2023中高能核核物理暑期学校

EIC核子自旋结构物理基础
Basics for Nucleon Spin Structure Physics at EIC

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2023粒子物理与核物理暑期学校

内容安排

第一部分：自旋状态的描写和高能反应过程的极化测量
Description of Spin States and Polarization Measurements in High Energy Reactions

第二部分：部分子分布函数和碎裂函数基础
Basics of Parton Distribution Functions (PDFs) and Fragmentation Functions (FFs)

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II. Description of the spin state in high energy reactions
- Spin 1/2 particles
  - Spin in non-relativistic quantum mechanics
  - Dirac equation and spin in relativistic QM
  - Helicity and chirality
  - Spin density matrix and polarization
- Spin-1 particles
  - Polarization vector and the spin polarization vector
  - Vector meson spin alignment

III. Polarization measurements in high energy reactions
- Hyperon polarization
- Vector meson spin alignment

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Introduction: The concept of electron spin

原子光谱学与量子物理理论发展

分立谱(Balmer's formulae)	→	量子力学
精细结构(fine structure)	→	电子自旋
超精细结构(hyper-fine structure)	→	质子自旋
兰姆位移(Lamb shift)	→	量子场论

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Introduction: The concept of the electron spin

电子自旋的发现

Die Naturwissenschaften 13, 953–954 (1925)

Heft 47. 20. II. 1925] Zuschriften und vorläufige Mitteilungen. 953

Ersetzung der Hypothese vom unmechanischen Zwang durch eine Forderung bezüglich des inneren Verhaltens jedes einzelnen Elektrons.

in Übereinstimmung zu kommen, muß man also diesem Modell die folgenden Forderungen stellen:

a) Das Verhältnis des magnetischen Momentes des Elektrons zum mechanischen muß für die Eigenrotation doppelt so groß sein als für die Umlaufsbewegung[2]).

b) Die verschiedenen Orientierungen vom R zur Bahnebene (oder K) des Elektrons muß, vielleicht in Zusammenhang mit einer HEISENBERG-WENTZELschen Mittelungsvorschrift[3]), die Erklärung des Relativitätsdoubletts liefern können.

G. E. UHLENBECK und S. GOUDSMIT.

Leiden, den 17. Oktober 1925.
Instituut voor Theoretische Natuurkunde.

Es ist mir ein Bedürfnis, festzustellen, daß Prof. W. J. DE HAAS mir schon vor einigen Monaten die Apparatur für ein sehr interessantes Experiment zeigte, das sich ebenfalls mit dem Problem der inneren Rotation des Elektrons beschäftigt. Obwohl mir die betreffenden Ideen von Prof. DE HAAS seit längerer Zeit bekannt waren, hatten die Herren UHLENBECK und GOUDSMIT, als sie mir kürzlich die obigen Überlegungen mitteilten, davon keinerlei Kenntnis.

P. EHRENFEST.

264 NATURE [FEBRUARY 20, 1926

Letters to the Editor.

[*The Editor does not hold himself responsible for opinions expressed by his correspondents. Neither can he undertake to return, nor to correspond with the writers of, rejected manuscripts intended for this or any other part of* NATURE. *No notice is taken of anonymous communications.*]

Spinning Electrons and the Structure of Spectra.

this moment of momentum is given by $Kh/2\pi$, where $K=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. The total angular momentum of the atom is $Jh/2\pi$, where $J=1, 2, 3$. The symbols K and J correspond to those used by Landé in his classification of the Zeeman effects of the optical multiplets. The letters S, P, D also relate to the analogy with the structure of optical spectra which we consider below. The dotted lines represent the position of the energy levels to be expected in the absence of the spin of the electron. As the arrows in-

electron.

In conclusion, we wish to acknowledge our indebtedness to Prof. Niels Bohr for an enlightening discussion, and for criticisms which helped us distinguish between the essential points and the more technical details of the new interpretation.

G. E. UHLENBECK.
S. GOUDSMIT.

Instituut voor Theoretische Natuurkunde,
Leyden, December 1925.

HAVING had the opportunity of reading this inter-

the correspondence between classical mechanics and the quantum theory. N. BOHR.

Copenhagen, January 1926.

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Introduction: The concept of the electron spin

GEORGE UHLENBECK AND THE DISCOVERY OF ELECTRON SPIN

How two young Dutchmen, one with only a master's degree, the other a graduate student, made a most important finding in theoretical atomic physics.

Abraham Pais

The owl depicted on the signet ring George Uhlenbeck used to wear—"Uhlenbeck" in German means "owl's brook"—derives from his family's coat of arms. The shield reads, in the language of heraldry: Azure, on a tree trunk proper rising from water argent, an owl contourné, head affronty. In plain language, it depicts an owl with its head turned toward you, sitting on a tree trunk in natural color, which rises up out of a silvery brook. (I owe the transcription of the Dutch blazon into English heraldry to Michael Maclagan, the Richmond Herald in the College of Arms, in London.)

Abraham Pais is Detlev Bronk Professor Emeritus at Rockefeller University, in New York. He based this article on his presentation at APS's Uhlenbeck Memorial Symposium, held in Baltimore on 3 May 1989.

34 PHYSICS TODAY DECEMBER 1989

假设	问题	解答
电子具有（绕自身转动的）额外的自由度 $s=1/2$	表面速度远大于光速？	量子效应
自旋磁矩 郎德因子$g_s=2$	电子磁矩与自身运动产生的磁场相互作用？	相对运动，外场
自旋轨道耦合 $V_{ls}(r)=-\frac{1}{2m^2}\frac{dV}{rdr}\hat{\vec{l}}\cdot\hat{\vec{s}}$	因子2的差别？	相对论运动学效应 托马斯进动 Thomas precession

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Introduction: The concept of the electron spin

在相对论量子力学中，这些问题得到完美的解决

Dirac equation: $i\partial_t\psi=\hat{H}\psi \qquad \hat{H}=\vec{\alpha}\cdot\hat{\vec{p}}+\beta m \qquad \psi=\begin{pmatrix}\varphi\\ \eta\end{pmatrix}$

（1）Dirac粒子是自旋1/2的费米子

Even for a free Dirac particle:

$$[\hat{H},\hat{\vec{L}}]=-i\vec{\alpha}\times\hat{\vec{p}}\neq 0$$

$$[\hat{H},\vec{\Sigma}]=2i\vec{\alpha}\times\hat{\vec{p}}\neq 0 \qquad \vec{\Sigma}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

$$[\hat{H},\hat{\vec{J}}]=0 \qquad \hat{\vec{J}}=\hat{\vec{L}}+\frac{\vec{\Sigma}}{2}$$

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Introduction: The concept of the electron spin

（2）Dirac粒子的磁矩$g_s = 2$

The magnetic moment: $\hat{\vec{v}} = \frac{d\vec{r}}{dt} = \frac{1}{i}[\vec{r}, \hat{H}] = \vec{\alpha}$

$$\hat{\vec{M}} = \frac{1}{2} q\vec{r}\times\hat{\vec{v}} = \frac{1}{2} q\vec{r}\times\vec{\alpha} = \frac{q}{2}\begin{pmatrix} 0 & \vec{r}\times\vec{\sigma} \\ \vec{r}\times\vec{\sigma} & 0 \end{pmatrix}$$

