Quark model for hadrons and searching for QCD exotics in various processes

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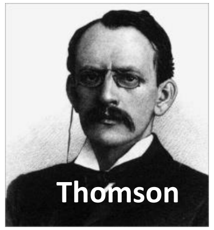
Outline

- Some facts about quarks; A brief review of hadron physics, introduction to non-relativistic constituent quark model (NRCQM) for mesons and baryons; the question of “missing resonances”
- QCD exotics and the search for exotic hadrons in various processes; Back to the quark model: where are the genuine color-singlet multiquark states?

1. Some facts about quarks

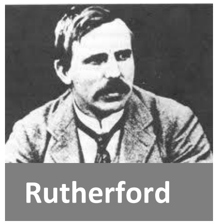
粒子物理标准模型建立之前，人类对亚原子结构的认知

- 1897： 电子



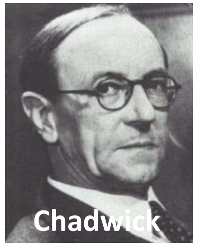
1906

- 1919： 质子



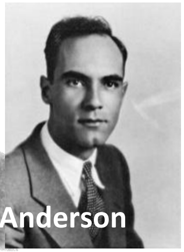
1908

- 1932： 中子



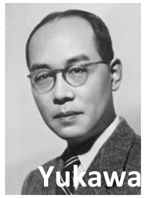
1935

- 1932： 正电子



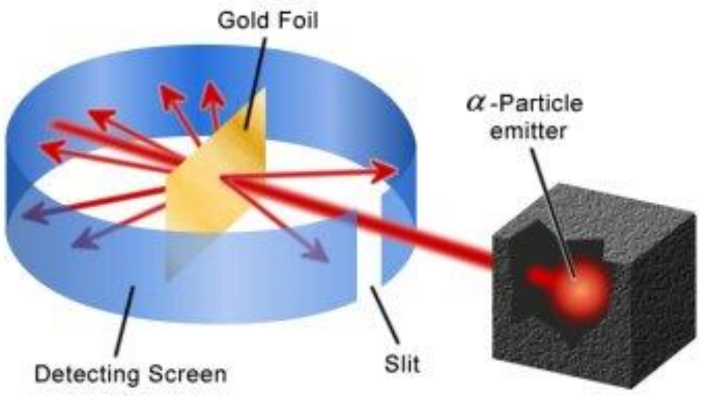
1936

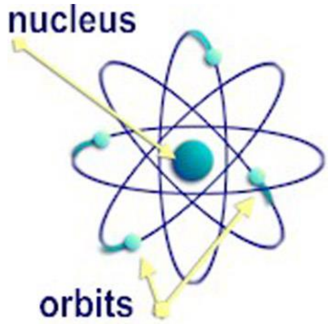
- 1935：Yukawa（汤川）预言π介子



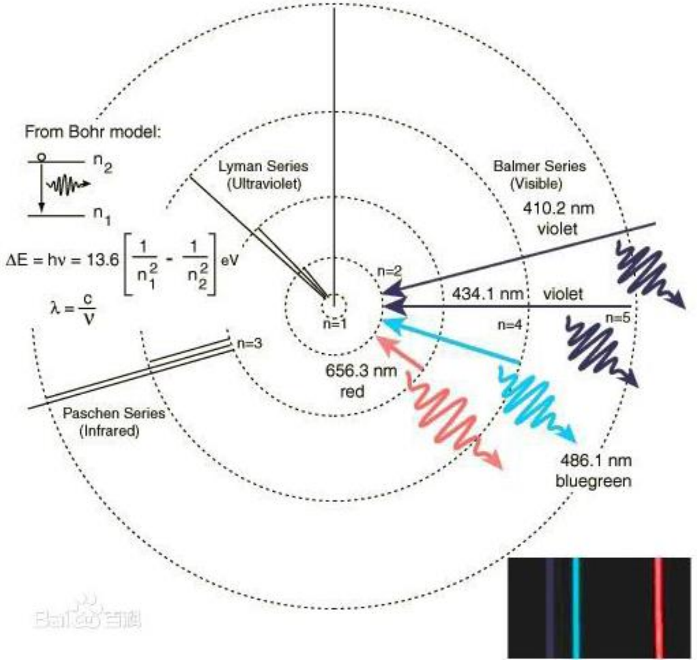
1949

1909: Geiger & Marsden exp., and Rutherford's atomic model





- 卢瑟福的原子核式结构对理解元素光谱具有重要意义
- 推动了量子力学的发展



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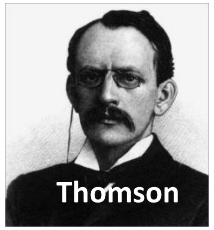

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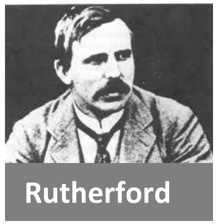

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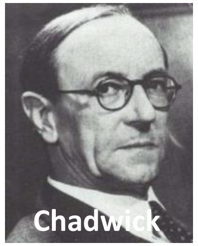

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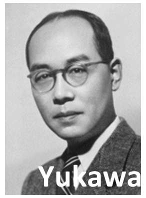
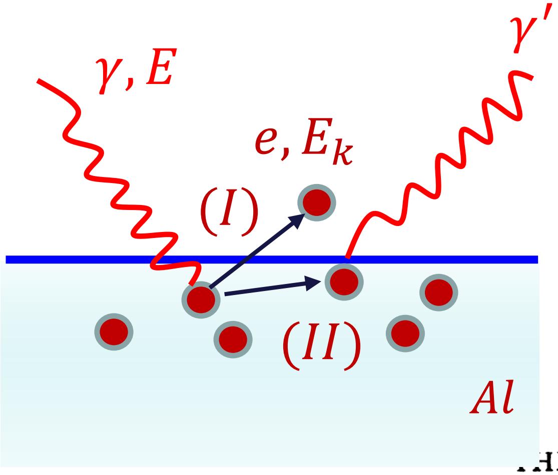
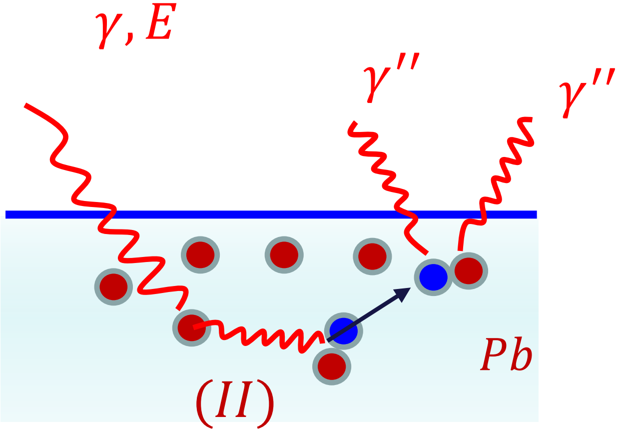


1949

正电子的发现

1929：C.-Y. Chao实验





THE

PHYSICAL REVIEW

SCATTERING OF HARD γ-RAYS

By C. Y. Chao*
Norman Bridge Laboratory of Physics, California Institute of Technology

(Received October 13, 1930)

Chao, Proc. Nat. Acad. Sci. 16, 431 (1930);
C.Y. Chao, Phys. Rev. 36, 1519 (1930)

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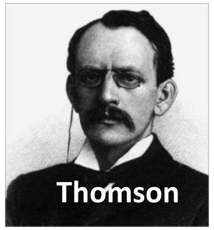

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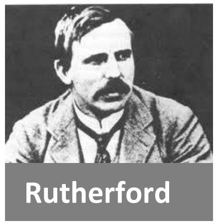

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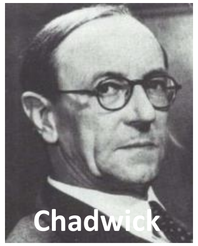

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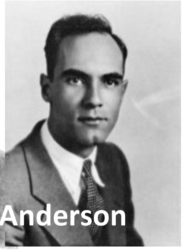

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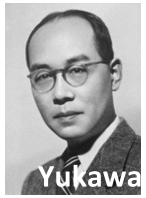


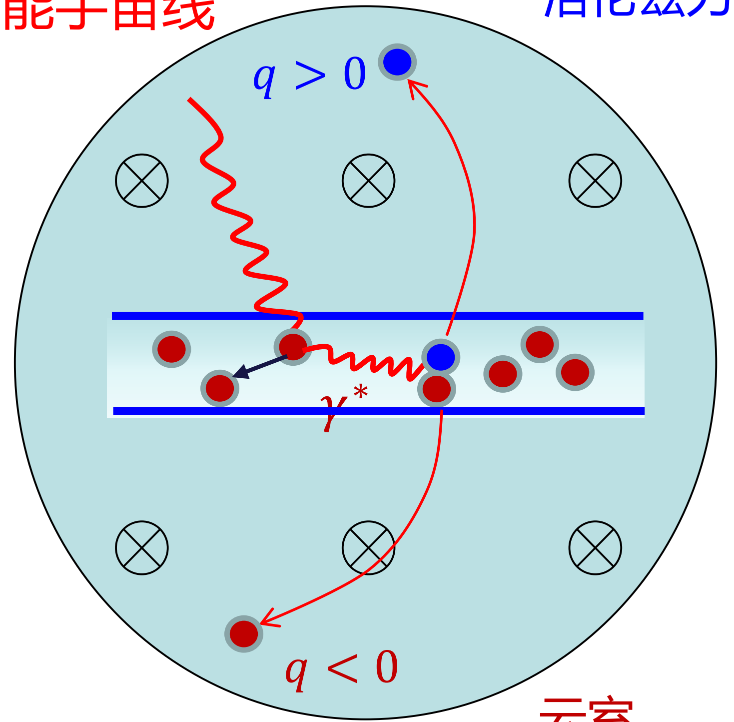
1949

正电子的发现

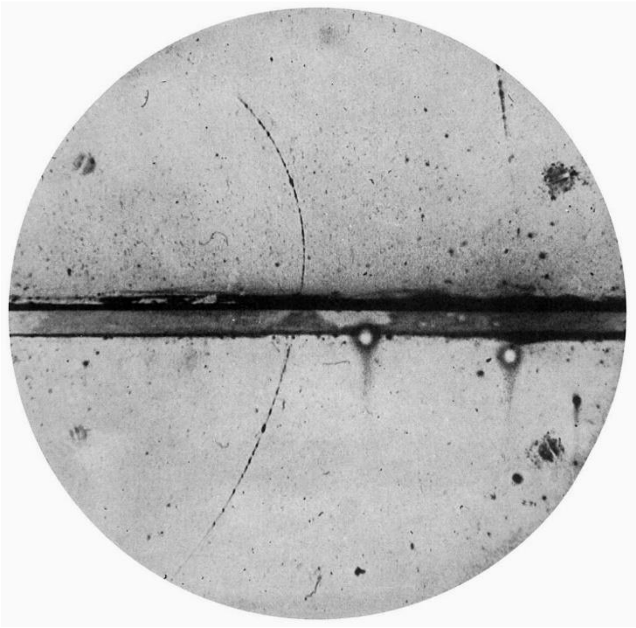
1932：Anderson实验

高能宇宙线

洛伦兹力：$F = q\vec{v} \times \vec{B}$



云室



C.D. Anderson, Science 76, 238 (1932)

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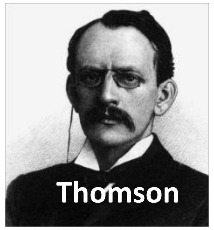


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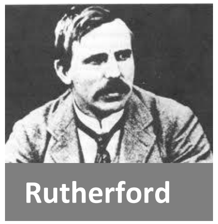


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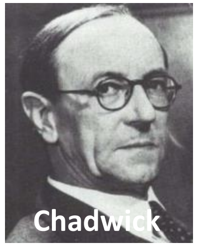


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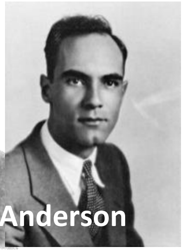


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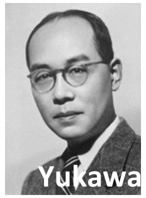



1949

正电子的发现
1928：Dirac equation

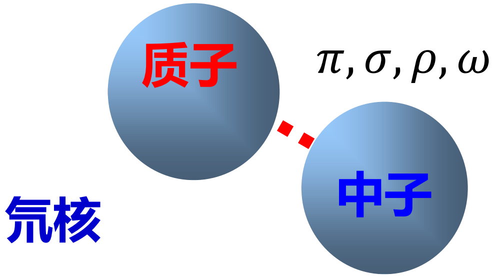
1928: Dirac (Nobel prize, 1933) equation was postulated, the existence of anti-electron, i.e. positron, was predicted.

1929: Chao, Proc. Nat. Acad. Sci. 16, 431 (1930);

1930: C.Y. Chao, Phys. Rev. 36, 1519 (1930);

1932: C.D. Anderson, Science 76, 238 (1932);

1936, Anderson and Neddermeyer discovered the muon



1963年的粒子数据表（Particle Data Group）Review of Modern Physics 35 (1963) 314-324, 共10页

REVIEWS OF MODERN PHYSICS VOLUME 35, NUMBER 2 APRIL 1963

Tables of Elementary Particles and Resonant States

Matts Roos
Nordisk Institut for Teoretisk Atomfysik, Copenhagen, Denmark

- 稳定粒子：$p, n, \Sigma, \Lambda, \Xi; \pi, K; \mu, e, \nu, \ldots$
- 介子共振态：$\rho, \omega, K^*, \ldots$
- 重子共振态：$\Delta, \Xi^*, \Lambda^*, \Sigma^*$ ...

"两类"粒子：

1. 非奇异（Non-strange）：强子对撞产生，衰变寿命短（ρ, ω, Δ ...）；
2. 奇异（Strange）：强子对撞产生，衰变寿命较长（$K, \Sigma, \Lambda, \Xi, \ldots$）

Table I. Elementary Particles, March 1963.

