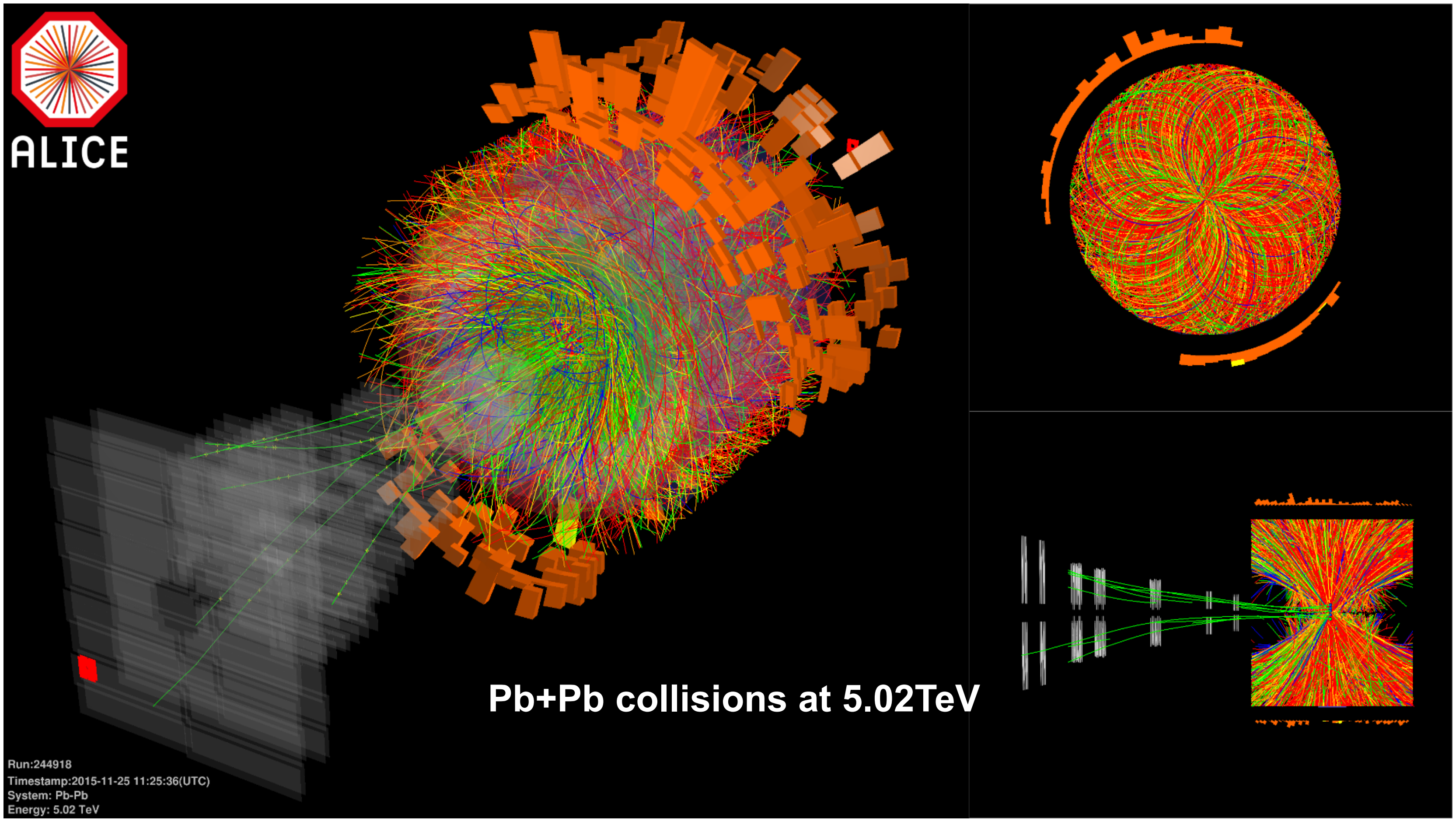


Problem: what is the center-of-mass energy per nucleon?

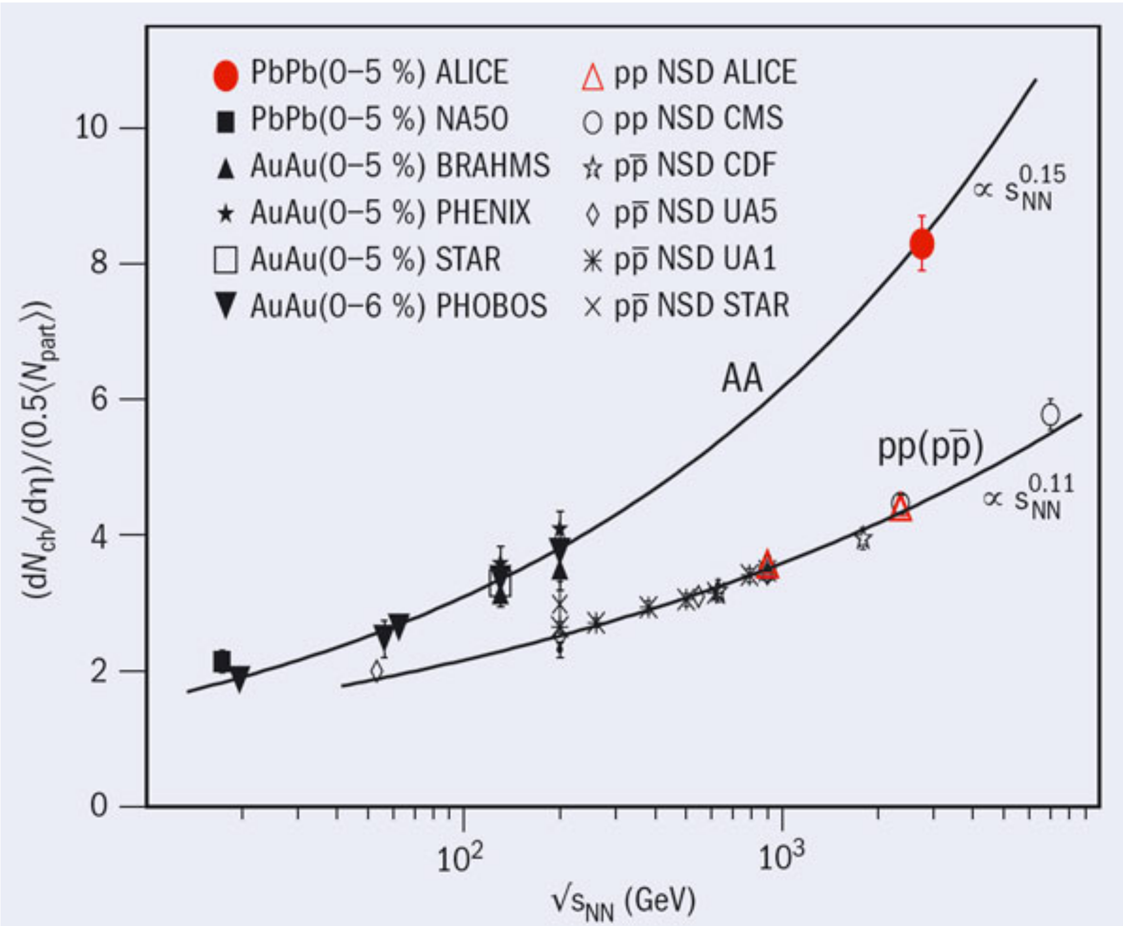
RHIC-STAR@BNL



LHC-ALICE@CERN



Multiplicity as functions of collision energy

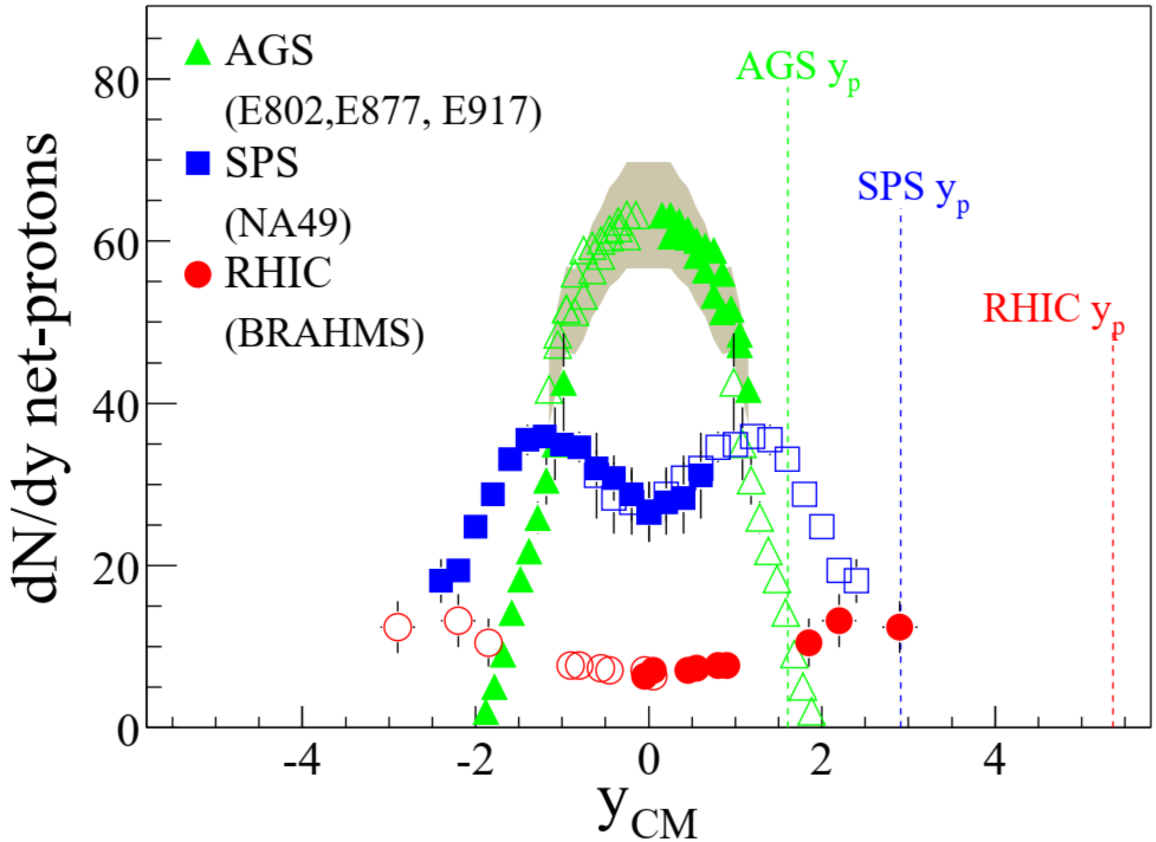


Problem: we see from the figure that

$$N_{ch} \sim \Delta\eta \cdot N_{part} \cdot s_{NN}^{0.15},$$
$$N_{part} \sim A^{2/3}$$

if there are 3000 tracks on average in Au+Au (A=197) collisions at $\sqrt{s_{NN}}$ = 200 GeV, how many tracks on average are there in Pb+Pb (A=207) collisions at $\sqrt{s_{NN}}$ = 2.76 TeV and 5.02 TeV?

Nuclear stopping power and nuclear transparency



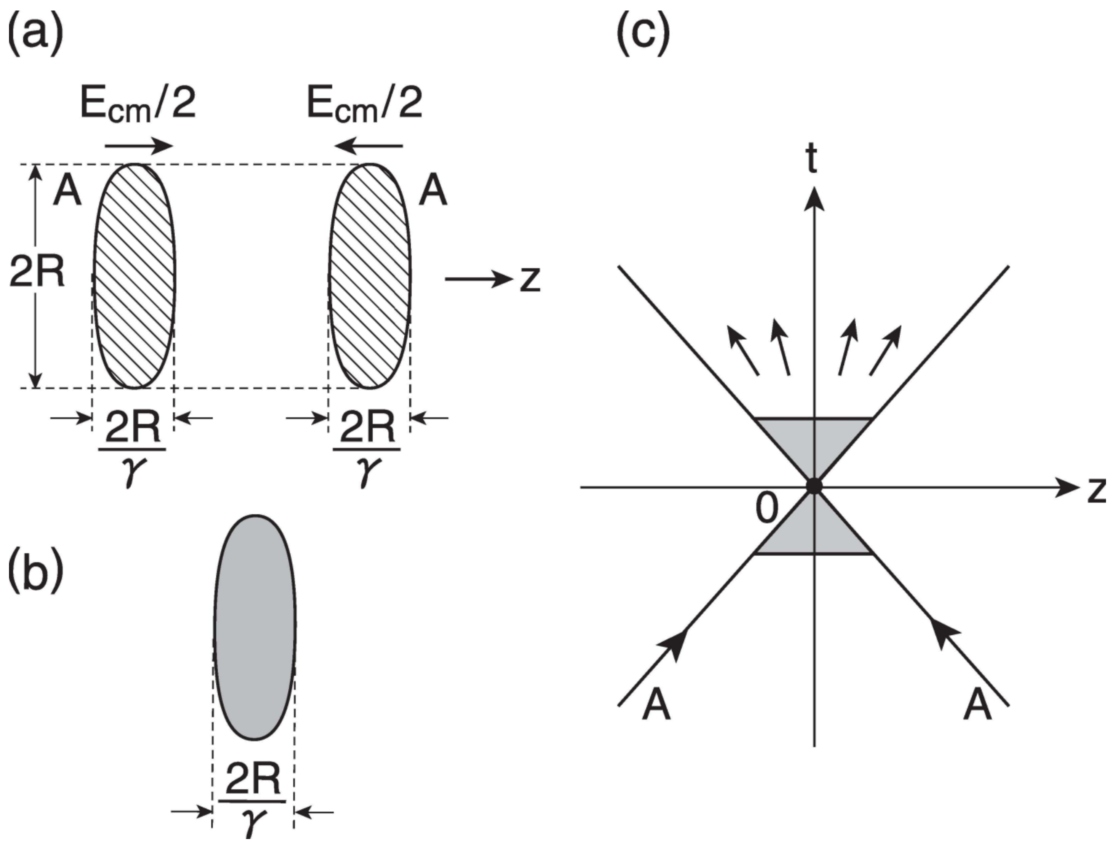
BRAHMS Collaboration, PRL 93, 102301 (2004)

The net–proton rapidity distribution at AGS (Au+Au at $\sqrt{s_{NN}}$= 5 GeV), SPS (Pb+Pb at $\sqrt{s_{NN}}$ = 17 GeV) and this measurement ($\sqrt{s_{NN}}$ = 200 GeV).

It is clear that the nuclear collision changes from stopping to transparent in these energies. In analogy to optics, one may say that the nucleus becomes transparent in high energy collisions.

Space-time picture of collisions

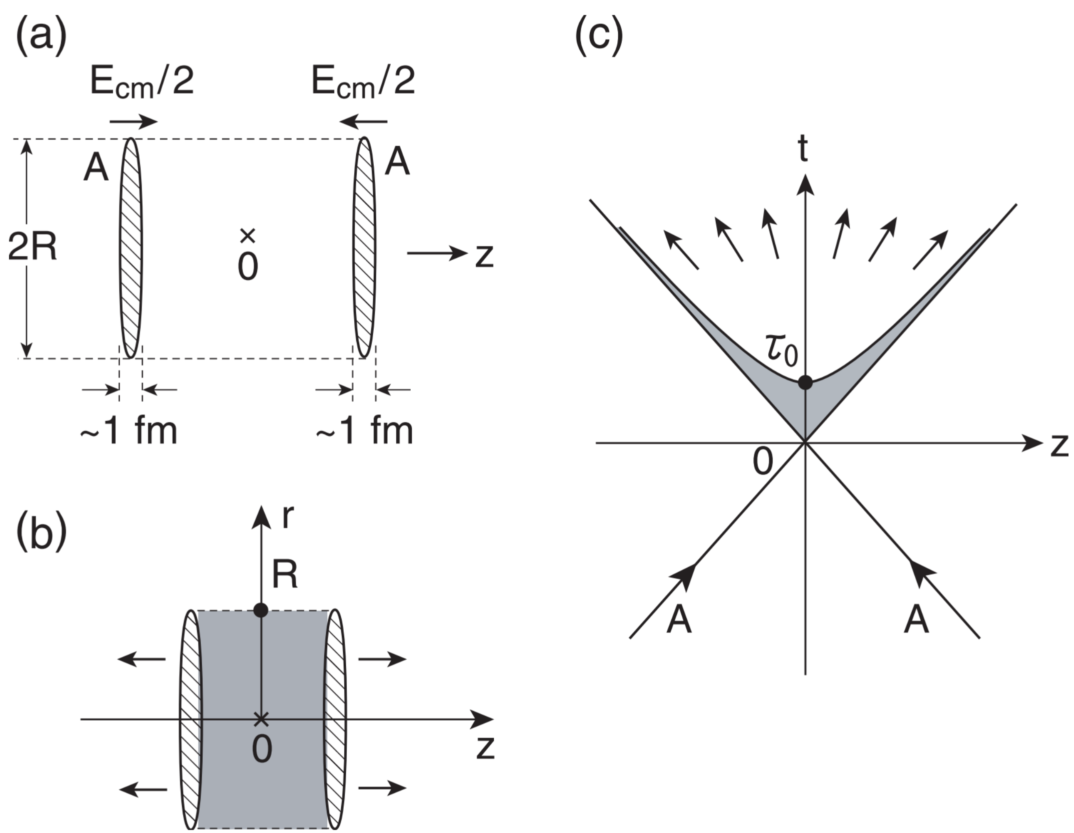
Space-time picture of collisions



Space-time view of a central collision of heavy nuclei A+A in the Landau picture.