考查自由的Dirac粒子

$$\psi = \begin{pmatrix} \varphi \\ \eta \end{pmatrix} \quad \hat{H}\psi = E\psi \quad \begin{cases} (E-m)\varphi = \vec{\sigma}\cdot\hat{\vec{p}}\eta \\ (E+m)\eta = \vec{\sigma}\cdot\hat{\vec{p}}\varphi \end{cases} \quad \eta = \frac{\vec{\sigma}\cdot\hat{\vec{p}}}{E+m}\xi$$

$$\langle\psi|\hat{\vec{M}}|\psi\rangle = \frac{q}{2}\int d^3r\,(\varphi^\dagger\vec{r}\times\vec{\sigma}\eta + \eta^\dagger\vec{r}\times\vec{\sigma}\varphi) = \frac{q}{E+m}\int d^3r\,\varphi^\dagger(\hat{\vec{L}} + \vec{\sigma})\varphi$$

Non-relativistic limit: $E \sim m \gg |\vec{p}| \sim V(r)$

Introduction: The concept of the electron spin

（3）Dirac粒子的spin-orbit coupling

考查中心力场中运动的Dirac粒子 $\hat{H} = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m + V(r) \quad \hat{H}_{eff}\varphi = E\varphi$

$$\hat{H}_{eff} = m + \frac{\hat{\vec{p}}^2}{E+m-V} + V + \frac{dV}{rdr}\frac{\vec{\sigma}\cdot\hat{\vec{L}}}{(E+m-V)^2} - i\frac{dV}{rdr}\frac{\vec{r}\cdot\hat{\vec{p}}}{(E+m-V)^2}$$

$$\approx m + \frac{\hat{\vec{p}}^2}{2m} + V + \frac{1}{4m^2}\frac{dV}{rdr}\vec{\sigma}\cdot\hat{\vec{L}} - i\frac{dV}{rdr}\frac{\vec{r}\cdot\hat{\vec{p}}}{4m^2}$$

but this is NOT the non-relativistic equation! ξ is not normalized, $\hat{H}_{eff}$ is not Hermitian
The correct form is obtained by using the Foldy-Wouthuysen transformation.

Introduction: The concept of the electron spin

Orbit angular momentum is non-zero even if the quark is in the ground state.

$$\hat{H}\psi = E\psi \qquad \hat{H} = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m + V(r)$$

The stationary state is taken as the eigenstate of $(\hat{H}, \hat{\vec{J}}^2, \hat{J}_z, \hat{\pi})$, where $\hat{\pi}$ is parity:

$$\psi_{E_n jm\pi}(r,\theta,\phi,S) = \begin{pmatrix} f_{nl}(r)\Omega^l_{jm}(\theta,\phi) \\ (-1)^{(l-l'+1)/2} g_{nl'}(r)\Omega^{l'}_{jm}(\theta,\phi) \end{pmatrix}$$

$j = l \pm \frac{1}{2}, l - l' = \mp 1$, and $\pi = (-1)^l$

$\Omega^l_{jm}(\theta,\phi)$ is the eigenstate of $(\hat{\vec{J}}^2, \hat{\vec{L}}^2, \hat{J}_z)$:

$$\Omega^l_{jm}(\theta,\varphi) = \sqrt{\frac{j+m}{2j}}Y_{l\,m-\frac{1}{2}}(\theta,\varphi)\xi\left(\frac{1}{2}\right) + \sqrt{\frac{j-m}{2j}}Y_{l\,m+\frac{1}{2}}(\theta,\varphi)\xi\left(-\frac{1}{2}\right) \qquad j = l + \frac{1}{2}$$

$$\Omega^l_{jm}(\theta,\varphi) = -\sqrt{\frac{j-m+1}{2j+2}}Y_{l\,m-\frac{1}{2}}(\theta,\varphi)\xi\left(\frac{1}{2}\right) + \sqrt{\frac{j+m+1}{2j+2}}Y_{l\,m+\frac{1}{2}}(\theta,\varphi)\xi\left(-\frac{1}{2}\right) \qquad j = l - \frac{1}{2}$$

Introduction: The concept of the electron spin

Ground state： $E = E_0, j = \frac{1}{2}, \pi = +1\ (l = 0), m = \pm\frac{1}{2}$

$$\psi_0 \equiv \psi_{E_0\frac{1}{2}m+}(r,\theta,\varphi,S) = \begin{pmatrix} f_{00}(r)\Omega^0_{\frac{1}{2}m}(\theta,\varphi) \\ -g_{01}(r)\Omega^1_{\frac{1}{2}m}(\theta,\varphi) \end{pmatrix}$$

$$\Omega^0_{\frac{1}{2}m}(\theta,\varphi) = Y_{00}(\theta,\varphi)\xi(m) = \frac{1}{\sqrt{4\pi}}\xi(m)$$

$$\Omega^1_{\frac{1}{2}m}(\theta,\varphi) = \sqrt{\frac{3+2m}{6}}Y_{1,m+\frac{1}{2}}(\theta,\varphi)\xi\left(-\frac{1}{2}\right) - \sqrt{\frac{3-2m}{6}}Y_{1,m-\frac{1}{2}}(\theta,\varphi)\xi\left(\frac{1}{2}\right)$$

The magnetic moment $\langle\psi_0|\hat{\vec{M}}|\psi_0\rangle = \mu\xi^+(m)\vec{\sigma}\xi(m) \qquad \mu = -\frac{2}{3}e\int dr\,r^3 f_{00}(r)g_{01}(r)$

The average value of the orbital angular momentum

$$\langle\psi_0|\hat{\vec{L}}^2|\psi_0\rangle = 2\int dr\,r^2 g^2_{01}(r) \qquad \langle\psi_0|\hat{L}_z|\psi_0\rangle = \frac{5m}{3}\int dr\,r^2 g^2_{01}(r)$$

ZTL and Meng Ta-chung（孟大中）, Z. Phys. A344, 177 (1992).

Spin-orbit coupling in systems under strong interaction

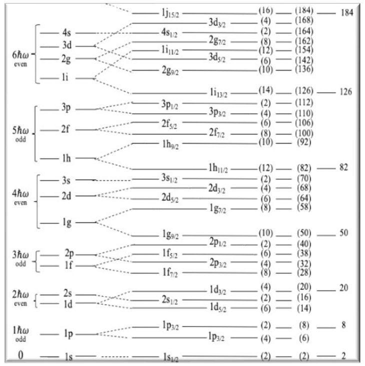
At the hadron level

Nuclear shell model

Nobel price 1963

M.G. Mayer, J.H.D. Jensen（1948）

LS-coupling ⟹ "magic numbers"

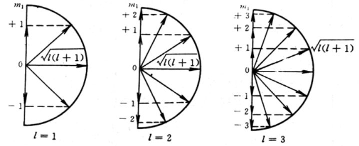


M.G. Mayer and J.H.D. Jensen,“Elementary Theory of Nuclear Shell Structure”, Wiley, New York and Chapman Hall, London, 1955.

Introduction: The concept of electron spin

Three characteristics of spin

- 量子
- 相对论
- 自旋轨道耦合



空间量子化示意图

By the way

$g = 2$, point-like; g-2 experiments, test of QED, new physics.

Anomalous magnetic moment:

$g \neq 2$ significantly different from 2, composite nature of particles;
e.g. $\mu_p = 2.97\mu_N$, $\mu_n = -1.91\mu_N$ the first signature of structure of nucleon.