Class	Symbol Charge	Antiparticle found	Isospin T	Isospin T_3	Spin Parity	Strangeness	Mass (MeV)	Mass $(m_{\pi^\pm})$	Magnetic moment $(e/2m_p)$	Mean life (sec)	Mean life $(1/m_{\pi^\pm})$	Common decay modes	Branching ratios (%)	References
Hyperons	Ξ^- Ξ^0	$\bar{\Xi}^+$	$\frac{1}{2}$	$-\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{2}$	−2 2 −2	1320.8 ± 0.4 1316	9.46 9.43		1.4 (+0.6/−0.2) × 10^{-10} 3.9 (+1.4/−0.9) × 10^{-10}	3 × 10^{13} 8 × 10^{13}	$\Lambda\pi^-$ $\Lambda\pi^0$	100 100	1 2
	Σ^- Σ^0 Σ^+	$\bar{\Sigma}^+$ $\bar{\Sigma}^0$ $\bar{\Sigma}^-$	1	−1 1 0 0 1 −1	$\frac{1}{2}$+ $\frac{1}{2}$+ $\frac{1}{2}$+	−1 1 −1 1 −1 1	1195.96 ± 0.30 1191.5 ± 0.5 1189.40 ± 0.20	8.57 8.54 8.52		(1.59 ± 0.05) × 10^{-10} $10^{-11} > \tau > 10^{-22}$ (0.78 ± 0.03) × 10^{-10}	3.4 × 10^{13} $10^{12} > \tau > 10$ 1.65 × 10^{13}	$n\pi^-$ $\Lambda\gamma$ $p\pi^0$ $n\pi^+$	100 100 50.7 ± 2.3 49.3 ± 2.3	3, 4, 19 3, 5, 19 3, 4, 19
	Λ^0	$\bar{\Lambda}^0$	0	0 0	$\frac{1}{2}$+	−1 1	1115.38 ± 0.10 1115.44 ± 0.32	7.991 7.991	−1.5 ± 0.5	(2.57 ± 0.30) × 10^{-10} (1.9 ± 1.0) × 10^{-10}	5.4 × 10^{13} 4 × 10^{13}	$p\pi^-$ $n\pi^0$	66(+4/−3) 34(+3/−4)	6, 20
Nucleons	n^0 p^+	$\bar{n}$ $\bar{p}$	$\frac{1}{2}$	$-\frac{1}{2}$ $\frac{1}{2}$ $\frac{1}{2}$ $-\frac{1}{2}$	$\frac{1}{2}$+ $\frac{1}{2}$+	0 0 0 0	939.507 ± 0.01 938.213 ± 0.01	6.731 6.722	−1.9128 2.792816 ± 0.000034 −1.8 ± 1.2	1013 ± 26 ∞	2.15 × 10^{26} ∞	$pe^-\bar{\nu}_e$	100	7, 8 7, 15 10
Mesons	K^+		$\frac{1}{2}$	$\frac{1}{2}$	0−	1	493.98 ± 0.14	3.539	0	(1.227 ± 0.008) × 10^{-8}	2.60 × 10^{15}	$\mu^+\nu_\mu(\mu 2)$ $\pi^+\pi^0(\pi 2)$ $\mu^+\pi^0\nu_\mu(\mu 3)$ $e^+\pi^0\nu_e(e3)$ $\pi^+\pi^+\pi^-(\tau^+)$ $\pi^+\pi^0\pi^0(\tau^{+\prime})$	64.2 ± 1.3 18.6 ± 0.9 4.8 ± 0.6 5.0 ± 0.5 5.7 ± 0.3 1.7 ± 0.2	9
	K^0	K^- $\bar{K}^0$		$-\frac{1}{2}$ $-\frac{1}{2}$ $\frac{1}{2}$	0−	−1 1 −1	497.9 ± 0.6	3.57	<0.04 $\hbar e/m_K$	K_1^0(0.90 ± 0.02) × 10^{-10} K_2^0 6.3(+1.6/−1.0) × 10^{-8}	1.9 × 10^{13} 1.3 × 10^{16}	$\pi^+\pi^-$ $\pi^0\pi^0$ $\pi^+\pi^-\pi^0$ $3\pi^0$ $\pi^+e^-\bar{\nu}_e$, $\pi^-e^+\nu_e$ $\pi^+\mu^-\bar{\nu}_\mu$, $\pi^-\mu^+\nu_\mu$	69.4 ± 1.0 30.6 ± 1.0 8.7 ± 2.3 38 ± 7 28.3 ± 5.9 25.0 ± 5.9	11 12
	π^+ π^0	π^- π^0	1	1 −1 0	0− 0−	0 0 0	139.58 ± 0.05 134.97 ± 0.05	1 0.967	0 0	(2.547 ± 0.027) × 10^{-8} (1.05 ± 0.18) × 10^{-16}	5.48 × 10^{15} 2.23 × 10^{7}	$\mu^+\nu_\mu$ 2γ γe^+e^-	100 98.8 1.2	18 13, 22
Leptons	μ^-	μ^+			$\frac{1}{2}$		105.65	206.765 ± 0.002 m_e	(1.001162 ± 0.000005) $e/2m_\mu$	(2.210 ± 0.002) × 10^{-6}	4.69 × 10^{17}	$e^-\bar{\nu}_e\nu_\mu$	100	14
	e^-	e^+			$\frac{1}{2}$		0.510976 ± 0.000007	$1m_e$	(1.0011609 ± 0.0000024) $e/2m_e$	∞	∞			7, 15
	$\nu_\mu{}^0$ $\nu_e{}^0$	$\bar{\nu}_\mu$ $\bar{\nu}_e$			$-\frac{1}{2}$ $+\frac{1}{2}$ $-\frac{1}{2}$ $+\frac{1}{2}$		< 2.5 < 0.00025	< $5m_e$ <5 × 10^{-4} m_e						16, 21 17, 21
Photon	γ^0				1	0	0	0						21

1961: Gell-Mann, Nishijima & Nee’man: The Eightfold Way (八重法)

The Nobel Prize in Physics 1969 was awarded to Murray Gell-Mann "for his contributions and discoveries concerning the classification of elementary particles and their interactions."

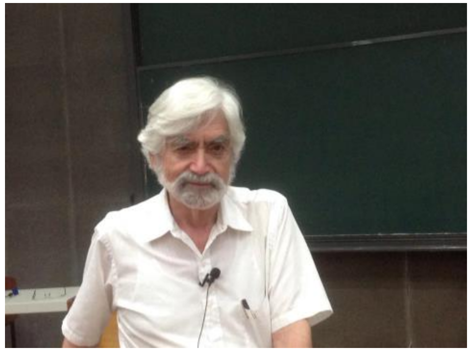
Gell-Mann

Nishijima

Nee’man

三类轻味夸克：

量子数\夸克	down	up	strange
电荷	−1/3	+2/3	−1/3
同位旋	1/2	1/2	0
同位旋z分量	−1/2	+1/2	0
奇异数	0	0	−1



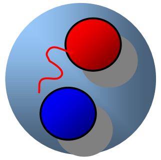
George Zweig

夸克的基本性质：

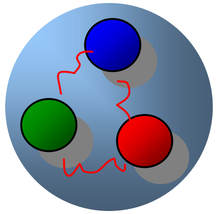
1）具有1/2自旋，重子数为1/3；

2）夸克具有正宇称，反夸克具有负宇称；

3）夸克“味道”量子数的符号与其电荷的符号一致。

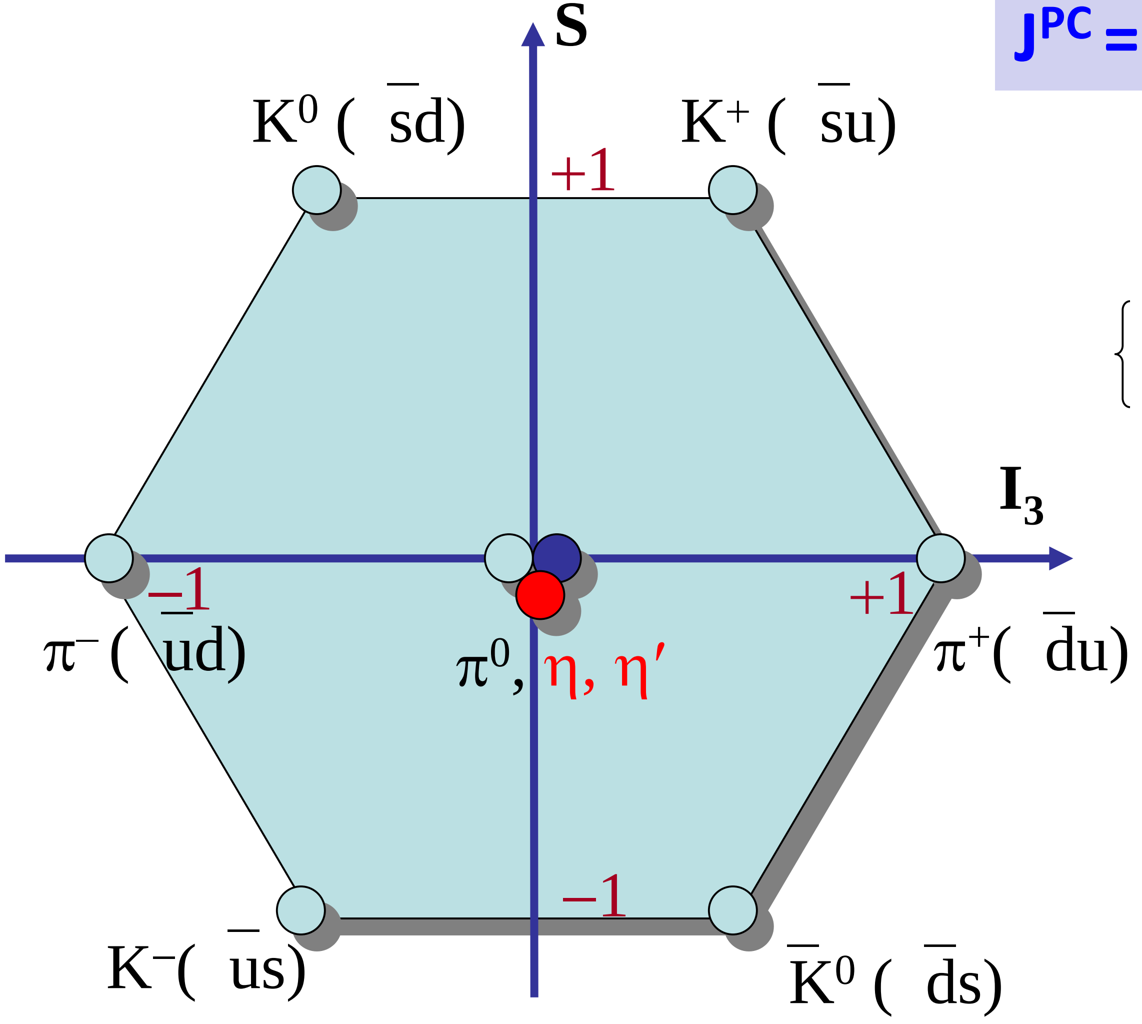


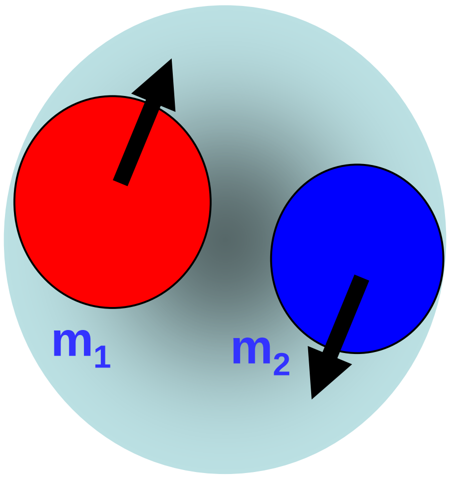
介子($q\bar{q}$)



重子(qqq)

$\bar{q}q$ SU(3) flavor nonet: $\bar{3}\otimes 3 = 1\oplus 8$





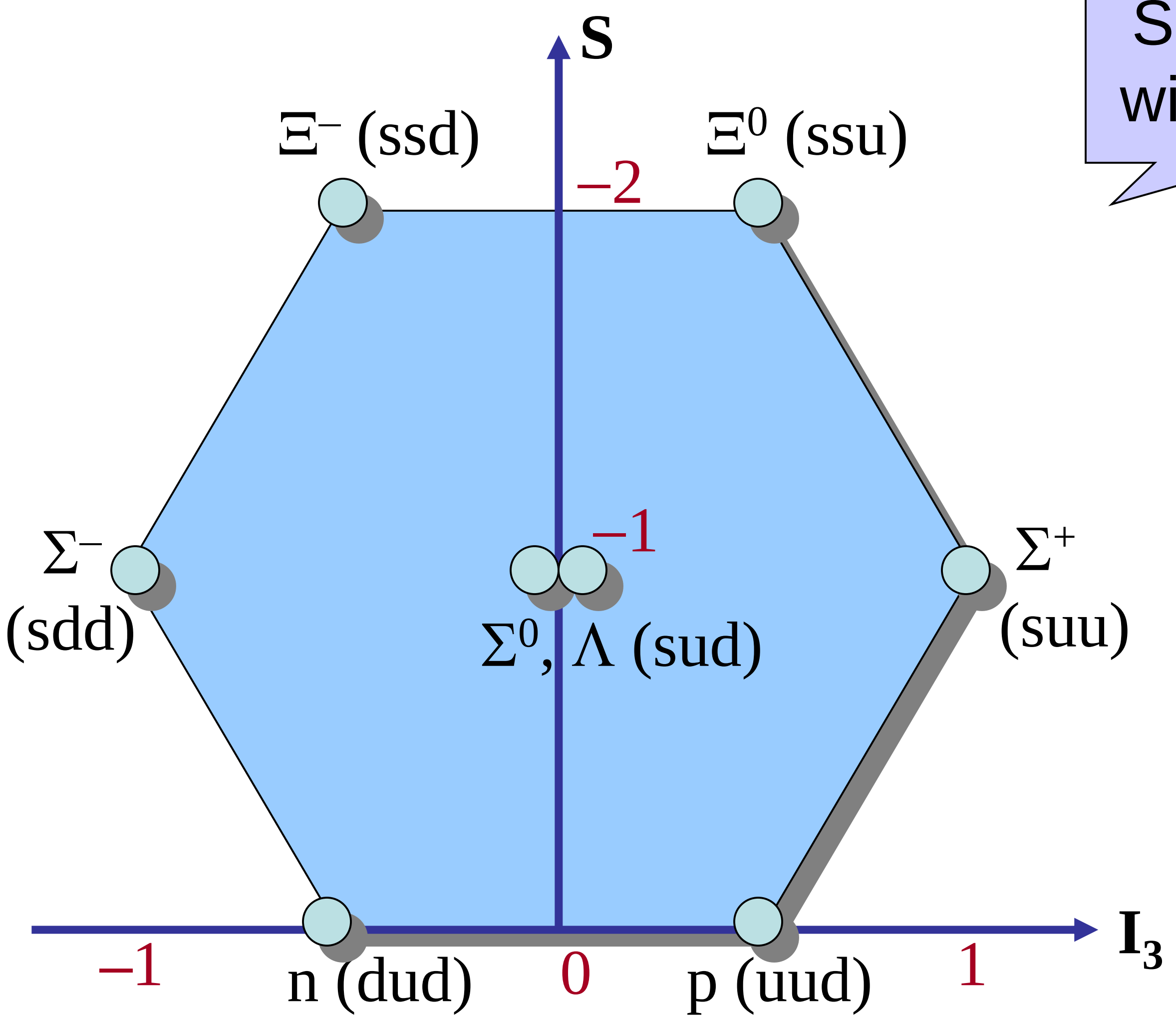
$J^{PC} = 0^{-+}$

$$\begin{cases} \eta = \cos\alpha_P |n\bar{n}\rangle - \sin\alpha_P |s\bar{s}\rangle \\ \eta' = \sin\alpha_P |n\bar{n}\rangle + \cos\alpha_P |s\bar{s}\rangle \end{cases}$$

