From One-Loop to Resummation

1 One-Loop Computation

Analogous to the QED process $e^+e^- \to \mu^+\mu^-$, which has been computed in QED, we can study the simplest QCD process

$$e^+e^- \to q\bar{q} \to \text{hadrons}.$$

The only modification that we need to make is as follows

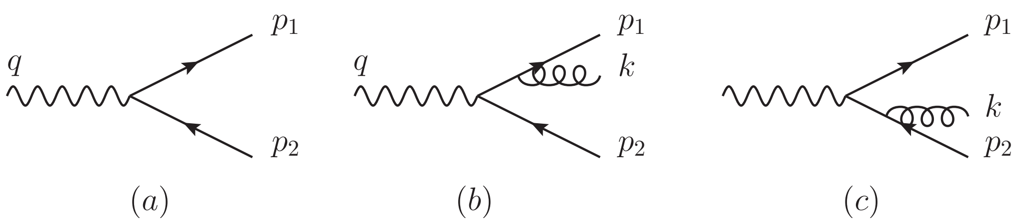
- Replace the muon charge e with the quark charge Qe.
- Count each quark three times, one for each color.
- Properly take into account the effects of strong interactions between $q\bar{q}$. Assume that $q\bar{q} \to hadrons$ process does not charge the total cross sections.

1.1 Leading order cross section

The process we consider is

$$e^+ + e^- \to \gamma^* \to q\bar{q}(\text{LO}) \quad \text{or} \quad q + \bar{q} + g(\text{NLO}). \tag{1}$$

To simplify the calculation we can compute this process as the decay of the virtual photon γ into $q\bar{q}$ and $q\bar{q}g$ as shown in the following figure.



Choose the centre of the mass frame of the virtual photon which gives $q = (Q, \vec{0})$ and neglect all masses for the quarks and gluon.

- According to energy momentum conservation $q^\mu = p_1^\mu + p_2^\mu$. We can define $q^2 = Q^2 = s$ as the center of mass energy. In fact, we can view Q as the mass of the virtual photon.
- Define the leading order cross section $\gamma^* \to q\bar{q}$, which is shown in Figure (a), as the product of the flux factor $\frac{1}{2\sqrt{s}}$, the amplitude square and the two body final state phase space R_2^n (see Page 107, Eq. 4.86 of Peskin)

$$\sigma_0 = \frac{1}{2Q} \sum_{q,c,s} |\mathcal{M}_{\gamma^* \to q\bar{q}}|^2 R_2^n \tag{2}$$

where R_2^n is defined as (Here we use $n = d = 4 - 2\epsilon$ as the number of dimension)

$$\begin{aligned}
R_2^n &= \int \frac{d^{n-1}p_1}{(2\pi)^{n-1}2p_1} \int \frac{d^{n-1}p_2}{(2\pi)^{n-1}2p_2} (2\pi)^n \delta^{(n)}(q - p_1 - p_2), &(3)\\
&= \frac{1}{4(2\pi)^{n-2}} \Omega_{n-1} \int_0^\infty \frac{dp_1 p_1^{n-2}}{E_1 E_2} \delta(Q - E_1 - E_2) &(4)\\
&\Downarrow \quad \text{with} \quad \frac{dw}{w} = \frac{dp_1 p_1}{E_1 E_2} \quad \text{and } w \equiv E_1 + E_2, &(5)\\
&= \frac{1}{8\pi} \frac{\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)} \left(\frac{Q^2}{4\pi}\right)^{-\epsilon}, &(6)
\end{aligned}$$

where in the last step, we have used the following gamma function identity

$$\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z), \quad \text{with} \quad z = 1-\epsilon, \tag{7}$$

where $\Gamma(t) \equiv \int_0^\infty dx x^{t-1} e^{-x}$. More gamma function identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad \text{with } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \tag{8}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \left(\frac{1}{2}\gamma_E^2 + \frac{\pi^2}{12}\right)\epsilon + \mathcal{O}(\epsilon^2), \tag{9}$$

$$\text{the Euler constant } \gamma_E \equiv \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m} - \ln n\right] \simeq 0.577,$$

$$\Gamma(x) = (x-1)\Gamma(x-1). \tag{10}$$

In the massless case, it is very simple to see that $E_1 = E_2 = p_1 = p_2$, which makes the above calculation straightforward.

- It is straightforward to write down the amplitude for $\gamma^* \to q\bar{q}$ as follows

$$i\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}} = (-iee_q\mu^\epsilon)\delta_{ij}\bar{u}(p_1)\gamma^\mu v(p_2)\epsilon_\mu(q), \tag{11}$$

where e_q is the charge number of the quark and δ_{ij} indicates that the quark and anti-quark should have opposite color. Remeber now that the electric coupling carries dimension of μ^ϵ in the dimensional regularization. Summing over the polarizations of the virtual photon when we square the amplitude, we can obtain

$$\sum_{q,c,s}|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2 = -g_{\mu\nu}\sum_{q,c,s}\mathcal{M}^{\mu}_{\gamma^*\to q\bar{q}}\mathcal{M}^{\nu *}_{\gamma^*\to q\bar{q}}, \tag{12}$$

$$= -e^2\sum_q e_q^2\mu^{2\epsilon}N_c\mathrm{Tr}\left[\not{p}_1\gamma^\mu\not{p}_2\gamma_\mu\right], \tag{13}$$

$$= 4(1-\epsilon)e^2\mu^{2\epsilon}N_cQ^2\sum_q e_q^2. \tag{14}$$

The evaluation of the above trace can be done directly or use the identity $\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu$.

- Therefore, we can obtain the LO cross section as follows

$$\sigma_0 = \alpha\sum_q e_q^2\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}N_cQ\sum_q e_q^2\frac{\Gamma(2-\epsilon)}{\Gamma(2-2\epsilon)}. \tag{15}$$

- Notice that in the case of $n=4$ dimension, $R_2 = \frac{1}{8\pi}$ and $\sigma_0 = \alpha QN_c\sum_q e_q^2$.

1.2 Next-to-leading order (NLO) Real Diagram

Now consider the process $\gamma^* \to q\bar{q}g$ as shown in Figure (b) and (c). Compute the corresponding cross section which is defined as

$$\sigma_3 = \frac{1}{2Q}|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2R_3\,, \tag{16}$$

where R_3 is the three body final state phase space defined as

$$R_3^n = \int\frac{d^{n-1}p_1}{(2\pi)^{n-1}2p_1}\int\frac{d^{n-1}p_2}{(2\pi)^{n-1}2p_2}\int\frac{d^{n-1}k}{(2\pi)^{n-1}2k}(2\pi)^n\delta^{(n)}(q-p_1-p_2-k). \tag{17}$$

- Now let us compute R_3^n

$$
\begin{aligned}
R_3^n &= \int \frac{d^{n-1}p_1}{(2\pi)^{n-1}2p_1}\int \frac{d^{n-1}p_2}{(2\pi)^{n-1}2p_2}\frac{2\pi}{2E_k}\delta(Q-E_1-E_2-E_k). \\
&\Downarrow \quad \text{with} \quad x_1=\frac{2p_1^0}{Q},\, x_2=\frac{2p_2^0}{Q},\, x_3=\frac{2k^0}{Q}=\frac{2E_k}{Q} \\
&= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{1}{Q^3}\int dx_1 d\Omega_{n-1}\int dx_2 d\Omega_{n-1} \\
&\quad \times \frac{(x_1x_2)^{n-2}}{x_1x_2x_3}\delta(x_1+x_2+x_3-2) \qquad (18)
\end{aligned}
$$

It is also interesting to notice that due to momentum conservation (as compared to $x_1+x_2+x_3=2$, which is due to energy conservation)

$$
x_3=\sqrt{x_1^2+x_2^2+2x_1x_2\cos\theta}, \qquad (19)
$$

which implies that the angle θ between $\vec{p}_1$ and $\vec{p}_2$ is determined by the delta function $\delta(x_1+x_2+x_3-2)$. At this moment, we need to use the explicit form for differential solid angle on a d-dimensional unit sphere

$$
\begin{aligned}
d\Omega_d &= \sin^{d-2}\phi_{d-1}\sin^{d-3}\phi_{d-2}\cdots\sin\phi_2 d\phi_{d-1}\cdots d\phi_2 d\phi_1 \qquad (20)\\
d\Omega_d &= d\Omega_{d-1}\sin^{d-2}\phi_{d-1}d\phi_{d-1}. \qquad (21)
\end{aligned}
$$

Ω_d can be interpreted geometrically as the $d-1$-dimensional surface area of a unit d-dimensional sphere. For example, $\Omega_3=4\pi$.

Now we let $\phi_{d-1}=\theta$ to be the angle between quark and antiquark, define $z\equiv\cos\theta$, and set $d=n-1$ in the above formula, which gives $d\Omega_{n-1}=d\Omega_{n-2}(1-z^2)^{(n-4)/2}$, therefore we can obtain

$$
\begin{aligned}
R_3^n &= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{\Omega_{n-1}\Omega_{n-2}}{Q^3}\int dx_1\int dx_2 (x_1x_2)^{n-3} \\
&\quad \times\int_{-1}^{+1}dz(1-z^2)^{(n-4)/2}\frac{1}{x_3}\delta(x_1+x_2+x_3-2) \qquad (22)\\
&\Downarrow \quad \text{note that} \int dz\delta(f(z))=\frac{1}{|f'(z)|}, \text{and} \quad \left|\frac{dx_3}{dz}\right|=\frac{x_1x_2}{x_3} \\
&= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{\Omega_{n-1}\Omega_{n-2}}{Q^3}\int dx_1dx_2\left[x_1x_2(1-z^2)^{1/2}\right]^{(n-4)}. \qquad (23)
\end{aligned}
$$

In addition, from Eq. (19) and energy conservation $x_1+x_2+x_3=2$, we also have

$$
z=\frac{x_3^2-x_1^2-x_2^2}{2x_1x_2}\Rightarrow x_1^2x_2^2(1-z^2)=4(1-x_1)(1-x_2)(x_1+x_2-1), \qquad (24)
$$

which eventually gives ($x_3<1$, $x_1+x_2>1$)

$$
\begin{aligned}
R_3^n &= \left(\frac{Q}{4\pi}\right)^{2n-3}\frac{\Omega_{n-1}\Omega_{n-2}}{Q^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2 \\
&\quad \times\left[4(1-x_1)(1-x_2)(x_1+x_2-1)\right]^{(n-4)/2}, \qquad (25)\\
&= \frac{Q^2\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}}{128\pi^3\Gamma(2-2\epsilon)}\int_0^1 dx_1\int_{1-x_1}^1 dx_2\frac{1}{[(1-x_1)(1-x_2)(1-x_3)]^{\epsilon}}.
\end{aligned}
$$

In the special case of $n=4$, one can show that $R_3=\frac{Q^2}{128\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2$.