(a) Two nuclei approaching each other with relativistic velocities and zero-impact parameters in the center-of-mass frame.
(b) They slow down, stick together at the center and produce particles.
(c) A light-cone diagram of the collision in the Landau picture, where the particle production takes place in the shaded area.

Space-time picture of collisions

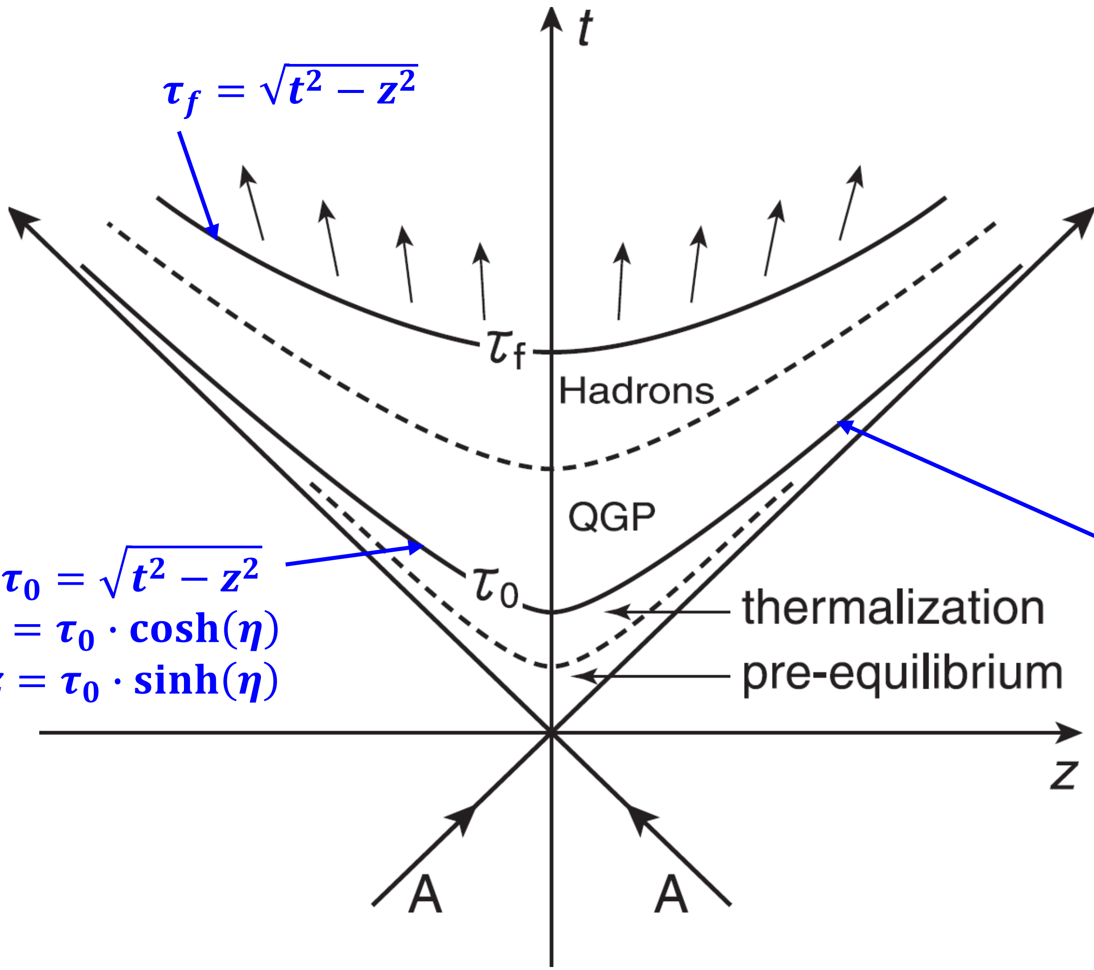


Space-time view of the central AA collisions in the Bjorken picture.

(a) Two nuclei approach each other with ultra-relativistic velocities and zero-impact parameters in the center-of-mass frame.
(b) They pass through each other, leaving highly excited matter with a small net baryon number (shaded area) between the nuclei.
(c) A light-cone diagram of the collision in the Bjorken picture: the highly excited matter is formed in the shaded area.

Space-time picture of collisions

$\tau_f = \sqrt{t^2 - z^2}$

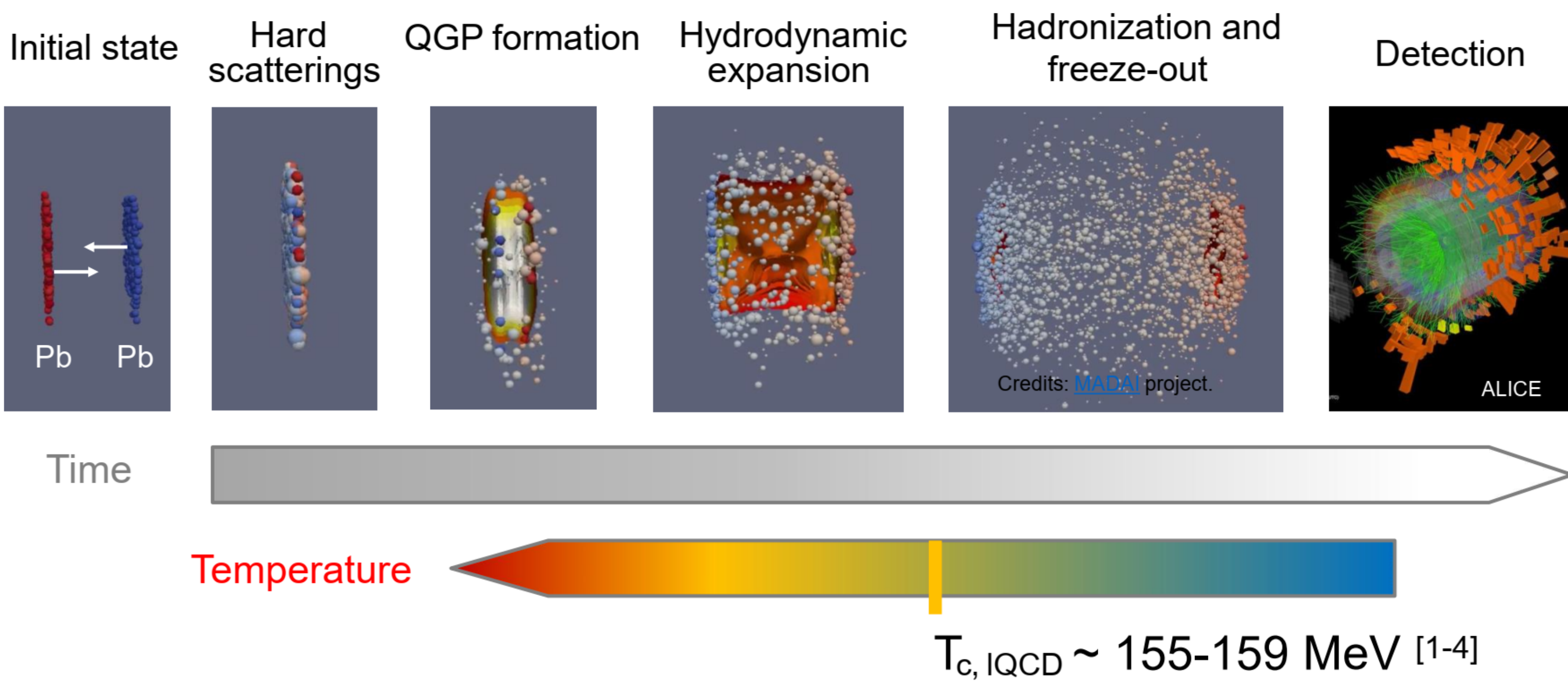


$\tau_0 = \sqrt{t^2 - z^2}$
$t = \tau_0 \cdot \cosh(\eta)$
$z = \tau_0 \cdot \sinh(\eta)$

Light-cone diagram showing the longitudinal evolution of an ultra-relativistic AA collision. Contours of constant proper time τ appear as hyperbolas, $\tau = \sqrt{t^2 - z^2}$.

Slow particles emerge first near the collision point, while the fast particles emerge last, far from the collision point.

Space-time picture of collisions



Space-time picture of collisions

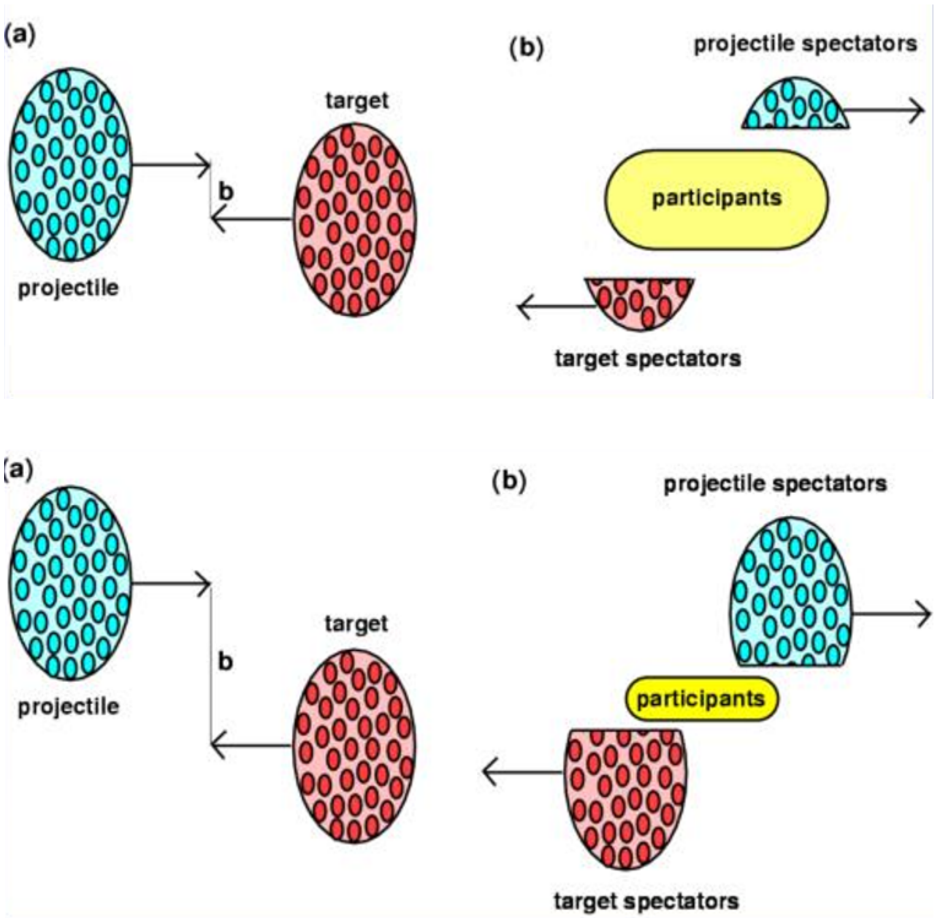
- The wee partons may be considered as vacuum fluctuations which couple to the fast-moving valence quarks passing through the QCD vacuum (Bjorken, 1976).
- Wee partons may be regarded as part of a coherent classical field created by the source of fast partons, which is called the color glass condensate (McLerran and Venugopalan, 2004).
- Since nucleons and nuclei are always associated with these low-momentum wee partons, the longitudinal size of hadrons or nuclei, Δz, can never be smaller than $1/\Lambda_{QCD}\sim 1$ fm owing to the uncertainty principle at ultra-high energies.
- So the two incoming nuclei in the center-of-mass frame before the collision wear the "fur coat of wee partons" (Bjorken, 1976) of typical size 1 fm, while the longitudinal size of the wave function of a valence quark is $\sim 2R/\gamma_{cm}$.

Space-time picture of collisions

- **It takes a certain proper time, τ_{de}, (de-excitation or de-coherence time), for these quanta to be de-excited to real quarks and gluons.**
- **The state of matter for $\tau \in [0, \tau_{de}]$ is said to be in the pre-equilibrium stage.**
- **Since τ_{de} is defined in the rest frame of each quantum, it experiences Lorentz dilation and becomes $\tau = \gamma\tau_{de}$ in the center-of-mass frame, where γ is the Lorentz factor of each quantum. This implies that slow particles emerge first near the collision point, while the fast particles emerge last, far from the collision point. This phenomenon, which is not taken into account in the Landau picture, is called the inside-outside cascade.**

Geometry of heavy ion collisions

Geometry of heavy ion collisions



Participant-spectator picture.

Since the spectator keeps its longitudinal velocity and emerges at nearly zero degrees in the collision, it is relatively easy to separate the spectator and the participant experimentally.

In experiments, information about the impact parameter, b, is obtained by measuring the sizes of the spectator and/or the participant.