$$n\bar{n} \equiv (u\bar{u} + d\bar{d})/\sqrt{2}$$

$$\alpha_P = \theta_P + \arctan\sqrt{2}$$

SU(3) multiplets of baryons made of u, d, and s



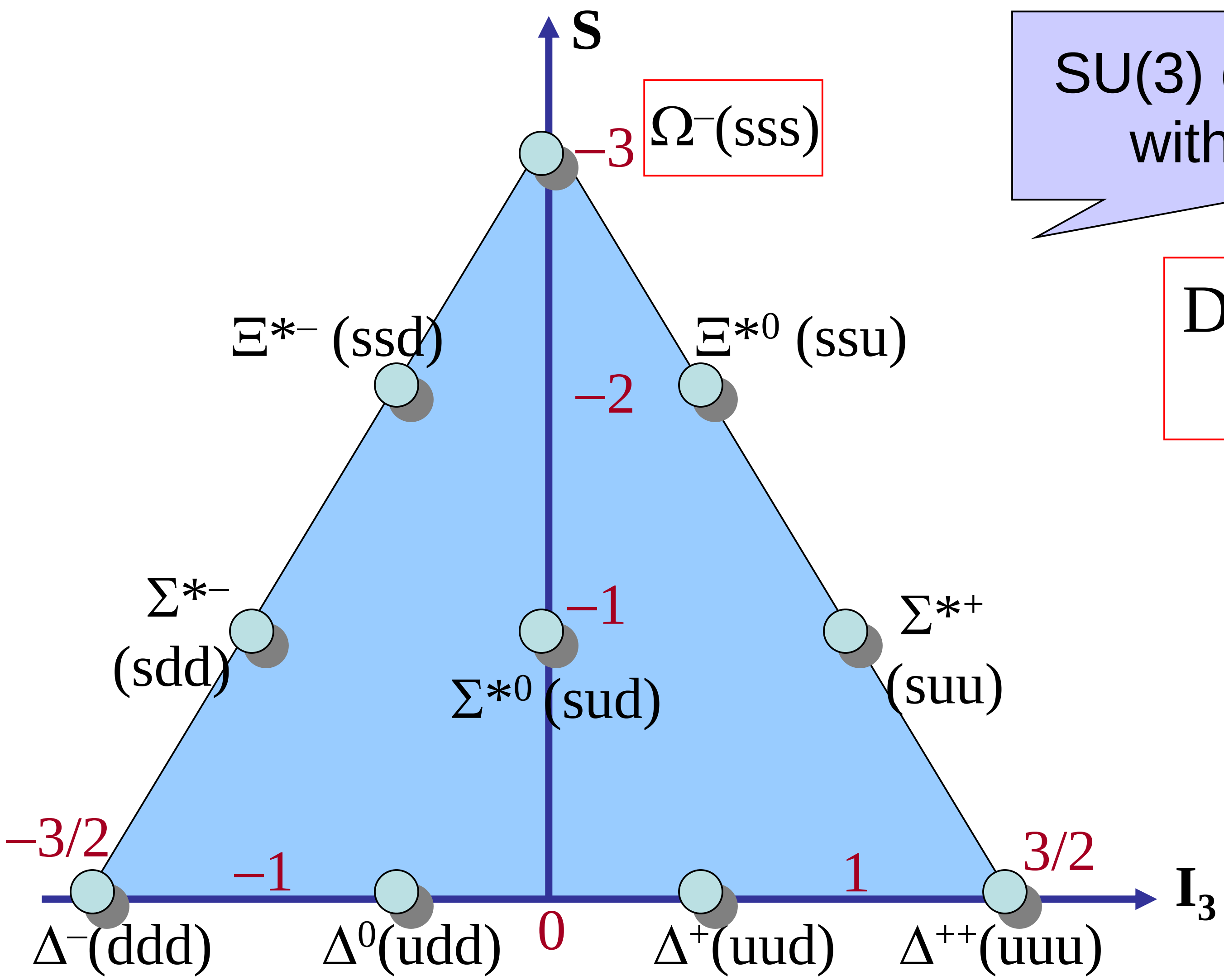
SU(3) octet with $J^P=1/2^+$

Gell-Mann - Nishijima:
$Q=I_3+Y/2=I_3+(B+S)/2$

$3 \otimes 3 \otimes 3$
$=(\bar{3} \oplus 6) \otimes 3$
$=(1 \oplus 8) \oplus (8 \oplus 10)$

$$\frac{1}{\sqrt{2}}(udu - duu)$$
$$\frac{1}{\sqrt{6}}(2uud - duu - udu)$$

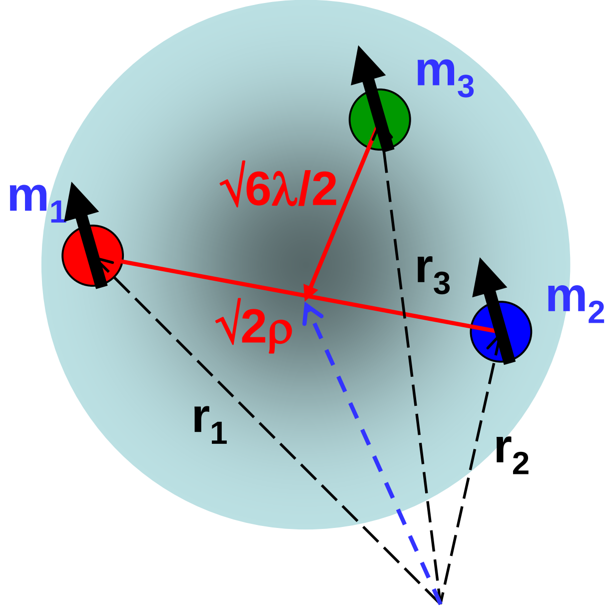
SU(3) multiplets of baryons made of u, d, and s



SU(3) decuplet 10 with $J^P=3/2^+$

Decuplet 10:
$\Omega^-(sss)$

- Symmetric spin wavefunction: S=3/2
- Symmetric flavor wavefunction: sss
- Symmetric spatial wavefunction: L=0

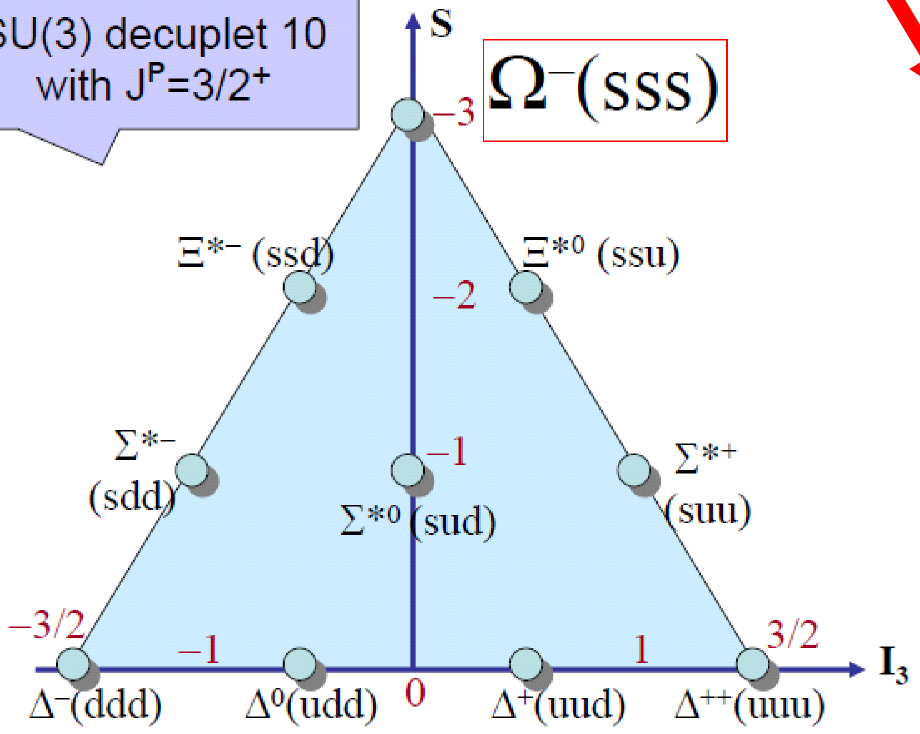


Jacobi coordinate

A problem encountered:

Violation of the Pauli principle and Fermi-Dirac statistics for the identical strange quark system?

SU(3) decuplet 10 with $J^P=3/2^+$

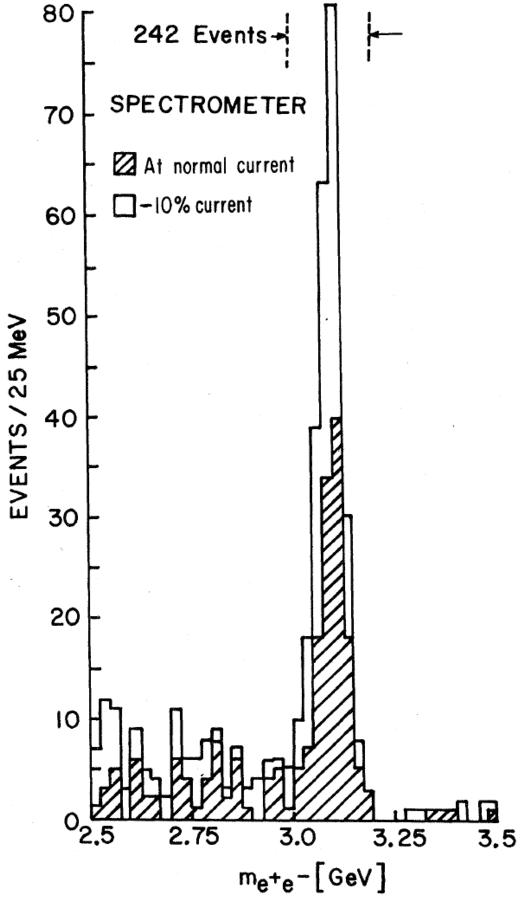


- An additional degrees of freedom, **Colour**, is introduced.
- Quark carries colour, while hadrons are colour neutral objects.

$$3 \otimes 3 \otimes 3$$
$$= (\bar{3} \oplus 6) \otimes 3$$
$$= (1 \oplus 8) \oplus (8 \oplus 10)$$

60-70年代

1970, Glashow-Iliopoulos-Maiani (GIM) mechanism, 味道改变中性流(FCNC)需要存在第四种味道的夸克。



1961年，电弱理论统一（Weinberg-Salam-Glashaw模型，1979 Nobel Prize）

1964年，Gell-Mann (1969 Nobel Prize)和Zweig提出夸克模型

1971-74年，量子色动力学 (t'Hooft, Veltman, 1999); Politzer, Wilczek, Gross, 2004)

1974年，丁肇中-Richter发现 J/ψ (1976 Nobel Prize)

1988

Leon Lederman, Melvin Schwartz and Jack Steinberger, "for … the discovery of the muon neutrino"

1977年，FermiLab发现底(bottom)夸克(Leon Lederman, et al.)

1995年，FermiLab发现顶(top)夸克

2012年，LHC发现希格斯(Higgs)玻色子…

The Nobel Prize in Physics 2013

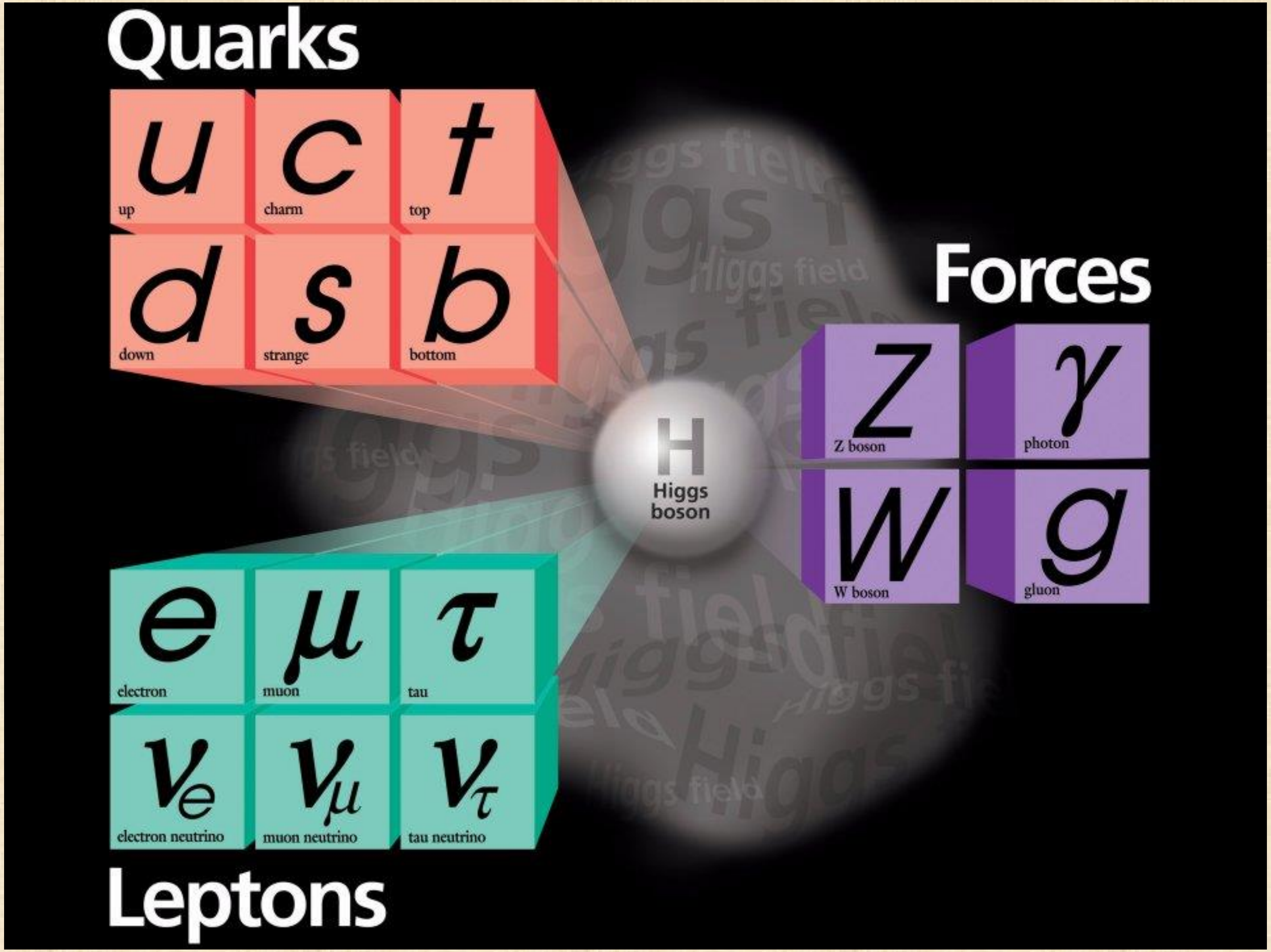
Photo: A. Mahmoud
François Englert
Prize share: 1/2