- Then our next task is to compute $|\mathcal{M}_{\gamma^*\to q\bar{q}g}|^2$. In this note, we use the Feynman gauge, which gives $-g_{\mu\nu}$ for the gluon propagator. The total result in the end is gauge independent, although the contribution from each diagram may change with different gauge choice (See Field's book). **The real diagram**

amplitudes can be written as

$$
\begin{aligned}
i\mathcal{M}_{\gamma^*\to q\bar{q}g} &= (-iee_q)(-ig)\mu^{2\epsilon}\epsilon^{\mu}(q)\epsilon^{*\nu}(k)t^a_{ij} \\
&\quad\times\bar{u}(p_1)\left[\gamma_\nu\frac{i(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma_\mu+\gamma_\mu\frac{i(-\not{p}_2-\not{k})}{(p_2+k)^2}\gamma_\nu\right]v(p_2).
\end{aligned}
\tag{26}
$$

It is important to remember that we have a minus sign when the momentum flow direction is in the opposite direction of the fermion charge flow. This results in a minus sign for the second term coming from figure (c). Let us call the first term in the square brackets as $i\mathcal{M}_1$ and the second as $i\mathcal{M}_2$. It is then straightforward to find that

$$
|\mathcal{M}_1|^2 = (ee_q g\mu^{2\epsilon})^2 C_F N_c \mathrm{Tr}\left[\gamma_\nu\frac{(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma_\mu\not{p}_2\gamma^\mu\frac{(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma^\nu\not{p}_1\right], \tag{27}
$$

$$
\Downarrow \qquad \gamma_\mu\not{p}_2\gamma^\mu=-2(1-\epsilon)\not{p}_2 \text{ and } \gamma_\nu\not{p}_1\gamma^\nu=-2(1-\epsilon)\not{p}_1
$$

$$
= \frac{(ee_q g\mu^{2\epsilon})^2 C_F N_c}{D_1^2}4(1-\epsilon)^2\mathrm{Tr}\left[(\not{p}_1+\not{k})\not{p}_2(\not{p}_1+\not{k})\not{p}_1\right] \tag{28}
$$

$$
= 8(ee_q g\mu^{2\epsilon})^2 C_F N_c(1-\epsilon)^2\frac{1-x_1}{1-x_2}. \tag{29}
$$

Here we have used

$$
D_1\equiv(p_1+k)^2=(q-p_2)^2=q^2-2p_2\cdot q=Q^2(1-x_2),
$$

and we will also use $D_2\equiv(p_2+k)^2=Q^2(1-x_1)$. Note that we can have similar relations as follows:

$$
\begin{aligned}
&(p_1+k)^2=2p_1\cdot k=Q^2(1-x_2),\\
&(p_2+k)^2=2p_2\cdot k=Q^2(1-x_1),\\
&(p_1+p_2)^2=2p_1\cdot p_2=Q^2(1-x_3).
\end{aligned}
$$

It is important notice that the case $x_1\to1$ corresponds to $k\|p_2$. This is to say that the radiated gluon and antiquark are parallel to each other, or they are collinear with each other. Furthermore, $x_2\to1$ implies to $k\|p_1$ and $x_3\to1$ indicates $p_1\|p_2$. In addition, the soft limit (gluon has vanishing energy) $x_3\to0$ is reached when both $x_1\to1$ and $x_2\to1$.

Noticing that $|\mathcal{M}_2|^2$ has the same structure as $|\mathcal{M}_1|^2$, it is then pretty easy to find

$$
|\mathcal{M}_2|^2=8(ee_q g\mu^{2\epsilon})^2 C_F N_c(1-\epsilon)^2\frac{1-x_2}{1-x_1}, \tag{30}
$$

by interchanging $x_1\Leftrightarrow x_2$, and therefore find

$$
|\mathcal{M}_1|^2+|\mathcal{M}_2|^2=8(ee_q g\mu^{2\epsilon})^2 C_F N_c(1-\epsilon)^2\left[\frac{1-x_1}{1-x_2}+\frac{1-x_2}{1-x_1}\right]. \tag{31}
$$

Next, let us compute the interference term $\mathcal{M}_1^*\mathcal{M}_2+\mathcal{M}_2^*\mathcal{M}_1=2|\mathcal{M}_1^*\mathcal{M}_2|$ as follows

$$
\begin{aligned}
2|\mathcal{M}_1^*\mathcal{M}_2| &= -\frac{2(ee_q g\mu^{2\epsilon})^2 C_F N_c}{Q^4(1-x_1)(1-x_2)}\mathrm{Tr}\left[\gamma_\mu(\not{p}_2+\not{k})\gamma_\nu\not{p}_2\gamma^\mu(\not{p}_1+\not{k})\gamma^\nu\not{p}_1\right]\\
&\qquad \Downarrow \quad \text{use } \gamma^\mu\gamma^\alpha\gamma^\rho\gamma^\sigma\gamma_\mu=-2\gamma^\sigma\gamma^\rho\gamma^\alpha+2\epsilon\gamma^\alpha\gamma^\rho\gamma^\sigma;\\
&\qquad \Downarrow \quad \text{and } \gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu=4g^{\rho\sigma}-2\epsilon\gamma^\rho\gamma^\sigma;\\
&\Downarrow \quad \mathrm{Tr}[\cdots]=-16(2-\epsilon-\epsilon^2)p_1\cdot p_2(p_1+k)\cdot(p_2+k)\\
&\qquad\qquad +16\epsilon(1-\epsilon)\left[(p_1+k)\cdot p_1(p_2+k)\cdot p_2\right]\\
&\qquad\qquad +16\epsilon(1-\epsilon)\left[(p_2+k)\cdot p_1(p_2+k)\cdot p_1\right];\\
&= -\frac{2(ee_q g\mu^{2\epsilon})^2 C_F N_c}{Q^4(1-x_1)(1-x_2)}\\
&\quad\times 8Q^4\left[-(1-x_3)+\epsilon x_1x_2-\epsilon^2(1-x_1)(1-x_2)\right]\\
&= -16(ee_q g\mu^{2\epsilon})^2 C_F N_c\left[\frac{1-x_1-x_2+\epsilon x_1x_2}{(1-x_1)(1-x_2)}-\epsilon^2\right].
\end{aligned}
\tag{32}
$$

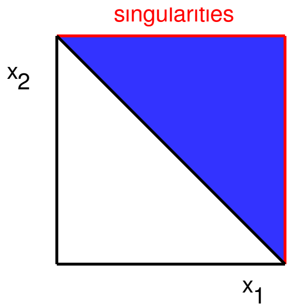
Therefore, at the end of the day, one gets for the total square of the amplitudes as

$$|\mathcal{M}|^2 = 8(ee_q g\mu^{2\epsilon})^2 C_F N_c (1-\epsilon)\frac{x_1^2 + x_2^2 - \epsilon x_3^2}{(1-x_1)(1-x_2)}. \tag{33}$$

Here $x_3 = 2 - x_1 - x_2$.

- Let us put the results back to $n = 4(\epsilon = 0)$ dimension, which gives

$$|\mathcal{M}_{\gamma^* \to q\bar{q}g}|^2 = 8(ee_q g)^2 C_F N_c \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}, \tag{34}$$



with the integration range indicated in the coloured area. When $x_1 \to 1$ or/and $x_2 \to 1$, there are divergences appearing in the cross section. When $x_3 \to 0$, we have the soft divergence, and when $x_1 \to 1$ or $x_2 \to 1$, we have the collinear divergence. Of course, these two types of IR divergences can coincide and happen at the same time.

- The logarithmic divergence $\int_0^1 \frac{dx}{1-x}$ in 4-d has been converted into a pole in ϵ in the $4-2\epsilon$ dimension as follows

$$\int_0^1 \frac{dx}{(1-x)^{1+\epsilon}} = \frac{1}{\epsilon}(1-x)^{-\epsilon}\Big|_0^1 = \frac{-1}{\epsilon}, \tag{35}$$

provided that $\epsilon < 0$. As we shall see immediately, we will obtain poles from integrations over x_1 and x_2 for σ_3.

- The appearance of the IR divergence in the real cross section is due to the fact that we are trying to distinguish the NLO $q\bar{q}g$ state from the LO $q\bar{q}$ state. According to the KLN theorem, we are going to get the IR divergences as expected.

Now eventually, we can compute the cross section of $\gamma^* \to q\bar{q}g$ by putting everything altogether according to Eq. (16)

$$\sigma_3 = \frac{Q\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}}{256\pi^3\Gamma(2-2\epsilon)}\int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \frac{|\mathcal{M}|^2}{[(1-x_1)(1-x_2)(1-x_3)]^{\epsilon}}. \tag{36}$$

The direct evaluation of the above integrals are not easy. One needs to use a trick to simplify the integrand. Let us define $x_2 = 1 - vx_1$, which gives

$$1 - x_3 = x_1 + x_2 - 1 = x_1(1-v), \tag{37}$$

$$1 - x_2 = vx_1, \quad \Rightarrow \int_{1-x_1}^1 dx_2 = \int_0^1 x_1 dv. \tag{38}$$

It is then clear that the integration can be separated into $\int dv$ and $\int dx_1$ completely. The objective of this change of variable is to be able to factorize the denominator of the integrand in Eq (36) into factors with independent separate variables.