Glauber model

- **Glauber model [Glauber 1959] is semi-classical model, treating the nucleus-nucleus collisions as multiple nucleon-interactions.**
- **Nucleons are assumed to travel in straight lines, and are not deflected after the collisions, which holds as a good approximation at very high energies. [Optical limit]**
- **Nuclear thickness function (number of nucleons per unit area)**

$$T_A(\mathbf{s}) = \int dz_A \underbrace{\rho_A(\mathbf{s}, z_A)}_{\text{nucleon number density}}, \quad \int d^2 s\, T_A(\mathbf{s}) = 1$$

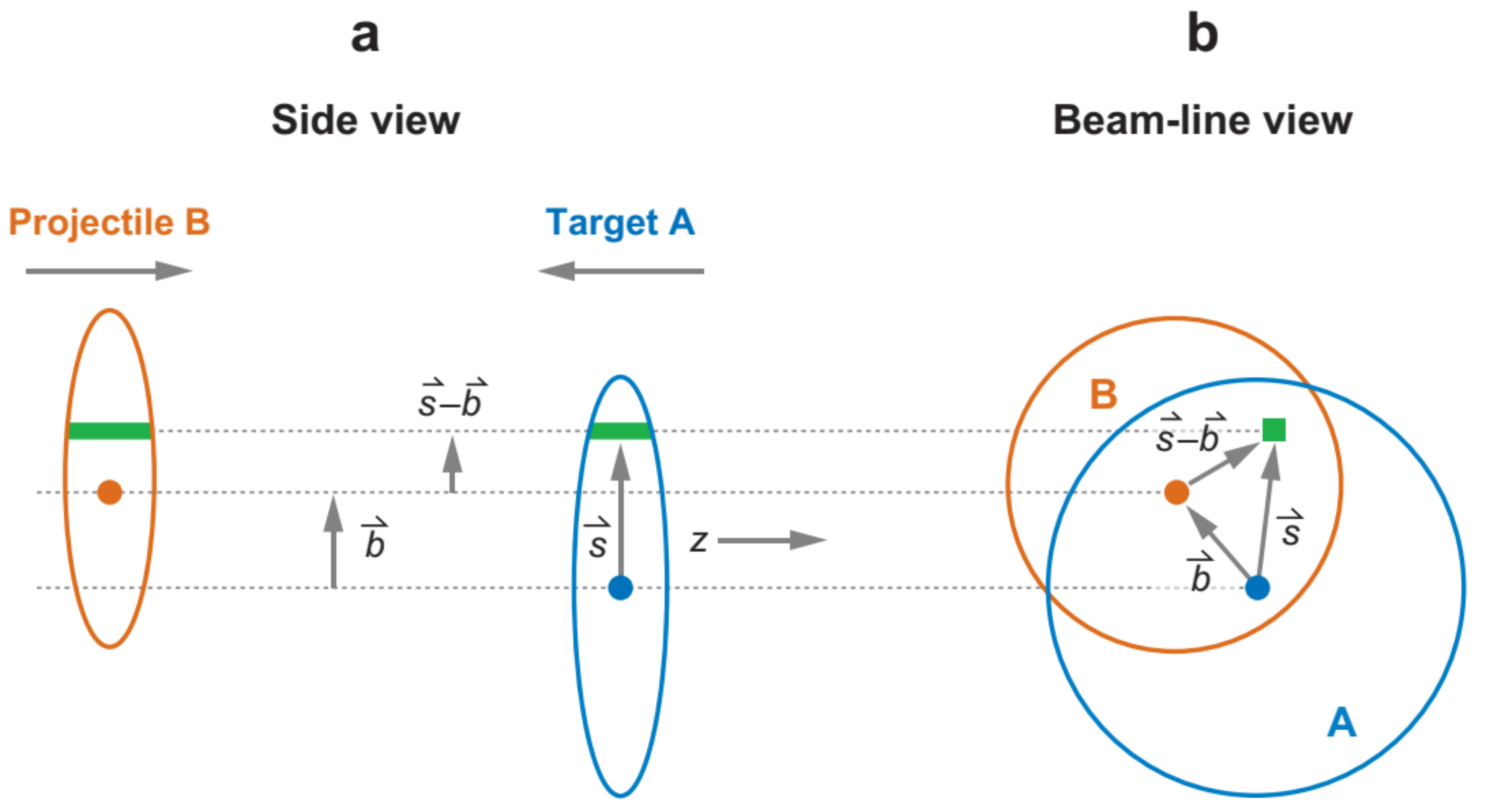
- **Nuclear overlap function (number of nucleon-pairs per unit overlapping area)**

$$T_{AB}(\mathbf{b}) = \int d^2 s\, T_A(\mathbf{s}) T_B(\mathbf{s} - \mathbf{b}), \quad \int d^2 b\, T_{AB}(\mathbf{b}) = 1$$

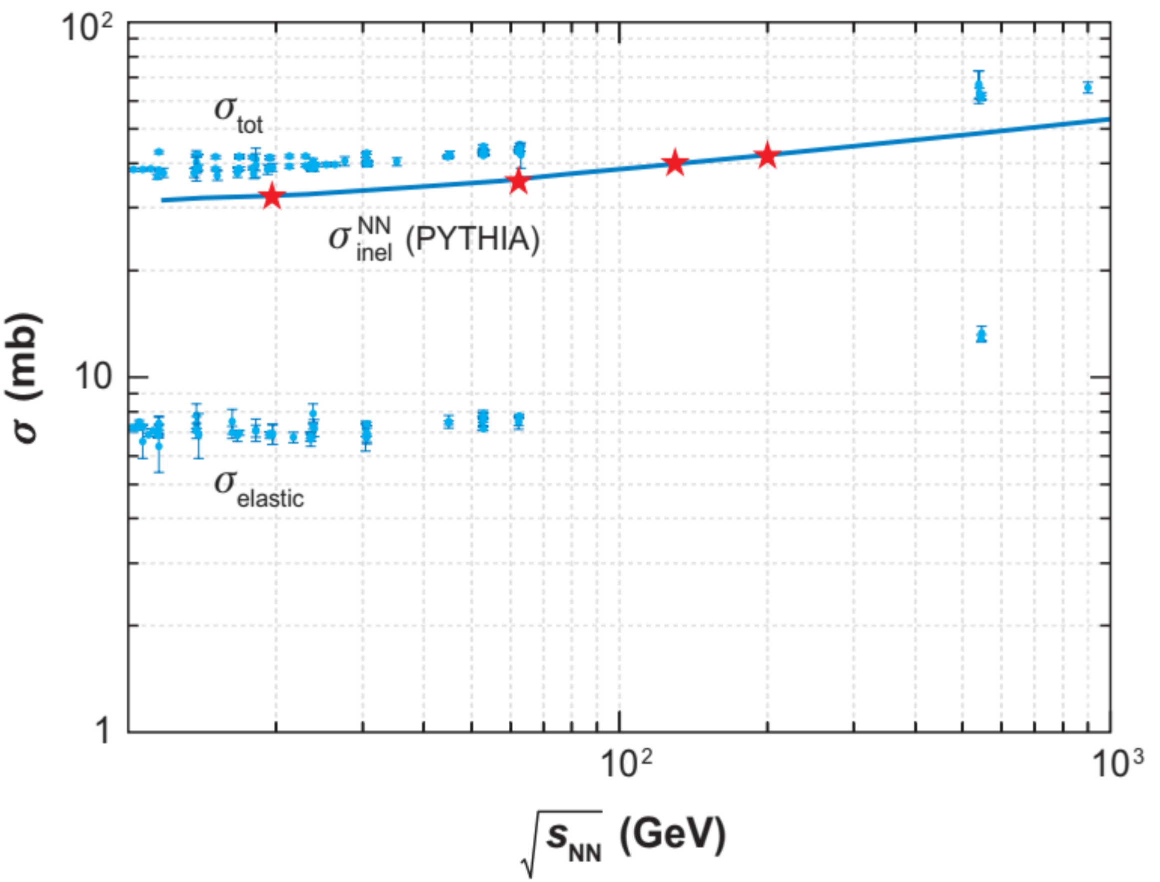
T_{AB} is proportional to joint probability per unit overlapping area

$\underline{T_{AB}(\mathbf{b})\sigma_{\mathrm{inel}}^{NN}}$ **probability of nucleon interaction (inelastic)**

Collision geometry



Inelastic nucleon-nucleon cross section



The inelastic nucleon-nucleon cross section as parameterized by PYTHIA in addition to data on total and elastic NN cross sections as a function of collision energy.

The stars indicate the nucleon-nucleon cross section used for Glauber Monte Carlo calculations at RHIC.

Miller, et al., Annu. Rev. Nucl. Part. Sci. 57, 205 (2007).

Inelastic collision probability

- **Elastic processes lead to very little energy loss and are consequently not considered in the Glauber model.**
- **The probability of having *n* such interactions between nuclei A and B is given as a binomial distribution**

$$P(n,b) = C_{AB}^{n}\left[T_{AB}(\mathbf{b})\underbrace{\sigma_{\rm inel}^{NN}}_{}\right]^{n}\left[1 - T_{AB}(\mathbf{b})\underbrace{\sigma_{\rm inel}^{NN}}_{}\right]^{AB-n}$$

inelastic cross section

- **The total probability of an interaction between A and B is**

$$P_{\rm inel}^{AB}(b) \equiv \frac{d^2\sigma_{\rm inel}^{AB}}{db^2} = \sum_{n=1}^{AB} P(n,\mathbf{b}) = 1 - \left[1 - T_{AB}(\mathbf{b})\sigma_{\rm inel}^{NN}\right]^{AB}$$

$$\underline{\sigma_{\rm inel}^{AB} = \int d^2\mathbf{b}\left\{1 - \left[1 - T_{AB}(\mathbf{b})\sigma_{\rm inel}^{NN}\right]^{AB}\right\}}$$

To determine N_part and centrality through N_ch

Binary collision and participant number

- **The total number of nucleon-nucleon collisions is**

$$N_{\mathrm{coll}}(\mathbf{b}) = \sum_{n=1}^{AB} nP(n,\mathbf{b}) = ABT_{AB}(\mathbf{b})\sigma_{\mathrm{inel}}^{NN}$$

Problem: prove this relation

- **The number of participants (or wounded nucleons) at impact parameter b**

Probability of pB or nB inelastic scattering

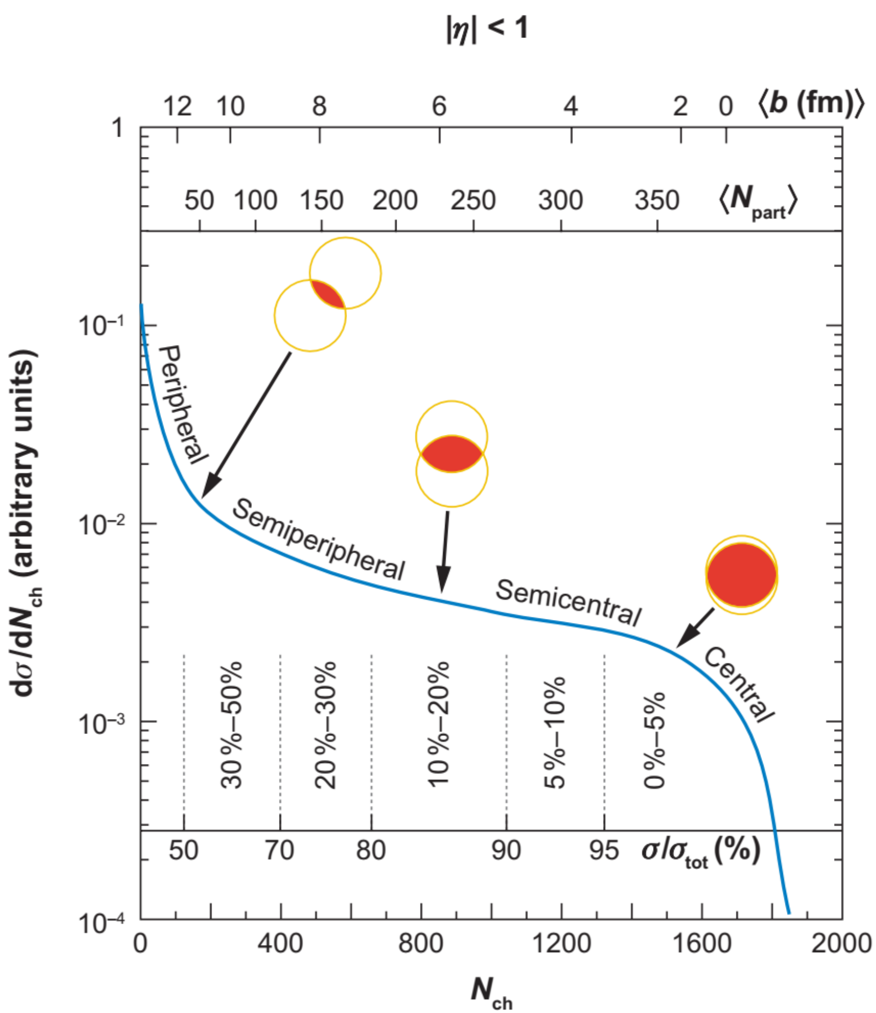
$$N_{\mathrm{part}} = A\int d^2s\, T_A(\mathbf{s})\left\{1-\left[1-T_B(\mathbf{s}-\mathbf{b})\sigma_{\mathrm{inel}}^{NN}\right]^B\right\}$$

Probability of pA or nA inelastic scattering

$$+ B\int d^2s\, T_B(\mathbf{s}-\mathbf{b})\left\{1-\left[1-T_A(\mathbf{s})\sigma_{\mathrm{inel}}^{NN}\right]^A\right\}$$

$$\sigma_{\mathrm{inel}}^{A} = \int d^2s\left\{1-\left[1-T_A(\mathbf{s})\sigma_{\mathrm{inel}}^{NN}\right]^A\right\}$$ **pA or nA inelastic cross section**

Centrality and participant number

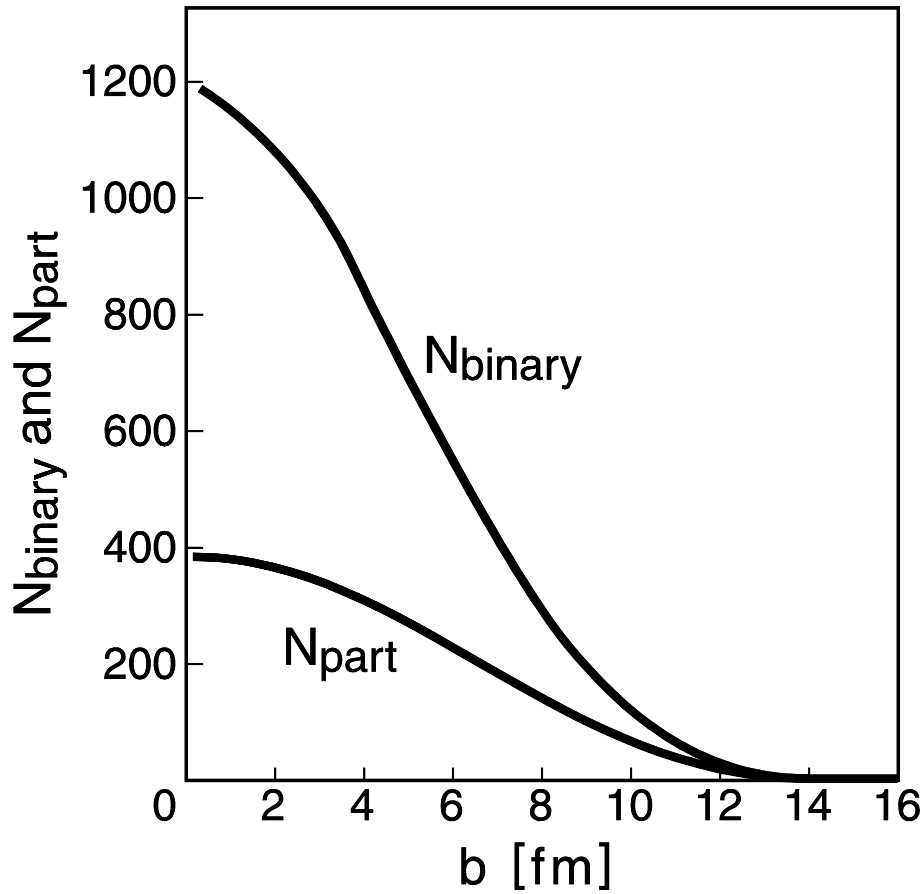


An illustrated example of the correlation of the final-state-observable total inclusive charged-particle multiplicity N_ch with Glauber-calculated quantities (b, N_part).

The plotted distribution and various values are illustrative and not actual measurements.

Miller, et al., Annu. Rev. Nucl. Part. Sci. 57, 205 (2007).

Binary collision and participant number as functions of impact parameter



Number of binary collisions and number of participant nucleons as a function of the Impact parameter in Au + Au collisions.

The Woods-Saxon distribution with parameters a = 0.53 fm. R_Au = 6.38 fm and σ_{NN}^{inel}=42 mb.