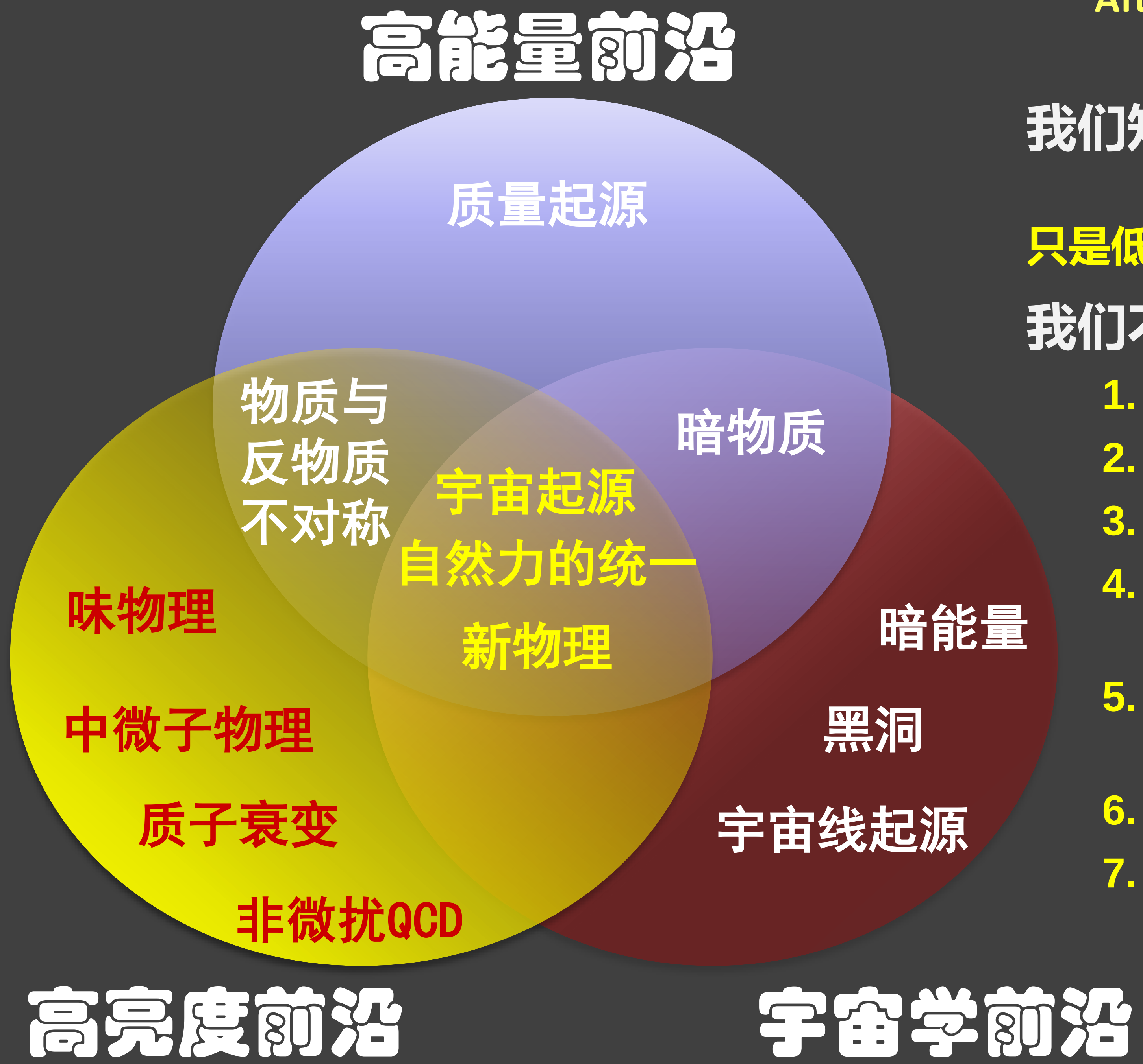
Photo: A. Mahmoud
Peter W. Higgs
Prize share: 1/2

Robert Brout
(1928-06-14, 2011-05-03)

“for the theoretical discovery of a mechanism that contributes to our understanding of the origin of mass of subatomic particles, and which recently was confirmed through the discovery of the predicted fundamental particle, by the ATLAS and CMS experiments at CERN‘s Large Hadron Collider”

基本粒子标准模型


Quarks
u
up
c
charm
t
top
d
down
s
strange
b
bottom
Forces
Z
Z boson
γ
photon
W
W boson
g
gluon
H
Higgs boson
e
electron
μ
muon
τ
tau
ν_e
electron neutrino
ν_μ
muon neutrino
ν_τ
tau neutrino
Leptons



After the discovery of Higgs …

我们知道的：

粒子物理标准模型虽然很成功，但它只是低能标下的一个有效理论。

我们不知道的：

1. Higgs粒子是否是基本粒子？
2. Higgs粒子的质量起源是什么？
3. Higgs粒子与宇宙起源有怎样的联系？
4. Higgs如何与暗物质粒子发生相互作用？
5. 粒子物理标准模型如何与引力自洽统一？
6. 暗能量的本质是什么？
7. 。。。

量子色动力学(Quantum Chromo-Dynamics)

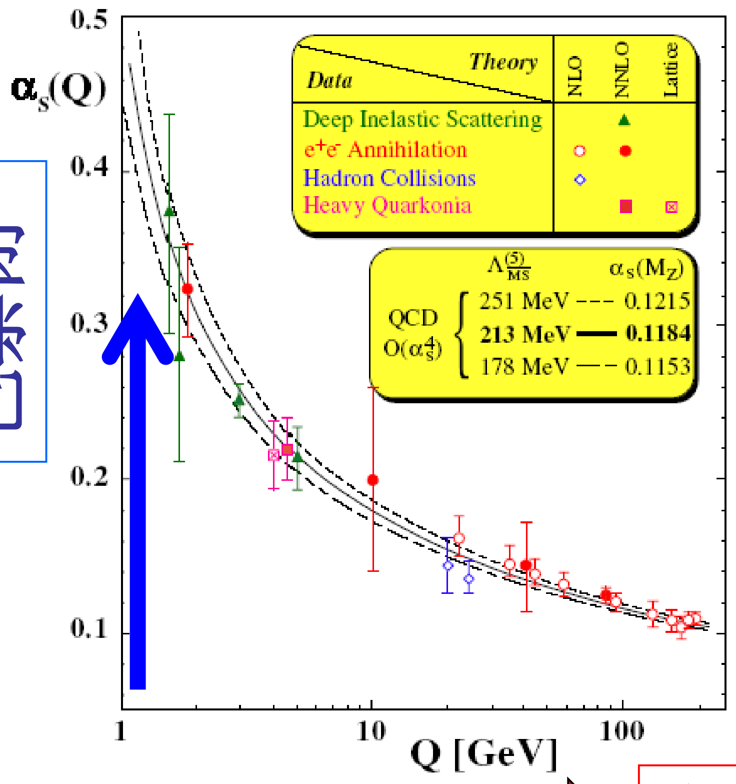
-- 目前最成功地描述强相互作用的基本理论

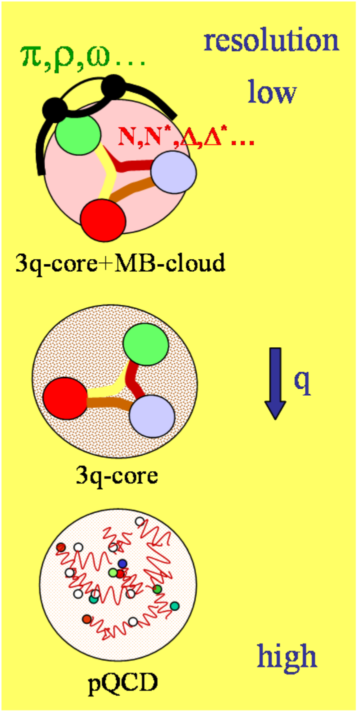
2004

David J. Gross　H. David Politzer　Frank Wilczek

SU(3)非阿贝尔规范理论

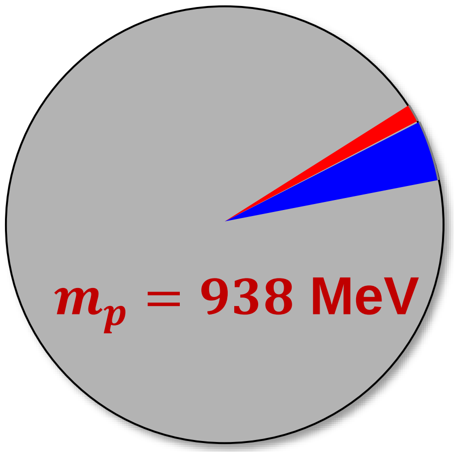
色禁闭





渐近自由

- Higgs机制使得夸克获得质量，但是不足以解释强子主要的质量来源



质子的质量来源：

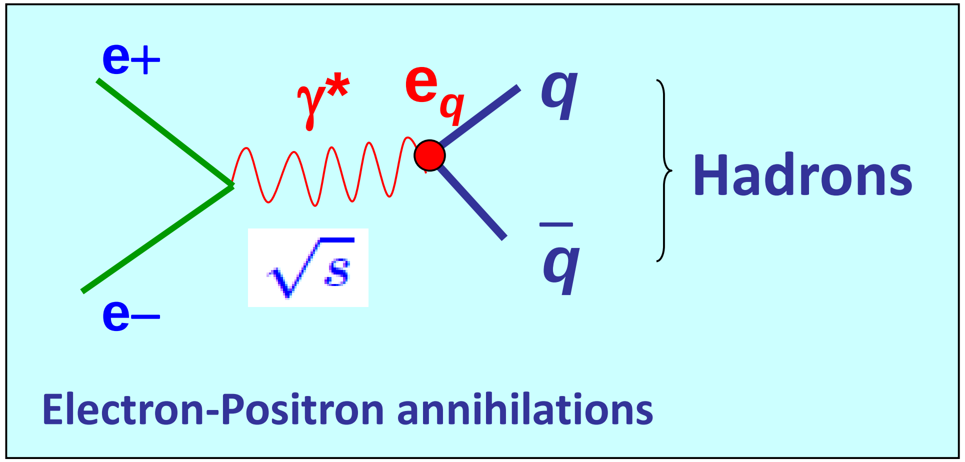
Higgs机制：～1%

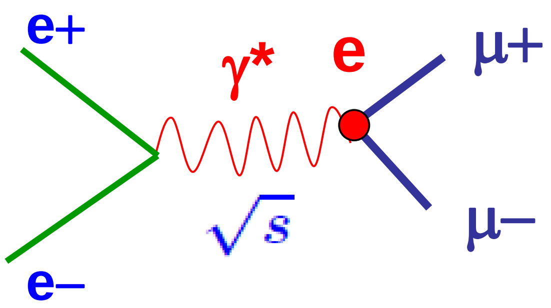
强相互作用：～95%

- 强相互作用如何将夸克胶子束缚在强子内部形成“稳定的”束缚态?
- “简单的”微观动力学规律如何衍生出复杂的物质形态?

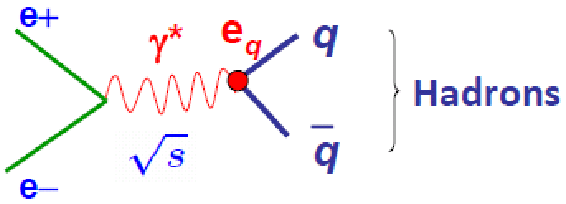
Again: Are quarks real objects?

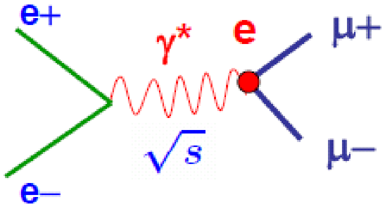
Probe coloured quarks in electron-positron collisions





R ≡ (numerator diagram) / (denominator diagram)





$$\cong \sum \hat{e}_q^{\,2}$$

$$= (2/3)^2 + (1/3)^2 + (1/3)^2 + \ldots$$

u *d* *s* …

$$R \cong \sum \hat{e}_q^{\,2}$$

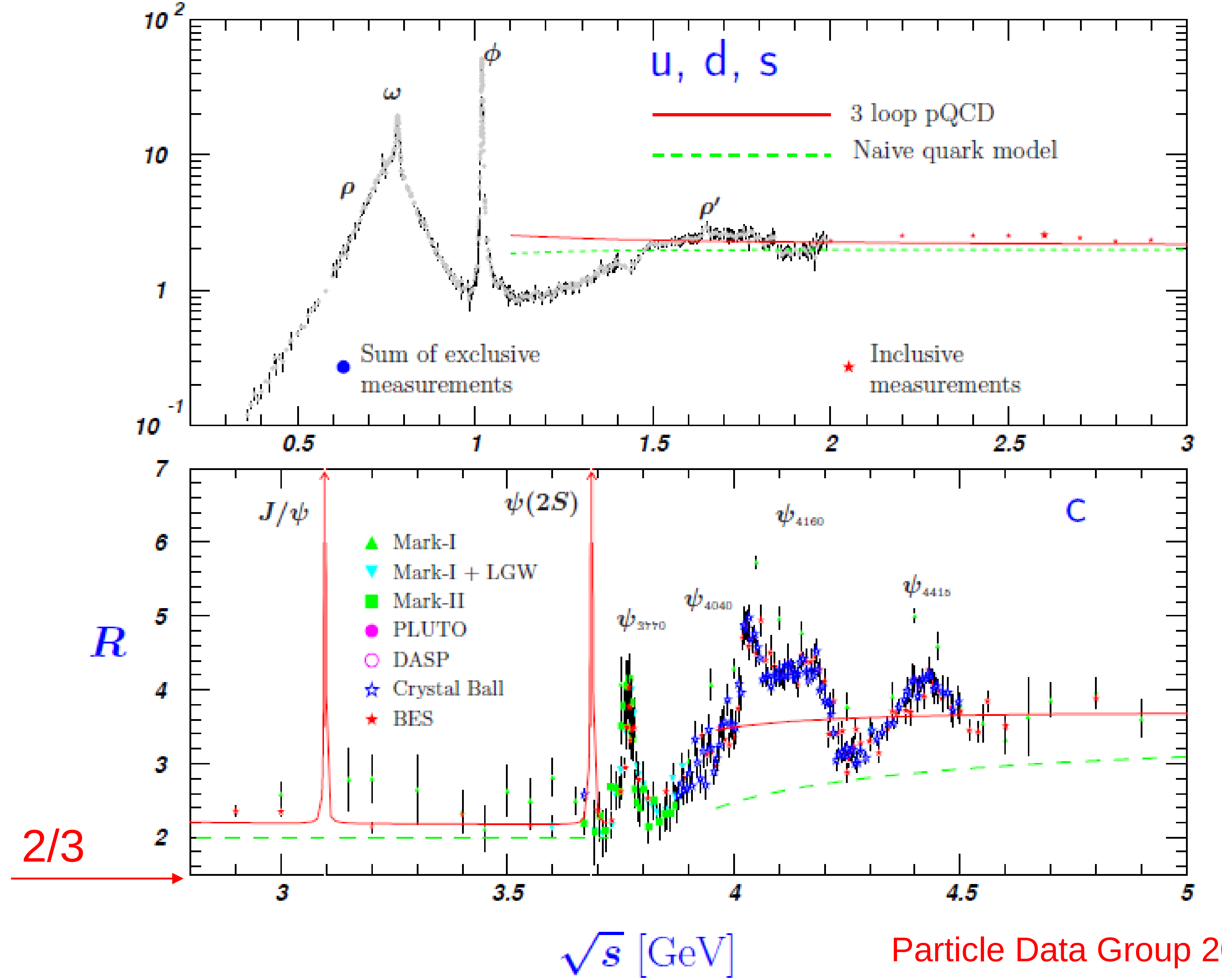
q :	u(3/2)	d(-1/3)	s(-1/3)	c(2/3)	b(-1/3)	t(2/3)	R
	$(2/3)^2$	$+ (1/3)^2$	$+ (1/3)^2$				$= [2/3]$
		[2/3]		$+ (2/3)^2$			$= [10/9]$
			[10/9]		$+(1/3)^2$		$= [11/9]$
				[11/9]		$+ (2/3)^2$	$= [15/9]$