Using the definition of β function (to save time, one always uses computer programs to do these integrations), one can now compute the three body final state real cross section σ_3 analytically and expand it in terms of ϵ up to constant terms as follows

$$\sigma_3 \;=\; \frac{Q\left(\frac{Q^2}{4\pi}\right)^{-2\epsilon}}{256\pi^3\Gamma(2-2\epsilon)}\int_0^1 dx_1\int_0^1 dv\frac{x_1|\mathcal{M}|^2}{[(1-x_1)(1-x_2)(1-x_3)]^{\epsilon}}, \tag{39}$$

$$= \frac{\sigma_0\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}}{\Gamma(2-\epsilon)}\frac{\alpha_s C_F}{2\pi}\int_0^1 dx_1\int_0^1 dv\frac{x_1|\mathcal{M}|^2}{[(1-x_1)(1-x_2)(1-x_3)]^{\epsilon}}, \tag{40}$$

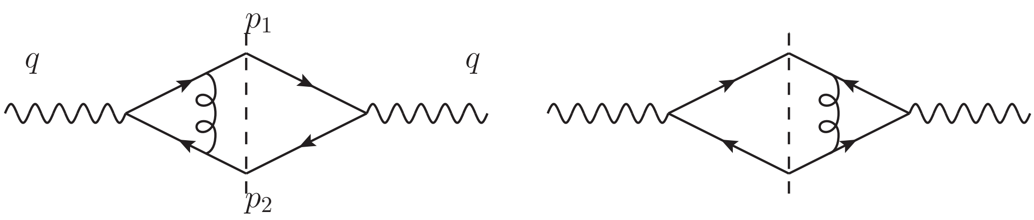
$$= \sigma_0\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}-\frac{2\pi^2}{3}+\mathcal{O}(\epsilon)\right]. \tag{41}$$

Several comments are in order:

1. As commented before, we are expecting to encounter singularities when we perform the above calculation in $n=4$-dimension. That is to say we are getting divergent real NLO cross sections when we take $\epsilon \to 0$ limit as shown above.
2. The $\dfrac{2}{\epsilon^2}$ term comes from the kinematical region where the radiated gluon is soft ($x_3 \to 0$) and collinear (parallel to q or $\bar{q}$). The pole of $\dfrac{3}{\epsilon}$ comes from the collinear singularities when the radiated gluon is parallel to q or $\bar{q}$.
3. Of course, σ_3 itself is not an experimental observable, therefore it is acceptable to be divergent. In high energy experiments, subject to the sensitivity of detectors, we can only measure particles whose energy is above some certain value E_0 and we can only distinguish particles which are separated at least by some minimum angle θ_0. Namely, we can never measure an arbitrary soft gluon or distinguish a quark from a quark plus a collinear gluon.

1.3 NLO virtual diagram

The virtual diagrams at NLO is shown as follows[1]

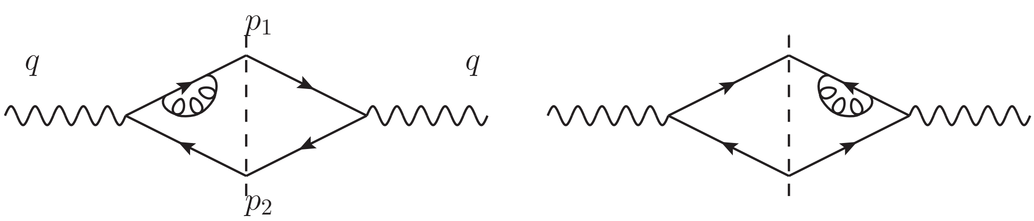


(d) (e)

Now consider the virtual process $\gamma^* \to q\bar{q}$ as shown in Figure (d) and (e). Compute the corresponding cross section which is defined as

$$\sigma_v = \frac{1}{2Q}R_2^n|\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}+\mathcal{M}^{(0)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}}|. \tag{42}$$

[1] In principle, we should also consider the following two virtual diagrams as well.



(f) (g)

However, in the dimensional regularization, the contributions from these two diagrams vanish. In other regularisations, one should of course consider the figures (f) and (g). We will discuss this issue in the next chapter when we cover the topic of renormalization.

We have studied the two body Lorentz invariant phase space R_2^n before. Now the only task left is to compute the corresponding amplitude square. The way to understand the virtual graph is to write the two body final state amplitude as

$$i\mathcal{M}_{\gamma^*\to q\bar{q}} = i\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}(\text{No}\, g) + i\mathcal{M}^{(1)}_{\gamma^*\to q\bar{q}}(1g) + i\mathcal{M}^{(2)}_{\gamma^*\to q\bar{q}}(2g) + \cdots, \tag{43}$$

the square of the above amplitudes can be viewed as an expansion in terms of α_s^m (m is the number of virtual gluons). Therefore, at NLO, the amplitude square gives $2\text{Re}\left[\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}\right]$. [2]

According to Feynman rules, summing over the virtual photon polarisations $\sum \epsilon_\mu \epsilon_\nu^* \Rightarrow -g_{\mu\nu}$, we can write it down as follows

$$\begin{aligned}
& 2\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}} \\
= \;& 2(ee_q g\mu^{2\epsilon})^2 C_F N_c (i)^{3+2}(-i)^2(-1) \\
\times \;& \int \frac{d^n k}{(2\pi)^n k^2} \text{Tr}\left[\not{p}_1 \gamma^\alpha \frac{(\not{p}_1+\not{k})}{(p_1+k)^2}\gamma^\mu \frac{-(\not{p}_2-\not{k})}{(p_2-k)^2}\gamma_\alpha \not{p}_2 \gamma_\mu\right] \\
= \;& 2i(ee_q g\mu^{2\epsilon})^2 C_F N_c \int \frac{d^n k}{(2\pi)^n}\frac{N}{k^2(p_1+k)^2(p_2-k)^2} \\
& \quad \text{with}\, N \equiv \text{Tr}\left[\not{p}_1\gamma^\alpha(\not{p}_1+\not{k})\gamma^\mu(\not{k}-\not{p}_2)\gamma_\alpha\not{p}_2\gamma_\mu\right].
\end{aligned} \tag{44}$$

The loop integral has the denominator of the form $k^2(p_1+k)^2(p_2-k)^2$, which vanishes when $k\to 0$ (soft) or when k is collinear with either p_1 (quark) or p_2 (antiquark). The singularities of the denominator corresponds the same types of singularities as observed in the real contribution for the $q\bar{q}g$ final state. **It is important to notice that, when the gluon is soft or collinear, the NLO contribution from real and virtual diagrams become the same but with opposite sign.** This is the reason that the singularity shall cancel between real and virtual diagrams for the total cross section as we shall see later. Graphically, the minus sign between the real and virtual diagrams can be easily seen by comparing the gluon vertices in these two cases ($i^2=-1$ for virtual graphs, while $-i\times i=1$ for real graphs.). When the gluon is soft in virtual graphs, it becomes on-shell as in the real graphs. When the gluon is collinear with quark or antiquark in virtual graphs, physically it is indistinguishable from the collinear real graphs.

Using the same technique for γ-matrices, it is straightforward to cast N into

$$\begin{aligned}
N \;=\; & -16(2-\epsilon)(p_1+k)\cdot p_2(k-p_2)\cdot p_1 - 16\epsilon^2 k^2 p_1\cdot p_2 \\
& +16\epsilon\left[(p_1+k)\cdot(k-p_2)p_1\cdot p_2 + p_1\cdot k p_2\cdot k\right] \\
=\; & 8(1-\epsilon)\left[Q^4 - 2Q^2(k\cdot p_1 - k\cdot p_2) - 4k\cdot p_1 k\cdot p_2 + \epsilon Q^2 k^2\right].
\end{aligned} \tag{45}$$

It is also worth mentioning that these four terms in N has different types of singularities. The first term has both soft and collinear (IR) singularities, while the second and third term which is proportional to k only have collinear singularity. The fourth term not only has collinear singularity (IR) but also have UV divergence. On the other hand, the UV and IR divergences cancel each other in the dimensional regularization which makes the fourth term finite. (For further discussion on this issue, see Chapter 20 of Schwartz.)

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[2] For convenience, we will just compute $2\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}$ directly in the following, and take the real part of the final results in the end.

Next, let us start to work on the integrations, which can be achieved by using the following identities

$$\begin{aligned}
\mathcal{I}_1 &= \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2(p_1+k)^2(p_2-k)^2} \\
&= \frac{-i\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}} \left(-\frac{1}{Q^2}\right)^{3-\frac{n}{2}} \frac{B(\frac{n}{2}-2,\frac{n}{2}-1)}{\frac{n}{2}-2}, && (46)\\
\mathcal{I}_2 &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu}{k^2(p_1+k)^2(p_2-k)^2} \\
&= \frac{i\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}} \left(-\frac{1}{Q^2}\right)^{3-\frac{n}{2}} \frac{B(\frac{n}{2}-1,\frac{n}{2}-1)}{\frac{n}{2}-2} (p_{1\mu}-p_{2\mu}), && (47)\\
\mathcal{I}_3 &= \int \frac{d^n k}{(2\pi)^n} \frac{k_\mu k_\nu}{k^2(p_1+k)^2(p_2-k)^2} \\
&= \frac{-i\Gamma(3-\frac{n}{2})}{(4\pi)^{n/2}} \left(-\frac{1}{Q^2}\right)^{3-\frac{n}{2}} \frac{B(\frac{n}{2}-1,\frac{n}{2})}{(\frac{n}{2}-1)(\frac{n}{2}-2)} && (48)\\
&\times \left[\frac{n-2}{2}(p_{1\mu}p_{1\nu}+p_{2\mu}p_{2\nu}) - \frac{n-4}{2}(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu}) - p_1\cdot p_2 g_{\mu\nu}\right].
\end{aligned}$$

Let me discuss the derivation of $\mathcal{I}_1$ in detail, and leave the proof of $\mathcal{I}_2$ and $\mathcal{I}_3$ to you if you wish to practice what you have learnt so far. [3]

Using the so-called Feynman parameter technique, we can cast $\mathcal{I}_1$ into the following form ($x,y,z\in[0,1]$)