Relativistic hydrodynamics for heavy ion collisions

Fermi and Landau pictures of multi-particle production

- **The total energy of the system in the center-of-mass frame**

$$W_{cm} = AE_{cm} = 2Am_N\underline{\gamma_{cm}}$$

Lorentz contraction factor

- **The initial energy density**

$$\epsilon = \frac{W_{cm}}{V} = \frac{2Am_N\gamma_{cm}}{V_{rest}/\gamma_{cm}} = 2\epsilon_{nm}\gamma_{cm}^2 \propto E_{cm}^2$$

$$\epsilon_{nm} \equiv m_N\rho_{nm} = 0.15\ \mathrm{GeV/fm}^3$$ Mass density of nuclear matter

$$\rho_{nm} \equiv \frac{A}{V_{rest}} = 0.16\ \mathrm{fm}^{-3}$$ Number density of nuclear matter

Ideal fluids

- If we assume the perfect fluid, only an equation of state of matter is necessary for a hydrodynamic description of the system. For ideal gas of relativistic particles (neglecting μ) the EOS is

$$\epsilon + P = Ts$$

$$P = \frac{1}{3}\epsilon \Longrightarrow \epsilon = \frac{3}{4}Ts$$

- The results are consistent with Stefan-Boltzmann's law

$$dP = d(Ts - \epsilon) = sdT$$

$$\Rightarrow d\epsilon = 3sdT$$

$$d\epsilon = Tds$$

$$\Rightarrow \frac{d\epsilon}{\epsilon} = 4\frac{dT}{T} \Rightarrow \epsilon \propto T^4$$

$$\Rightarrow s \propto T^3 \propto \epsilon^{3/4} \propto E_{cm}^{3/2}$$

The number of particles produced in HIC

- **By definition the perfect fluid has no viscosity and does not produce entropy. The total entropy stays constant during the hydrodynamical expansion. The number density of the produced particles (pions) is proportional to the entropy according to the black body formula**

$$E_{cm} \propto E_{lab}^{1/2}$$

$$N_\pi \propto sV \propto E_{cm}^{3/2} V_{rest}/\gamma_{cm} \propto A E_{cm}^{1/2} \propto A E_{lab}^{1/4}$$

- **In Landau picture, the nucleons of colliding nuclei must lose all their kinetic energy in the center-of-mass frame while traversing the other nucleus. This demands that the average energy loss of nucleons per unit length be greater than**

$$\left(\frac{dE}{dz}\right)_{\mathrm{cr}} = \frac{E_{\mathrm{cm}}/2}{2R/\gamma_{\mathrm{cm}}} \approx \frac{106}{9.6A^{1/3}} \left(\frac{E_{\mathrm{cm}}}{10\mathrm{GeV}}\right)^2 \frac{\mathrm{GeV}}{\mathrm{fm}}$$

For E_cm = 200 GeV,
The energy loss is too large!!

$$\sim 800\ \mathrm{GeV/fm}$$

Relativistic Hydrodynamics

- **Fluid dynamics is equivalent to the conservation of energy, momentum and net charges.**

Charge current density

$$\partial_\mu J^\mu = 0$$ **No. of equations: 1**

$$\partial_\mu T^{\mu\nu} = 0$$ **No. of equations: 4**

- **Choose u^μ, an arbitrary, time-like, normalized 4-vector, $u \cdot u = 1$**

Charge density

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu,\ u_\mu \Delta^{\mu\nu} = 0$$

$$J^\mu = n u^\mu + \nu^\mu$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \pi)\Delta^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \pi^{\mu\nu}$$

Energy density **Pressure**

- **So there are 5 equations but 14 variables:**

$$u^\mu \pi_{\mu\nu} = 0$$

$$\pi^{\mu\nu}\Delta_{\mu\nu} = 0$$

variables	n	ν^μ	ϵ	$P+\pi$	q^μ	$\pi^{\mu\nu}$
No. dof	1	3	1	1	3	5

Relativistic Hydrodynamics

- **Since u^μ is arbitrary, there are many choices:**

$$u_N^\mu = \frac{J^\mu}{\sqrt{J \cdot J}}$$

Eckart or particle frame

$$u_E^\mu = \frac{T_\nu^\mu u_E^\nu}{\sqrt{u_E^\alpha T_\alpha^\beta T_{\beta\gamma} u_E^\beta}}$$

Landau or energy frame

- **In the Eckart or particle frame, u_N^μ is the physical velocity of the flow of the conserved charge**

$$J^\mu = \sqrt{J \cdot J}\, u_N^\mu \longrightarrow \nu^\mu = 0$$

- **In the Landau or energy frame, the velocity u_E^μ is actually the eigenvector of T_ν^μ and we have $q^\mu = 0$.**

Relativistic Hydrodynamics

- **We can check** $q_\mu = 0$

$$
\begin{aligned}
T^{\beta}_{\nu} u^{\nu}_{E} &= (\epsilon u^{\beta}_{E} u^{E}_{\nu} - P\Delta^{\beta}_{\nu} + q^{\beta} u^{E}_{\nu} + u^{\beta}_{E} q_{\nu} + \pi^{\beta}_{\nu}) u^{\nu}_{E} \\
&= \epsilon u^{\beta}_{E} + q^{\beta} = \sqrt{u^{\alpha}_{E} T^{\beta}_{\alpha} T_{\beta\gamma} u^{\beta}_{E} u^{\mu}_{E}}
\end{aligned}
$$

- **Consider an ideal gas in local thermodynamical equilibrium. The single particle phase space distributions for fermions and bosons are**

$$
f_0(x,k) = \frac{g}{(2\pi)^3} \frac{1}{\exp[(k\cdot u(x) - \mu(x))/T(x)] \pm 1}
$$

- **The chemical potential for anti-particles is $-\mu$. We denote the anti-particle distribution as $\overline{f}_0(x,k)$.**

Relativistic Hydrodynamics

- **The charge density and energy-momentum tensor can be expressed as**

$$
\begin{aligned}
J^{\mu} &= Q\int \frac{d^3k}{(2\pi)^3}\frac{k^{\mu}}{E}\left[f_0(x,k)-\overline{f}_0(x,k)\right] \\
&= Q\int d^3k\left(1,\frac{\mathbf{k}}{E}\right)\left[f_0(x,k)-\overline{f}_0(x,k)\right] \\
&= n\gamma(1,\mathbf{v}) \\
n &= Q\int \frac{d^3k}{(2\pi)^3}\left[f_0(x,k)-\overline{f}_0(x,k)\right]
\end{aligned}
$$

$$
\begin{aligned}
T^{\mu\nu} &= \int \frac{d^3k}{(2\pi)^3}\frac{k^{\mu}k^{\nu}}{E}\left[f_0(x,k)+\overline{f}_0(x,k)\right] \\
T^{00} &= \int \frac{d^3k}{(2\pi)^3}E\left[f_0(x,k)+\overline{f}_0(x,k)\right]=\epsilon \\
T^{ij} &= \int \frac{d^3k}{(2\pi)^3}\frac{k^{i}k^{j}}{E}\left[f_0(x,k)+\overline{f}_0(x,k)\right] \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{k^2}{3E}\delta^{ij}\left[f_0(x,k)+\overline{f}_0(x,k)\right]=P
\end{aligned}
$$

- **Bridge between hydrodynamics and kinetic theory**

Ideal fluids

- For ideal fluids, there are 5 equations but 6 variables (n, P, ϵ, u^μ). One needs the equation of state (EOS) $P(\epsilon, n)$ to close the systems of equations. But the EOS of this type is not complete, the complete EOS is $s(\epsilon, n)$ or $P(T, \mu)$.
- From entropy density $s(\epsilon, n)$ one can determine $\left(\frac{1}{T}, \frac{\mu}{T}\right)$ through

$$ds = \frac{1}{T}d\epsilon - \frac{\mu}{T}dn$$

- From the thermodynamical relation, one can determine the unknown function P as function of ϵ and n

$$\begin{aligned} P(\epsilon, n) =& Ts + \mu n - \epsilon \\ =& s\left(\frac{\partial s}{\partial \epsilon}\right)^{-1} - n\frac{\partial s}{\partial n}\left(\frac{\partial s}{\partial \epsilon}\right)^{-1} - \epsilon \end{aligned}$$

Ideal fluids

- **From another type of EOS, the pressure as function of T and μ, $P(T,\mu)$, one can determine ϵ and n as functions of T and μ, so T and μ can be expressed as functions of ϵ and n, therefore the pressure can be expressed as a function ϵ and n , $P(\epsilon,n)$:**

$$\left.\begin{aligned} P(T,\mu) \rightarrow & s(T,\mu) = \frac{\partial P(T,\mu)}{\partial T} \\ \rightarrow & n(T,\mu) = \frac{\partial P(T,\mu)}{\partial \mu} \end{aligned}\right\} \quad dP = s dT + n d\mu$$

$$\rightarrow \epsilon(T,\mu) = Ts(T,\mu) + \mu n(T,\mu) - P(T,\mu)$$

$$\rightarrow T(\epsilon,n),\ \mu(\epsilon,n)$$

$$\rightarrow P\left[T(\epsilon,n),\mu(\epsilon,n)\right]$$

Ideal fluids

- **The energy-momentum conservation is**

$$P = Ts + \mu n - \epsilon$$

$$\begin{aligned} 0 = \partial_\beta T_0^{\alpha\beta} &= -\partial^\alpha P + \partial_\beta [(\epsilon + P) u^\alpha u^\beta] \\ &= -\partial^\alpha P + \partial_\beta [(Ts + \mu n) u^\alpha u^\beta] \\ &= -\partial^\alpha P + \partial_\beta \left(Ts u^\alpha u^\beta \right) + \partial_\beta \left(\mu u^\alpha n u^\beta \right) \\ &= -\partial^\alpha P + \partial_\beta \left(Ts u^\alpha u^\beta \right) + n u^\beta \partial_\beta \left(\mu u^\alpha \right) \end{aligned}$$

$$T_0^{\alpha\beta} = (\epsilon + P) u^\alpha u^\beta - g^{\alpha\beta} P$$

- **The continuity equation is**

$$\partial_\beta (n u^\beta) = u^\beta \partial_\beta n + n \partial_\beta u^\beta = 0$$

- **The 4-velocity satisfies**

$$\partial_\beta (u \cdot u) = 2 u_\alpha \partial_\beta u^\alpha = 0$$

Ideal fluids and entropy current

- **The energy-momentum conservation projected onto the velocity gives**

$$
\begin{aligned}
u_\alpha \partial_\beta T_0^{\alpha\beta} &= -u_\alpha \partial^\alpha P + u_\alpha \partial_\beta \left(T s u^\alpha u^\beta\right) + n u_\alpha u^\beta \partial_\beta \left(\mu u^\alpha\right) \\
&= -u^\alpha \partial_\alpha P + s u^\beta \partial_\beta T + T u^\beta \partial_\beta s + T s \partial_\beta u^\beta + n u^\beta \partial_\beta \mu \\
&= u^\beta \left(\underline{-\partial_\beta P} + \underline{s \partial_\beta T} + T \partial_\beta s + \underline{n \partial_\beta \mu}\right) + T s \partial_\beta u^\beta \\
&= \underline{T \partial_\beta (s u^\beta) = 0}
\end{aligned}
$$

$$\underline{dP = s dT + n d\mu}$$

Entropy is conserved

- **The entropy current density is su^μ, where the fluid velocity is the same as defined in the charge density $J^\mu = nu^\mu$.**

Ideal fluids and entropy current

- **The energy-momentum conservation projected onto $\Delta^{\mu\nu}$ gives**

$$
\begin{aligned}
\Delta^{\mu}{}_{\alpha}\partial_\beta T_0^{\alpha\beta} &= \left(g_\alpha^\mu - u^\mu u_\alpha\right)\partial_\beta T_0^{\alpha\beta} \\
&= \left(g_\alpha^\mu - u^\mu u_\alpha\right)\partial_\beta\left[(\epsilon+P)u^\alpha u^\beta - g^{\alpha\beta}P\right] \\
&= \left(g_\alpha^\mu - u^\mu u_\alpha\right)(\epsilon+P)u^\beta\partial_\beta u^\alpha - \left(g_\alpha^\mu - u^\mu u_\alpha\right)g^{\alpha\beta}\partial_\beta P \\
&= (\epsilon+P)u^\beta\partial_\beta u^\mu - \Delta^{\mu\beta}\partial_\beta P = 0
\end{aligned}
$$

$$\frac{Du^\mu}{Dt}$$

- **With $u^\mu = \gamma(1, \boldsymbol{v})$, the above equation can be put into 3-dim form (Navier-Stokes equation)**

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1-v^2}{\epsilon+P}\left(\nabla P + \boldsymbol{v}\frac{\partial P}{\partial t}\right)$$

Problem: prove this equation

Viscous fluid: first order theory

- **The current density and energy-momentum tensor are**

$$\begin{aligned} J^\mu &= nu^\mu + \nu^\mu \\ T^{\mu\nu} &= \epsilon u^\mu u^\nu - (P+\pi)\Delta^{\mu\nu} + q^\mu u^\nu + u^\mu q^\nu + \pi^{\mu\nu} \end{aligned}$$

- **The conservation equations:** $\partial_\mu J^\mu = 0$ **and** $\partial_\mu T^{\mu\nu} = 0$
- **Projecting EM conservation equation onto fluid velocity gives**

$$\begin{aligned} 0 = u_\alpha \partial_\beta T^{\alpha\beta} &= \underline{T\partial_\beta(su^\beta)} - \mu\partial_\beta \nu^\beta \\ &\quad + u_\alpha\partial_\beta(q^\alpha u^\beta + u^\alpha q^\beta + \pi^{\alpha\beta}) - u_\alpha\partial_\beta(\pi\Delta^{\alpha\beta}) \\ \boxed{u_\alpha\partial_\beta T_0^{\alpha\beta}} &= T\left[\partial_\beta(su^\beta) - \frac{\mu}{T}\partial_\beta\nu^\beta \right. \\ &\quad \left. + \frac{1}{T}u_\alpha\partial_\beta(q^\alpha u^\beta + u^\alpha q^\beta + \pi^{\alpha\beta}) - \frac{1}{T}u_\alpha\partial_\beta(\pi\Delta^{\alpha\beta})\right] \end{aligned}$$

Viscous fluid: first order theory

- **We can rewrite the above equation into this form**

$$
\begin{aligned}
\partial_\beta(su^\beta) &= -\partial_\beta\left[\frac{1}{T}u_\alpha(q^\alpha u^\beta + u^\alpha q^\beta + \pi^{\alpha\beta})\right] + \frac{1}{T}(q^\alpha u^\beta + u^\alpha q^\beta + \pi^{\alpha\beta})\partial_\beta u_\alpha \\
&\quad + u_\alpha(q^\alpha u^\beta + u^\alpha q^\beta + \pi^{\alpha\beta})\partial_\beta \frac{1}{T} + \partial_\beta\left(\frac{\mu}{T}\nu^\beta\right) - \nu^\beta\partial_\beta\frac{\mu}{T} + \frac{\pi}{T}\partial_\beta u^\beta \\
&= -\partial_\beta\left(\frac{1}{T}q^\beta - \frac{\mu}{T}\nu^\beta\right) + \frac{1}{T}(q^\alpha u^\beta + \pi^{\alpha\beta})\partial_\beta u_\alpha + q^\beta\partial_\beta\frac{1}{T} - \nu^\beta\partial_\beta\frac{\mu}{T} + \frac{\pi}{T}\partial_\beta u^\beta
\end{aligned}
$$

- **which leads to**

$$\pi^{\alpha\beta}\Delta_{\alpha\beta} = 0, \ \pi^{\alpha\beta} = \pi^{\beta\alpha}$$

$$
\begin{aligned}
\partial_\beta\left(su^\beta + \frac{1}{T}q^\beta - \frac{\mu}{T}\nu^\beta\right) &= \frac{1}{T}\pi^{\alpha\beta}\partial_\beta u_\alpha + \frac{1}{T}q^\alpha\left(u^\beta\partial_\beta u_\alpha + T\partial_\alpha\frac{1}{T}\right) \\
&\quad -\nu^\beta\partial_\beta\frac{\mu}{T} + \frac{\pi}{T}\partial_\beta u^\beta
\end{aligned}
$$

$$q^\alpha = q_\mu\Delta^{\mu\alpha}$$

$$\nu^\beta = \Delta^{\beta\rho}\nu_\rho$$

Viscous fluid: first order theory

- **By assuming**

$$\pi_{\alpha\beta} = \eta\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{3}\Delta_{\alpha\beta}\partial\cdot u\right)$$ **Shear stress tensor**

$$q^\mu = \kappa T\Delta^{\mu\alpha}\left(u^\beta\partial_\beta u_\alpha + T\partial_\alpha\frac{1}{T}\right)$$ **Heat flow**

$$\nu^\mu = -\sigma T^2\Delta^{\mu\alpha}\partial_\alpha\frac{\mu}{T}$$ **Difusion flow**

$$\pi = \zeta\partial_\beta u^\beta$$ **Bulk viscosity**

- **The entropy equation can be put into a quadratic form**

$$\partial_\beta\underbrace{\left(su^\beta + \frac{1}{T}q^\beta - \frac{\mu}{T}\nu^\beta\right)}_{\text{Entropy flow}} = \underbrace{\frac{\pi^{\alpha\beta}\pi_{\alpha\beta}}{2\eta T} + \frac{q^\alpha q_\alpha}{\kappa T^2} + \frac{\nu^\beta\nu_\beta}{\sigma T^2} + \frac{\pi^2}{\zeta T}}_{\text{Positive definite}}$$