Contents of Part I

Description of spin states: spin-1/2 particles

单粒子状态　　非相对论情形

$$\hat{\vec{s}} = \frac{1}{2}\vec{\sigma} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_z\xi_z(m) = m\xi_z(m) \qquad \xi_z(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi_z(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For any $\vec{n} = \sin\theta\cos\varphi\,\vec{e}_x + \sin\theta\sin\varphi\,\vec{e}_y + \cos\theta\,\vec{e}_z$, we have

$$\sigma_n = \vec{\sigma}\cdot\vec{n} \qquad \sigma_n\xi_n(m) = m\xi_n(m) \qquad \xi_n(+) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$$

For any $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$, we have $\sigma_n\xi = \xi$ $\quad \tan\frac{\theta}{2} = \frac{|b|}{|a|} \quad e^{i\varphi} = \frac{|a|b}{|b|a}$

For any $\hat{O}$, we have $\hat{O} = \hat{O}_s I + \hat{\vec{O}}_V\cdot\vec{\sigma} \quad \hat{O}_s = \frac{1}{2}\mathrm{Tr}\hat{O} \quad \hat{\vec{O}}_V = \frac{1}{2}\mathrm{Tr}(\vec{\sigma}\hat{O})$

Description of spin states: spin-1/2 particles

单粒子状态 **相对论情形**

$$\psi_{pS}(x) = u(p,s)e^{ipx} \qquad u(p,s) = N\begin{pmatrix} \xi_z(m) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi_z(m) \end{pmatrix} \qquad \sigma_z\xi_z(m) = m\xi_z(m)$$

$$\Sigma_z u(p,S) \neq mu(p,S)$$

Helicity（螺旋度） $\hat{h} \equiv \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$ $\qquad \hat{h}u(p,\lambda) = \lambda u(p,\lambda)$

$$u(p,\lambda) = N\begin{pmatrix} \xi_h(\lambda) \\ \lambda\sqrt{\frac{E-m}{E+m}}\xi_h(\lambda) \end{pmatrix} \qquad \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_h(\lambda) = \lambda\xi_h(\lambda)$$

$$\xi_h(+) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{-i\varphi} \end{pmatrix} \qquad \xi_h(-) = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}e^{-i\varphi} \end{pmatrix}$$

$$\xrightarrow{m=0} \quad u(p,\lambda) = \begin{pmatrix} \xi_h(\lambda) \\ \lambda\xi_h(\lambda) \end{pmatrix}$$

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Description of spin states: spin-1/2 particles

Helicity（螺旋度） $\hat{h} \equiv \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$

ANNALS OF PHYSICS: 7, 404–428 (1959)

On the General Theory of Collisions for Particles with Spin*

M. JACOB† AND G. C. WICK

This has been done by Stapp (6) for collisions between spin-½ particles and by Chao and Shirokov (7)[1] for particles of arbitrary spin. In either case, the authors

7. CHOU KUANG-CHAO AND M. I. SHIROKOV, *J. Exptl. Theoret. Phys. (U.S.S.R.)* **34,** 1230 (1958); translation: *Soviet Phys. JETP* **7,** 851 (1958).

* *Note added in proof.* We have recently received a copy of a paper by Chou Kuang-Chao [*J. Exptl. Theoret. Phys.* (U.S.S.R.) **36,** 909 (1959)] in which a treatment is given which applies when one of the incident particles has zero mass.

周光召

① Only for particles with given $\vec{p}$

② Neither additive nor multiplicative

③ Lorentz invariant for $m = 0$, helicity=chirality

④ Helicity conservation:
- scattering: $h_{in} = h_{final}$
- pair creation/annihilation: $h_{particle} = -h_{anti-particle}$

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Description of spin states: spin-1/2 particles

Chirality and helicity

（1）Chirality（手征性）定义与性质

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \qquad \gamma_5^\dagger = \gamma_5 \qquad \{\gamma_5,\gamma_\mu\} = 0 \qquad \gamma_5^2 = 1 \qquad \gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\gamma_5\psi = \lambda\psi \qquad \lambda = \pm 1 \Longleftrightarrow \psi_{L/R} = \frac{1}{2}(1\pm\gamma_5)\psi \qquad \psi = \psi_L + \psi_R$$

$$\psi^\dagger\psi = \psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R \qquad \psi_L^\dagger\psi_R = \psi_R^\dagger\psi_L = 0$$

$$\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L \qquad \bar{\psi}_L\psi_L = \bar{\psi}_R\psi_R = 0$$

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R \qquad \bar{\psi}_L\gamma^\mu\psi_R = \bar{\psi}_R\gamma^\mu\psi_L = 0$$

（2）当$m = 0$时， chirality=helicity

$$u(p,\lambda) = \begin{pmatrix} \xi(\lambda) \\ \lambda\xi(\lambda) \end{pmatrix} \qquad \gamma_5 u(p,\lambda) = \begin{pmatrix} \lambda\xi(\lambda) \\ \xi(\lambda) \end{pmatrix}$$

$$u(p,R) = \begin{pmatrix} \xi(+) \\ \xi(+) \end{pmatrix} = u(p,+) \qquad u(p,L) = \begin{pmatrix} \xi(-) \\ \xi(-) \end{pmatrix} = u(p,-)$$

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Description of spin states: spin-1/2 particles

Dirac spinor的bilinear covariants（双线性协变量）

（1）The independent Γ-matrices

In the 2x2 case: $(I,\sigma_x,\sigma_y,\sigma_z)$ $\qquad \{\sigma_i,\sigma_j\} = 2\delta_{ij}$ $\qquad \mathrm{Tr}\sigma_i = 0$ $\qquad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$

For a given $\hat{O}$: $\quad \hat{O} = \hat{O}_s I + \hat{\vec{O}}_V\cdot\vec{\sigma} \qquad \hat{O}_s = \frac{1}{2}\mathrm{Tr}(\hat{O}) \qquad \hat{\vec{O}}_V = \frac{1}{2}\mathrm{Tr}(\hat{O}\vec{\sigma})$

In the 4x4 case: $\quad \Gamma_n = \{I,\gamma_5,\gamma_\mu,\gamma_5\gamma_\mu,\sigma_{\mu\nu}\}$ $\qquad$ 16 independent Γ-matrices

$$\mathrm{Tr}\Gamma_n = 0 \text{ besides } \Gamma_1 = I. \qquad \Gamma_n^2 = \pm I \qquad \mathrm{Tr}\{\Gamma_a\Gamma_b\} = \pm 4\delta_{ab}$$

$$\gamma_5^\dagger = \gamma_5,\ \gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0,\ (\gamma_5\gamma_\mu)^\dagger = \gamma_0(\gamma_5\gamma_\mu)\gamma_0,\ \sigma_{\mu\nu}^\dagger = \gamma_0\sigma_{\mu\nu}\gamma_0$$

For a given $\hat{O}$: $\quad \hat{O} = \hat{O}_s I + \hat{O}_P\gamma_5 + \hat{O}_{V\mu}\gamma^\mu + \hat{O}_{A\mu}\gamma_5\gamma^\mu + \hat{O}_{T\mu\nu}\sigma^{\mu\nu}$

$$\hat{O}_n = \pm\frac{1}{4}\mathrm{Tr}(\hat{O}\Gamma_n)$$

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Description of spin states: spin-1/2 particles

Dirac spinor的bilinear covariants（双线性协变量）

（2）The bilinear covariants $\psi\Gamma_n\psi$

$\psi\psi$	scalar	$\hat{P}\psi\psi = \psi\psi$
$\psi\gamma_5\psi$	pseudo-scalar	$\hat{P}\psi\gamma_5\psi = -\psi\gamma_5\psi$
$\psi\gamma_\mu\psi$	vector	$\hat{P}\psi\gamma_\mu\psi = \psi\gamma^\mu\psi$
$\psi\gamma_5\gamma_\mu\psi$	axial vector	$\hat{P}\psi\gamma_5\gamma_\mu\psi = -\psi\gamma_5\gamma^\mu\psi$
$\psi\sigma_{\mu\nu}\psi$	tensor	$\hat{P}\psi\sigma_{\mu\nu}\psi = \psi\sigma^{\mu\nu}\psi$