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 dxdydz\delta(x+y+z-1)\int \frac{d^n k}{(2\pi)^n}\frac{2}{D^3} && (49)\\
&\quad \text{with} \quad D \equiv xk^2 + y(p_1+k)^2 + z(p_2-k)^2 \\
&\qquad\qquad = l^2 - (yp_1 - zp_2)^2 = l^2 + yzQ^2, && (50)
\end{aligned}$$

where $l \equiv k + yp_1 - zp_2$ allows us to shift the integration from k to l. Let us also define $\Delta = -yzQ^2$, and then use the Wick rotation to write $\mathcal{I}_1$ as

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^1 dydz \int \frac{d^n l}{(2\pi)^n}\frac{2}{(l^2-\Delta)^3} && (51)\\
&= -i\int_0^1 dydz \int \frac{d^n l_E}{(2\pi)^n}\frac{2}{(l_E^2+\Delta)^3} && (52)\\
&= \int_0^1 dydz \frac{-2i}{(4\pi)^{n/2}}\frac{\Gamma(3-\frac{n}{2})}{\Gamma(3)}\left(\frac{1}{\Delta}\right)^{3-\frac{n}{2}}, && (53)
\end{aligned}$$

where we have use $\int \frac{d^d l_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^m} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(m-\frac{d}{2})}{\Gamma(m)}\left(\frac{1}{\Delta}\right)^{m-\frac{d}{2}}$ in the last step. In the end, it is straightforward to finish the rest of the $dydz$ integrations by noticing that $0<x<1 \Rightarrow y+z<1$ which sets the upper limit of dz integration to be $1-y$. Therefore, we arrive at the $\mathcal{I}_1$ identity after taking into account $\Delta = -yzQ^2$ and the result

$$\int_0^1 dy \int_0^{1-y} dz y^{\frac{n}{2}-3} z^{\frac{n}{2}-3} = \int_0^1 dy y^{\frac{n}{2}-3}\frac{(1-y)^{\frac{n}{2}-2}}{\frac{n}{2}-2} = \frac{B(\frac{n}{2}-2,\frac{n}{2}-1)}{\frac{n}{2}-2}. \tag{54}$$

Now we are ready to evaluate all the integrals with $\mathcal{I}_{1,2,3}$ identities, and assemble all the pieces altogether

[3] In principle, you do not have to start from the beginning to compute $\mathcal{I}_2$ and $\mathcal{I}_3$. There is a trick from the book Quantum Field Theory by Lewis H. Ryder. (See page 382, chapter 9, Appendix A). The idea is that one can always get factors of k_μ by differentiating with respect to p_1^μ or p_2^μ.

and find the virtual contribution to the NLO cross section can be explicitly written as

$$\begin{aligned}
\sigma_V &= \sigma_0 \frac{2\mathcal{M}^{(1)*}_{\gamma^*\to q\bar{q}}\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}}{|\mathcal{M}^{(0)}_{\gamma^*\to q\bar{q}}|^2}, && (55)\\
&= -\sigma_0 \frac{\alpha_s C_F}{\pi}\left(-\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\Gamma(1+\epsilon)\\
&\quad\times\left[-\frac{1}{\epsilon}\frac{\Gamma(-\epsilon)\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}+\frac{2}{\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(1-\epsilon)}{\Gamma(2-2\epsilon)}-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}+2\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\right], && (56)\\
&= \sigma_0 \frac{\alpha_s C_F}{\pi}\left(-\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\times\left[-\frac{1}{\epsilon^2}-\left(\frac{3}{2\epsilon}+1\right)\frac{1}{1-2\epsilon}\right]. && (57)
\end{aligned}$$

The last step is to take the real part of the above expression as mentioned earlier in the beginning of this subsection. With the assistance of

$$\mathrm{Re}(-1)^{-\epsilon} = \mathrm{Re}[e^{-i\epsilon\pi}] = 1-\frac{1}{2}\pi^2\epsilon^2+\mathcal{O}(\epsilon^4), \tag{58}$$

$$\Gamma(1-\epsilon)\Gamma(1+\epsilon) = 1+\frac{1}{6}\pi^2\epsilon^2+\mathcal{O}(\epsilon^4), \tag{59}$$

we can obtain

$$\sigma_V = \sigma_0\frac{\alpha_s C_F}{2\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right]. \tag{60}$$

1.4 Total cross section

At the end of the day, one finds that the NLO real and virtual contributions are

$$\begin{aligned}
\sigma_3 &= \sigma_0\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+\frac{19}{2}-\frac{2\pi^2}{3}\right],\\
\sigma_V &= \sigma_0\frac{\alpha_s}{2\pi}C_F\left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}-8+\frac{2\pi^2}{3}\right],
\end{aligned}$$

respectively. Therefore, by summing over the LO and NLO contributions to the cross section of $\gamma^* \to X$, we can obtain

$$\lim_{\epsilon\to 0}\sigma^{\text{tot}}_{\gamma^*\to X} = \sigma_0\left[1+\frac{3}{4}C_F\frac{\alpha_s(\mu)}{\pi}+\mathcal{O}(\alpha_s^2)\right], \tag{61}$$

which is finite in 4-dimension when we take $\epsilon \to 0$. [4] In general, we can use the following standard procedures to compute all the diagrams

Some more comments are in order:

- The ratio between the $e^+e^- \to$ hadrons total cross section and the $e^+e^- \to \mu^+\mu^-$ cross section. $N_c\sum_{u,d,s} e_i^2 = 2$, $N_c\sum_{u,d,s,c} e_i^2 = \frac{10}{3}$, $N_c\sum_{u,d,s,c,b} e_i^2 = \frac{11}{3}$.

$$R = \frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)} = N_c\sum_{u,d,s,\ldots} e_i^2\left[1+\frac{\alpha_s(Q^2)}{\pi}\right]$$

- The cancellation of soft and collinear singularities between the real and virtual gluon diagrams for total cross section is no accident. We have the **KLN theorem**, which states that suitably defined inclusive quantities will indeed be free of singularities in the massless limit. The total hadronic cross section of $e^+e^- \to \gamma^* \to X$(hadrons) is an example of such **infrared safe** quantities. In fact, any experimental observable must be infrared safe in order to be sensibly computed in field theory.

[4] For this specific process, one can give gluons a small mass m_g and use it as a regulator for IR divergence, which allows us to obtain exactly the same total cross section. See Application of pQCD by R. D. Field for details.

2 Introduction to Resummation

In this lecture note, we discuss the physical origin of the Sudakov factor[5] by using the thrust distribution as an example and the application of the Sudakov resummation in hadronic collisions. Sudakov factors, especially the double logarithmic terms, can appear in many physical processes as a result of the incomplete cancellation of soft (soft-collinear) divergences between real and virtual contributions.

This lecture note is intended to be an informal and intuitive note on the Sudakov resummation in QCD for graduate students to learn some basics on this topic and for the purpose of learning only. It is by no means complete or rigorous. Please use it with caution.

In the lecture, I presume that you are familiar with QFT and Peskin's book on QFT. From time to time, I will refer to certain chapters in Peskin in case that you are interested in more details. A lot of material in this note is based on what I learnt over the past few years from Prof. Al Mueller, Prof. J. Owens, Dr. F. Yuan and other colleagues, as well as from some QCD textbooks and material published online.

2.1 Resummation

Resummation is a vague and broad concept, which implies that summation needs to be done once more. The question is that what summation has been performed in the first place. In perturbative QCD, we expand cross section σ in terms of powers of α_s

$$\sigma = \sigma_0 + \alpha_s \sigma_1 + \alpha_s^2 \sigma_2 + \cdots = \sum_{i=0}^{\infty} \alpha_s^i \sigma_i, \tag{62}$$

$$\sigma_0 \sum_{i=0}^{\infty} \alpha_s^i \left(L^i + C^{(i)}\right) \quad \text{ideal QCD expansion}$$

$$\begin{array}{c|c} \sigma_0 \sum_{i=0}^{n-1} \alpha_s^i L^i & \sigma_0 \sum_{i=0}^{n-1} \alpha_s^i C^{(i)} \\ \hline \sigma_0 \sum_{i=n}^{\infty} \alpha_s^i L^i & \sigma_0 \sum_{i=n}^{\infty} \alpha_s^i C^{(i)} \end{array} \quad \Leftarrow \text{pQCD}$$

$\Uparrow$ resummation $\qquad\qquad \nwarrow$ negligible

where $\sigma_i = \sigma_0 \left(L^i + C^{(i)}\right)$ represents the cross section computed from the i-th order perturbative calculation. Sometimes, this expansion is convergent without the appearance of large logarithms L, then the above series can be truncated at certain order. Often, this series is not convergent, since large logarithms can appear in higher order expansions $\sigma_n \sim L^n$, where L stands for the logarithmic term.

More specifically, for inclusive total cross section in e^+e^- annihilation, we will see that higher order corrections are just constants without large logarithms. In this case, the pQCD expansion is convergent. For the thrust distribution, as we will see, the appearance of large logarithms can cause the breakdown of pQCD expansion, since $\alpha_s^n \sigma_n$ can increase as n increases. This means that we can no longer truncate the above series in terms of pQCD expansion, and we no longer have reliable predictions if we only rely on finite fixed order results.

Obviously, we can not do all order calculations exactly in pQCD. However, we can systematically resum $\alpha_s^n L^n$ terms up to all order and neglect the constant higher order corrections. Examples: the well-known DGLAP evolution equation resums $\left(\alpha_s \ln Q^2/\mu^2\right)^n$ and BFKL evolution equation resums $(\alpha_s \ln 1/x)^n$ at the leading logarithmic (LL) level. Usually we categorize the level of the resummation of $\sum_n \alpha_s^k (\alpha_s L)^n$ as follows

- LL: leading log $\Leftrightarrow \sum_n (\alpha_s L)^n$ with $k = 0$;

[5] See Peskin for discussions on conventional Sudakov double logarithmic form factor.

- NLL: next to leading log $\Leftrightarrow \alpha_s \sum_n (\alpha_s L)^n$;
- N^{k}LL: $\sum_n \alpha_s^k (\alpha_s L)^n$.

The purpose of resummation is to restore predictive power in theoretical calculations and describe the relevant physics better from an overall perspective.

2.2 Infrared safety

In higher order caculations in QFT, we often encounter two kinds collinear divergence and soft divergence. Both of them are of the Infrared divergence type. That is to say, they both involve long distance.