Bjorken scaling solution

- **In HIC, the reaction volume is strongly expanded in the longitudinal beam direction (z-axis). In the 1st approximation, it is therefore reasonable to drop transverse spatial dim (x, y) and to describe the reaction in z and t. We use (τ, η) to replace (t, z)**

$$\tau = \sqrt{t^2 - z^2}, \ \eta = \frac{1}{2} \ln \frac{t+z}{t-z}$$

$$t = \tau \cosh \eta, \ \ z = \tau \sinh \eta$$

- **An ansatz such that the local velocity u^μ of the perfect fluid has the same form as the free stream of particles from the origin**

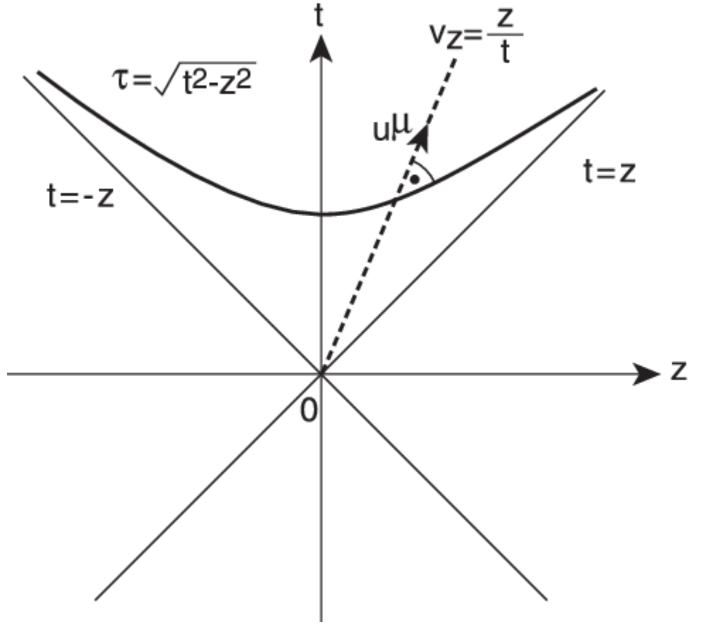
$$\begin{aligned} u^\mu =& \gamma(1, 0, 0, v_z) \\ & \to \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau} \right) = (\cosh \eta, 0, 0, \sinh \eta) \end{aligned}$$

$$v_z = \frac{z}{t} = \frac{\sinh \eta}{\cosh \eta}$$

$$\gamma = \cosh \eta$$

Bjorken scaling solution

- **Definition of the (1 + 1) coordinates. The hyperbola shown by the solid line corresponds to a curve with a constant proper time τ. The dashed line represents the direction of the local flow velocity u^μ.**



$$\tau = \sqrt{t^2 - z^2}, \ \eta = \frac{1}{2} \ln \frac{t+z}{t-z}$$

$$t = \tau \cosh \eta, \ \ z = \tau \sinh \eta$$

$$\begin{aligned} u^\mu =& \gamma(1, 0, 0, v_z) \\ \rightarrow& \left(\frac{t}{\tau}, 0, 0, \frac{z}{\tau} \right) = (\cosh \eta, 0, 0, \sinh \eta) \end{aligned}$$

$$v_z = \frac{z}{t} = \frac{\sinh \eta}{\cosh \eta}$$

$$\gamma = \cosh \eta$$

Transformation

- **Transformation rule between (τ, η) and (t, z)**

$$\begin{pmatrix} \frac{\partial}{\partial t} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial \tau}{\partial t} & \frac{\partial \eta}{\partial t} \\ \frac{\partial \tau}{\partial z} & \frac{\partial \eta}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\tau \partial \eta} \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\eta & -\sinh\eta \\ -\sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \tau} \\ \frac{\partial}{\tau \partial \eta} \end{pmatrix}$$

- **Then we have**

$$\begin{aligned} u_\mu \partial^\mu =& \gamma \frac{\partial}{\partial t} + \gamma v_z \frac{\partial}{\partial z} \\ =& \gamma \left(\cosh\eta \frac{\partial}{\partial \tau} - \sinh\eta \frac{\partial}{\tau \partial \eta} \right) \\ &+ \gamma \frac{\sinh\eta}{\cosh\eta} \left(-\sinh\eta \frac{\partial}{\partial \tau} + \cosh\eta \frac{\partial}{\tau \partial \eta} \right) \\ =& \frac{\partial}{\partial \tau} \end{aligned}$$

$$\gamma = \cosh\eta$$

$$\begin{aligned} \partial^\mu u_\mu =& \frac{\partial \gamma}{\partial t} + \frac{\partial}{\partial z}(\gamma v_z) \\ =& \frac{\partial}{\partial t} \cosh\eta + \frac{\partial}{\partial z} \sinh\eta \\ =& -\frac{1}{\tau} \sinh\eta \frac{\partial \cosh\eta}{\partial \eta} + \frac{1}{\tau} \cosh\eta \frac{\partial \sinh\eta}{\partial \eta} \\ =& \frac{1}{\tau} \end{aligned}$$

Hydrodynamic equations of Bjorken fluids

- **The pressure is Boost invariant (a constant on the hyperbola)**

$$\frac{\partial P(\tau,\eta)}{\partial\eta}=0$$

Lorentz boost is linear in η

$$\eta\rightarrow\eta-\tanh^{-1}(v_{\mathrm{boost}})$$

- **The proof**

$$\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}=-\frac{1-v^2}{\epsilon+P}\left(\nabla P+\boldsymbol{v}\frac{\partial P}{\partial t}\right)$$

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}&=\frac{\partial v_z}{\partial t}+v_z\frac{\partial v_z}{\partial z}\\&=-\frac{z}{t^2}+\frac{z}{t^2}=0\end{aligned}$$

$$\begin{aligned}\nabla P+\boldsymbol{v}\frac{\partial P}{\partial t}&=\frac{\partial P}{\partial z}+v_z\frac{\partial P}{\partial t}=(v_z,1)\begin{pmatrix}\frac{\partial P}{\partial t}\\ \frac{\partial P}{\partial z}\end{pmatrix}\\&=(v_z,1)\begin{pmatrix}\cosh\eta & -\sinh\eta\\ -\sinh\eta & \cosh\eta\end{pmatrix}\begin{pmatrix}\frac{\partial P}{\partial\tau}\\ \frac{\partial P}{\tau\partial\eta}\end{pmatrix}\\&=\cosh\eta\frac{\partial P}{\tau\partial\eta}-\sinh\eta\frac{\partial P}{\partial\tau}\\&\quad-v_z\sinh\eta\frac{\partial P}{\tau\partial\eta}+v_z\cosh\eta\frac{\partial P}{\partial\tau}\\&=\left(\cosh\eta-\frac{\sinh^2\eta}{\cosh\eta}\right)\frac{\partial P}{\tau\partial\eta}\\&=\frac{1}{\tau\cosh\eta}\frac{\partial P}{\partial\eta}=0\end{aligned}$$

Hydrodynamic equations of Bjorken fluids

- **The entropy equation**

$$\partial_\mu (s u^\mu) = 0 \quad \Longrightarrow \quad \frac{\partial s(\tau)}{\partial \tau} = -\frac{s(\tau)}{\tau}$$

- **giving the solution**

$$s(\tau) = \frac{\tau_0}{\tau} s(\tau_0)$$

- **The energy equation**

$$u_\nu \partial_\mu \left[\underbrace{(\epsilon + P) u^\mu u^\nu - g^{\mu\nu} P}_{\text{ideal fluid } T^{\mu\nu}} \right] = 0$$

$$\Downarrow$$

$$u^\mu \partial_\mu \epsilon + (\epsilon + P) \partial_\mu u^\mu = 0 \quad \Longrightarrow \quad \frac{\partial \epsilon(\tau)}{\partial \tau} = -\frac{\epsilon + P}{\tau}$$

Thermodynamic quantities

- **Consider a simple form of the equation of state with $\mu_B = 0$**

$$P = \lambda\epsilon, \quad \lambda = c_s^2 = \frac{\partial P}{\partial \epsilon}$$

- **A special case is** $\lambda = \frac{1}{3} \Rightarrow c_s = \frac{1}{\sqrt{3}}$

- **With** $\underline{\epsilon + P = (1+\lambda)\epsilon = Ts}$, $\underline{dP = \lambda d\epsilon = sdT}$, $\underline{d\epsilon = Tds}$**, we obtain**

$$d\epsilon = \frac{1}{\lambda} sdT = Tds$$

$$\rightarrow \frac{ds}{s} = \frac{1}{\lambda}\frac{dT}{T} \rightarrow s = aT^{1/\lambda}$$

- **Solution** $\boxed{s = aT^{1/\lambda}, \quad \epsilon = \frac{1}{\lambda}P = \frac{Ts}{1+\lambda} = \frac{a}{1+\lambda}T^{1+1/\lambda}}$

Hydrodynamic equations of Bjorken fluids

- **The proper time behavior of entropy density, energy density and temperature**

$$
\begin{aligned}
s(\tau) &= s_0 \frac{\tau_0}{\tau} \\
\epsilon(\tau) &= \epsilon_0 \left(\frac{\tau_0}{\tau}\right)^{1+\lambda} \\
T(\tau) &= T_0 \left(\frac{\tau_0}{\tau}\right)^{\lambda}
\end{aligned}
$$

The energy density and pressure decrease faster than the entropy under the scaling expansion of the fluid.

- **where s_0, ϵ_0, T_0 are values at the initial time τ_0.**

Relations to observables

- Let us construct the relations between s_0 and ϵ_0 to $\frac{dN}{dy}$ and $\frac{dE_T}{dy}$ at freeze-out time τ_f (y is momentum rapidity). Since the volume element on the freeze-out hyper-surface at τ_f in (1+1)-dim expansion is $\pi R^2 \tau_f d\eta \approx \pi R^2 \tau_f dy$, we have

$$\frac{dN}{dy} = \pi R^2 \tau_f \underbrace{n_f}_{\text{number density}} \quad \overset{s_f = \xi n_f}{\Longrightarrow} \quad s_0 \tau_0 = s_f \tau_f = \frac{\xi}{\pi R^2} \frac{dN}{dy}$$

Problem: the relationship between (momentum) rapidity and pseudo-rapidity?

- Similarly the total energy produced per unit rapidity is given by

$$\frac{dE}{dy} = \pi R^2 \tau_f \epsilon_f = \pi R^2 \tau_0 \epsilon_0 \left(\frac{\tau_0}{\tau_f}\right)^\lambda \Longrightarrow \epsilon_0 = \frac{1}{\pi R^2 \tau_0} \left(\frac{\tau_f}{\tau_0}\right)^\lambda \left.\frac{dE_T}{dy}\right|_{y=0}$$

$$\left.\frac{dE}{dy}\right|_{y=0} = \left.\frac{dE_T}{dy}\right|_{y=0}$$

A measure of the energy transfer due to the work done by pressure during expansion

Relations to observables

- Another way to estimate ϵ_0 is to use the entropy density and convert this to the energy density by using the equation of state

$$\epsilon = \frac{1}{(1+\lambda)a^{\lambda}} s^{1+\lambda}$$

$$\epsilon_0 = \frac{1}{(1+\lambda)a^{\lambda}} s_0^{1+\lambda} = \frac{1}{(1+\lambda)a^{\lambda}} \left(\frac{\xi}{\pi R^2 \tau_0} \frac{dN}{dy} \right)^{1+\lambda}$$

- Using these two formula we can estimate ϵ_0 by the observed particle number per unit rapidity in the central rapidity region. By equating two formula we can determine a (treating τ_0 as parameter).

Transport for pre-equilibrium processes

Classical Boltzmann equation

- **One-particle distribution function in phase space $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{p})$. For simplicity, we do not consider spin or any other internal degrees of freedom.**
- **The particle density and current can be expressed in terms of $\boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{p})$**

$$n(t, \boldsymbol{x}) = \int d^3p f(t, \boldsymbol{x}, \boldsymbol{p})$$

$$\boldsymbol{J}(t, \boldsymbol{x}) = \int d^3p \boldsymbol{v} f(t, \boldsymbol{x}, \boldsymbol{p})$$

- **The change in distribution with time takes place through two different processes: drift and collision**

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{drift}} + \left(\frac{\partial f}{\partial t}\right)_{\text{collision}}$$

Classical Boltzmann equation

- **The drift term describes the change in particle distribution through single-particle motion flowing into and out of phase-space volume**

$$\left(\frac{\partial f}{\partial t}\right)_{\text{drift}} = -\left(\boldsymbol{v}\cdot\nabla_x + \boldsymbol{F}\cdot\nabla_p\right)$$

- **The collision term describes the change through kicking-in (gain) and kicking-out (loss) processes due to particle collisions in the phase-space volume**

$$\left(\frac{\partial f}{\partial t}\right)_{\text{collision}} = C_{\text{gain}} - C_{\text{loss}}$$

$$C_{\text{gain}} = \underline{\frac{1}{2}}\int d^3p_2 d^3p_1' d^3p_2' w(1'2' \to 12)\underline{f^{(2)}(t,\boldsymbol{x},\boldsymbol{p}_1',\boldsymbol{p}_2')}$$

$$C_{\text{loss}} = \underline{\frac{1}{2}}\int d^3p_2 \underline{d^3p_1' d^3p_2' w(12 \to 1'2')} f^{(2)}(t,\boldsymbol{x},\underline{\boldsymbol{p}_1},\boldsymbol{p}_2)$$

$$\boldsymbol{p} \equiv \boldsymbol{p}_1$$

identical particle

Transition rate for two particles in $(\boldsymbol{p}_1,\boldsymbol{p}_2)$ to the range of $(\boldsymbol{p}_1',\boldsymbol{p}_2')\sim(\boldsymbol{p}_1'+d\boldsymbol{p}_1',\boldsymbol{p}_2'+d\boldsymbol{p}_2')$

two-particle distribution

Classical Boltzmann equation

- **The detailed balance relation results from the time-reversal and rotational invariance of the two-body scattering**

$$w(1'2' \to 12) = w(12 \to 1'2')$$

- **Boltzmann proposed "Strosszahl Ansatz“ (1872): the correlation between the two particles before the collision is neglected:** $f^{(2)}(t, \boldsymbol{x}, \boldsymbol{p}_1, \boldsymbol{p}_2) = f(t, \boldsymbol{x}, \boldsymbol{p}_1) f(t, \boldsymbol{x}, \boldsymbol{p}_2)$
- **Boltzmann equation (celebrated non-linear integro-differential equation)**

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla_x + \boldsymbol{F} \cdot \nabla_p = C[f]$$

$$C[f] = \frac{1}{2} \int d^3p_2 d^3p_1' d^3p_2' w(12 \to 1'2') \times [f(t, \boldsymbol{x}, \boldsymbol{p}_1') f(t, \boldsymbol{x}, \boldsymbol{p}_2') - f(t, \boldsymbol{x}, \boldsymbol{p}_1) f(t, \boldsymbol{x}, \boldsymbol{p}_2)]$$

Collision term

- The differential cross-section is related to the transition rate

$$|\boldsymbol{v}_1 - \boldsymbol{v}_2| d\sigma = w(12 \to 1'2') d^3 p_1' d^3 p_2'$$

- The collision term can be put into the form

$$C[f] = \frac{1}{2} \int d^3 p_2 \int d\Omega |\boldsymbol{v}_1 - \boldsymbol{v}_2| \frac{d\sigma}{d\Omega} [f_{1'} f_{2'} - f_1 f_2]$$

scattering solid angle between $\boldsymbol{p}_1 - \boldsymbol{p}_2$ and $\boldsymbol{p}_1' - \boldsymbol{p}_2'$

- Note that most of the integrals over momenta can be carried out due to implicit delta-functions in $w(12 \to 1'2')$ representing the conservation of both total energy and total momentum.
- Maxwell-Boltzmann distribution in equilibrium can be derived as a unique stationary solution of the transport equation.