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Description of spin states: spin-1/2 particles

Polarization vector of a spin-1/2 particle system

The spin density matrix $\hat{\rho} = \sum_\alpha g_\alpha |\alpha\rangle\langle\alpha|$ normalization $\mathrm{Tr}\hat{\rho} = \sum_\alpha g_\alpha = 1$

Average value of $\hat{O}$: $\langle\hat{O}\rangle = \mathrm{Tr}\,\hat{\rho}\hat{O}$ Probability in the state $|\psi\rangle$: $P_\psi = \langle\psi|\hat{\rho}|\psi\rangle$

We choose a basis, e.g., the helicity basis $|\lambda\rangle$, where $\lambda = \pm 1$,

$$\rho_{\lambda\lambda'} = \langle\lambda|\hat{\rho}|\lambda'\rangle = \sum_\alpha g_\alpha \langle\lambda|\alpha\rangle\langle\alpha|\lambda'\rangle$$

$\hat{\rho} = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix}$ is a 2x2 Hermitian matrix.

We decompose it as $\hat{\rho} = \frac{1}{2}(1 + \vec{P}\cdot\vec{\sigma})$

$\vec{P} = \mathrm{Tr}(\hat{\rho}\vec{\sigma}) = \langle\vec{\sigma}\rangle$ is the polarization vector of the system.

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Description of spin states: spin-1/2 particles

Polarization vector of a spin-1/2 particle system in a pure state $|p,n\rangle$

Non-relativistic, the spin state is given by the Pauli spinor $\xi(n)$

$$\vec{\sigma}\cdot\vec{n}\xi(n) = \xi(n) \qquad \xi(n) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$$

The helicity state is given by $\xi(\lambda)$ where $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_h(\lambda) = \lambda\xi_h(\lambda)$

$$\hat{\rho} = |n\rangle\langle n| \qquad \rho_{\lambda\lambda'} = \langle\lambda|\hat{\rho}|\lambda'\rangle = \langle\lambda|n\rangle\langle n|\lambda'\rangle \qquad \langle\lambda|n\rangle = \xi_h^\dagger(\lambda)\xi(n)$$

take $\vec{p} = |\vec{p}|\vec{e}_z$ as an example where we have $\xi_h(+) = \begin{pmatrix}1\\0\end{pmatrix}$ $\xi_h(-) = \begin{pmatrix}0\\1\end{pmatrix}$

$$\xi_h^\dagger(+)\xi(n) = \cos\frac{\theta}{2} \qquad \xi_h^\dagger(-)\xi(n) = \sin\frac{\theta}{2}e^{i\varphi}$$

$$\hat{\rho} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta e^{i\varphi} \\ \frac{1}{2}\sin\theta e^{i\varphi} & \sin^2\frac{\theta}{2} \end{pmatrix} \qquad \vec{P} = \mathrm{Tr}(\hat{\rho}\vec{\sigma}) = \vec{n}$$

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Description of spin states: spin-1/2 particles

Polarization vector of a spin-1/2 particle system in a pure state $|p,n\rangle$

Relativistic, the spin state is given by the Dirac spinor $|p,n\rangle$

$$|p,n\rangle = N\begin{pmatrix} \xi(n) \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(n) \end{pmatrix} \quad \text{where } \vec{\sigma}\cdot\vec{n}\xi(n) = \xi(n) \quad \xi(n) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\varphi} \end{pmatrix}$$

The helicity state $|p,\lambda\rangle$ $|p,\lambda\rangle = N\begin{pmatrix} \xi_h(\lambda) \\ \frac{\lambda|\vec{p}|}{E+m}\xi_h(\lambda) \end{pmatrix}$ where $\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_h(\lambda) = \lambda\xi_h(\lambda)$

$$\langle\lambda|n\rangle = \langle p,\lambda|p,n\rangle = N^2\left[\xi_h^\dagger(\lambda)\xi(n) + \xi_h^\dagger(\lambda)\frac{\lambda|\vec{p}|}{E+m}\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi(n)\right] = \xi_h^\dagger(\lambda)\xi(n)$$

$$\Longrightarrow \quad \hat{\rho} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \frac{1}{2}\sin\theta e^{i\varphi} \\ \frac{1}{2}\sin\theta e^{i\varphi} & \sin^2\frac{\theta}{2} \end{pmatrix} \qquad \vec{P} = \mathrm{Tr}(\hat{\rho}\vec{\sigma}) = \vec{n}$$

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Description of spin states: spin-1/2 particles

Four dimensional polarization vector and spin projection operator of a spin-1/2 particle

In the rest frame $\quad s = (0, \vec{s}) \qquad p \cdot s = 0$

In the moving frame $\quad s = \left(\frac{\vec{p}\cdot\vec{s}}{m}, \vec{s} + \frac{(\vec{p}\cdot\vec{s})\vec{p}}{m(E+m)}\right)$

Longitudinal polarization $\vec{s} \parallel \vec{p}$: $\quad s_{||} = \lambda \frac{1}{m}\left(|\vec{p}|, E\frac{\vec{p}}{|\vec{p}|}\right) = \lambda v \frac{p}{m} + \lambda \frac{m}{E}\left(0, \frac{\vec{p}}{|\vec{p}|}\right) \to \lambda \frac{p}{m}$

Transverse polarization $\vec{s} \perp \vec{p}$: $\quad s_\perp = (0, \vec{s}_\perp, 0)$

$$s = s_{||} + s_\perp = \lambda v \frac{p}{m} + \lambda \frac{m}{E}\left(0, \frac{\vec{p}}{|\vec{p}|}\right) + s_\perp$$

Space reflection: $\quad p^\mu = (p_0, \vec{p}) \to \tilde{p}^\mu = p_\mu = (p_0, -\vec{p})$

$\quad s^\mu = (s_0, \vec{s}) \to -\tilde{s}^\mu = -s_\mu = (-s_0, \vec{s})$

Description of spin states: spin-1/2 particles

The spin projection operator $\quad u(p,s)u(p,s) = (\not{p} + m)\frac{1}{2}(1 + \gamma_5 \not{s})$

$u(p,s) = N\begin{pmatrix} \xi(s) \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(s) \end{pmatrix}$ where $\vec{\sigma}\cdot\vec{s}\xi(s) = \xi(s)$ $\qquad \xi(s)\xi^\dagger(s) = \frac{1}{2}(1 + \vec{\sigma}\cdot\vec{s})$

$$u(p,s)u(p,s) = \begin{pmatrix} \xi(s)\xi^\dagger(s) & -\xi(s)\xi^\dagger(s)\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(s)\xi^\dagger(s) & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\xi(s)\xi^\dagger(s)\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(1+\vec{\sigma}\cdot\vec{s}) & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m}\frac{1}{2}[1 - \vec{\sigma}\cdot\vec{s} + \frac{2}{p^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})] \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m}\frac{1}{2}(1+\vec{\sigma}\cdot\vec{s}) & -\frac{E-m}{E+m}\frac{1}{2}[1 - \vec{\sigma}\cdot\vec{s} + \frac{2}{p^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})] \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{E-m}{E+m} \end{pmatrix}\begin{pmatrix} \frac{1}{2}(1+\vec{\sigma}\cdot\vec{s}) & 0 \\ 0 & \frac{1}{2}[1 - \vec{\sigma}\cdot\vec{s} + \frac{2}{p^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma})] \end{pmatrix}$$