- According to uncertainty principle, soft $\longleftrightarrow$ long distance;
- Also one needs an infinite time in order to specify accurately the particle momenta, and therefore their directions.

However, physical observables, due to infrared safety, are always finite when we measure them. Infrared safety is the property of any experimental observable, which can be computed reliably in perturbative QCD (order by order perturbative expansion of α_s with finite coefficients at every order).

- **Kinoshita-Lee-Nauenberg theorem**: For a suitable defined inclusive observable (e.g., $\sigma_{e^+e^-\to\text{hadrons}}$), there is a cancellation between the soft and collinear singularities occurring in the real and virtual contributions. Physical observables always requires the cancellation.
- Any new observables must have a definition which does not distinguish between

$$\begin{gathered}\text{parton} \longleftrightarrow \text{parton + soft gluon}\\ \text{parton} \longleftrightarrow \text{two collinear partons}\end{gathered}$$

- Observables that respect the above constraint are called infrared safe observables. Infrared safety is a requirement that the observable is calculable in pQCD.
- Other infrared safe observables, for example, jets and the event shape observable thrust: $T = \max \dfrac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$.

2.3 Inclusive total cross section in e^+e^- annihilation

Consider the process (for detailed derivations, please read my lecture note on the e^+e^- annihilation)

$$e^+ + e^- \to \gamma^* \to q\bar{q}(\text{LO}) \quad \text{or} \quad q + \bar{q} + g(\text{NLO}). \tag{63}$$

We have seen “almost complete" cancellation between real and virtual contributions for the total cross section as suggested by the KLN theorem with a small constant NLO correction of the order $\frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi}$. Here I used the loose term “almost complete" cancellation to describe the situation that only a small constant correction survives the cancellation in the total cross section at NLO.

In contrast, for other more differential observables, due to additional cuts made to the real contributions (or due to different constraints made to the real and virtual contributions), the cancellation between the real and virtual diagrams often is “incomplete" in the sense that large logarithms can appear as the result of the soft and collinear divergence cancellation between real and virtual graphs. In general, large logarithms can show up as the expansion in terms of ϵ

$$\frac{1}{\epsilon} a^\epsilon = \frac{1}{\epsilon} e^{\epsilon \ln a} = \frac{1}{\epsilon} + \ln a + \cdots \tag{64}$$

$$\frac{1}{\epsilon^2} a^\epsilon = \frac{1}{\epsilon^2} e^{\epsilon \ln a} = \frac{1}{\epsilon^2} + \ln a \frac{1}{\epsilon} + \frac{1}{2} \ln^2 a + \cdots . \tag{65}$$

Next, let us take a look at the example of the thrust distribution in e^+e^- annihilation where large logarithms start to appear.

3 Sudakov Resummation for Thrust Distribution

Thrust is an event shape observable reflecting the structure of the hadronic events in e^+e^- annihilation. The thrust T[1] is defined as

$$T = \max_{\vec{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|},$$

where $\vec{p}_i$ are the final-state hadron (or parton) momenta in the center of mass frame of e^+e^- collisional system and $\vec{n}$ is an arbitrary unit vector which maximizes $\frac{\sum_i |\vec{p}_i \cdot \vec{n}|}{\sum_i |\vec{p}_i|}$. The direction of $\vec{n}$ vector is given by the direction of the largest momentum particle. It is straightforward to find $T = 1$ for back-to-back pencil-like events and $T = 1/2$ for spherically symmetric events.

Furthermore, T is infrared safe, i.e. insensitive to the emission of soft or collinear gluons, since T is invariant under the branching $\vec{P}_i \to \vec{P}_j + \vec{P}_k$, whenever $\vec{P}_j \parallel \vec{P}_k$ or one of them is soft.

3.1 Thrust in pQCD

Now let us study the thrust distribution in terms of the pQCD expansion.

(a) At zeroth order, we have the born process $e^- + e^- \to q + \bar{q}$, which gives the pencil like events, it is easy to show that $T = 1$ in this case (this is true for NLO virtual graph as well). Due to momentum conservation, $p_1 = p_2$, therefore, $T = 1$ by definition when $\vec{n}$ is chosen along $\vec{p}_1$ or $\vec{p}_2$. Therefore, we can write the normalized differential cross section as

$$\frac{1}{\sigma_0}\frac{d\sigma_0}{dT} = \delta(T-1).$$

(b) At the first order, we consider the $2 \to 3$ ($e^- + e^- \to q + \bar{q} + g$) process, which generates three particle final state events. We have derived the cross section in class which reads

$$\frac{1}{\sigma_0}\frac{d\sigma}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)},$$

where $x_1 = \frac{2p_1}{Q}$ and $x_2 = \frac{2p_2}{Q}$ are for the quark and antiquark, respectively. The energy conservation indicates $x_1 + x_2 + x_3 = 2$ where $x_3 = \frac{2p_3}{Q}$. Using geometry and momentum conservation, we should be able to find $\underline{T = \max[x_1, x_2, x_3]}$ in this case. For three-particle events, minimum value of T is 2/3 when all three momentum are equal, while the maximum value of T is 1.

(c) Use the delta function trick, we can write the differential cross section of thrust as

$$\frac{1}{\sigma_0}\frac{d\sigma}{dT} = \frac{C_F\alpha_s}{2\pi}\int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}\delta(T - \max[x_1, x_2, x_3]),$$

where $x_3 = 2 - x_1 - x_2$. We can perform the above integrations and find $\frac{1}{\sigma_0}\frac{d\sigma}{dT}$ as the function of T as follows

$$\frac{d\sigma}{\sigma_0 dT} = 2\frac{C_F\alpha_s}{2\pi}\int_{1-T/2}^{T} \mathrm{d}x_2 \left[\frac{T^2 + x_2^2}{(1-T)(1-x_2)}\right]_{x_1>x_2>x_3;\,\mathrm{or},\,x_2>x_1>x_3} \tag{66}$$

$$+2\frac{C_F\alpha_s}{2\pi}\int_{2-2T}^{1-T/2} \mathrm{d}x_2 \left[\frac{T^2 + x_2^2}{(1-T)(1-x_2)}\right]_{x_1>x_3>x_2;\,\mathrm{or},\,x_2>x_3>x_1} \tag{67}$$

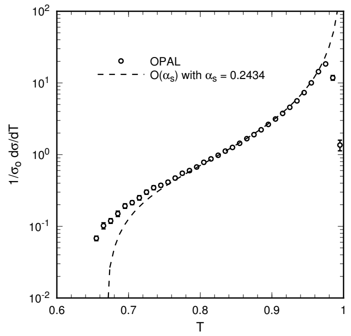
$$+2\frac{C_F\alpha_s}{2\pi}\int_{1-T/2}^{T} \mathrm{d}x_2 \left[\frac{(2-T-x_2)^2 + x_2^2}{(T+x_2-1)(1-x_2)}\right]_{x_3>x_2>x_1;\,\mathrm{or},\,x_3>x_1>x_2} \tag{68}$$

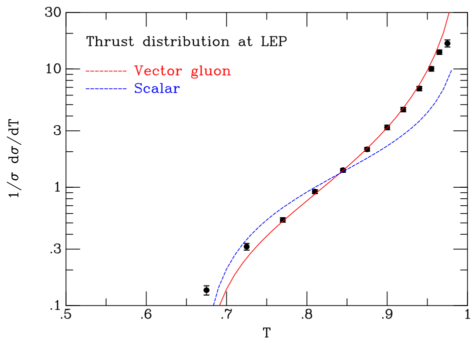
As shown above, first we consider the region in which $x_1 > x_2 > x_3$, and note that the delta function sets $T = x_1$ in this region, then determine the range of the integration for x_2 according to energy momentum conservation before we integrate over x_2. Next we can consider the other five different regions and sum all of them together which gives the following final expression at the first non-trivial order

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F\alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right]. \tag{69}$$

Homework: Derive the above thrust distribution in Eq. (69) for $T < 1$.

(d) We can compare the above expression to the experimental data as shown below.





- Left figure: Deficiency at low T due to kinematics. $T > 2/3$ at this order. By continuing the pQCD expansion to higher order, we expect that the agreement at low T will get improved.
- Left figure: Miss the data when $T \to 1$ due to divergence as seen below.
$$\left.\frac{d\sigma}{\sigma_0 dT}\right|_{T\to 1} \sim \frac{C_F\alpha_s}{2\pi}\left[\frac{4}{(1-T)}\ln\frac{1}{1-T} - \frac{3}{1-T}\right] \to \infty.$$
In this region, pQCD expansion fails due to the divergent behavior. This is to say that we have to resum the large logarithms up to all order: Sudakov factor!
- Right figure: Indication of gluon being a vector boson instead of a scalar. [2]

3.2 Resummation of the thrust distribution

To perform the resummation near $T = 1$, we can use the following intuitive steps.

(a) Let us include the Born and virtual as well as real contributions, and write

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F\alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T}\right] + C\delta(1-T), \tag{70}$$

where C is a divergent constant, which can be determined by the following integral according to Eq. (**??**)

$$\int_{T_{\min}}^{1} dT\frac{d\sigma}{\sigma_0 dT} = 1 + C_F\frac{3\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2). \tag{71}$$

Homework:
From Eq. (69), show that the following exact expression of the thrust distribution at one-loop satisfies Eq. (71) with $T_{\min} = 2/3$

$$\begin{aligned}\frac{d\sigma}{\sigma_0 dT} &= \delta(1-T) + \frac{C_F\alpha_s}{2\pi}\left[\delta(1-T)\left(\frac{\pi^2}{3}-1\right) - \frac{3(3T-2)(2-T)}{(1-T)_+}\right] \\ &\quad + \frac{C_F\alpha_s}{2\pi}\frac{2(3T^2-3T+2)}{T}\left[\frac{\ln(2T-1)}{(1-T)_+} - \left(\frac{\ln(1-T)}{1-T}\right)_+\right],\end{aligned} \tag{72}$$

where the plus distribution is defined as $\int_a^1 dx f(x)(g(x))_+ = \int_a^1 dx f(x)g(x) - f(1)\int_0^1 dx g(x)$.

Since we are only interested in the large logarithm resummation, we can neglect the $\frac{\alpha_s}{\pi}$ term in the above expression as far as the resummation is concerned in our later derivations.