Equilibrium condition and relaxation-time approximation

- **The necessary and sufficient condition for $C[f_{MB}] = 0$ is**

$$f_{MB}(\boldsymbol{p}_1)f_{MB}(\boldsymbol{p}_2) = f_{MB}(\boldsymbol{p}'_1)f_{MB}(\boldsymbol{p}'_2)$$

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}'_1 + \boldsymbol{p}'_2$$

- **We see that $\ln f_{MB}$ is an additive and a conserved quantity. So it can be written as a linear combination of E_p and $\boldsymbol{p}$**

$$\ln f_{MB}(\boldsymbol{p}) = a + b_0 E_p + \boldsymbol{b}\cdot\boldsymbol{p}$$

- **For example, for non-relativistic particles with averaged momentum $\boldsymbol{p}_0$, the standard Maxwell-Boltzmann distribution**

$$f_{MB}^{\text{nonrel}}(\boldsymbol{p}) = \frac{n}{(2\pi m T)^{3/2}} \exp\left[-\frac{(\boldsymbol{p}-\boldsymbol{p}_0)^2}{2mT}\right]$$

- **Relaxation time approximation: the collision term can be linearized as**

$$C[f] \approx -\frac{1}{\tau}(f - f_{\text{eq}}) = -\frac{1}{\tau}\delta f$$

relaxation time

$$\tau = \frac{1}{n\sigma_{\text{tot}}\langle v\rangle}$$

Relaxation-time approximation

- The spatially uniform distribution (no x dependence)

$$\frac{\partial \delta f}{\partial t} \approx -\frac{1}{\tau}\delta f \quad \Longrightarrow \quad f(t,\boldsymbol{p}) = f_{\rm eq}(\boldsymbol{p}) + [f(t=0,\boldsymbol{p}) - f_{\rm eq}(\boldsymbol{p})]\exp\left(-\frac{t}{\tau}\right)$$

- which shows that the system approaches the equilibrium distribution with a typical time scale τ

$$f_{\rm eq,1} f_{\rm eq,2} = f_{\rm eq,1'} f_{\rm eq,2'}$$

equilibrium condition

$$C[f] = \frac{1}{2}\int d^3p_2 d^3p_1' d^3p_2' w(12\to 1'2')$$
$$\times [(f_{\rm eq,1'}+\delta f_{1'})(f_{\rm eq,2'}+\delta f_{2'}) - (f_{\rm eq,1}+\delta f_1)(f_{\rm eq,2}+\delta f_2)]$$
$$\approx -\boxed{\frac{1}{2}}\delta f_1 \int d^3p_2 d^3p_1' d^3p_2' w(12\to 1'2')$$
$$\times \left[f_{\rm eq,2} + f_{\rm eq,1}\frac{\delta f_2}{\delta f_1} - f_{\rm eq,1'}\frac{\delta f_{2'}}{\delta f_1} - f_{\rm eq,2'}\frac{\delta f_{1'}}{\delta f_1}\right]$$

relaxation time

$$\Longrightarrow \quad \tau = \frac{1}{n\sigma_{\rm tot}\langle v\rangle}$$

set to zero

Boltzmann's H-theorem

- **The entropy density and entropy current**

$$s(t,\boldsymbol{x}) = \int d^3p f(t,\boldsymbol{x},\boldsymbol{p})\left[1-\ln f(t,\boldsymbol{x},\boldsymbol{p})\right]$$

$$\boldsymbol{s}(t,\boldsymbol{x}) = \int d^3p \boldsymbol{v} f(t,\boldsymbol{x},\boldsymbol{p})\left[1-\ln f(t,\boldsymbol{x},\boldsymbol{p})\right]$$

$$\frac{\partial s(t,\boldsymbol{x})}{\partial t} = -\int d^3p \frac{\partial f}{\partial t}\ln f$$

$$\nabla\cdot\boldsymbol{s}(t,\boldsymbol{x}) = -\int d^3p \boldsymbol{v}\cdot\nabla f \ln f$$

- **The time variation of the entropy density and entropy flow are related**

$$\frac{\partial s}{\partial t} + \nabla\cdot\boldsymbol{s} = -\int d^3p C[f]\ln f$$

For equilibrium dist. $C[f]=0$, the entropy is conserved

- **The Boltzmann entropy and the associated H-function for a non-equilibrium system**

$$S(t) = -H(t) = \int d^3x s(t,\boldsymbol{x})$$

Boltzmann's H-theorem

- **The production rate of the entropy**

$$\boxed{w \geq 0, \quad (x-y)\ln\frac{x}{y} \geq 0}$$

$$\frac{dS}{dt} = \frac{1}{8}\int d^3x d^3p_1 d^3p_2 d^3p_1' d^3p_2' \underline{w(12 \to 1'2')}$$
$$\times \underline{(f_{1'}f_{2'} - f_1 f_2)\left[\ln(f_{1'}f_{2'}) - \ln(f_1 f_2)\right] \geq 0}$$

- **The entropy is conserved for** $f_{\mathrm{eq},1}f_{\mathrm{eq},2} = f_{\mathrm{eq},1'}f_{\mathrm{eq},2'}$
- **The proof of Boltzmann's H-theorem**

$$\int d^3x \nabla \cdot \boldsymbol{s}(t,x) = 0$$

$$\frac{dS}{dt} = \int d^3x \frac{\partial}{\partial t} s(t,\boldsymbol{x}) = -\int d^3x \int \underline{d^3p} C[f] \underline{\ln f}$$
$$= -\frac{1}{2}\int d^3x \underline{d^3p_1} d^3p_2 d^3p_1' d^3p_2' w(12 \to 1'2')(f_{1'}f_{2'} - f_1 f_2) \underline{\ln f_1}$$
$$= -\frac{1}{8}\int d^3x d^3p_1 d^3p_2 d^3p_1' d^3p_2' w(12 \to 1'2')$$
$$\times (f_{1'}f_{2'} - f_1 f_2) \underline{\left[\ln(f_1 f_2) - \ln(f_{1'}f_{2'})\right]}$$

use the symmetry

$$1 \leftrightarrow 2,\ 1' \leftrightarrow 2',\ 12 \leftrightarrow (-)1'2'$$

Covariant form of classical transport equation

- **Covariant form of transport equation (de Groot, van Leeuwen, van Weert, Relativistic Kinetic Theory, 1980)**

$$x^\mu = (t, \boldsymbol{x}),\ p^\mu = (p^0 = E_p, \boldsymbol{p})$$
$$f(x,p)|_{p^0=E_p} = f(t, \boldsymbol{x}, \boldsymbol{p})$$

- **Particle-number current, energy-momentum tensor and entropy current**

$$J^\mu = (n, \boldsymbol{J}) = \int \frac{d^3p}{p_0} p^\mu f(x,p),\quad T^{\mu\nu} = \int \frac{d^3p}{p_0} p^\mu p^\nu f(x,p)$$
$$s^\mu = (s, \boldsymbol{s}) = \int \frac{d^3p}{p_0} p^\mu f(x,p) \left[1 - \ln f(x,p)\right]$$

- **Lorentz-invariant volume element**

$$\frac{d^3p}{2p_0} = d^4p\theta(p_0)\delta(p^2 - m^2)$$

Covariant form of classical transport equation

- **We obtain a covariant form of transport equation**

$$F^\mu(x,p) = p_0(\boldsymbol{v}\cdot\boldsymbol{F},\boldsymbol{F})$$

$$\left[p^\mu \partial_\mu^x + F^\mu(x,p)\partial_\mu^p\right] f(x,p) = p_0 C[f]$$

$$p_{10} \equiv p_0$$

$$p_0 C[f] = \frac{1}{2}\int \frac{d^3p_2}{p_{20}}\frac{d^3p_1'}{p_{10}'}\frac{d^3p_2'}{p_{20}'}\underline{[p_{10}p_{20}p_{10}'p_{20}' w(12\to 1'2')]}$$
$$\times\left[f(x,p_1')f(x,p_2') - f(x,p_1)f(x,p_2)\right]$$

$$\widetilde{w}(12\to 1'2')$$

- **Conservation of energy, momentum and other quantum numbers can be obtained by momentum integrals with weight** $\boldsymbol{\chi(p) = 1, p^\alpha}$

$$\int \frac{d^3p}{p_0}\underline{\chi(p)}\left[p^\mu \partial_\mu^x + F^\mu(x,p)\partial_\mu^p\right] f(x,p) = \int \frac{d^3p}{p_0}\underline{\chi(p)}p_0 C[f]$$

Conservation laws

- **We obtain the identity by applying a similar step as in proof of the H-theorem**

$$\int \frac{d^3p_1}{p_{10}} \chi(p_1) p_{10} C[f_1]$$

$$A \equiv \chi(p_1) + \chi(p_2) - \chi(p_1') - \chi(p_2') = 0$$

$$= \frac{1}{8} \int \frac{d^3p_1}{p_{10}} \frac{d^3p_2}{p_{20}} \frac{d^3p_1'}{p_{10}'} \frac{d^3p_2'}{p_{20}'} \widetilde{w}(12 \to 1'2') \left(f_{1'} f_{2'} - f_1 f_2 \right) A = 0$$

- **Using the Boltzmann transport equation and the relation $\int d^3p\, \partial_\mu^p \left[p_0^{-1} p^\alpha F^\mu(x,p) \right] f = 0$ after the partial integration, we arrive at the macroscopic conservation laws for $\chi = 1, p^\alpha$ for particle number and energy-momentum**

$$\partial_\mu J^\mu(x,p) = 0, \quad \partial_\nu T^{\nu\mu} = 0$$

Local H-theorem and local equilibrium

- **The entropy production rate can be written as**

$$\partial_\mu s^\mu(x) = -\int d^3p C[f] \ln f \geq 0 \xrightarrow{C[f]=0} \partial_\mu s^\mu(x) = 0$$

- **In local equilibrium, we have**

$$f(x,p_1)f(x,p_2) = f(x,p_1')f(x,p_2')$$

$$\ln f(x,p) = \underline{a(x) + b_\mu(x)p^\mu}$$

$$a(x) = \frac{\mu(x)}{T(x)}, \quad b^\mu(x) = \frac{u^\mu(x)}{T(x)}$$

$$f_{\mathrm{B}}(x,p) = N \exp\left[-\beta(x)\left(p_\mu u^\mu(x) - \mu(x)\right)\right]$$

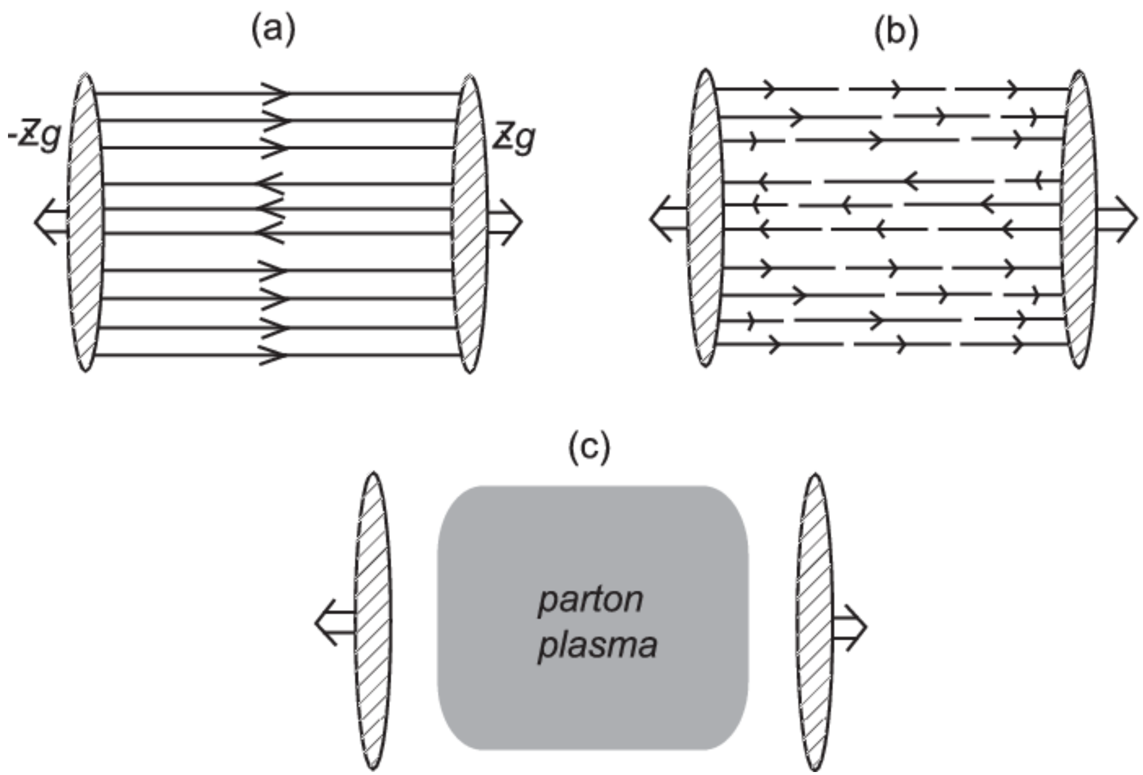
- **Extension to Bose-Einstein and Fermi-Dirac distributions**

$$f_{1'}f_{2'}(1\pm f_1)(1\pm f_2) = (1\pm f_{1'})(1\pm f_{2'})f_1 f_2$$

$$f_{BE(FD)} = \frac{1}{(2\pi)^3}\frac{1}{\exp\left[\beta(x)\left(p_\mu u^\mu(x) - \mu(x)\right)\right] \mp 1}$$

Formation and evolution of QGP

The initial condition: color-string breaking model



(a) The color strings formed between two nuclei passing through each other. An average color charge, $\pm gZ$, is accumulated in each nucleus due to the exchange of multiple gluons at the time of the collision.

(b) Decay of the strings and the production of quark and gluon pairs due to the Schwinger mechanism

(c) The formation of the quark-gluon plasma due to the mutual interaction of the produced partons.

The initial condition: color-string breaking model

- Two nuclei collide and pass through each other. Wounded nucleons in nuclei have color excitations and become a source of color strings and color ropes between the two nuclei. The color-string is assumed to be a coherent and classical color electric field.
- Schwinger mechanism: $q\overline{q}$ and gluon pairs are created under the influence of strong color electric field between two nuclei. The general pair creation rate per unit space-time volume is given by

$$w_{q/g}(\sigma) = -\frac{\sigma}{4\pi^2}\int_0^\infty dp_T^2 \ln\left[1 \mp \exp\left(-\frac{\pi p_T^2}{\sigma}\right)\right]$$

$$w_q(\sigma \sim gE_c) \sim N_f \frac{(gE_c)^2}{24\pi}, \quad w_g(\sigma \sim gE_c) \sim N_c \frac{(gE_c)^2}{48\pi}$$

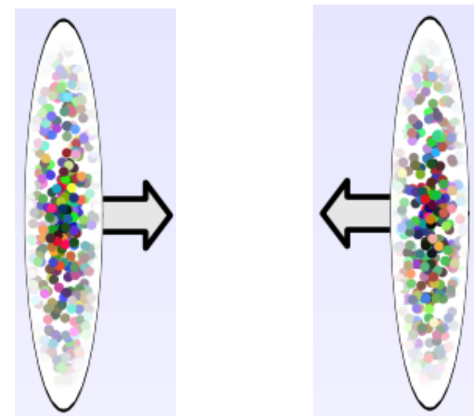
- The quark-gluon plasma with local thermal equilibrium is expected to be produced through the mutual interactions of the quarks and gluons just formed.