But if quark carries color, one should have

$$R \cong 3 \times \sum \hat{e}_q^{\,2}$$

$$R \cong 3 \times \sum \hat{e}_q^{\,2}$$

q :	u(3/2)	d(-1/3)	s(-1/3)	c(2/3)	b(-1/3)	t(2/3)	R
	$(2/3)^2$	$+ (1/3)^2$	$+ (1/3)^2$				= 2
		[2/3]		$+ (2/3)^2$			= 10/3
			[10/9]		$+(1/3)^2$		= 11/3
				[11/9]		$+ (2/3)^2$	= 5

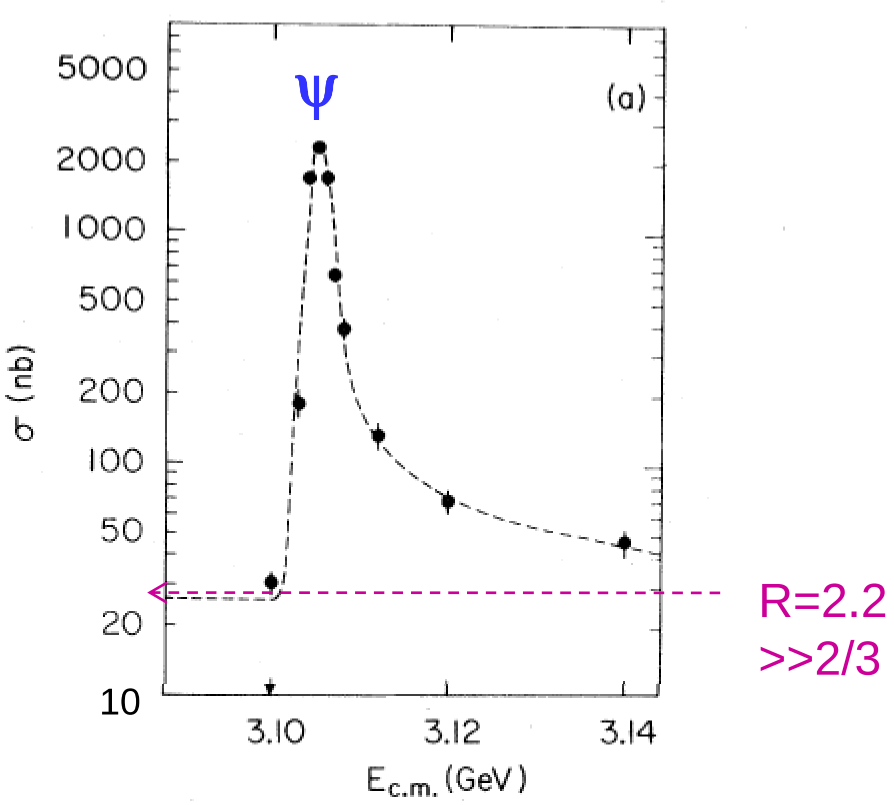


Particle Data Group 2010

思考：列举一些你认为能证明“夸克”是真实微观粒子的证据或方法。

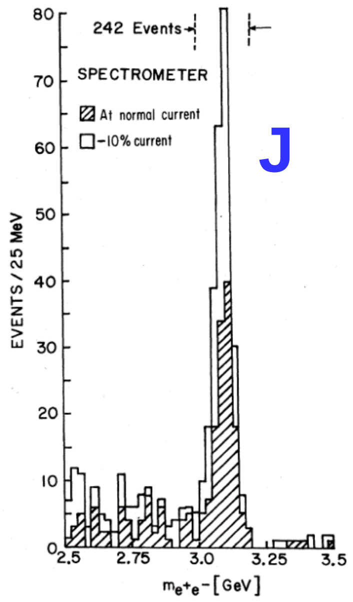
1976 Nobel Prize:
B. Richter and S. C.-C. Ting

"for their pioneering work in the discovery of a heavy elementary particle of a new kind"



J.E. Augustine et al., PRL 33, 1406 (1974)

Also seen in $pN \rightarrow e^+e^-X$



J.J. Aubert et al., PRL 33, 1404 (1974)

Quarks are real building blocks of hadrons: meson ($q\bar{q}$), baryon (qqq)

Property \ Quark	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$
I – isospin	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0	0
I_z – isospin z-component	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0
S – strangeness	0	0	-1	0	0	0
C – charm	0	0	0	$+1$	0	0
B – bottomness	0	0	0	0	-1	0
T – topness	0	0	0	0	0	$+1$

Convention (Particle Data Group):

1) Quark has spin 1/2 and baryon number 1/3;

2) Quark has positive parity and antiquark has negative parity;

3) The flavor of a quark has the same sign as its charge.

Quarks are real building blocks of hadrons: meson ($q\bar{q}$), baryon (qqq)

Property \ Quark	d	u	s	c	b	t
Q – electric charge	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$	$-\frac{1}{3}$	$+\frac{2}{3}$

- **Quarks are not free due to QCD colour force (colour confinement).**
- **Chiral symmetry spontaneous breaking gives masses to quarks.**
- **Hadrons, with rich internal structures, are the smallest objects in Nature that cannot be separated to be further finer free particles.**

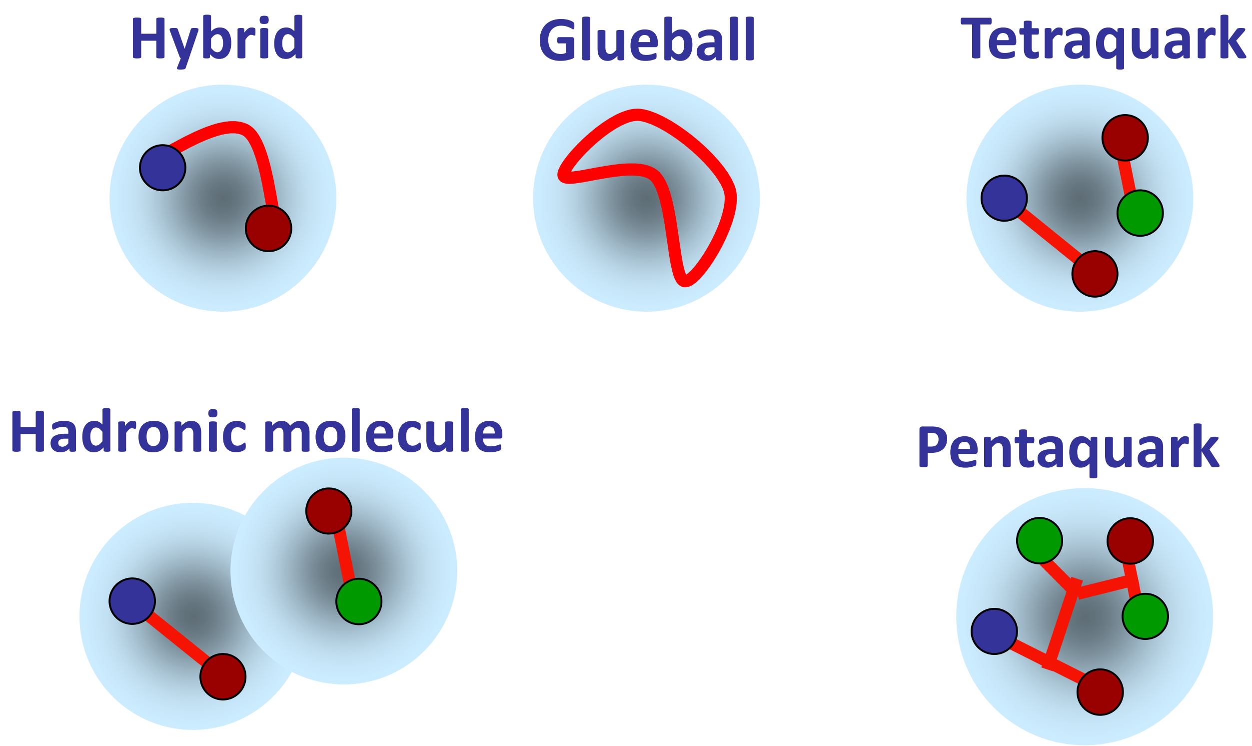
Convention (Particle Data Group):
1) Quark has spin 1/2 and baryon number 1/3;
2) Quark has positive parity and antiquark has negative parity;
3) The <u>flavor</u> of a quark has the same sign as its <u>charge.</u>

Remaining questions:

- What are the proper effective degrees of freedom for hadron internal structures? -- How the constituent degrees of freedom manifest themselves and how to describe them?
- What are the possible color-singlet hadrons apart from the simplest conventional mesons ($q\bar{q}$) and baryons (qqq)?
- What's happening in between "perturbative" and "non-perturbative"?
- How to probe the hadron structures in experiment?
- … …

Multi-faces of QCD:

Exotic hadrons beyond conventional QM



The study of hadron structures and hadron spectroscopy should deepen our insights into the Nature of strong QCD.

Recent reviews on hadron spectroscopy and QCD exotics:

1. F.K. Guo, C. Hanhart, U. G. Meißner, Q. Wang, Q. Zhao and B.S. Zou, **Rev. Mod. Phys. 90**, no.1, 015004 (2018);
2. H.X. Chen, W. Chen, X. Liu and S.L. Zhu, **Phys. Rept. 639**, 1-121 (2016);
3. A. Esposito, A. Pilloni and A. D. Polosa, **Phys. Rept. 668**, 1-97 (2017);
4. A. Ali, J.S. Lange and S. Stone, **Prog. Part. Nucl. Phys. 97**, 123-198 (2017);
5. Y.R. Liu, H. X. Chen, W. Chen, X. Liu and S.L. Zhu, **Prog. Part. Nucl. Phys. 107**, 237-320 (2019);
6. N. Brambilla, S. Eidelman, C. Hanhart, A. Nefediev, C.P. Shen, C.E. Thomas, A. Vairo and C.Z. Yuan, **Phys. Rept. 873**, 1-154 (2020);
7. R.F. Lebed, R.E. Mitchell and E.S. Swanson, **Prog. Part. Nucl. Phys. 93**, 143-194 (2017);
8. Y. Yamaguchi, A. Hosaka, S. Takeuchi and M. Takizawa, **J. Phys. G 47**, no.5, 053001 (2020).
9. R.M. Albuquerque, J.M. Dias, K.P. Khemchandani, A. Martinez Torres, F.S. Navarra, M. Nielsen and C. M. Zanetti, **J. Phys. G 46**, no.9, 093002 (2019).

I. A brief review of hadron physics and introduction to non-relativistic constituent quark model (NRCQM)

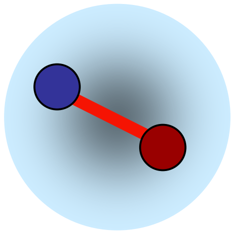
Basic assumptions of NRCQM

i) **Chiral symmetry spantaneous breaking leads to the presence of massive constituent quarks as effective degrees of freedom inside hadrons.**

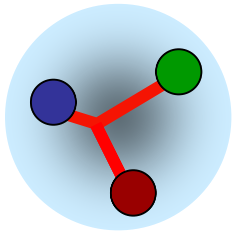
ii) **Hadrons can be viewed as quark systems in which the gluon fields generate effective potentials that depend on the spins and positions of the massive quarks.**

Thus, meson is a $q\bar{q}$ system and baryon is made of qqq.

$q\bar{q}$ meson



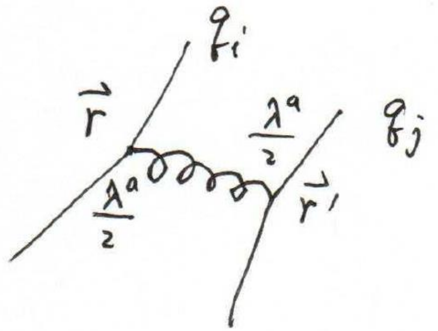
qqq baryon



Quark-quark (-antiquark) interaction potential

Following the basic assumptions for NRCQM, the quarks (antiquarks) inside hadrons can be described by the non-relativistic Schrödinger equation:

$$\hat{H} = \int d\vec{r} \sum_i q_i^\dagger(\vec{r})\, \beta \left(m_i - \frac{\nabla^2}{2m_i}\right) q_i(\vec{r}) + \frac{1}{2}\int d\vec{r}\, d\vec{r}\,'\, V_0(\vec{r}-\vec{r}\,') \sum_{ija} q_i^\dagger(\vec{r}) \frac{\lambda^a}{2} q_i(\vec{r})\, q_j^\dagger(\vec{r}\,') \frac{\lambda^a}{2} q_j(\vec{r}\,')$$



$V_0(\vec{r}-\vec{r}\,')$ can take forms of the spin-independent confinement potential

- The number of quarks (anti-quarks) does not change, i.e. quarks and anti-quarks are “constituents”.
- The implementation of the spin-independent confinement potential means that the total spin and total orbital angular momentum are conserved.
- The spin-dependent interactions are relatively small “perturbations” in comparison with the confinement potential.