Description of spin states: spin-1/2 particles

The spin projection operator $\quad u(p,s)u(p,s) = (\not{p} + m)\frac{1}{2}(1 + \gamma_5 \not{s})$

$$N^2\begin{pmatrix} 1 & -\frac{\vec{p}\cdot\vec{\sigma}}{E+m} \\ \frac{\vec{p}\cdot\vec{\sigma}}{E+m} & -\frac{E-m}{E+m} \end{pmatrix} = \frac{N^2}{E+m}\begin{pmatrix} E+m & -\vec{p}\cdot\vec{\sigma} \\ \vec{p}\cdot\vec{\sigma} & -E+m \end{pmatrix} = \frac{N^2}{E+m}(\gamma\cdot p + m)$$

$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_5\gamma_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_5\vec{\gamma} = \begin{pmatrix} -\vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \qquad s = (\frac{\vec{p}\cdot\vec{s}}{m}, \vec{s} + \frac{(\vec{p}\cdot\vec{s})\vec{p}}{m(E+m)})$

$$\gamma_5\gamma\cdot s = \begin{pmatrix} \vec{\sigma}\cdot\vec{s} + \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} & -\frac{\vec{p}\cdot\vec{s}}{m} \\ \frac{\vec{p}\cdot\vec{s}}{m} & -\vec{\sigma}\cdot\vec{s} - \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} \end{pmatrix}$$

$$\begin{pmatrix} 1+\vec{\sigma}\cdot\vec{s} & 0 \\ 0 & 1 - \vec{\sigma}\cdot\vec{s} + \frac{2}{\vec{p}^2}(\vec{p}\cdot\vec{\sigma})(\vec{s}\cdot\vec{\sigma}) \end{pmatrix} = 1 + \gamma_5\gamma\cdot s + \begin{pmatrix} -\frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} & \frac{\vec{p}\cdot\vec{s}}{m} \\ -\frac{\vec{p}\cdot\vec{s}}{m} & \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E-m)} \end{pmatrix}$$

$$(\gamma\cdot p + m)\begin{pmatrix} -\frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E+m)} & \frac{\vec{p}\cdot\vec{s}}{m} \\ -\frac{\vec{p}\cdot\vec{s}}{m} & \frac{(\vec{p}\cdot\vec{s})(\vec{p}\cdot\vec{\sigma})}{m(E-m)} \end{pmatrix} = 0$$

Description of spin states: spin-1 particles

Spin operator

$\hat{\vec{S}} = \vec{\Sigma} \qquad \Sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \Sigma_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

The helicity basis $\quad |\lambda = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |\lambda = -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad |\lambda = 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

The polarization vector $\vec{\varepsilon}$

The basis: $|\varepsilon_{(x)}\rangle \equiv \frac{1}{\sqrt{2}}(|\lambda = -1\rangle - |\lambda = 1\rangle) \qquad |\varepsilon_{(y)}\rangle \equiv \frac{i}{\sqrt{2}}(|\lambda = -1\rangle + |\lambda = 1\rangle)$

$|\varepsilon_{(z)}\rangle \equiv |\lambda = 0\rangle$

The general form of a pure state: $\quad |\varepsilon\rangle = \varepsilon_x|\varepsilon_{(x)}\rangle + \varepsilon_y|\varepsilon_{(y)}\rangle + \varepsilon_z|\varepsilon_{(z)}\rangle$

The polarization vector is defined as: $\quad \vec{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z)$

The spin polarization vector $\vec{S}$ $\quad \vec{S} \equiv \langle\vec{S}\rangle = (\vec{S}_T, \lambda)$ in the rest frame of V

Description of spin states: spin-1 particles

The spin density matrix

$$\hat{\rho} = \begin{pmatrix} \rho_{11} & \rho_{10} & \rho_{1-1} \\ \rho_{01} & \rho_{00} & \rho_{0-1} \\ \rho_{-11} & \rho_{-10} & \rho_{-1-1} \end{pmatrix}$$

Decomposition $\hat{\rho} = \frac{1}{3}\left[1 + \frac{3}{2}S^i\Sigma^i + 3T^{ij}\Sigma^{ij}\right]$ $\quad \Sigma^{ij} = \frac{1}{2}(\Sigma^i\Sigma^j + \Sigma^j\Sigma^i) - \frac{2}{3}I\delta_{ij}$

$$T = \frac{1}{2}\begin{pmatrix} -\frac{2}{3}S_{LL} + S_{TT}^{xx} & S_{TT}^{xx} & S_{LT}^{x} \\ S_{TT}^{xy} & -\frac{2}{3}S_{LL} - S_{TT}^{xx} & S_{LT}^{y} \\ S_{LT}^{x} & S_{LT}^{y} & \frac{4}{3}S_{LL} \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} S_{LL} & 0 & 0 \\ 0 & S_{LL} & 0 \\ 0 & 0 & -2S_{LL} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & S_{LT}^{x} \\ 0 & 0 & S_{LT}^{y} \\ S_{LT}^{x} & S_{LT}^{y} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} S_{TT}^{xx} & S_{TT}^{xx} & 0 \\ S_{TT}^{xy} & S_{TT}^{xx} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Description of spin states: spin-1 particles

Spin density matrix in terms of Lorentz covariants

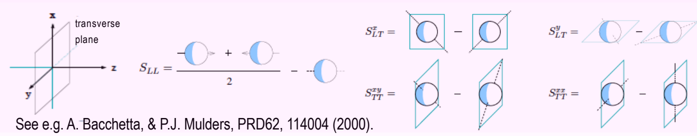
Spin polarization vector: $S = (0, S_x, S_y, S_z) = (0, S_T^x, S_T^y, S_L)$

Tensor polarization: Scalar S_{LL}

Vector $S_{LT} = (0, S_{LT}^x, S_{LT}^y, 0)$

Tensor $S_{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & S_{TT}^{xx} & S_{TT}^{xy} & 0 \\ 0 & S_{TT}^{xy} & S_{TT}^{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\hat{\rho} = \begin{pmatrix} \frac{1+S_{LL}}{3} + \frac{S_L}{2} & \frac{(S_{LT}^x - iS_{LT}^y) + (S_T^x - iS_T^y)}{2\sqrt{2}} & \frac{S_{TT}^{xx} - iS_{TT}^{xy}}{2} \\ \frac{(S_{LT}^x + iS_{LT}^y) + (S_T^x + iS_T^y)}{2\sqrt{2}} & \frac{1-2S_{LL}}{3} & \frac{(-S_{LT}^x + iS_{LT}^y) + (S_T^x - iS_T^y)}{2\sqrt{2}} \\ \frac{S_{TT}^{xx} + iS_{TT}^{xy}}{2} & \frac{(-S_{LT}^x - iS_{LT}^y) + (S_T^x + iS_T^y)}{2\sqrt{2}} & \frac{1+S_{LL}}{3} - \frac{S_L}{2} \end{pmatrix}$$



See e.g. A. Bacchetta, & P.J. Mulders, PRD62, 114004 (2000).