(b) With the help of the cumulative distribution method, we can first define $F(T) = \int_T^1 dT' \frac{d\sigma}{\sigma_0 dT'}$, and find

$$1 = \int_{T_{\min}}^1 dT \frac{d\sigma}{\sigma_0 dT} = \int_{T_{\min}}^T dT \frac{d\sigma}{\sigma_0 dT} + \int_T^1 dT \frac{d\sigma}{\sigma_0 dT} = \int_{T_{\min}}^T dT \frac{d\sigma}{\sigma_0 dT} + F(T) \quad \Rightarrow \tag{73}$$

$$. \quad F(T) \simeq 1 - \frac{C_F\alpha_s}{\pi}\ln^2(1-T) - \frac{C_F\alpha_s}{\pi}\frac{3}{2}\ln(1-T), \tag{74}$$

where the double log term comes from soft gluon region while the single log term is due to collinear gluon emissions.

(c) Assuming that soft and collinear contributions factorize (as we will show later in this lecture notes), we can resum these logarithms and obtain

$$\sum_{n=0}^{\infty}\frac{1}{n!}x^n = e^x \qquad \Rightarrow \qquad F(T) = \exp\left[-\frac{C_F\alpha_s}{\pi}\ln^2(1-T) - \frac{C_F\alpha_s}{\pi}\frac{3}{2}\ln(1-T)\right], \tag{75}$$

which is also known as the Sudakov factor for the thrust distribution. The exponentiation of the Sudakov factor can be understood as the result of resumming arbitrary number n soft gluon emission.

Alternatively, one can derive the above result by solving a differential equation involving $F(T)$. The cumulative distribution $F(T)$ represents the probability of thrust T in the interval $[T, 1]$. From fixed order result in Eq. (73), we find that $F(T)$ must satisfy

$$\frac{dF(T)}{dT} = -\frac{d\sigma}{\sigma_0 dT} = -\frac{C_F\alpha_s}{2\pi}\left[\frac{4}{(1-T)}\ln\frac{1}{1-T} - \frac{3}{1-T}\right]. \tag{76}$$

However, physically $F(T)$ should always be positive and therefore the above differential equation must be modified to insure its positivity. Since the gluon emission described by the right hand side of the above equation, can only decrease T (More gluon radiation tends to make the event shape more spherical), we expect that the change of $F(T)$ should also depend on itself. Therefore, we obtain

$$\frac{dF(T)}{dT} = -\frac{C_F\alpha_s}{2\pi}\left[\frac{4}{(1-T)}\ln\frac{1}{1-T} - \frac{3}{1-T}\right]F(T), \tag{77}$$

which can also be interpreted as the result of iteration due to multiple gluon emission. The solution of the above equation gives the resummed result in Eq. (75).

In the end, taking a derivative respect to T yields the Sudakov resummed thrust distribution

$$\frac{d\sigma}{\sigma_0 dT} = -\frac{dF(T)}{dT} = \frac{C_F\alpha_s}{2\pi}\left[\frac{4}{(1-T)}\ln\frac{1}{1-T} - \frac{3}{1-T}\right]\exp\left[-\frac{C_F\alpha_s}{\pi}\ln^2(1-T) - \frac{C_F\alpha_s}{\pi}\frac{3}{2}\ln(1-T)\right]. \tag{78}$$

This result can describe the measured thrust distribution reasonably well in the $T \to 1$ limit by taming the divergence. Modern technique such as the renomalization group equation (RGE) method in soft collinear effective theory (SCET) has been developed not too long ago, this allow us to perform the Sudakov resummation systematically. (See more discussion on the thrust distribution in Ref. [3].)

(d) The above resummation techniques will be used repeatedly in the following discussions. You will be able to also observe the pattern of Sudakov factors when they appear.

4 Sudakov Resummation in Drell-Yan Process

4.1 Transverse momentum (Q_T) distribution resummation in DY process

Now let us consider Q_T distribution of the lepton pair (γ^*) in hadronic collisions in the $Q_T \ll Q$ limit. We choose the kinematic region where Q^2 is not too close to S, so that threshold logs are not important.

At leading order, $Q_T = 0$ since q and $\bar{q}$ carries no k_T in collinear factorization. This can be cast into the following form

$$\frac{d\sigma_{DY}^{(LO)}}{dQ^2 d^2 Q_T} = \sum_q \int \frac{dx_1}{x_1} q(x_1) \int \frac{dx_2}{x_2} \bar{q}(x_2) \frac{\sigma(q\bar{q})}{S} \delta(1-z) \frac{1}{(2\pi)^2} \int d^2 b_\perp e^{-iQ_T \cdot b_\perp}, \tag{79}$$

where the last integral yields $\delta^{(2)}(Q_T)$ as expected.

4.1.1 Momentum space analysis

Additional gluon radiation can generate non-zero Q_T. In particular, soft-gluon emissions are the dominant contribution in the $Q_T \ll Q$ limit. To simplify the calculation, let us consider the emission of a gluon from an energetic quark, which allows us to write the rate of the $q \to qg$ splitting as

$$P = \frac{\alpha_s C_F}{2\pi^2} \int \frac{d^2 k_\perp}{k_\perp} \int d\xi \frac{1+\xi^2}{1-\xi} \bigg|_{\xi \neq 1}, \tag{80}$$

where ξ is the longitudinal momentum fraction of the final state quark w.r.t. the parent quark. When $\xi = 1$, which implies that the radiated gluon carries zero momentum $(1-\xi)$, we need to include the virtual contribution and replace $\frac{1+\xi^2}{1-\xi}$ by $\left(\frac{1+\xi^2}{1-\xi}\right)_+$. The detail derivation of the above splitting function can be found in Peskin and other textbook on the DGLAP equation.

If we consider the gluon radiation at given $k_\perp$ on top of the Born process $q\bar{q} \to \gamma^*$ together with the kinematic constraint on the soft gluon radiation ($\frac{k_\perp^2}{(1-\xi)p^+ x_1} \leq x_2 p^- \to \xi \leq 1 - \frac{k_\perp^2}{Q^2}$), we can ontain

$$\frac{dP}{d^2 k_\perp} = \frac{\alpha_s C_F}{2\pi^2} \frac{1}{k_\perp} \int_0^{1-\frac{k_\perp^2}{Q^2}} d\xi \frac{1+\xi^2}{1-\xi} = \frac{\alpha_s C_F}{\pi^2} \frac{1}{k_\perp} \ln \frac{Q^2}{k_\perp^2} + \cdots. \tag{81}$$

Next, let us use the same trick as employed in the thrust calculation by considering a partially integrated rate

$$F(Q_T) = \int_0^{Q_T} d^2 k_\perp \frac{dP}{d^2 k_\perp} \simeq 1 - \int_{Q_T}^{Q} d^2 k_\perp \frac{dP}{d^2 k_\perp} \simeq 1 - \frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}. \tag{82}$$

Again we have used the fact that $\int_{\text{full space}} d^2 k_\perp \frac{dP}{d^2 k_\perp} = 1 + \mathcal{O}(\alpha_s)$ after taking the Born contribution into account as in the thrust case. Now the above result should look pretty familiar to you.

Furthermore, I wish to convince you that soft gluon radiations factorize kinematically and factorize in color space. The former factorization in kinematics can be easily seen by applying the Dirac equation as in Page 202 of Peskin if the soft gluon is radiated from an energetic quark. (If the soft gluon is radiated from an energetic gluon, then we need to apply the Ward identity and Eikonal approximation to the triple gluon vertex. The rest of the derivation is identical to the quark case.) The latter factorization can be understood if you try to compute the color factor of two gluon radiations and find the color factor of the leading logarithmic contribution is C_F^2 as a simple exercise. For more complete discussion on this issue, see Refs. [6, 7].

In the end, after summing over arbitrary number of identical gluon emissions, we obtain

$$F(Q_T) = \sum_0^\infty \frac{(-1)^n}{n!} \left[\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}\right]^n = \exp\left[-\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}\right]. \tag{83}$$

Similar to the thrust distribution, we can differentiate $F(Q_T)$ and get

$$\frac{dP}{dQ_T^2} = \frac{1}{\sigma} \frac{d\sigma}{dQ_T^2} = \frac{\alpha_s C_F}{\pi} \frac{\ln \frac{Q^2}{Q_T^2}}{Q_T^2} \exp\left[-\frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2}{Q_T^2}\right]. \tag{84}$$

Several comments are in order.

- $F(Q_T)$ is usually referred as the Sudakov form factor for which can be interpreted as the probability for emitting no gluons with transverse momentum greater than Q_T.
- When $Q_T \to 0$, $F(Q_T) \to 0$ means that lepton pairs always have non-zero transverse momentum since gluon radiation is inevitable. When $q\bar{q}$ annihilate into a virtual photon, the gluon clouds as part of the quark wave function have to be release due to the annihilation.
- Although this result is qualitatively correct, there is one problem with it! We should take into account the transverse momentum conservation for arbitrary number of gluon radiations. This can be achieved in coordinate space.

4.1.2 Resummation in coordinate space

The key problem now is to transform everything to the coordinate space which naturally conserves the transverse momentum as shown below.

$$\begin{aligned}
&\text{0th order} \quad \frac{dP_0}{d^2Q_T} = \delta^{(2)}(Q_T) \\
&\text{1th order} \quad \frac{dP_1}{d^2Q_T} = \frac{\alpha_s C_F}{\pi^2}\int d^2k_\perp \ln\frac{Q^2}{k_\perp^2}\delta^{(2)}(Q_T - k_\perp) \\
&\text{2th order} \quad \frac{dP_2}{d^2Q_T} = \frac{1}{2!}\left(\frac{\alpha_s C_F}{\pi^2}\right)^2\int d^2k_{1\perp} \ln\frac{Q^2}{k_{1\perp}^2}\int d^2k_{2\perp} \ln\frac{Q^2}{k_{2\perp}^2}\delta^{(2)}(Q_T - k_{1\perp} - k_{2\perp}) \\
&\cdots,
\end{aligned}$$

where we have taken the soft gluon limit and assumed that soft gluons factorize. Now use the identity

$$\delta^{(2)}(Q_T - k_{1\perp} - k_{2\perp} - \cdots - k_{n\perp}) = \frac{1}{(2\pi)^2}\int d^2b_\perp e^{-i(Q_T - k_{1\perp} - k_{2\perp}\cdots - k_{n\perp})\cdot b_\perp}, \tag{85}$$

and define the Sudakov factor $S(b_\perp) = -\frac{\alpha_s C_F}{\pi^2}\int d^2k_\perp \frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2}e^{ik_\perp\cdot b_\perp}$, eventually one can arrive at

$$\frac{dP}{d^2Q_T} = \frac{1}{(2\pi)^2}\int d^2b_\perp e^{-iQ_T\cdot b_\perp}\sum_{n=0}^{\infty}\frac{1}{n!}\left[-S(b_\perp)\right]^n = \frac{1}{(2\pi)^2}\int d^2b_\perp e^{-iQ_T\cdot b_\perp}e^{-S(b_\perp)}. \tag{86}$$

The above result is one of the main results which shows the exponentiation of the Sudakov factor in the coordinate space together with the consideration of transverse momentum conservation.