The initial condition: color glass condensate

- **Valence quarks carry color sources ρ_1 and ρ_2 which are located on the light cone**

McLerran, Venugopalan (2004)

$$
\begin{aligned}
J^\mu &= J_1^\mu + J_2^\mu \\
&= \delta^{\mu+}\delta(x^-)\rho_1(\boldsymbol{x}_T) + \delta^{\mu-}\delta(x^+)\rho_2(\boldsymbol{x}_T)
\end{aligned}
$$



$$
\begin{aligned}
L &= -\frac{1}{2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + J^\mu A_\mu \\
&\rightarrow [D_\mu, F^{\mu\nu}] = J^\nu
\end{aligned}
$$

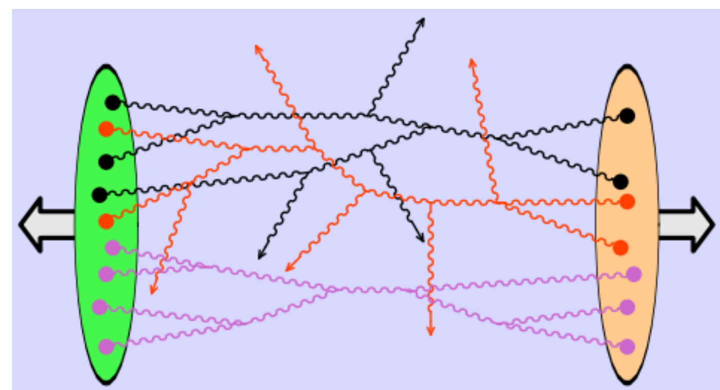
$$A^\mu = A_a^\mu t^a$$

$$F^{\mu\nu} = F_a^{\mu\nu} t^a$$

$$D^\mu = \partial^\mu - igA^\mu$$

$$F^{\mu\nu} = \frac{i}{g}[D^\mu, D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu + g f_{abc} A_b^\mu A_c^\nu$$



Figures taken from F. Gelis' lecture in Schleching, Germany, February 2014

Color glass condensate

- **Light-cone variables ($\mu = +, -, 1, 2$)**

$$x^\mu = (x^+, x^-, \mathbf{x}_T)$$

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \qquad \mathbf{x}_T = (x^1, x^2)$$

- **The Minkowski metric tensor and some formula**

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^-, x^+, -\mathbf{x}_T)$$

$$d^4x = dx^+ dx^- d^2x_T$$

$$x \cdot y = x^+ y^- + x^- y^+ - x_T^i y_T^i$$

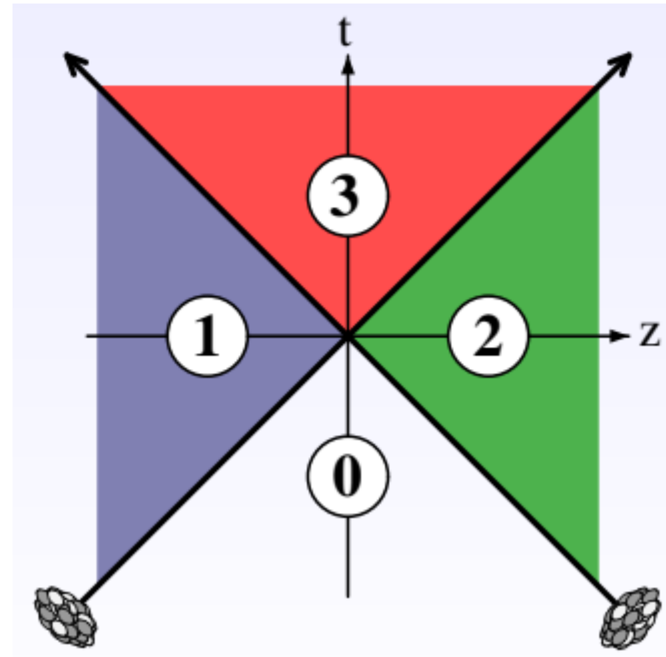
$$\partial^\mu = (\partial^+, \partial^-, -\partial_T) = \left(\frac{\partial}{\partial x^-}, \frac{\partial}{\partial x^+}, -\partial_T\right)$$

$$\partial_\mu = (\partial_+, \partial_-, \partial_T) = \left(\frac{\partial}{\partial x^+}, \frac{\partial}{\partial x^-}, \partial_T\right)$$

Color glass condensate

- Region 0 $(x^+ < 0,\ x^- < 0)$: $A^\mu = 0$
- Region 1 $(x^+ < 0,\ x^- > 0)$: the field depends on ρ_1
- Region 2 $(x^+ > 0,\ x^- < 0)$: the field depends on ρ_2
- Region 3 $(x^+ > 0,\ x^- > 0)$: the field depends on ρ_1 and ρ_2
- The continuity equation

$$[D_\mu, J^\mu] = \partial_\mu J^\mu - ig\,[A_\mu, J^\mu] = 0$$



Figures taken from F. Gelis' lecture in Schleching, Germany, February 2014

- We choose an axial gauge which satisfies the continuity equation

$$x^+ A^- + x^- A^+ = 0$$

Color glass condensate

- The gluon fields in the axial gauge are

$$
\begin{aligned}
A^+(x) &= \Theta(x^+)\Theta(x^-)\underline{x^+ A(\tau,\mathbf{x}_T)} \\
A^-(x) &= -\Theta(x^+)\Theta(x^-)\underline{x^- A(\tau,\mathbf{x}_T)} \\
A^i(x) &= \Theta(-x^+)\Theta(x^-)\underline{A_1^i(\mathbf{x}_T)} \\
&\quad + \Theta(x^+)\Theta(-x^-)\underline{A_2^i(\mathbf{x}_T)} \\
&\quad + \Theta(x^+)\Theta(x^-)\underline{A_T^i(\tau,\mathbf{x}_T)}
\end{aligned}
$$

Fields after collisions

Fields from ρ_1 before collisions

Fields from ρ_2 before collisions

- In the forward light cone the fields have the simple form

$$
\begin{aligned}
A^+(x) &= x^+ A(\tau,\mathbf{x}_T) \\
A^-(x) &= -x^- A(\tau,\mathbf{x}_T) \\
A^i(x) &= A_T^i(\tau,\mathbf{x}_T)
\end{aligned}
$$

There is no explicit dependence on the space-time rapidity η reflecting the boost-invariance of the system

Color glass condensate

- **By the above gauge potential, we can express the strength tensor**

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$$

$$F^{+-} = -2A - \tau \frac{\partial A}{\partial \tau}$$

$$F^{ij} = \partial^i A_T^j - \partial^j A_T^i - ig[A_T^i, A_T^j]$$

$$F^{i+} = x^+ \left(-\frac{1}{\tau} \frac{\partial A_T^i}{\partial \tau} + [D^i, A] \right)$$

$$F^{i-} = x^- \left(-\frac{1}{\tau} \frac{\partial A_T^i}{\partial \tau} - [D^i, A] \right)$$

- **Solve EOM for gluon fields**

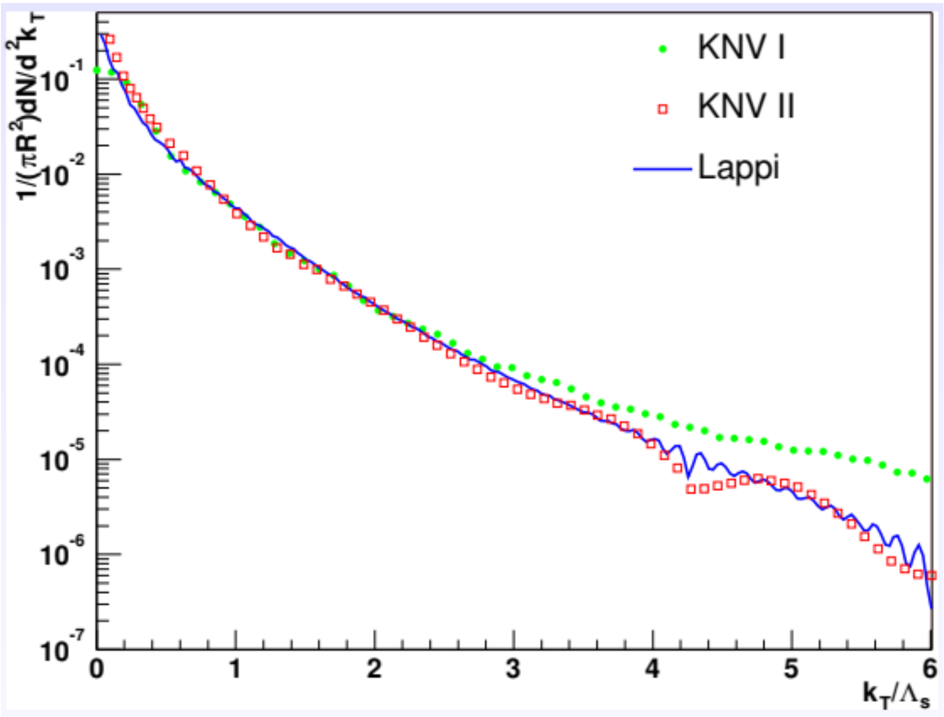
$$\begin{aligned} 0 &= [D_\mu, F^{\mu\nu}] = [\partial_\mu - igA_\mu, F^{\mu\nu}] \\ &= \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}] \end{aligned}$$

Gluons' momentum spectra in the early stage of HIC

$$\Longrightarrow \quad A_\mu^a(\tau, \mathbf{x}_T) \Longrightarrow |a(\tau, \mathbf{k}_T)|^2$$

Color glass condensate

- **Single gluon transverse momentum spectra**



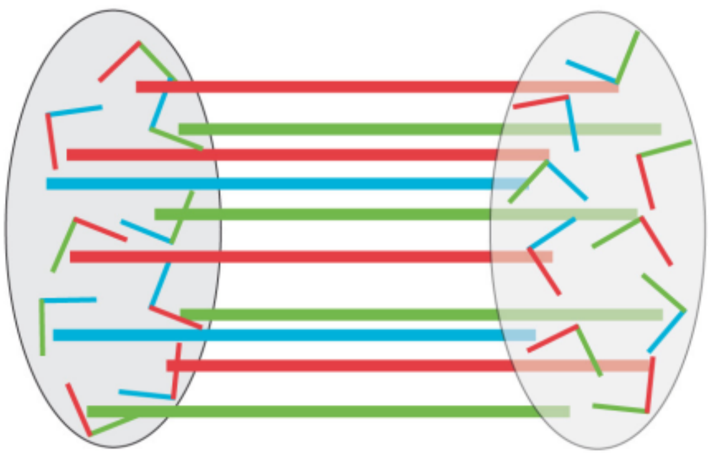
Real time lattice calculation

Important softening at small k_T compared to PQCD (saturation effect)

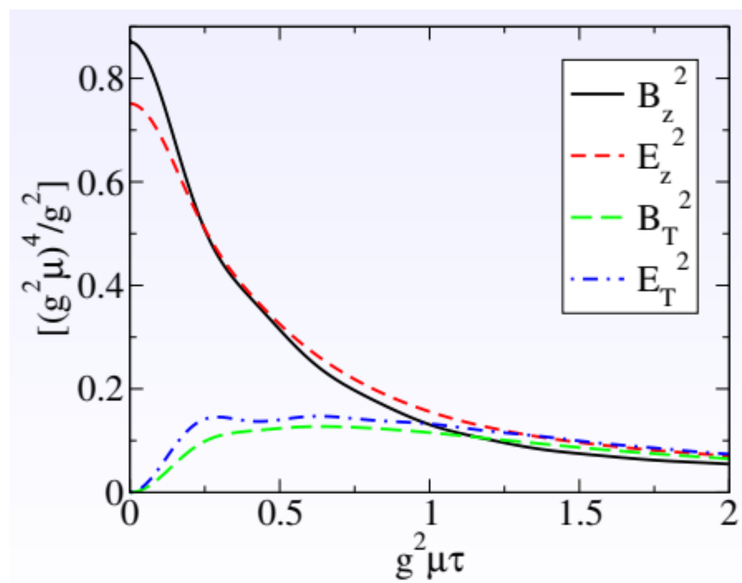
Figures taken from F. Gelis' lecture at the 45th 'Arbeitstreffen Kernphysik', Schleching, Germany, February 2014

Color glass condensate

- **Glassma = Glass Plasma** **[Lappi, MecLerran (2006)]**
- **Before the collision, the chromo-electric and chromo-magnetic fields are localized in two sheets transverse to the beam axis**
- **Immediately after the collision ($\tau = 0$), the chromo-electric and chromo-magnetic fields have become longitudinal**

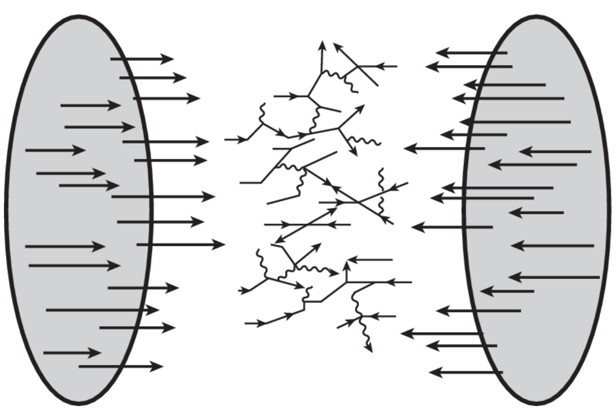


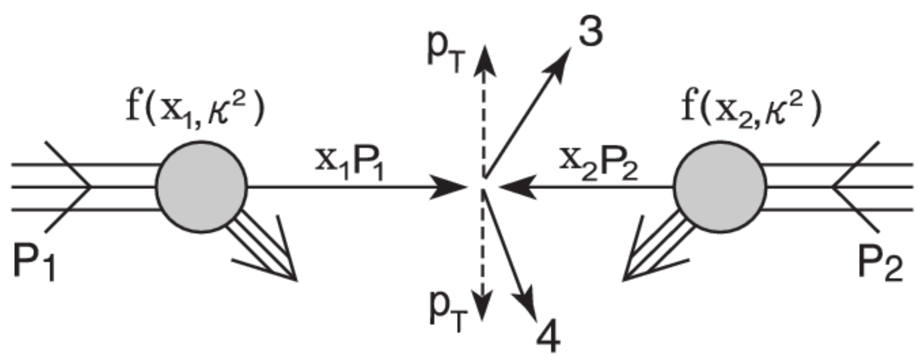
$$E^z = ig\left[A_1^i, A_2^i\right],\ B^z = ig\epsilon^{ij}\left[A_1^i, A_2^j\right]$$



The initial condition: mini-jets in PQCD models

- In high-energy HIC, hard or semi-hard parton scatterings in the initial stage may result in a large amount of jet production.
- Mini-jets have typical transverse momentum of a few GeV and can give rise to an important fraction of the transverse energy, which are good candidates for initial seeds of QGP.
- The mini-jet production can be estimated by models based on Monte Carlo event generators such as HIJING [Wang, Gyulassy,1994; Wang,1997].