The algebra of infinitesimal SU(3) transformations is spanned by the 8 Gell-Mann's matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \sqrt{\frac{1}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The Gell-Mann matrices are the generators of the SU(3) group which will act on quarks. Quark is the SU(3)-group representation of **3.**

$$\lambda_a = \lambda_a^\dagger,$$

$$\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab},$$

$$\mathrm{Tr}(\lambda_a) = 0.$$

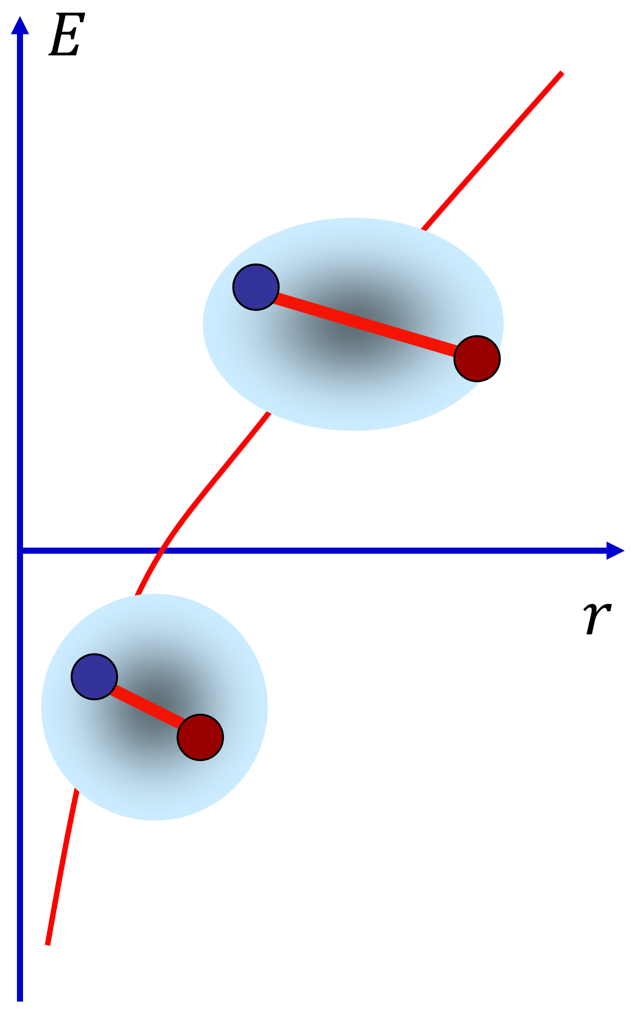
$$[\lambda_a, \lambda_b] = \frac{i}{2} \sum_c f_{abc} \lambda_c$$

Note: $\lambda_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not part of $SU(3)$. It corresponds to a $U(1)$ symmetry. Namely,

$$U(3) = SU(3) \otimes U(1)$$

Color charges of quarks and gluons in QCD

- QCD has an exact $SU(3)_{color}$ local symmetry with a corresponding gauge invariance and 8 associated gauge bosons, i.e. gluons.
- Due to the non-Abelian property of QCD, the QCD coupling will cause an increase of the energy involved in separating two colored charges. Therefore, free observable particles must correspond to states with no color, or "colorless".
- A colorless state, i.e. a color singlet, will keep unchanged under the gauge transformation: $U = \exp\left(\frac{ia\lambda}{2}\right)$, where a is the unit of the color charge, and λ is the $SU(3)$ generator.



Denote the color charge (base vectors of SU(3)) as follows:

Red: $|r\rangle = (1, 0, 0)$

Blue: $|b\rangle = (0, 1, 0)$

Green: $|g\rangle = (0, 0, 1)$

The color states of hadrons

Consider a color state: $|r\rangle|b\rangle|g\rangle$ under the infinitesimal transformation:

$$U = \exp(i\alpha\lambda_1) \simeq 1 + i\alpha\lambda_1$$

We have

$$|r\rangle|b\rangle|g\rangle \rightarrow (|r\rangle + i\alpha|b\rangle)(|b\rangle + i\alpha|r\rangle)|g\rangle$$

$$= |r\rangle|b\rangle|g\rangle + i\alpha|b\rangle|b\rangle|g\rangle + i\alpha|r\rangle|r\rangle|g\rangle$$

$$\neq |r\rangle|b\rangle|g\rangle$$

An invariant state of three color sources (baryon made of three quarks) can be obtained by constructing

$$|B\rangle = \frac{1}{\sqrt{6}}\sum_{ijk}\epsilon_{ijk}|i\rangle|j\rangle|k\rangle$$

$$= \frac{1}{\sqrt{6}}[|r\rangle|b\rangle|g\rangle + |g\rangle|r\rangle|b\rangle + |b\rangle|g\rangle|r\rangle - |b\rangle|r\rangle|g\rangle - |r\rangle|g\rangle|b\rangle - |g\rangle|b\rangle|r\rangle]$$

An invariant state of color-anti-color sources (meson made of $q\bar{q}$)

The anti-color transforms as

$\langle r| \to \langle r'| = \langle r|U^H,$

where U^H is the Hermitian matrix of U.

The meson color wavefunction can be expressed as: $|r\rangle\langle r|$

$\to (|r\rangle + i\alpha|b\rangle)(\langle r| - i\alpha\langle b|)$

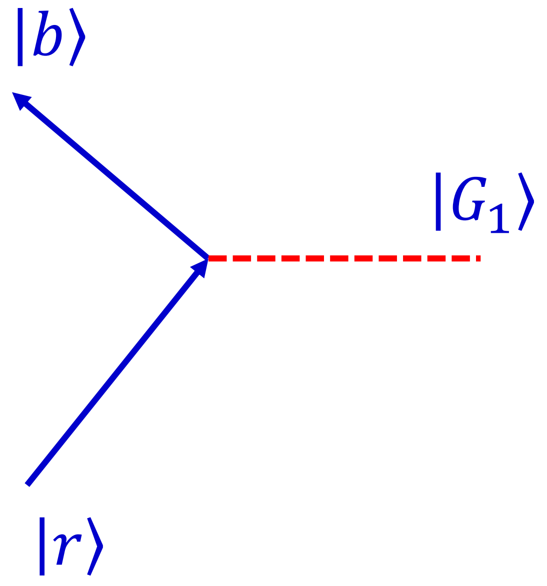
$= |r\rangle\langle r| + i\alpha|b\rangle\langle r| - i\alpha|r\rangle\langle b|$

$\neq |r\rangle\langle r|$

A colorless state of $q\bar{q}$ should be

$$|M\rangle = \frac{1}{\sqrt{3}}(|r\rangle\langle r| + |b\rangle\langle b| + |g\rangle\langle g|)$$

Color wavefunctions for octet gluons



$G_1 = \frac{1}{\sqrt{2}}(r\bar{b} + b\bar{r}),\ \ G_2 = \frac{i}{\sqrt{2}}(r\bar{b} - b\bar{r}),\ \ G_3 = \frac{1}{\sqrt{2}}(r\bar{r} - b\bar{b}),$

$G_4 = \frac{1}{\sqrt{2}}(b\bar{g} + g\bar{b}),\ \ G_5 = \frac{i}{\sqrt{2}}(b\bar{g} - g\bar{b}),$

$G_6 = \frac{1}{\sqrt{2}}(g\bar{r} + r\bar{g}),\ \ G_7 = \frac{i}{\sqrt{2}}(g\bar{r} - r\bar{g}),$

$G_8 = \frac{1}{\sqrt{6}}(r\bar{r} + b\bar{b} - 2g\bar{g})$

We can derive the color wavefunctions for the octet gluons by the color conservation. For instance, under the operation of the SU(3) generator λ_1, the red and blue color will transform mutually to each other by radiating a gluon. The coupling vertex in the color space can be expressed as:

$$\bar{\psi}\lambda_1 G_1 \psi = (\bar{r}, \bar{b}, \bar{g})\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} G_1 \begin{pmatrix} r \\ b \\ g \end{pmatrix} = \bar{r}G_1 b + \bar{b}G_1 r,$$

where G_1 is the octet gluon state. To keep the color conserved, it means that the average value of $\bar{\psi}\lambda_1 G_1 \psi$ in vacuum must be a constant: $\langle 0|\bar{r}G_1 b + \bar{b}G_1 r|0\rangle = const. \to G_1 = \frac{1}{\sqrt{2}}(r\bar{b} + b\bar{r})$

With the interaction Hamiltonian one can calculate the interaction element between a quark and antiquark within a quark-antiquark system.

A quark and antiquark system:

$$|q_{is}(\vec{r})\,\bar{q}_{js'}(\vec{r}\,')\rangle = \frac{1}{(2\pi)^3}\left[\int d\vec{p}\, e^{-i\vec{p}\cdot\vec{r}}\, b^{\dagger}_{is}(\vec{p})\right]\left[\int d\vec{p}\,'\, e^{-i\vec{p}\,'\cdot\vec{r}\,'}\, d^{\dagger}_{js'}(\vec{p}\,')\right]|0\rangle$$

Interaction element between a quark and antiquark:

$$\langle q(\vec{r}_0)\,\bar{q}(\vec{r}_0\,')|V|q(\vec{r})\,\bar{q}(\vec{r}\,')\rangle = -\sum_a \frac{\lambda^a_q}{2}\frac{\tilde{\lambda}^a_{\bar{q}}}{2} V_0(\vec{r}-\vec{r}\,')\,\delta(\vec{r}-\vec{r}_0)\,\delta(\vec{r}\,'-\vec{r}_0\,') \qquad (5)$$

$\tilde{\lambda}^a_{\bar{q}}$ is the transpose of $\lambda^a_{\bar{q}}$.

For a system of two quarks, i.e. qq, the potential yields

$$\langle q(\vec{r}_0)\,q(\vec{r}_0\,')|\hat{V}|q(\vec{r})\,q(\vec{r}\,')\rangle = \sum_a \frac{\lambda^a_q}{2}\frac{\lambda^a_q}{2} V_0(\vec{r}-\vec{r}\,')\,\delta(\vec{r}-\vec{r}_0)\,\delta(\vec{r}\,'-\vec{r}_0\,')$$

The SU(3) color matrix elements $\sum_a \frac{\lambda_q^a}{2}\frac{\tilde{\lambda}_{\bar{q}}^a}{2}$ and $\sum_a \frac{\lambda_q^a}{2}\frac{\lambda_q^a}{2}$ can be calculated for representations of the $q\bar{q}$ and qq systems in the color space, respectively.

$$\begin{cases} 3 \otimes \bar{3} = 1 + 8 \\ 3 \otimes 3 = \bar{3} + 6 \end{cases}$$

Question 1:

Given a representation of $|R\rangle (= 1, \bar{3}, 6, 8)$, we can directly calculate

$$\langle R| \sum_a \frac{\lambda_1^a}{2}\frac{\lambda_2^a}{2} |R\rangle = \frac{1}{2}\left\{ \langle R| \sum_a \left(\frac{\lambda_1^a}{2} + \frac{\lambda_2^a}{2}\right)^2 |R\rangle - \langle R| \left[\sum_a \left(\frac{\lambda_1^a}{2}\right)^2 + \sum_a \left(\frac{\lambda_2^a}{2}\right)^2\right] |R\rangle \right\}$$

$$= \frac{1}{2}\left[C(R) - 2C(3)\right]$$

where the index 1 means a quark and index 2 represents a quark or an antiquark ($\lambda_2^a \to -\tilde{\lambda}_2^a$).

C(R) is the color Casimir of qq or $q\bar{q}$ which are of the $SU(3)_c$ representation $\mathbf{R}$:

$$\begin{cases} C(\mathbf{1}) = 0 \\ C(8) = 3 \\ C(3) = C(\bar{3}) = \frac{4}{3} \\ C(6) = \frac{10}{3} \end{cases}$$

The potential strengths for different color states can be worked out

$$\langle (q\bar{q})_1 | \hat{V} | (q\bar{q})_1 \rangle = -\frac{4}{3} V_0$$

$$\langle (q\bar{q})_8 | \hat{V} | (q\bar{q})_8 \rangle = +\frac{1}{6} V_0$$

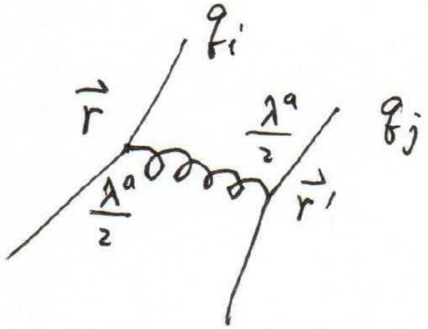
$$\langle (qq)_{\bar{3}} | \hat{V} | (qq)_{\bar{3}} \rangle = -\frac{2}{3} V_0$$

$$\langle (qq)_6 | \hat{V} | (qq)_6 \rangle = +\frac{1}{3} V_0$$

If one takes the Lorentz-scalar potential, i.e.

$$\hat{V}_s = \frac{1}{2}\int d\vec{r}\, d\vec{r}\,'\, V_0(\vec{r}-\vec{r}\,') \sum_{ija} \bar{q}_i(\vec{r})\frac{\lambda^a}{2} q_i(\vec{r})\, \bar{q}_j(\vec{r}\,')\frac{\lambda^a}{2} q_j(\vec{r}\,')$$

an additional minus sign for the $q\bar{q}$ coupling will be obtained compared with Eq.(5). As a result, it means that a scalar potential cannot confine meson and baryon simultaneously, although more elaborate calculations suggest that a small mixture of the Lorentz-scalar potential should be present.

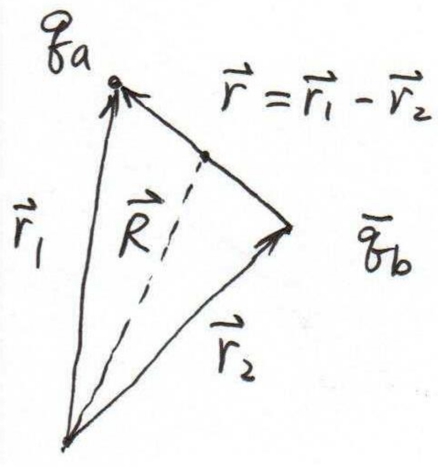


Questions:
1) Prove that the Lorentz-scalar potential leads to a repulsive interaction between a quark and antiquark in the color singlet representation.