Description of spin states: spin-1 particles

Relationship between the spin polarization vector $\vec{S}$ and the polarization vector $\vec{\varepsilon}$

$$\vec{S} = \text{Im}(\vec{\varepsilon}^* \times \vec{\varepsilon})$$

$\vec{S} = 0$ for any pure state with a real $\vec{\varepsilon}$

$\vec{\varepsilon}^{(\pm)} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0) \Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, \pm 1)$

$\vec{\varepsilon}^{(0)} = (0, 0, 1) \Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, 0)$

pure state with $\rho_{00} = 1$ $\Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, 0) \Longleftrightarrow \vec{\varepsilon}^{(0)} = (0, 0, 1)$ in OZ direction

pure state with $\rho_{00} = 0$ $\Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0, 0, \sin 2\theta \sin\varphi) \Longleftrightarrow \vec{\varepsilon} = (\cos\theta, \sin\theta e^{i\varphi}, 0)$ perpendicular to OZ direction

Description of spin states: spin-1 particles

The pure state:

$|\lambda = 1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |\lambda = -1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad |\lambda = 0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$|\varepsilon_{(x)}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad |\varepsilon_{(y)}\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad |\varepsilon_{(z)}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$|\varepsilon\rangle = \varepsilon_x|\varepsilon_{(x)}\rangle + \varepsilon_y|\varepsilon_{(y)}\rangle + \varepsilon_z|\varepsilon_{(z)}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}}(-\varepsilon_x + i\varepsilon_y) \\ \varepsilon_z \\ \frac{1}{\sqrt{2}}(\varepsilon_x + i\varepsilon_y) \end{pmatrix}$$

$$\langle\varepsilon| = \left(\frac{1}{\sqrt{2}}(-\varepsilon_x^* - i\varepsilon_y^*) \quad \varepsilon_z^* \quad \frac{1}{\sqrt{2}}(\varepsilon_x^* - i\varepsilon_y^*)\right)$$

$$\Sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Sigma_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Sigma_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\vec{S} = \langle\vec{S}\rangle = (\langle\Sigma_x\rangle, \langle\Sigma_y\rangle, \langle\Sigma_z\rangle) = \frac{1}{2}\left(i(\varepsilon_z^*\varepsilon_y - \varepsilon_y^*\varepsilon_z),\ i(\varepsilon_z^*\varepsilon_x - \varepsilon_x^*\varepsilon_z), i(\varepsilon_x^*\varepsilon_y - \varepsilon_y^*\varepsilon_x)\right) = \text{Im}(\vec{\varepsilon}^* \times \vec{\varepsilon})$$

Description of spin states: spin-1 particles

The spin density matrix:

$$\hat{\rho}(\varepsilon) = |\varepsilon\rangle\langle\varepsilon| = \begin{pmatrix} \frac{1}{2}|\varepsilon_x - i\varepsilon_y|^2 & \frac{1}{\sqrt{2}}(-\varepsilon_x + i\varepsilon_y)\varepsilon_z^* & \frac{1}{2}(-\varepsilon_x + i\varepsilon_y)(\varepsilon_x^* - i\varepsilon_y^*) \\ \frac{1}{\sqrt{2}}\varepsilon_z(-\varepsilon_x^* - i\varepsilon_y^*) & |\varepsilon_z|^2 & \frac{1}{\sqrt{2}}\varepsilon_z(\varepsilon_x^* - i\varepsilon_y^*) \\ \frac{1}{2}(\varepsilon_x + i\varepsilon_y)(-\varepsilon_x^* - i\varepsilon_y^*) & \frac{1}{\sqrt{2}}(\varepsilon_x + i\varepsilon_y)\varepsilon_z^* & \frac{1}{2}|\varepsilon_x + i\varepsilon_y|^2 \end{pmatrix}$$

$$\rho_{00} = 1 \Longleftrightarrow \hat{\rho}(\varepsilon) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longleftrightarrow \begin{matrix} \varepsilon_z = 1 \\ \varepsilon_x = \varepsilon_y = 0 \end{matrix} \Longleftrightarrow \vec{\varepsilon} = (0,0,1) \Longleftrightarrow \vec{S} = \langle\vec{S}\rangle = (0,0,0)$$

$$\rho_{00} = 0 \Longleftrightarrow \varepsilon_z = 0 \Longleftrightarrow \hat{\rho}(\varepsilon) = \begin{pmatrix} \frac{1}{2}|\varepsilon_x - i\varepsilon_y|^2 & 0 & \frac{1}{2}(-\varepsilon_x + i\varepsilon_y)(\varepsilon_x^* - i\varepsilon_y^*) \\ 0 & 0 & 0 \\ \frac{1}{2}(\varepsilon_x + i\varepsilon_y)(-\varepsilon_x^* - i\varepsilon_y^*) & 0 & \frac{1}{2}|\varepsilon_x + i\varepsilon_y|^2 \end{pmatrix}$$

Normalization $|\varepsilon_x|^2 + |\varepsilon_y|^2 = 1$ $\quad \varepsilon_x = \cos\theta \quad \varepsilon_y = \sin\theta e^{i\varphi} \quad \vec{\varepsilon} = (\cos\theta, \sin\theta e^{i\varphi}, 0)$

$$\hat{\rho}(\varepsilon) = \begin{pmatrix} \frac{1}{2}(1 + \sin2\theta\sin\varphi) & 0 & \frac{1}{2}(-\cos2\theta + i\sin2\theta\cos\varphi) \\ 0 & 0 & 0 \\ \frac{1}{2}(-\cos2\theta - i\sin2\theta\cos\varphi) & 0 & \frac{1}{2}(1 - \sin2\theta\sin\varphi) \end{pmatrix} \qquad \vec{S} = \langle\vec{S}\rangle = (0,0,\sin2\theta\sin\varphi)$$

Description of polarization of the photon

Circular polarization of the photon:

in the helicity state
$$\rho_\gamma^{circ} = \frac{1}{2}\begin{pmatrix} 1 + P_{circ} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - P_{circ} \end{pmatrix}$$

Linear polarization of the photon:

linearly polarized along OX or OY $\Longleftrightarrow$ in state $|\varepsilon_{(x)}\rangle$ or $|\varepsilon_{(y)}\rangle$

$$\rho_\gamma^{lin(x)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -P_{lin} \\ 0 & 0 & 0 \\ -P_{lin} & 0 & 1 \end{pmatrix} \qquad \rho_\gamma^{lin(y)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & P_{lin} \\ 0 & 0 & 0 \\ P_{lin} & 0 & 1 \end{pmatrix}$$

in OXY plane at an angle γ to OX
$$\rho_\gamma^{lin} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -P_{lin}e^{-2i\gamma} \\ 0 & 0 & 0 \\ -P_{lin}e^{2i\gamma} & 0 & 1 \end{pmatrix}$$

$$\rho_\gamma^{lin} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -P_{lin}e^{-2i\gamma} \\ 0 & 0 & 0 \\ -P_{lin}e^{2i\gamma} & 0 & 1 \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} 1 & -P_{lin}e^{-2i\gamma} \\ -P_{lin}e^{2i\gamma} & 1 \end{pmatrix} = \frac{1}{2}[1 - P_{lin}(\sigma_x\cos2\gamma + \sigma_y\sin2\gamma)]$$

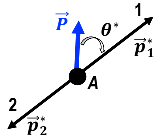
Contents of Part I

Polarization measurements: hyperon polarization

Two body decay $A \to 1 + 2$ In the rest frame of A

$$p_A = (M_A, 0, 0, 0) \quad p_1 = (E_1^*, \vec{p}_1^*) \quad p_2 = (E_2^*, \vec{p}_2^*) \quad \vec{p}_1^* = -\vec{p}_2^* = \vec{p}^*$$

$$p_A = p_1 + p_2 \quad E_1^* = (M_A^2 + m_1^2 - m_2^2)/2M_A$$



For unpolarized (or spinless) A, the decay product is isotropic.

$$\frac{d^3N}{d^3p_1} = \frac{1}{4\pi p^{*2}}\delta(|\vec{p}_1| - |\vec{p}^*|) \qquad \frac{dN}{d\Omega} = \frac{1}{4\pi}$$

For parity conserved decays of A, the decay product is isotropic.

