1 QFT Summer Homework problems:

Homework Problem 1: Consider the $2 \to 3$ ($e^+ + e^- \to q + \bar{q} + g$) process and derive the following thrust distribution for $T < 1$

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right]. \tag{1}$$

Hint:
Use the delta function trick, we can write the differential cross section of thrust as

$$\frac{1}{\sigma_0} \frac{d\sigma}{dT} = \frac{C_F \alpha_s}{2\pi} \int dx_1 \int dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \delta(T - \max[x_1, x_2, x_3]),$$

where $x_3 = 2 - x_1 - x_2$.

Homework Problem 2:
First note that the total cross section reads

$$\lim_{\epsilon \to 0} \sigma^{\text{tot}}_{\gamma^* \to X} = \sigma_0 \left[1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2) \right]. \tag{2}$$

After including the Born and virtual as well as real contributions, the differential cross section becomes

$$\frac{d\sigma}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right] + C\delta(1-T), \tag{3}$$

where C is a divergent constant, which can be determined by the following integral according to the total cross section and

$$\int_{T_{\min}}^{1} dT \frac{d\sigma}{\sigma_0 dT} = 1 + C_F \frac{3\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2). \tag{4}$$

1. Show that $T_{\min} = 2/3$ for $2 \to 3$ processes.

2. From Eq. (1), show that the following exact expression of the thrust distribution at one-loop satisfies Eq. (4)

$$\begin{aligned} \frac{d\sigma}{\sigma_0 dT} &= \delta(1-T) + \frac{C_F \alpha_s}{2\pi} \left[\delta(1-T) \left(\frac{\pi^2}{3} - 1 \right) - \frac{3(3T-2)(2-T)}{(1-T)_+} \right] \\ &\quad + \frac{C_F \alpha_s}{2\pi} \frac{2(3T^2 - 3T + 2)}{T} \left[\frac{\ln(2T-1)}{(1-T)_+} - \left(\frac{\ln(1-T)}{1-T} \right)_+ \right], \end{aligned} \tag{5}$$

   where the plus distribution is defined as $\int_a^1 dx g(x) (f(x))_+ = \int_a^1 dx g(x) f(x) - g(1) \int_0^1 dx f(x)$.

Homework Problem 3:
Use the dimensional regularization ($\overline{MS}$ scheme, multiplying a factor of $S_\epsilon^{-1} = (4\pi e^{-\gamma_E})^{-\epsilon}$ with $\gamma_E \simeq 0.577$

the Euler constant) and show the following identities

$$S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}=\frac{1}{4\pi}\left[-\frac{1}{\epsilon}+\ln\frac{c_0^2}{\mu^2 b_\perp^2}\right], \tag{6}$$

$$S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}k_\perp}{(2\pi)^{2-2\epsilon}}e^{ik_\perp\cdot b_\perp}\frac{1}{k_\perp^2}\ln\frac{Q^2}{k_\perp^2}=\frac{1}{4\pi}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}+\frac{1}{2}\ln^2\frac{Q^2}{\mu^2}-\frac{1}{2}\ln^2\frac{Q^2b_\perp^2}{c_0^2}-\frac{\pi^2}{12}\right], \tag{7}$$

$$S_\epsilon^{-1}\,\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}l_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{l_\perp^2}\ln\frac{Q^2}{l_\perp^2}\bigg|_{l_\perp<Q}=\frac{1}{4\pi}\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{Q^2}{\mu^2}+\frac{1}{2}\ln^2\frac{Q^2}{\mu^2}-\frac{\pi^2}{12}\right], \tag{8}$$

where $c_0\equiv 2e^{-\gamma_E}$. Hints: see the appendix in $[arXiv:1308.2993]$.