PQCD is applicable for semi-hard processes with $p_T > p_0 \sim$1-2 GeV

The initial condition: minijets production

- The semi-inclusive cross-section of a dijet with a transverse momentum $p_T \geq p_0 \sim$ 1-2 GeV in pp collisions
- The cross-section may be factorized into a long-distance part and a short-distance part:

transverse momentum of out-going partons

$$\frac{d^3\sigma_{\rm jet}^{\rm pp}}{dp_T^2 dy_1 dy_2} = K \sum_{a,b=q,\bar{q},g} x_1 f_a(x_1, p_T^2) x_2 f_b(x_2, p_T^2) \boxed{\frac{d\sigma^{ab}(\hat{s},\hat{t},\hat{u})}{d\hat{t}}}$$

Diff. cross section of partons scatterings

rapidities of out-going partons

$$\sigma_{\rm jet}^{\rm pp} = \frac{1}{1+\delta} \int_{p_0^2}^{s/4} dp_T^2 dy_1 dy_2 \frac{d^3\sigma_{\rm jet}^{\rm pp}}{dp_T^2 dy_1 dy_2}$$

distributions of incident partons

symmetry factor $\delta = 1$ for out-going identical partons

momentum fractions

$$x_1 = \frac{1}{2} x_T (e^{y_1} + e^{y_2})$$

$$x_2 = \frac{1}{2} x_T (e^{-y_1} + e^{-y_2})$$

$$x_T = \frac{2p_T}{\sqrt{s}}$$

Mandelstam variables

$$\hat{s} = x_1 x_2 s$$

$$\hat{t} = -p_T^2 [1 + \exp(y_2 - y_1)]$$

$$\hat{u} = -p_T^2 [1 + \exp(y_1 - y_2)]$$

Problem: derive these formula

The initial condition: minijets production

- **Assuming independent binary parton collisions, the total number of jets in a central AA collision**

$$N_{\text{jet}}^{AA}(\sqrt{s};p_0,|y|\leq\Delta y)\approx A^2T_{AA}(b=0)\sigma_{\text{jet}}^{\text{pp}}(\sqrt{s};p_0,|y|\leq\Delta y)$$

- **For a Au+Au collision, we have [$\boldsymbol{T_{AB}(b)}$ is normalized to 1]**

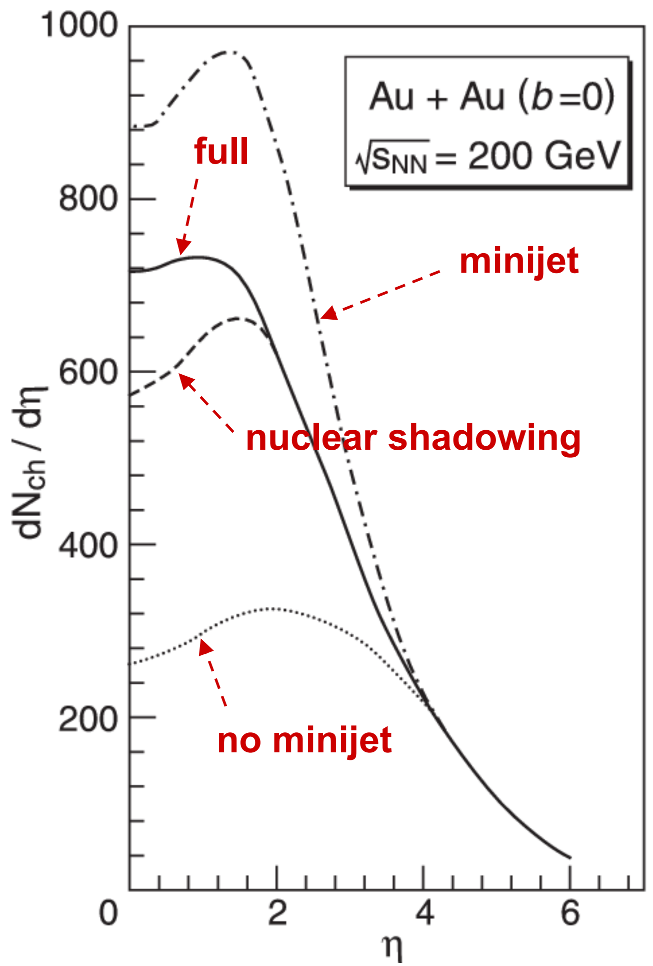
$$A^2T_{\text{Au+Au}}(0)=\left.\frac{9A^2}{8\pi R_A^2}\right|_{R_A\approx 7\text{fm}}\approx 28.4\,\text{mb}^{-1}$$

- **Partons produced in the central rapidity region probe the gluon distributions at**

$$x=\frac{2p_T}{\sqrt{s}}\sim\begin{cases}10^{-3}\text{ for LHC }(\sqrt{s_{NN}}=5.6\text{ TeV})\\10^{-2}\text{ for RHIC }(\sqrt{s_{NN}}=200\text{ GeV})\end{cases}$$

- **There are two nuclear effects which are not included in the above simple formula: the initial state and final state interactions: nuclear shadowing and the energy loss or jet quenching.**

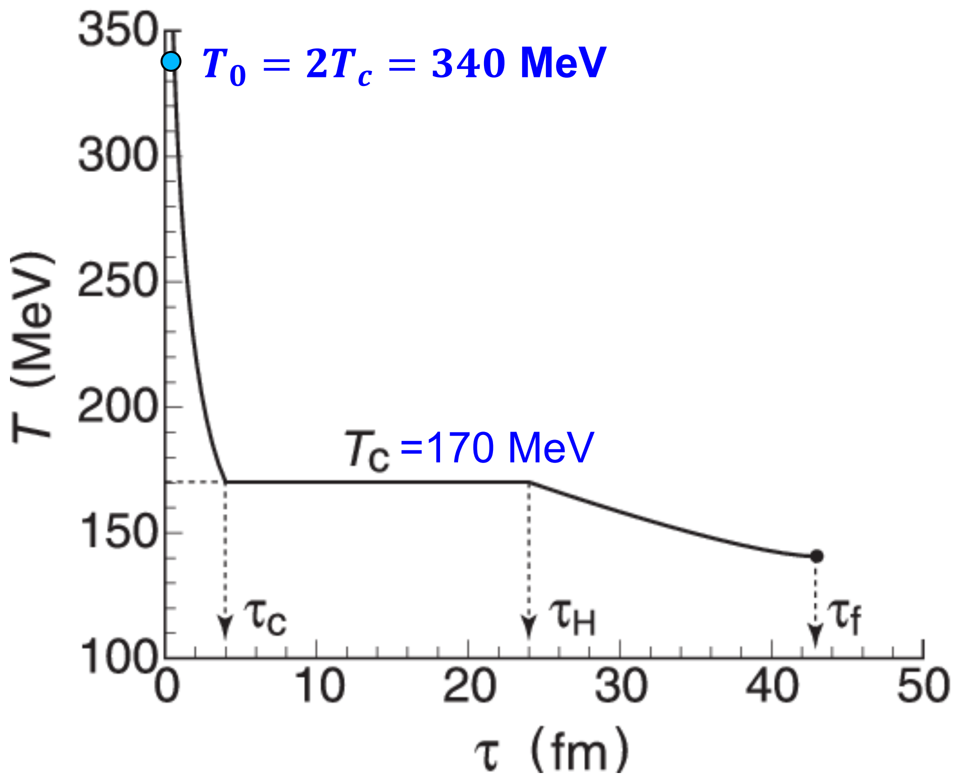
The initial condition: minijets production



- Figure adapted from the HIJING simulation (Wang, Gyulassy 1994; Wang 1997) on the charge multiplicity per unit pseudo-rapidity $dN_{ch}/d\eta$, for Au+Au collisions with a zero impact parameter at $\sqrt{s_{NN}}$=200 GeV.
- The solid line includes all possible effects, namely the soft production, semi-hard mini-jets, nuclear shadowing and jet quenching.
- The dotted line denotes the case where only soft interactions are considered without mini-jet production.
- The dash-dotted line corresponds to the case for mini-jets with $p_T > 2$ GeV.
- The dashed line includes the effect of the nuclear shadowing.

Longitudinal expansion

- **A model for longitudinal plasma expansion with a first-order QCD phase transition**



Time evolution of the temperature of hot matter with a first-order QCD phase transition (solid line) at T_c = 170 MeV created in the central region of an ultra-relativistic heavy nucleus-nucleus collision. The initial temperature is taken to be $T_0 = 2T_c$ at τ_0=0.5 fm and the freeze-out time is given by τ_H/τ_c = 5.9.

Longitudinal expansion

- In the Bjorken picture, the longitudinal expansion of QGP obeys the simple scaling solution

$$s = \frac{s_0 \tau_0}{\tau}, \quad v_z = \frac{z}{t}$$

- In the Stefan-Boltzmann limit of the QGP entropy, the temperature in the QGP period behaves as

$$T = T_c \left(\frac{\tau_c}{\tau} \right)^{1/3}, \quad (\tau_0 < \tau < \tau_c)$$

- In the first order phase transition, the system becomes a mixture of QGP and hadronic plasma during the phase transition, we introduce the volume fraction $f(\tau)$ of the hadronic phase

$$s(\tau) = s_H(\tau)\underline{f(\tau)} + s_{QGP}(\tau)\left[1 - f(\tau)\right] = \frac{s_0 \tau_0}{\tau}$$

$$f(\tau_c) = 0, \; f(\tau_H) = 1$$

Longitudinal expansion

- **In the Stefan-Boltzmann limi, the lifetime of the mixed phase is given by the ratio**

$$\frac{\tau_H}{\tau_c}=\frac{s_{QGP}}{s_H}=\frac{d_{QGP}}{d_M}=\begin{cases}12.3 & (N_f=2)\\ 5.9 & (N_f=3)\end{cases}$$

Problem: derive this formula

$$S_H = 4d_M\frac{\pi^2}{90}T^3$$

$$S_{QGP} = 4d_{QGP}\frac{\pi^2}{90}T^3$$

$$d_M = N_f^2-1$$

$$d_{QGP} = d_g+\frac{7}{8}d_q$$

$$= 2_{\text{spin}}\times(N_c^2-1)+[2_{\text{spin}}\times 2_{q\bar{q}}\times N_cN_f]$$

Difference in Fermi and Bose statistics

- **After the phase transition is over at the interacting hadron plasma undergoes a hydrodynamic expansion. In the Stefan-Boltzmann limit, we have**

$$T=T_c\left(\frac{\tau_H}{\tau}\right)^{1/3},\quad (\tau_H<\tau<\tau_f)$$

Freeze-out time

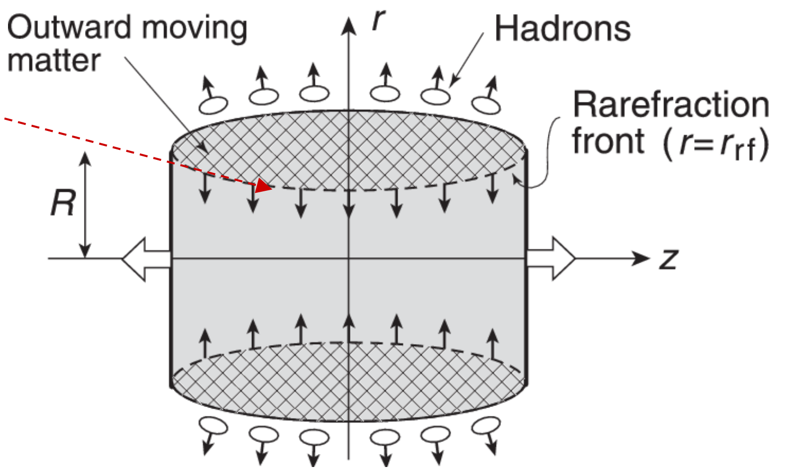
Transverse expansion

- **A transverse hydrodynamic expansion is caused by a transverse pressure gradient which is significant near the transverse edge of the system ($r \approx R$) [Bjorken (1983); Blaizot, Ollitrault (1990); Rischke (1999)]**
- **A rarefaction wave is created at the transverse edge: a flow where the fluid is continually rarefied as it moves. A boundary of the rarefaction wave (the wave front) propagates inwards at the velocity of sound in the local rest frame of the fluid.**

$$r_{\mathrm{rarefaction}} = R - c_s\sqrt{t^2 - z^2}$$

$$r_T \sim \frac{R}{c_s}$$

time scale for transverse expansion



Transverse expansion

- The invariant momentum spectrum of hadrons emitted at freeze-out is given by a local thermal distribution $f(x,p)$ at the freeze-out temperature T_f boosted by a local velocity field u^μ at the freeze-out hypersurface Σ_f [Cooper, Frye (1974)]

$$\boxed{\begin{aligned} E\frac{d^3N}{d^3p} &= \frac{d^3N}{m_T dm_T dy d\phi_p} = \int_{\Sigma_f} d\Sigma_\mu p^\mu f(x,p) \\ f(x,p) &= \frac{g}{(2\pi)^3}\frac{1}{\exp\{\beta(x)\left[p_\mu u^\mu(x) - \mu(x)\right]\} \mp 1} \end{aligned}}$$

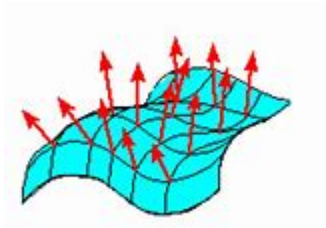
$$p^\mu = \begin{pmatrix} m_T \cosh y \\ p_T \cos\phi_p \\ p_T \sin\phi_p \\ m_T \sinh y \end{pmatrix}$$

$$y = \frac{1}{2}\ln\frac{E+p_z}{E-p_z}, \quad m_T = \sqrt{p_T^2 + m^2}$$

$$\phi_s \approx \phi_b$$

$$\begin{aligned} x^\mu =& (\tau\cosh\eta, \mathbf{x}_T, \tau\sinh\eta) \\ u^\mu(x) =& [\cosh\eta\cosh\alpha(x_T,\phi_s), \sinh\alpha(x_T,\phi_s)\cos\phi_b, \\ & \sinh\alpha(x_T,\phi_s)\sin\phi_b, \sinh\eta\cosh\alpha(x_T,\phi_s)] \\ p\cdot u =& m_T\cosh\alpha(r,\phi_s)\cosh(\eta - Y) \\ & - p_T\sinh\alpha(r,\phi_s)\cos(\phi_b - \phi_p) \end{aligned}$$

$$d\Sigma^\mu = \tau d\eta d^2x_T \frac{\partial x^\mu}{\partial\tau} = \tau d\eta d^2x_T(\cosh\eta, 0, 0, \sinh\eta)$$



Normal time-like vector on freeze-out hyper-surface Σ_f.
A special static case:
$d\Sigma_\mu = (dV, 0, 0, 0)$.

Transverse expansion

- **A cylindrical thermal source expanding in both the longitudinal (z) and transverse (r) directions with boost-invariance in the z-direction leads to the following qualitative formula for the transverse mass spectrum**

$$\frac{dN}{m_T dm_T} \sim \frac{V_f}{2\pi^2} m_T \underbrace{K_1(\xi_m) I_0(\xi_p)}_{\text{Bessel functions}}$$

volume at freeze-out

Bessel functions

Problem: derive this formula

$$\xi_m \equiv \frac{1}{T_f} m_T \cosh \alpha_f \qquad \xi_p \equiv \frac{1}{T_f} m_T \sinh \alpha_f \qquad \alpha_f = \frac{1}{2} \ln \frac{1+v_r}{1-v_r}$$

transverse rapidity

- **For $m_T \sim p_T \gg T_f$, the arguments of K_1 and I_0 are large, we can utilize the asymptotic forms of $K_1(\xi_m \to \infty)$ and $I_0(\xi_p \to \infty)$ to obtain**

$$\frac{dN}{m_T dm_T} \sim \exp\left(-\frac{m_T}{T_f^{\text{eff}}}\right), \quad T_f^{\text{eff}} \approx T_f \sqrt{\frac{1+v_r}{1-v_r}}$$

- **$T_f^{eff} > T_f$ by a blue shift factor implies that a rapidly expanding source shifts emitted particles to higher momenta.**

Transverse expansion

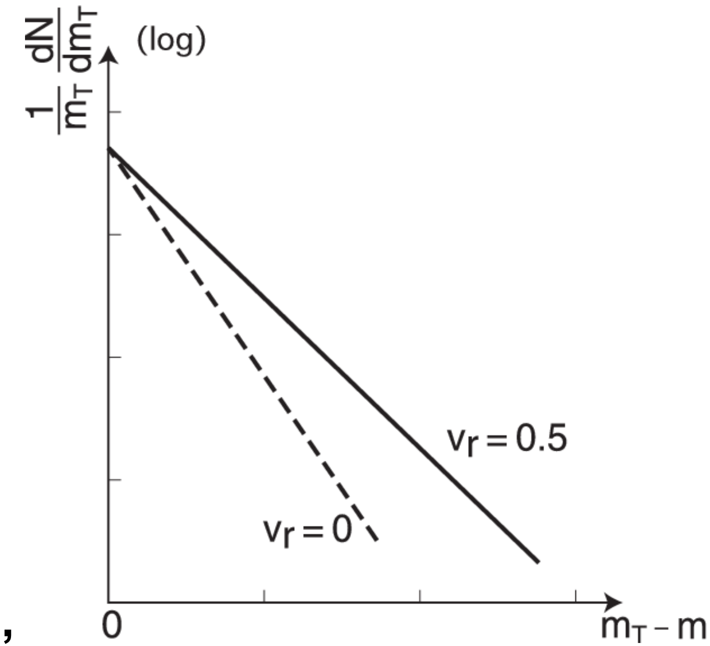
- **For moderate values of m_T, the effective freeze-out temperature is defined as**

$$T_f^{\text{eff}} = -\left[\frac{d}{dm_T}\ln\left(\frac{dN}{m_T dm_T}\right)\right]^{-1}$$

- **At the limit $m \gg T_f, p_T$ and $T_f \gg m v_r^2$**

$$T_f^{\text{eff}} \approx T_f + \frac{1}{2} m v_r^2$$

- **This shows that the heavier the particles, the more they gain momenta/energy from the flow velocity, and hence the larger the effective temperature.**



Transverse mass spectra with transverse flows of $v_r = 0$ and $v_r = 0.5$

Exercises

- For references, please read Chapters 10-13 in the book.
- Solve problems for exercises for Chapters 11-13 on page 261, 279, 293.
- For experimental results, please read Chap. 15-16 in the book.