Mesons in SU(6)⊗O(3) symmetric quark model

Hamiltonian for color-singlet $q\bar{q}$

$$\hat{H} = \frac{\vec{p}_1^{\,2}}{2m_1} + \frac{\vec{p}_2^{\,2}}{2m_2} - \frac{4}{3} V_0(\vec{r}_1 - \vec{r}_2) \qquad (1)$$



Taking the coordinate:

$$\begin{cases} \vec{r} \equiv \vec{r}_1 - \vec{r}_2 \\ \vec{R} \equiv \frac{1}{2}(\vec{r}_1 + \vec{r}_2) \end{cases}$$

$$\left[\text{rigorously}: \quad \vec{R} \equiv \frac{1}{m_1 + m_2}(m_1 \vec{r}_1 + m_2 \vec{r}_2)\right]$$

$$\begin{cases} \vec{P} \equiv \vec{p}_1 + \vec{p}_2 \\ \vec{p} \equiv \frac{1}{2}(\vec{p}_1 - \vec{p}_2) \end{cases}$$

Assuming $V(\vec{r}) \equiv -\frac{4}{3} V_0(\vec{r})$ is a spin-independent central potential, we rewrite $\hat{H}$ as

$$\hat{H} = \frac{\vec{P}^2}{4m_q} + \frac{\vec{p}^2}{m_q} + V(\vec{r}) \qquad \text{with } m_1 = m_2 \equiv m_q \qquad (2)$$

$$\left\{ \text{rigorously: } \hat{H} \equiv \frac{\vec{P}^2}{2M_R} + \frac{\vec{p}^2}{2\mu_q} + V(\vec{r}) \quad \text{with} \begin{cases} M_R \equiv m_1 + m_2 \\ \mu_q \equiv \dfrac{m_1 m_2}{m_1 + m_2} \end{cases} \right\}$$

The c.m. motion is described by a plane wave with

$$\hat{H}_0 = \frac{\vec{P}^2}{2M_R}$$

$$\therefore \quad \hat{H}_0 \psi_R(\vec{R}) = \frac{\vec{P}^2}{2M_R} \psi_R(\vec{R})$$

$$= -\frac{\hbar^2}{2M_R} \nabla_R^2 \psi_R(\vec{R}) = E_0 \psi_R(\vec{R})$$

$$\Rightarrow \quad \psi_R(\vec{R}) = \frac{1}{N_R} e^{\pm i \vec{P}\cdot\vec{R}}$$

For the relative motion of m_1 and m_2, the eigen equation is

$$\hat{H}_I \psi(\vec{r}) = \left(\frac{\vec{p}^2}{2\mu_{\bar{q}}} + V(\vec{r}) \right) \psi(\vec{r}) = E\, \psi(\vec{r})$$

with $\quad \psi_L^M(\vec{r}) = Y_{LM}(\hat{r}) \dfrac{U_L(r)}{r}$

$U_L(r)$ is the radial wave function.

The main quantum number N defined by the quantization condition is $N \geq L+1$. Thus, the energy level for each eigenstate $\psi_L^M(\vec{r})$ can be labelled by $N\, {}^{2S+1}L_J$ with $\vec{J} = \vec{L} + \vec{S}$ as the total angular momentum of the $q\bar{q}$ system.

For the central potential $\hat{V}$, both total spin $\vec{S}$ and total orbital angular momentum $\vec{L}$ are conserved. Thus, in the light quark sector, the spin-flavor-sptial wavefuncti are representations of $SU(3)_f \otimes SU(2) \otimes O(3) \subset SU(6) \otimes O(3)$ symmetry.

spin : $2 \otimes 2 = 1 + 3$

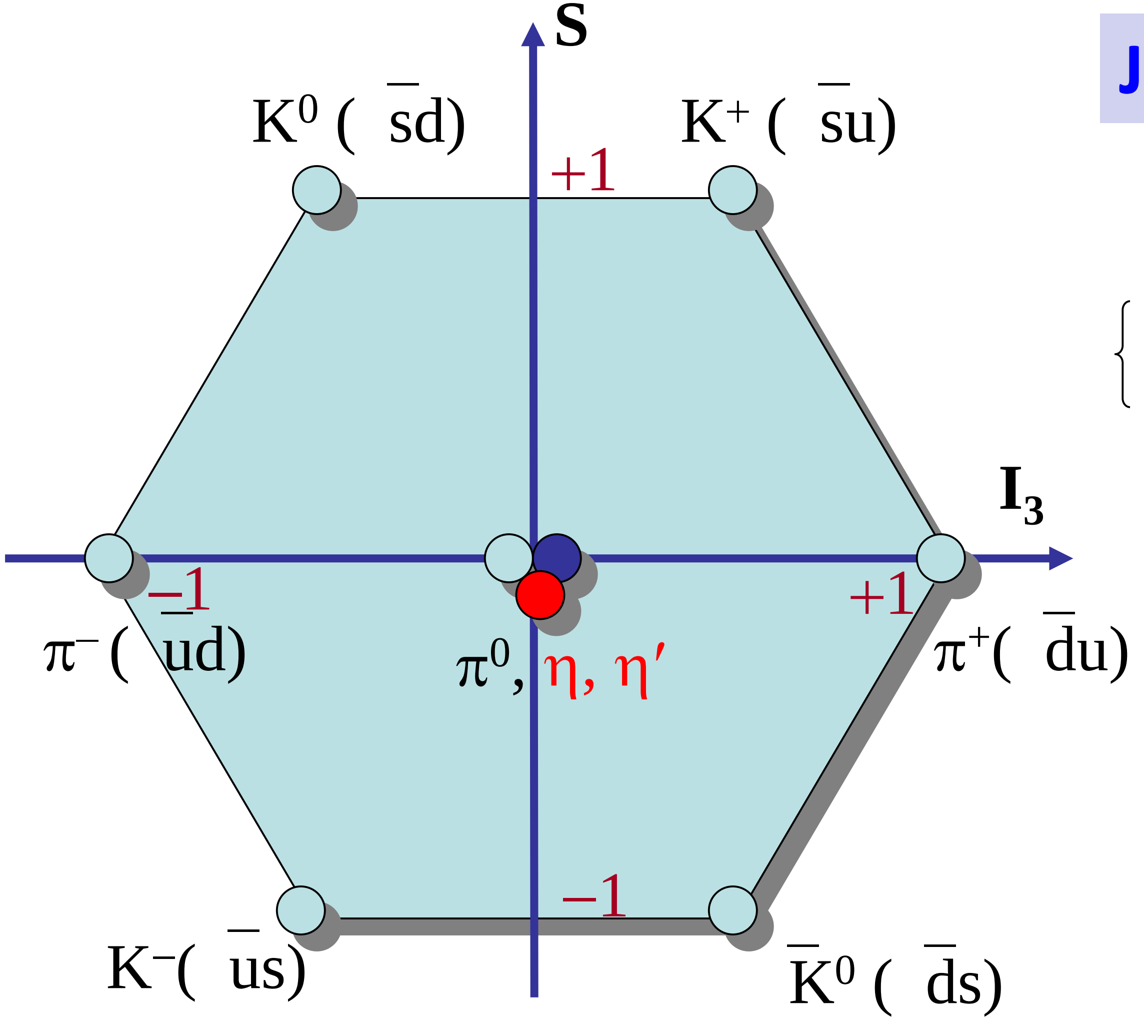
flavor : $3 \otimes \bar{3} = 1 + 8$

spin-flavor : $6 \otimes \bar{6} = 1 + 35$

Or $(2 \otimes 2) \otimes (3 \otimes \bar{3}) = (1 + 3) \otimes (1 + 8)$

$$= \underbrace{(1,1)}_{\text{singlet}} + \underbrace{(1,8) + (3,1) + (3,8)}_{\text{35 multiplets}}$$

$\bar{q}q$ SU(3) flavor nonet: $\bar{3}\otimes 3 = 1\oplus 8$



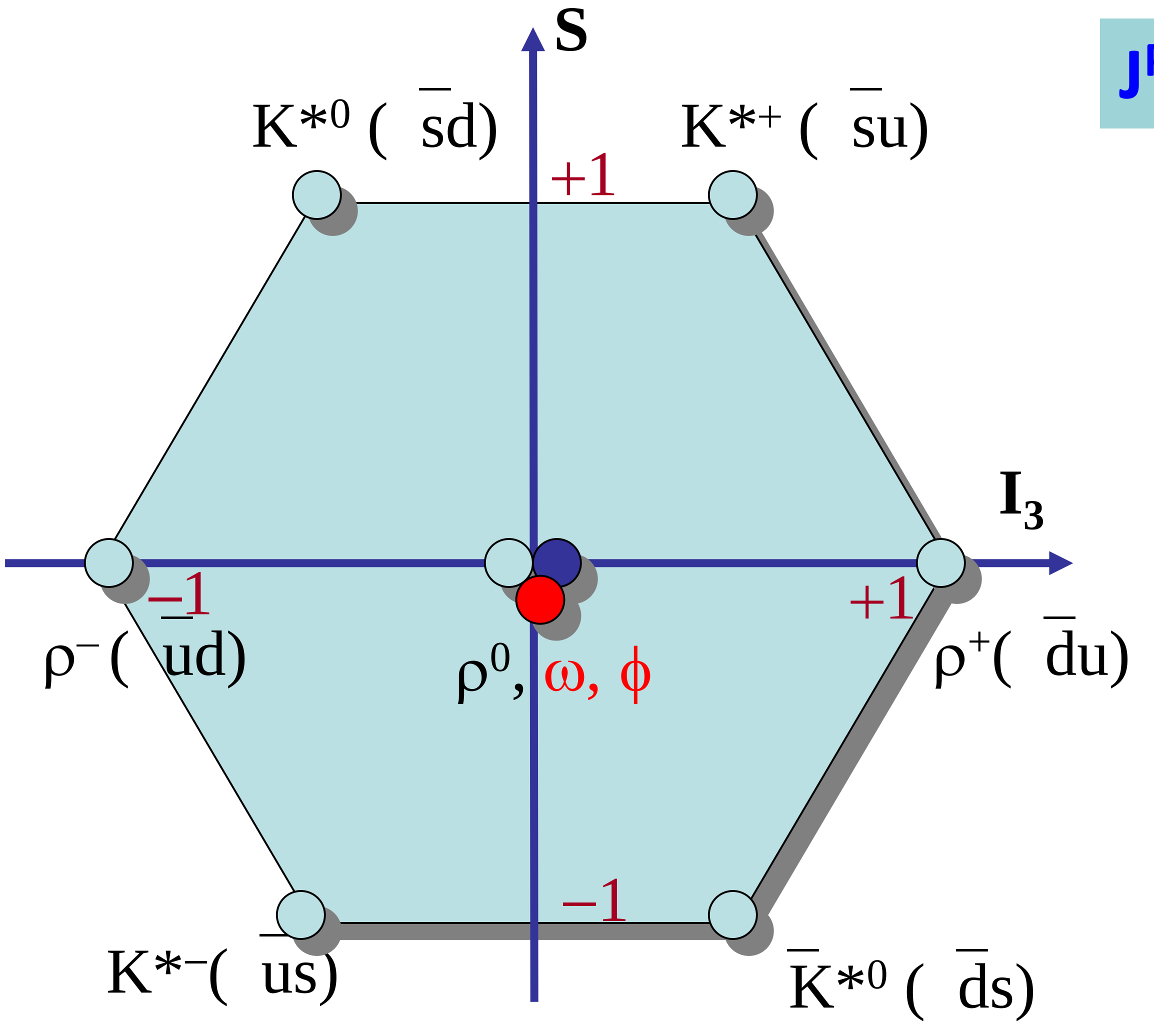
$J^{PC} = 0^{-+}$

$$\begin{cases} \eta = \cos\alpha_P |n\bar{n}\rangle - \sin\alpha_P |s\bar{s}\rangle \\ \eta' = \sin\alpha_P |n\bar{n}\rangle + \cos\alpha_P |s\bar{s}\rangle \end{cases}$$

$$n\bar{n} \equiv (u\bar{u} + d\bar{d})/\sqrt{2}$$

$$\alpha_P = \theta_P + \arctan\sqrt{2}$$

$\bar{q}q$ SU(3) flavor nonet: $\bar{3}\otimes 3 = 1\oplus 8$



$J^{PC} = 1^{--}$

$\rho^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$

$\omega = (u\bar{u} + d\bar{d})/\sqrt{2}$

$\phi = s\bar{s}$

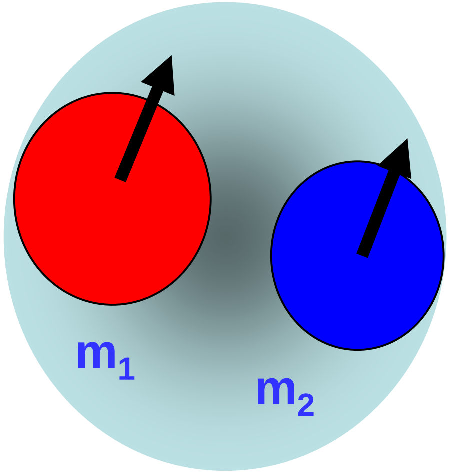
Typical wavefunction for ($q\bar{q}$) with fixed total angular momentum J can be written as

$$\psi_{NJM}(\vec{r}) = \varphi_c \varphi_F \sum_m \langle L(M-m), sm | JM \rangle \chi_{sm} \psi_{NL(M-m)}(\vec{r})$$

where N is the main quantum number, S and L are the total spin and relative orbital angular momentum, respectively.

The overall wavefunctions for $q\bar{q}$ (spin, flavor, spatial, and color) should be antisymmetrized as required by Fermi statistics.

Taking into account the negative intrinsic parity of the antiquark, the wavefunctions for the ground state mesons (L=0):



	ϕ_S (ϕ_A)
$\frac{1}{\sqrt{2}}(u\bar{s} \pm \bar{s}u)$	K^+ (K^{*+})
$\frac{1}{\sqrt{2}}(d\bar{s} \pm \bar{s}d)$	K^0 (K^{*0})
$-\frac{1}{\sqrt{2}}(s\bar{u} \pm \bar{u}s)$	K^- (K^{*-})
$-\frac{1}{\sqrt{2}}(s\bar{d} \pm \bar{d}s)$	$\bar{K}^0$ ($\bar{K}^{*0}$)
$\frac{1}{\sqrt{2}}(u\bar{d} \pm \bar{d}u)$	π^+ (ρ^+)
$-\frac{1}{\sqrt{2}}(d\bar{u} \pm \bar{u}d)$	π^- (ρ^-)
$\frac{1}{2}[(d\bar{d}-u\bar{u}) \pm (\bar{d}d-\bar{u}u)]$	π^0 (ρ^0)
$\frac{1}{2\sqrt{3}}[(u\bar{u}+d\bar{d}-2s\bar{s}) \pm (\bar{u}u+\bar{d}d-2\bar{s}s)]$	η_8^0 (ω_8^0)
$\frac{1}{\sqrt{6}}[(u\bar{u}+d\bar{d}+s\bar{s}) \pm (\bar{u}u+\bar{d}d+\bar{s}s)]$	η_1^0 (ω_1^0)

$$\begin{cases} \phi_S \chi_A & \text{for } 0^- \text{ states} \\ \phi_A \chi_S & \text{for } 1^- \text{ states.} \end{cases}$$

S: symmetric after interchange the quark labels.

A: antisymmetric after interchange the quark labels.

- 1964- Gell-Mann, “Quark model”
- 1970- Deep Inelastic Scattering, Feymann’s partons, Bjorken scaling; Gell-Mann: “***What are these put-ons (sic); are they quarks?***”

  -- The strong force also might be described by a theory based on quarks and gluons began to take hold…

  -- Yang-Mills gauge theory can be a candidate.
- 1972- David Polizter, David Gross, Frank Wilczek found a negative beta function; (t’ Hooft has found it in 1971).
- 1973- SPEAR/SLAC experiment found charm flavor
- 1974- Sam Ting and Richter found J/psi
- 1975- QCD established

Baryons in SU(6)⊗O(3) symmetric quark model

We concentrate on the baryons made of u, d, s quarks.