For parity violating decay of the hyperon,

$$\frac{dN}{d\Omega} = \frac{1}{4\pi}\left(1 + \alpha\vec{P}\cdot\frac{\vec{p}_1^*}{|\vec{p}_1^*|}\right) = \frac{1}{4\pi}(1 + \alpha P\cos\theta^*)$$

Spin self analyzing parity violating weak decay of the hyperon A.

α: the decay polarization parameter,
measured experimentally and can be found in PDG.

Polarization measurements: vector meson

Consider $A \to 1 + 2$ in the rest frame of A

Suppose A is in the spin state $|S_A, M_A\rangle$, the final state particles have helicities λ_1 and λ_2.

The decay amplitude is $A_m(\vec{p}; \lambda_1, \lambda_2) = \langle \vec{p}; \lambda_1, \lambda_2 | \hat{U} | S_A, M_A \rangle$

Applying total angular momentum conservation

$$A_m(\vec{p}; \lambda_1, \lambda_2) = \langle \vec{p}; \lambda_1, \lambda_2 | E, S_A, M_A; \lambda_1, \lambda_2 \rangle \langle E, J = S_A, M = M_A; \lambda_1, \lambda_2 | \hat{U} | S_A, M_A \rangle$$

Space rotation invariance demands

$$\langle S_A, M_A; \lambda_1, \lambda_2 | \hat{U} | S_A, M_A \rangle = \langle S_A; \lambda_1, \lambda_2 | \hat{U} | S_A \rangle = H_{S_A}(\lambda_1, \lambda_2)$$

Helicity amplitude, independent of M_A, independent of angles (θ, φ).

Hence $A_m(\vec{p}; \lambda_1, \lambda_2) = \langle \vec{p}; \lambda_1, \lambda_2 | E, S_A, M_A; \lambda_1, \lambda_2 \rangle \, H_{S_A}(\lambda_1, \lambda_2)$

The angular dependence is determined by the calculable state projection $\langle \vec{p}; \lambda_1, \lambda_2 | E, S_A, M_A; \lambda_1, \lambda_2 \rangle$

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Polarization measurements: vector meson

Calculation of $\langle p, \theta, \varphi; \lambda_1, \lambda_2 | p, J, M; \lambda_1, \lambda_2 \rangle$

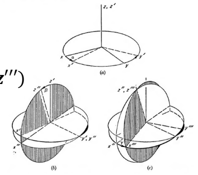
It can be shown that $\langle p, 0, 0; \lambda_1, \lambda_2 | p, J, M; \lambda_1, \lambda_2 \rangle = \left(\frac{2J+1}{4\pi} \right)^{1/2}$

Any rotation can be described by three Euler angles (α, β, γ)

(1) a rotation of angle α around z-axis $(x, y, z) \to (x', y', z')$

(2) a rotation of angle β around y'-axis $(x', y', z') \to (x'', y'', z'')$

(3) a rotation of angle γ around z''-axis $(x'', y'', z'') \to (x''', y''', z''')$



The rotation operator $\hat{R}_n(\alpha) = e^{-i\alpha \hat{J}_n}$

$$\hat{R}(\alpha, \beta, \gamma) = \hat{R}_{z''}(\gamma) \hat{R}_{y'}(\beta) \hat{R}_z(\alpha)$$

$$\hat{R}_{y'}(\beta) = \hat{R}_z(\alpha) \hat{R}_y(\beta) \hat{R}_z^{-1}(\alpha)$$

$$\hat{R}(\alpha, \beta, \gamma) = \hat{R}_z(\alpha) \hat{R}_y(\beta) \hat{R}_z(\gamma) = e^{-i\alpha \hat{J}_z} e^{-i\beta \hat{J}_y} e^{-i\gamma \hat{J}_z}$$

$$|p, \theta, \varphi; \lambda_1, \lambda_2\rangle = \hat{R}(\varphi, \theta, -\varphi) |p, 0, 0; \lambda_1, \lambda_2\rangle$$

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Polarization measurements: vector meson

The Wigner rotation matrix $\langle jm' | \hat{R}(\alpha, \beta, \gamma) | jm \rangle$

$$\hat{R}(\alpha, \beta, \gamma) |jm\rangle = \sum_{m'} |jm'\rangle \langle jm' | \hat{R}(\alpha, \beta, \gamma) | jm \rangle = \sum_{m'} D^j_{mm'}(\alpha, \beta, \gamma) \, |jm'\rangle$$

$$D^j_{mm'}(\alpha, \beta, \gamma) = \langle jm' | \hat{R}(\alpha, \beta, \gamma) | jm \rangle = e^{-im'\alpha} e^{-im\gamma} \left\langle jm' \middle| e^{-i\beta \hat{J}_y} \middle| jm \right\rangle = e^{-im'\alpha - im\gamma} d^j_{mm'}(\beta)$$

$$d^j_{mm'}(\beta) = \langle jm' | e^{-i\beta \hat{J}_y} | jm \rangle$$

$$= [(j+m)!\,(j-m)!\,(j+m')!\,(j-m')!]^{1/2} \times \sum_{k=max\{m-m',0\}}^{min\{j+m,j-m'\}} \frac{\left(\cos\frac{\beta}{2}\right)^{2j} \left(\tan\frac{\beta}{2}\right)^{2k-m+m'}}{(j+m-k)!\,(j-m'-k)!\,k!\,(k-m'+m)!}$$

$$d^{1/2}(\beta) = \begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \qquad d^1(\beta) = \begin{pmatrix} \frac{1+\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ \frac{\sin\beta}{\sqrt{2}} & \cos\beta & -\frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}$$

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Polarization measurements: vector meson

The inner product

$$\langle \vec{p}; \lambda_1, \lambda_2 | S_A, M_A; \lambda_1, \lambda_2 \rangle$$
$$= \langle p, 0, 0; \lambda_1, \lambda_2 | R^\dagger(\varphi, \theta, -\varphi) | S_A, M_A; \lambda_1, \lambda_2 \rangle$$
$$= \sum_{M'_A} \langle p, 0, 0; \lambda_1, \lambda_2 | S_A, M'_A; \lambda_1, \lambda_2 \rangle \langle S_A, M'_A; \lambda_1, \lambda_2 | R^\dagger(\varphi, \theta, -\varphi) | S_A, M_A; \lambda_1, \lambda_2 \rangle$$

$M'_A = \lambda = \lambda_1 - \lambda_2$

$$= \langle p, 0, 0; \lambda_1, \lambda_2 | S_A, \lambda; \lambda_1, \lambda_2 \rangle \langle S_A, \lambda; \lambda_1, \lambda_2 | R^\dagger(\varphi, \theta, -\varphi) | S_A, M_A; \lambda_1, \lambda_2 \rangle$$
$$= \left(\frac{2J+1}{4\pi} \right)^{\frac{1}{2}} D^{S_A *}_{M_A \lambda}(\varphi, \theta, -\varphi)$$

The decay amplitude $A_m(\vec{p}; \lambda_1, \lambda_2) = \langle \vec{p}; \lambda_1, \lambda_2 | \hat{U} | S_A, M_A \rangle$

$$= \langle \vec{p}; \lambda_1, \lambda_2 | S_A, M_A; \lambda_1, \lambda_2 \rangle \, H_{S_A}(\lambda_1, \lambda_2)$$
$$= \left(\frac{2J+1}{4\pi} \right)^{\frac{1}{2}} D^{S_A *}_{M_A \lambda}(\varphi, \theta, -\varphi) H_A(\lambda_1, \lambda_2)$$