The rest of the task is then to compute the Sudakov factor $S(b_\perp)$. Immediately, we can realize that the above definition of $S(b_\perp)$ is not convergent at $k_\perp \to 0$ limit when we perform the Fourier transform. The divergence is simply due to the fact that we forget to include the virtual contribution. By adding the virtual contribution, we arrive at

$$S(b_\perp) = -\frac{\alpha_s C_F}{\pi^2}\int d^2k_\perp e^{ik_\perp\cdot b_\perp}\left[\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2} - \delta^{(2)}(k_\perp)\int^{Q^2} d^2l_\perp \frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\right], \tag{87}$$

where the upper limit of the virtual contribution is set by Q^2 due to the cancellation of UV divergences between two types of virtual diagrams in Drell-Yan processes. The evaluate the above integrals, we can adopt the dimensional regularization and use the following identities

$$\mu^{2\epsilon}\int \frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2} = \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu^2} - \frac{1}{2}\ln^2\frac{Q^2b_\perp^2}{c_0^2} - \frac{\pi^2}{12}\right], \tag{88}$$

$$\mu^{2\epsilon}\int \frac{d^{2-2\epsilon}l_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\bigg|_{l_\perp<Q} = \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2} + \frac{1}{2}\ln^2\frac{Q^2}{\mu^2} - \frac{\pi^2}{12}\right], \tag{89}$$

where more details regarding this can be found in the appendix of Ref. [8]. Thus, due to the incomplete cancellation between real and virtual contributions (The real part has constraints due to the measurement at fixed Q_T, while no constraint is imposed on the virtual diagram.), a potentially large logarithmic contribution remains

$$S(b_\perp) = \frac{\alpha_s C_F}{2\pi} \ln^2 \frac{Q^2 b_\perp^2}{c_0^2}, \tag{90}$$

where $c_0 = 2e^{-\gamma_E}$. At the end of the day, we can then modify Eq. (79) and obtain the Sudakov resummation formula for DY processes at leading double logarithmic level

$$\frac{d\sigma_{DY}^{(LL)}}{dQ^2 d^2 Q_T} = \sum_q \frac{\sigma(q\bar{q})}{S} \frac{1}{(2\pi)^2} \int d^2 b_\perp e^{-iQ_T \cdot b_\perp} e^{-S(b_\perp)} \int \frac{dx_1}{x_1} q(x_1, \mu) \int \frac{dx_2}{x_2} \bar{q}(x_2, \mu) \delta(1-z). \tag{91}$$

Again, a few remarks are in order regarding the Sudakov resummation in DY processes:

- The original and more complete derivation at next-to-leading-logarithmic level by Collins, Soper and Sterman can be found in Ref. [9]. What we have done above is just an intuitive way of understanding the Sudakov double logarithm term which is also known as the A term. Again the physics behind the A term is the incomplete cancellation of soft divergences due to the imposed constraint.

- We can also include the single log term known as the B term $-\frac{3\alpha_s C_F}{2\pi} \ln \frac{Q^2 b_\perp^2}{c_0^2}$, which arises due to the incomplete cancellation of the collinear divergence. In addition, to simplify the above resummed expression, it is the common practice to set $\mu = c_0/b_\perp$ in quark distributions $q(x_1, \mu)$ and $\bar{q}(x_2 \mu)$.

- Putting everything altogether including the running coupling effect, the Sudakov factor usually is written as

$$S(b_\perp) = \int_{c_0^2/b_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left[\ln \frac{Q^2}{\bar{\mu}^2} A(\alpha_s) + B(\alpha_s) \right], \tag{92}$$

  where $A(\alpha_s) = \sum_n \left(\frac{\alpha_s}{2\pi}\right)^n A_n$ and $B(\alpha_s) = \sum_n \left(\frac{\alpha_s}{2\pi}\right)^n B_n$ with $A_1 = 2C_F$ and $B_1 = -3C_F$ for the quark channel. Higher order (for example two-loop) calculations yield the coefficients A_n and B_n with $n > 1$.

  The above expression is usually derived from the CSS evolution[9] of the $W(Q, b_\perp)$ function

$$\frac{\partial}{\partial \ln Q^2} W(Q, b_\perp) = [K(\mu, b_\perp) + G(Q, \mu)]\, W(Q, b_\perp), \tag{93}$$

  where $K + G = -\frac{\alpha_s C_F}{\pi} \left[\ln \frac{Q^2 b_\perp^2}{c_0^2} - \frac{3}{2} \right]$ at one-loop level with $K = -\frac{\alpha_s C_F}{\pi} \ln \frac{\mu^2 b_\perp^2}{c_0^2}$ the soft part of the evolution kernel and $G = -\frac{\alpha_s C_F}{\pi} \left[\ln \frac{Q^2}{\mu^2} - \frac{3}{2} \right]$ the hard part of the evolution kernel. These two kernels are both related to the cusp anomalous dimension γ_K

$$\frac{\partial}{\partial \ln \mu} K(\mu, b_\perp) = -\gamma_K = -\frac{\partial}{\partial \ln \mu} G(Q, \mu). \tag{94}$$

  In fact, the solutions to the above evolution equations resums the Sudakov logarithms.

- One interesting pattern regarding the Sudakov resummation is that it always appear in the case of multiple distinct scales (Q and Q_T in this example). If one integrates over Q_T by considering the total rate, then the Sudakov factor disappears. As discussed before, the Sudakov logs are due to the incomplete cancellation of real and virtual graphs when Q_T is fixed. If one integrates Q_T, such constraint imposed on the real graphs is then removed, thus we expect the “complete" cancellation as in inclusive observables.

- The full one-loop calculation also produce the constant correction known as the C term $\frac{\alpha_s}{2\pi} C_F \left[\pi^2 - 8\right]$. Usually the C term is not resummed.

- Non-perturvative Sudakov factor is also employed in phenomenology in order to regularize the large $b_\perp$ (small momentum) region. We have computed only the perturbative part in the above calculation which corresponds to the small $b_\perp$ region. When $b_\perp$ is as large as $1/\Lambda_{QCD}$, we should turn on the non-perturbative Sudakov factor and adopt the so-called $b_* \equiv \frac{b_\perp}{\sqrt{b_\perp^2/b_{max}^2+1}}$ prescription[9]. Therefore, in practice, Sudakov factor is usually the sum of the perturbative and NP part as follows

$$S^{tot}(b_\perp) = S_P(b_*) + S_{NP}(b_\perp), \tag{95}$$

  where $S_P(b_*)$ is computed from perturbations while $S_{NP}(b_\perp)$ is usually fitted from DY and SIDIS experimental data. Nevertheless, $S_{NP}(b_perp)$ is not important in very high energy collisions.

  One important related question: Can we trust the perturbative Sudakov factor in the $Q_T \sim 0$ region? The answer is yes. The reason is that the integrand before we perform the Fourier transform is strongly peaked at small-$b_\perp$ region, which implies that the dominant contribution comes from the small-$b_\perp$ region, i.e., the perturbative region.

- Sudakov resummation becomes insufficient and unnecessary in the region $Q_T \sim Q$, where the pQCD expansion is sufficient and accurate.

4.2 Sudakov Resummation in Dijet Productions

At last, I would like to briefly mention the Sudakov resummation in dijet angular (azimuthal) correlation in hadronic collisions[12].

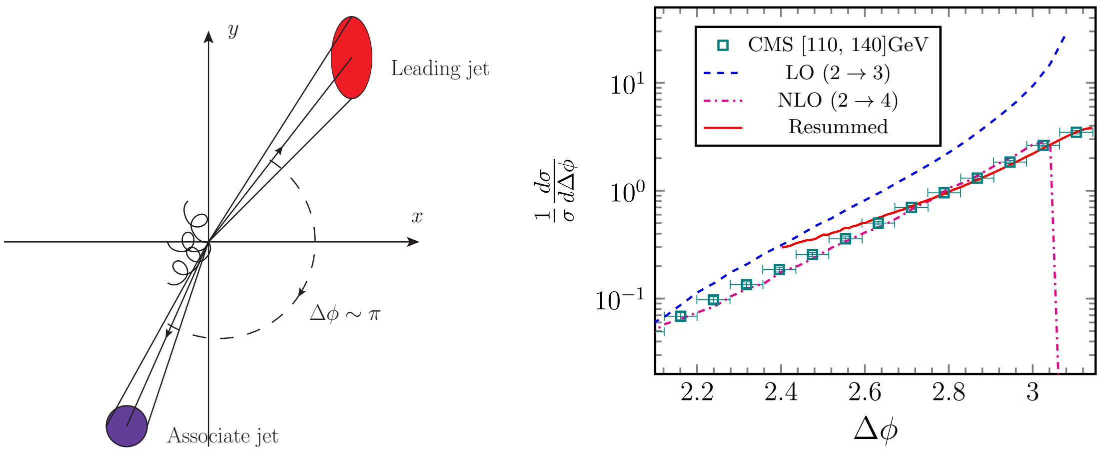


Figure 1: The angular correlation of inclusive dijet data compared with theoretical calculations.