Homework Problem 4: BFKL equation in the momentum and coordinate space

(a) As we mentioned in class, the BFKL equation in the dipole model can be written as

$$\partial_Y T(x,y;Y)=\frac{\bar\alpha_s}{2\pi}\int d^2z\frac{(x-y)^2}{(x-z)^2(z-y)^2}\left[T(x,z;Y)+T(z,y;Y)-T(x,y;Y)\right], \tag{9}$$

with $\bar\alpha_s=\dfrac{\alpha_s N_c}{\pi}$. Let us look for angular independent solution (the dominant one) and introduce the shorthand notation $x_{10}=x_1-x_0$, where $x_{0,1}$ are 2-d vectors, thus we can cast the equation into

$$\partial_Y T(x_{10};Y)=\frac{\bar\alpha_s}{2\pi}\int d^2x_2\frac{x_{10}^2}{x_{12}^2x_{20}^2}\left[T(x_{12};Y)+T(x_{20};Y)-T(x_{10};Y)\right]. \tag{10}$$

Suppose one can define

$$T(x;Y)=\int_{c-i\infty}^{c+i\infty}\frac{d\gamma}{2\pi i}\left(\frac{x^2}{x_{10}^2}\right)^\gamma T_\gamma(Y) \tag{11}$$

with x_{10} the initial dipole size, <u>show</u> that the BFKL equation can be converted into $dT_\gamma/dY=\bar\alpha_s\chi(\gamma)T_\gamma$, where the BFKL characteristic function $\chi(\gamma)=2\psi(1)-\psi(1-\gamma)-\psi(\gamma)$ with $\psi(x)$ the digamma function. Hint: First show that

$$\chi(\gamma)=\frac{1}{2\pi}\int d^2x_2\frac{x_{10}^2}{x_{12}^2x_{20}^2}\left[\left(\frac{x_{12}^2}{x_{10}^2}\right)^\gamma+\left(\frac{x_{20}^2}{x_{10}^2}\right)^\gamma-1\right] \tag{12}$$

and use the integral identity

$$\int_0^{2\pi}\frac{d\theta}{1-a\cos\theta}=\frac{1}{\sqrt{1-a^2}}\quad \text{with}\quad a<1$$

and the identity regarding the digamma function

$$\psi(\gamma)=-\gamma_E+\int_0^1 du\frac{1-u^{\gamma-1}}{1-u}. \tag{13}$$

with $\gamma_E\simeq 0.577$ the Euler constant.

(b) In the momentum space, the BFKL equation reads

$$\partial_Y G(l_\perp,l'_\perp;Y)=\frac{\bar\alpha_s}{\pi}\int\frac{d^2q_\perp}{(q_\perp-l_\perp)^2}\left[G(q_\perp,l'_\perp;Y)-\frac{l_\perp^2}{2q_\perp^2}G(l_\perp,l'_\perp;Y)\right], \tag{14}$$

where $G(l_\perp,l'_\perp;Y)$ is known as the BFKL propagator. In the Mellin space, <u>show</u> that the solution $G_\gamma(Y)$ has the same BFKL characteristic function, i.e., $G_\gamma(Y)=G_\gamma(0)\exp[\bar\alpha_s\chi(\gamma)Y]$.

<u>Hint</u>: Use the dimensional regularization ($\overline{MS}$ scheme with $S_\epsilon^{-1}=(4\pi e^{-\gamma_E})^{-\epsilon}$) and the following identity (see the appendix A in $[arXiv:1607.04726]$)

$$J(\gamma)=S_\epsilon^{-1}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}\frac{1}{(k_\perp+q_\perp)^2}\left(\frac{k_\perp^2}{q_\perp^2}\right)^\gamma=\frac{1}{4\pi}\left(\frac{e^{\gamma_E}\mu^2}{k_\perp^2}\right)^\epsilon\frac{\Gamma(\epsilon+\gamma)}{\Gamma(\gamma)}\frac{\Gamma(-\epsilon)\Gamma(-\epsilon-\gamma+1)}{\Gamma(-2\epsilon-\gamma+1)}. \tag{15}$$