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Polarization measurements: vector meson

Suppose the spin density matrix of *A* is $\hat{\rho}_A = \sum_{M_A} g_{M_A} |S_A, M_A\rangle\langle S_A, M_A|$

The spin density matrix of the system (1,2) is $\hat{\rho}_{12} = \sum_{M_A} g_{M_A} \hat{U} |S_A, M_A\rangle\langle S_A, M_A| \hat{U}^\dagger = \hat{U}\hat{\rho}_A \hat{U}^\dagger$

The angular distribution

$$W(\theta,\varphi) = N \sum_{\lambda_1,\lambda_2} \langle \vec{p};\lambda_1\lambda_2 | \hat{\rho}_{12} | \vec{p};\lambda_1\lambda_2\rangle = N \sum_{\lambda_1,\lambda_2} \langle \vec{p};\lambda_1\lambda_2 | \hat{U}\,\hat{\rho}_A\,\hat{U}^\dagger | \vec{p};\lambda_1\lambda_2\rangle$$

$$= N \sum_{\lambda_1,\lambda_2;M_A,M'_A} \langle \vec{p};\lambda_1\lambda_2 | \hat{U}\,|S_A, M_A\rangle\langle S_A, M_A|\hat{\rho}_A\,|S_A, M'_A\rangle\langle S_A, M'_A\,|\hat{U}^\dagger | \vec{p};\lambda_1\lambda_2\rangle$$

$$= N \sum_{\lambda_1,\lambda_2;M_A,M'_A} A_{M_A}(\vec{p};\lambda_1\lambda_2)\, A^*_{M'_A}(\vec{p};\lambda_1\lambda_2)\, \langle S_A, M_A|\hat{\rho}_A\,|S_A, M'_A\rangle$$

$$= N' \sum_{\lambda_1,\lambda_2;M_A,M'_A} |H_A(\lambda_1,\lambda_2)|^2\, D^{S_A*}_{M_A\lambda}(\varphi,\theta,-\varphi) D^{S_A}_{M'_A\lambda}(\varphi,\theta,-\varphi)\langle M_A|\hat{\rho}_A\,|M'_A\rangle$$

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Polarization measurements: vector meson

For $V \to 1+2$, where 1 and 2 are two pseudoscalar mesons, we have $S_A = 1, \lambda_1 = \lambda_2 = 0$

e.g., $\rho \to \pi\pi$

$$W(\theta,\varphi) = N \sum_{M_A,M'_A} |H_A|^2\, D^{1*}_{M_A0}(\varphi,\theta,-\varphi) D^{1}_{M'_A0}(\varphi,\theta,-\varphi)\langle M_A|\hat{\rho}_A|M'_A\rangle$$

$$= \frac{3}{4\pi}\{\frac{1}{2}(\rho_{11} + \rho_{-1-1})\sin^2\theta + \rho_{00}\cos^2\theta$$

$$-\frac{1}{\sqrt{2}}\sin 2\theta\,[\cos\varphi\,(\mathrm{Re}\rho_{10} - Re\rho_{-10}) - \sin\varphi\,(\mathrm{Im}\rho_{10} + \mathrm{Im}\rho_{-10})]$$

$$-\sin^2\theta\,(\cos 2\varphi\,\mathrm{Re}\rho_{1-1} - \sin 2\varphi\,\mathrm{Im}\rho_{1-1})\}$$

$$\int_0^{2\pi} d\varphi\, W(\theta,\varphi) = \frac{3}{4}[(1-\rho_{00}) + (3\rho_{00} - 1)\cos^2\theta]$$

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Polarization measurements: vector meson

For $V \to 1+2$, where 1 and 2 are two spin-1/2 Fermions, i.e., $S_A = 1, \lambda_1 = \pm\frac{1}{2},\ \lambda_2 = \pm\frac{1}{2}$

consider the case: (1) Helicity conservation: $\lambda_1 = -\lambda_2\,, \lambda = \pm 1$

(2) Space reflection invariance: $H_A(\lambda_1,\lambda_2) = H_A(-\lambda_1,-\lambda_2)$

only one independent helicity amplitude

e.g., $\mathrm{J}/\psi \to e^+e^-$

$$W(\theta,\varphi) = \frac{3}{8\pi(1+\rho_{00})}[1 + \lambda_\theta\cos^2\theta + \lambda_\varphi\sin^2\theta\cos 2\varphi + \lambda_{\theta\varphi}\sin 2\theta\cos\varphi$$

$$+\lambda^\perp_\varphi\sin^2\theta\sin 2\varphi + \lambda^\perp_{\theta\varphi}\sin 2\theta\sin\varphi]$$

$$\lambda_\theta = \frac{1-3\rho_{00}}{1+\rho_{00}} \qquad \lambda_\varphi = \frac{4\mathrm{Re}\rho_{1-1}}{1+\rho_{00}} \qquad \lambda_{\theta\varphi} = \frac{\sqrt{2}\mathrm{Re}(\rho_{10} - \rho_{-10})}{1+\rho_{00}}$$

$$\lambda^\perp_\varphi = \frac{4\mathrm{Im}\rho_{1-1}}{1+\rho_{00}} \qquad \lambda^\perp_{\theta\varphi} = \frac{\sqrt{2}\mathrm{Im}(\rho_{10} - \rho_{-10})}{1+\rho_{00}}$$

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Polarization measurements: Hyperon decay

For $H \to N\pi$, $S_A = \frac{1}{2}, \lambda_1 = \pm\frac{1}{2},\ \lambda_2 = 0$

$$W(\theta,\varphi) = \frac{1}{4\pi}(1 + \alpha P\cos\theta + \alpha\sin\theta\cos\varphi\,\mathrm{Re}\rho_{+-} + \alpha\sin\theta\sin\varphi\,\mathrm{Im}\rho_{+-})$$

$$\alpha = \frac{\left|H_A\left(\frac{1}{2}\right)\right|^2 - \left|H_A\left(-\frac{1}{2}\right)\right|^2}{\left|H_A\left(\frac{1}{2}\right)\right|^2 + \left|H_A\left(-\frac{1}{2}\right)\right|^2}$$

$$P = \rho_{++} - \rho_{--}$$

If space reflection invariance $H_A\left(\frac{1}{2}\right) = H_A\left(-\frac{1}{2}\right)$ $\alpha = 0$

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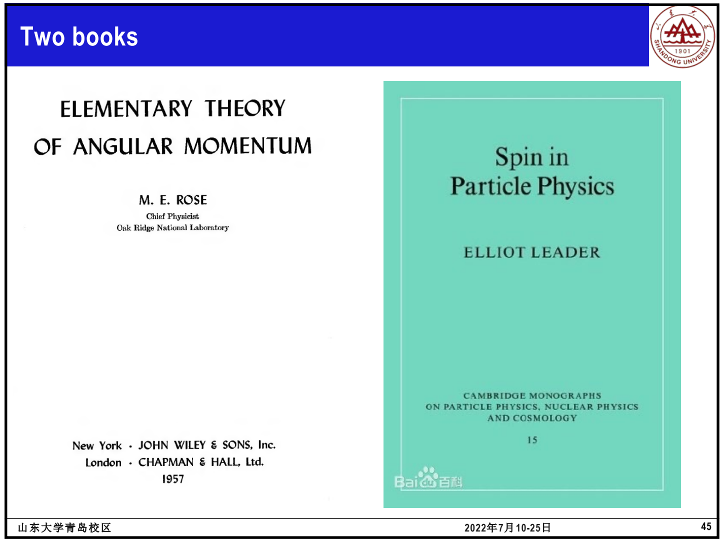
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