As shown above, the angular correlation between the leading jet (the jet with the largest P_T) and the associate jet (the jet with the second largest P_T) can be measured in proton-proton collisions at the LHC. In the right plot of Fig. 2, one can see that the perturbative QCD framework, i.e., the collinear factorization can describe the large angle region. However, due to the appearance of large logarithms such as $L \sim \ln^2 \frac{P_\perp^2}{q_\perp^2}$ with $P_\perp \gg q_\perp$, pQCD expansion eventually breaks down in the back-to-back region. Here $P_\perp$ is approximately the leading jet P_T, while $q_\perp$ is the transverse momentum imbalance between the leading jet and the associate jet. One needs to employ the Sudakov resummation formalism in order to resum those large logarithms and describe the data in the back-to-back region.

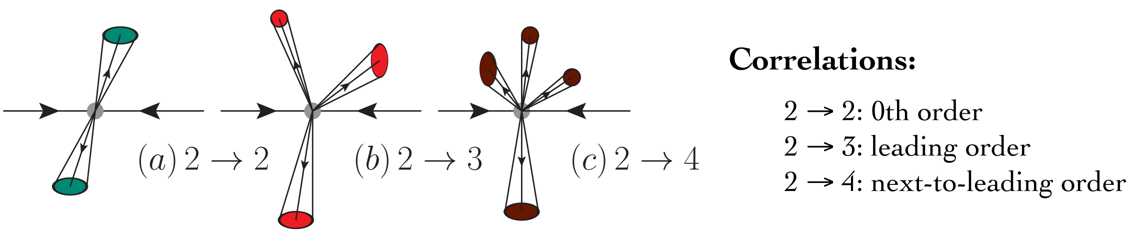


Figure 2: Inclusive dijet productions in terms of perturbative expansions.

In the collinear factorization, the incoming partons carry no transverse momentum, therefore the $2 \to 2$ process as shown in Fig. 2 gives no contribution to dijet angular correlations other than a delta function. The first non-trivial contribution comes from the $2 \to 3$ process and the NLO correction to dijet correlations starts at the $2 \to 4$ order together with $2 \to 3$ virtual graphs. However, no matter how high order we go, we always get divergent results for dijet productions in the back-to-back region due to soft (and/or collinear) gluon emissions. Let us consider the $2 \to 3$ case in which the leading jet and the associate jet are exactly back-to-back, i.e., $\Delta\phi = \pi$. It is easy to see that the unobserved parton (the third parton in the final state) can be soft and therefore introduce large double logarithms. The large Sudakov logarithms in the $\Delta\phi \sim \pi$ region make the pQCD expansion insufficient and eventually break down.

There are two types of soft gluon emission in this process, i.e., the initial state and final state gluon radiations. If a gluon is radiated by the incoming parton before the hard collision, it is considered as an initial state gluon. The final state gluon is radiated from the outgoing final state parton after the hard collision. The divergent behavior is caused by the soft and collinear gluon emission in the initial state as well as the soft gluon emission in the final state subject to the jet-cone regularization. It is important to notice that the collinear final state gluon contribution is removed due to the jet cone regularization.

For simplicity, let us consider the $q_i(k_1) + q_j(k_2) \to q_i(p_1) + q_j(p_2)$ channel in dijet productions, and define momentum imbalance $\vec{q}_\perp \equiv \vec{p}_{1\perp} + \vec{p}_{2\perp}$, jet momenta $P_\perp \sim p_{1\perp} \sim p_{2\perp}$. In the limit $P_\perp \gg q_\perp$, we need to resum multiple soft gluon emission and use the following resummed formula

$$\begin{aligned}
\frac{d\sigma^{ij}_{\text{dijet}}}{dy_1 dy_2 d^2p_{1\perp} d^2p_{2\perp}} &= \sigma_{ij} \int \frac{d^2b_\perp}{(2\pi)^2} e^{-iq_\perp \cdot b_\perp} W(Q, b_\perp), && (96)\\
\text{with} \quad W(Q, b_\perp) &= x_1 f_i(x_1, \mu_b) x_2 f_j(x_2, \mu_b) e^{-S(Q, b_\perp)}, && (97)\\
S(Q, b_\perp) &= S_{pert}(Q, b_*) + S_{NP}(Q, b_\perp) && (98)\\
S_{pert}(Q, b_*) &= \int_{\mu_b^2 = c_0^2/b_*^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[A \ln \frac{Q^2}{\mu^2} + B + (D_1 + D_2) \ln \frac{1}{R^2} \right]. && (99)
\end{aligned}$$

Several comments regarding the above Sudakov resummed formula are in order.

- As before, soft gluon emissions factorize from the Born cross section $\sigma_{ij} = \frac{\alpha_s^2 \pi}{s} \frac{4}{9} \frac{s^2 + u^2}{t^2}$, where s, t, u are the normal partonic Mandelstam variables. For other channels, the Born cross section can be found in Chapter 17 of Peskin.
- The kinematics give $Q^2 = x_1 x_2 S$ and $x_{1,2} = \frac{P_\perp}{S}\left(e^{\pm y_1} + e^{\pm y_2}\right)$ with the produced jet rapidity y_1 and y_2.
- All the $A = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n A^{(n)}$, $B = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n B^{(n)}$, $D = \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{2\pi}\right)^n D^{(n)}$ coefficients can be computed perturbatively. One-loop calculation gives the contribution of $n = 1$ coefficients, etc. A and B coefficients are associated with initial state Sudakov radiations. For each initial state quark, there is a contribution of $\frac{\alpha_s}{2\pi} C_F$ to A ($A^{(1)} = C_F$) and a contribution of $-\frac{\alpha_s}{2\pi} \frac{3}{2} C_F$ to B ($B^{(1)} = -\frac{3}{2} C_F$). For each incoming gluon, the corresponding contributions to A and B are $\frac{\alpha_s}{2\pi} C_A$ and $-\frac{\alpha_s}{2\pi} 2\beta_0 C_A$.
- The D term is due to final state radiation and it is $\frac{\alpha_s}{2\pi} C_F$ for final state quark jets and $\frac{\alpha_s}{2\pi} C_A$ for final state gluon jets. In deriving this result, the so-called narrow cone approximation has been made which requires $R^2 \ll 1$.
- The C term, which is not written in above equation, is just constant higher order α_s correction and it is not resummed usually.
- Other complication such as the soft factor will not be discussed here. For more information, see Ref. [12]. Again, we use b_* prescription to separate perturbative and non-perturbative regions.

The procedure of deriving the above result is very similar to what we have done before for DY process. The main new ingredient in this calculation is the final state gluon radiation. First of all, the virtual contribution to the Born diagrams, which is proportional to $\frac{1}{\epsilon^2}$ in dim-reg, is universal and it has been computed in Ref. [13]. Second, the contribution from the final state gluon radiation inside the jet cone is similar to the virtual contribution since it is also proportional to the Born cross section and $\frac{1}{\epsilon^2}$. Third, soft gluon radiations also has soft divergence $\frac{1}{\epsilon^2}$ as expected. Last, we can also have gluon radiation collinear to the incoming partons. It is worth noting that the final state collinear divergence is regularized by the jet cone algorithm.

At the end of the day, the $\frac{1}{\epsilon^2}$ divergences from the first three contributions exactly cancel and the finite remaining terms yield the Sudakov logarithms in the A and D terms. The collinear divergence in the last contribution should be renormalized into the incoming parton distribution and the leftover single logarithm gives contribution to the B term. Again, these logarithms arise due to the incomplete cancellation between the real and virtual contributions.

Recently, in order to probe the properties of the cold and hot nuclear medium, there have been increasing interests in applying Sudakov resummation to hard processes in heavy ion collisions. The joint resummation of small-x logarithms and Sudakov logarithms is discussed in Refs. [8, 14]. Dijets can be used to probe the quark gluon plasma created in heavy ion collision[15, 16]. The Sudakov factor which takes vacuum parton shower into account plays an important role, since it helps to establish the baseline in proton-proton collisions.

References

[1] E. Farhi, Phys. Rev. Lett. **39**, 1587 (1977).

[2] J. R. Ellis and I. Karliner, Nucl. Phys. B **148**, 141 (1979).

[3] M. D. Schwartz, “Quantum Field Theory and the Standard Model.”

[4] S. Catani and L. Trentadue, Nucl. Phys. B **327**, 323 (1989).

[5] T. Becher and M. Neubert, Phys. Rev. Lett. **97**, 082001 (2006) [hep-ph/0605050].

[6] J. G. M. Gatheral, Phys. Lett. **133B**, 90 (1983).

[7] J. Frenkel and J. C. Taylor, Nucl. Phys. B **246**, 231 (1984).

[8] A. H. Mueller, B. W. Xiao and F. Yuan, Phys. Rev. D **88**, no. 11, 114010 (2013) [arXiv:1308.2993 [hep-ph]].

[9] J. C. Collins, D. E. Soper and G. F. Sterman, Nucl. Phys. B **250**, 199 (1985).

[10] A. J. Larkoski, arXiv:1709.06195 [hep-ph].

[11] S. Marzani, G. Soyez and M. Spannowsky, Lect. Notes Phys. **958**, pp. (2019) [arXiv:1901.10342 [hep-ph]].

[12] P. Sun, C.-P. Yuan and F. Yuan, Phys. Rev. D **92**, no. 9, 094007 (2015) [arXiv:1506.06170 [hep-ph]].

[13] R. K. Ellis and J. C. Sexton, Nucl. Phys. B **269**, 445 (1986).

[14] A. H. Mueller, B. W. Xiao and F. Yuan, Phys. Rev. Lett. **110**, no. 8, 082301 (2013) [arXiv:1210.5792 [hep-ph]].

[15] A. H. Mueller, B. Wu, B. W. Xiao and F. Yuan, Phys. Lett. B **763**, 208 (2016) [arXiv:1604.04250 [hep-ph]]; Phys. Rev. D **95**, no. 3, 034007 (2017) [arXiv:1608.07339 [hep-ph]].

[16] L. Chen, G. Y. Qin, S. Y. Wei, B. W. Xiao and H. Z. Zhang, Phys. Lett. B **773**, 672 (2017) [arXiv:1607.01932 [hep-ph]]; Phys. Lett. B **782**, 773 (2018) [arXiv:1612.04202 [hep-ph]].