Color	$SU(3)$	$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$	$=\mathbf{10}_s+\mathbf{8}_\rho+\mathbf{8}_\lambda+\boxed{\mathbf{1}_a}$
Spin	$SU(2)$	$\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}$	$=\mathbf{4}_s+\mathbf{2}_\rho+\mathbf{2}_\lambda,$
Flavor	$SU(3)$	$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3}$	$=\mathbf{10}_s+\mathbf{8}_\rho+\mathbf{8}_\lambda+\mathbf{1}_a,$
Spin-flavor	$SU(6)$	$\mathbf{6}\otimes\mathbf{6}\otimes\mathbf{6}$	$=\mathbf{56}_s+\mathbf{70}_\rho+\mathbf{70}_\lambda+\mathbf{20}_a,$
Spatial	$\mathrm{O}(3)$	L^P	s, ρ, λ, a

For a three-quark Fermion system, the Pauli principle requires that the total wavefunction is antisymmetric under exchange of any two quarks. Therefore, the total wavefunction must be antisymmetrized.

Baryon wavefunction as representation of 3-dimension permutation group:

$$\phi_c|\mathrm{SU}(6)\otimes\mathrm{O}(3)\rangle=\phi_c|\mathrm{N}_6,\ {}^{2S+1}\mathrm{N}_3,\ N,L,J\rangle$$

symmetric

Property of dimension-3 permutation group

Generally, there are four representations of S_3: total symmetric basis e^s, total antisymmetric e^a, and two mixed-symmetry bases e^λ and e^ρ, which are defined under permutation transformations:

$$P_{12}\begin{pmatrix} e^\lambda \\ e^\rho \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^\lambda \\ e^\rho \end{pmatrix}, \tag{2}$$

and

$$P_{13}\begin{pmatrix} e^\lambda \\ e^\rho \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} e^\lambda \\ e^\rho \end{pmatrix}, \tag{3}$$

where P_{12} and P_{13} are permutation operators for exchange of $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$, respectively.

e.g. The SU(2) spin wavefunction for a three-quark system

$$\begin{cases} \chi^s(S_z = \frac{3}{2}) = \uparrow\uparrow\uparrow \\ \chi^\rho(S_z = \frac{1}{2}) = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow) \\ \chi^\lambda(S_z = \frac{1}{2}) = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow - \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow) \end{cases}$$

Spin and flavor wavefunctions of S3 representations

Spin	$SU(2)$	$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_s + \mathbf{2}_\rho + \mathbf{2}_\lambda,$
Flavor	$SU(3)$	$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_s + \mathbf{8}_\rho + \mathbf{8}_\lambda + \mathbf{1}_a,$

Flavor wavefunctions:

The symmetric wavefunction is

$$\phi^s(\mathbf{10}) = \sum_{permu.} a(1)b(2)c(3) \ ,$$

where a, b and c represent flavor u, d or s.

The anti-symmetric singlet is:

$$\phi^a(\mathbf{1}) = \frac{1}{\sqrt{6}}(uds + dsu + sud - dus - usd - sdu) \ .$$

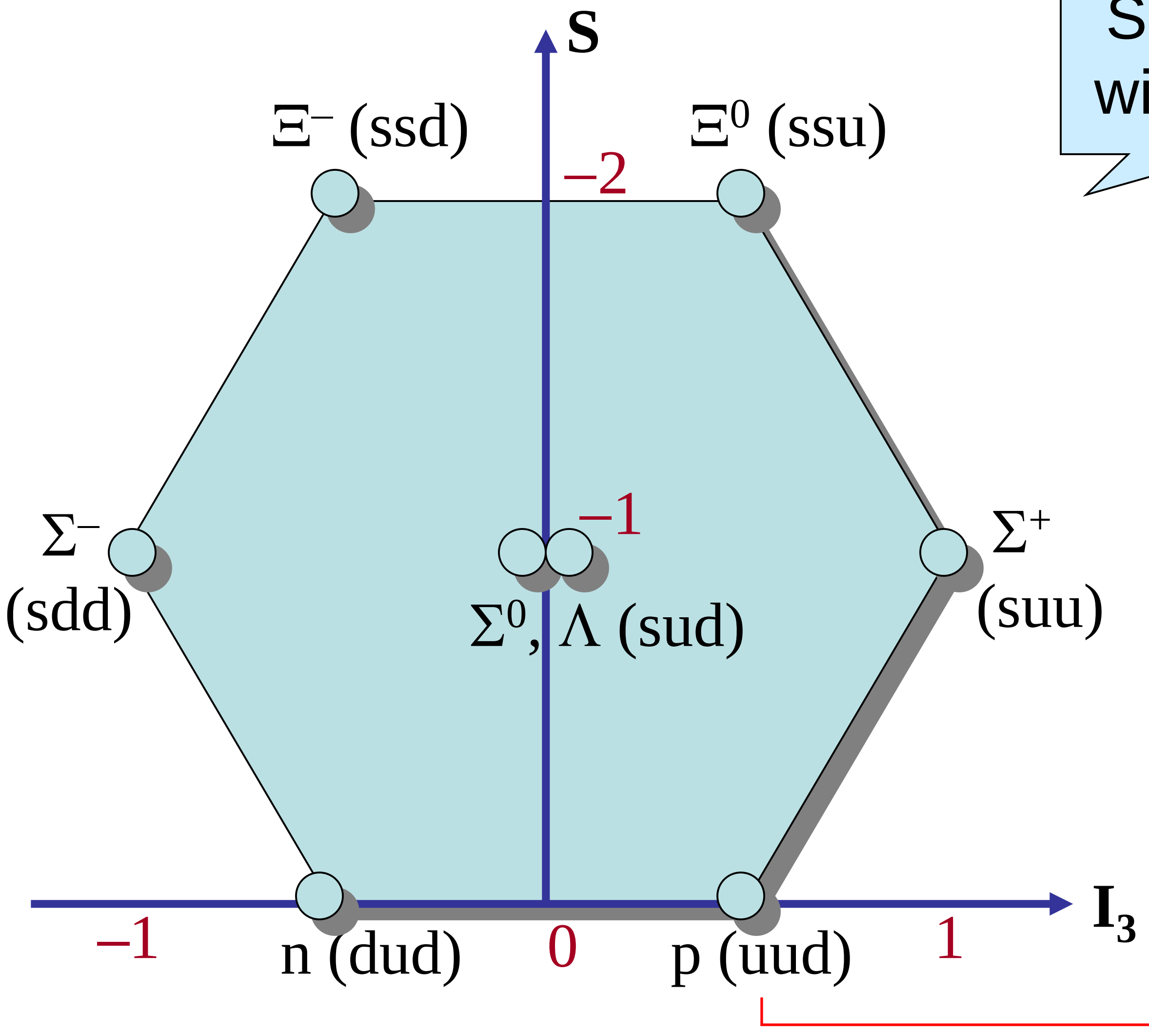
The mixed octet wavefunctions are

$$\phi^{\lambda}(\mathbf{8}) = \begin{cases} \frac{1}{\sqrt{6}}(2uud - duu - udu) & \text{for } p \\ \frac{1}{\sqrt{6}}(dud + udd - 2ddu) & \text{for } n \\ \frac{1}{\sqrt{6}}(2uus - suu - usu) & \text{for } \Sigma^+ \\ \frac{1}{2\sqrt{3}}(sdu + sud + usd + dsu - 2uds - 2dus) & \text{for } \Sigma^0 \\ \frac{1}{\sqrt{6}}(sdd + dsd - 2dds) & \text{for } \Sigma^- \\ \frac{1}{2}(sud + usd - sdu - dsu) & \text{for } \Lambda \\ \frac{1}{\sqrt{6}}(2ssu - sus - uss) & \text{for } \Xi^0 \\ \frac{1}{\sqrt{6}}(2ssd - sds - dss) & \text{for } \Xi^- \end{cases},$$

and

$$\phi^{\rho}(\mathbf{8}) = \begin{cases} \frac{1}{\sqrt{2}}(udu - duu) & \text{for } p \\ \frac{1}{\sqrt{2}}(udd - dud) & \text{for } n \\ \frac{1}{\sqrt{2}}(suu - usu) & \text{for } \Sigma^+ \\ \frac{1}{2}(sud + sdu - usd - dsu) & \text{for } \Sigma^0 \\ \frac{1}{\sqrt{2}}(sdd - dsd) & \text{for } \Sigma^- \\ \frac{1}{2\sqrt{3}}(usd + sdu - sud - dsu - 2dus + 2uds) & \text{for } \Lambda \\ \frac{1}{\sqrt{2}}(sus - uss) & \text{for } \Xi^0 \\ \frac{1}{\sqrt{2}}(sds - dss) & \text{for } \Xi^- \end{cases}.$$

SU(3) multiplets of baryons made of u, d, and s



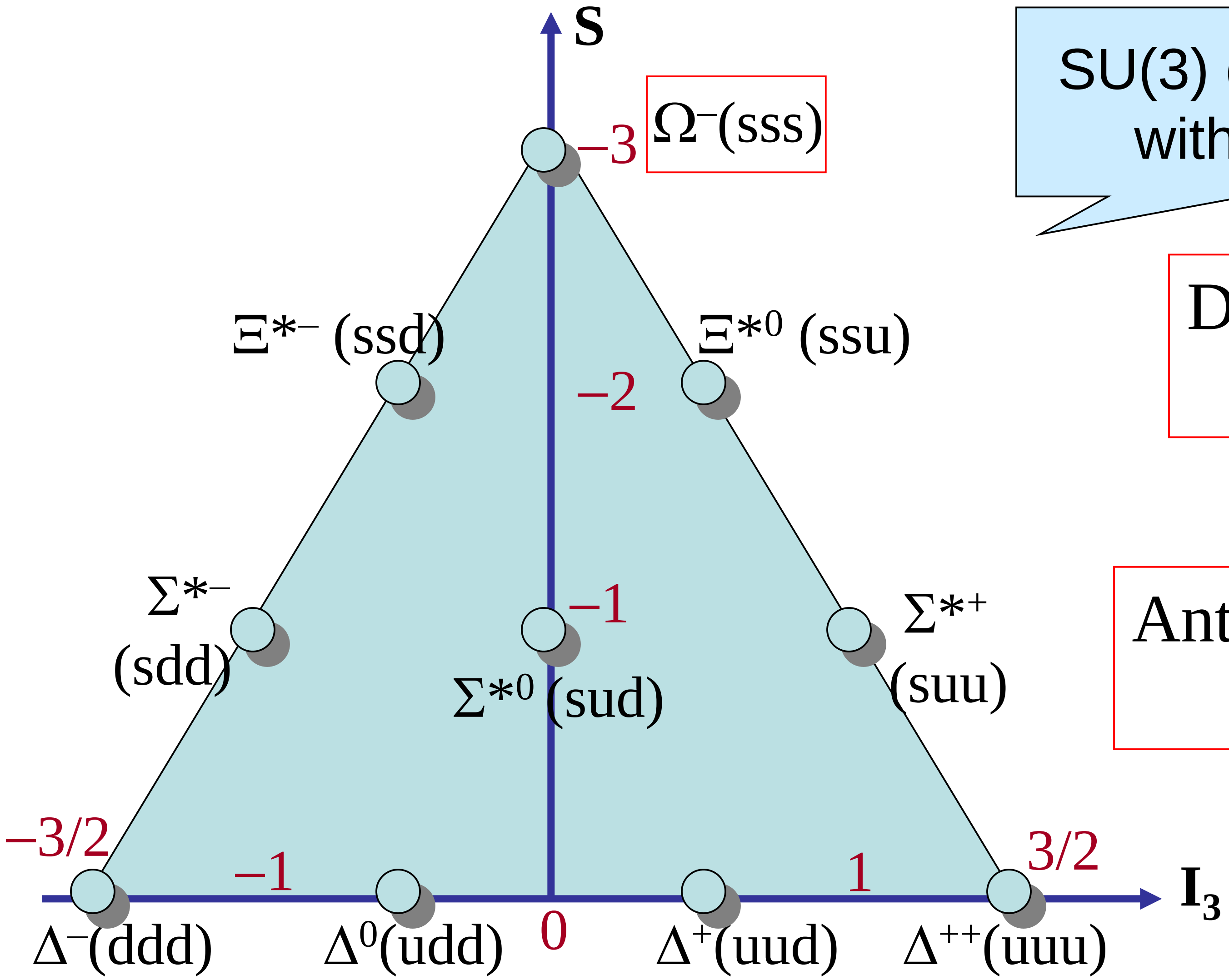
SU(3) octet with $J^P=1/2^+$

Gell-Mann - Nishijima:
$Q=I_3+Y/2=I_3+(B+S)/2$

$3\otimes3\otimes3$
$=(3^*\oplus 6)\otimes 3$
$=(1\oplus8)\oplus(8\oplus10)$

$$\frac{1}{\sqrt{2}}(udu - duu)$$
$$\frac{1}{\sqrt{6}}(2uud - duu - udu)$$

SU(3) multiplets of baryons made of u, d, and s



SU(3) decuplet 10
with $J^P=3/2^+$

Decuplet 10:
$\Omega^-(sss)$

⇩

Anti-decuplet $\overline{10}$:
$\overline{\Omega}^+(\overline{s}\overline{s}\overline{s})$

⇩

$\overline{3}\otimes\overline{3}\otimes\overline{3}$

SU(6) Spin-flavor wavefunctions of S3 representations

$$SU(6) \quad \mathbf{6} \otimes \mathbf{6} \otimes \mathbf{6} \quad = \mathbf{56}_s + \mathbf{70}_\rho + \mathbf{70}_\lambda + \mathbf{20}_a$$

The spin-flavor wavefunctions thus can be expressed as $|\mathbf{N}_6, {}^{2S+1}\mathbf{N}_3\rangle$, where $\mathbf{N}_6$ and $\mathbf{N}_3$ denote the SU(6) and SU(3) representation and S stands for the total spin of the wavefunction. More explicitly, we have the SU(6) wavefunctions:

$$\begin{aligned}
|\mathbf{56}, {}^2\mathbf{8}\rangle^s &= \frac{1}{\sqrt{2}}(\phi^\rho\chi^\rho + \phi^\lambda\chi^\lambda) , \\
|\mathbf{56}, {}^4\mathbf{10}\rangle^s &= \phi^s\chi^s , \\
|\mathbf{70}, {}^2\mathbf{8}\rangle^\rho &= \frac{1}{\sqrt{2}}(\phi^\rho\chi^\lambda + \phi^\lambda\chi^\rho) , \\
|\mathbf{70}, {}^4\mathbf{8}\rangle^\rho &= \phi^\rho\chi^s, \\
|\mathbf{70}, {}^2\mathbf{10}\rangle^\rho &= \phi^s\chi^\rho , \\
|\mathbf{70}, {}^2\mathbf{1}\rangle^\rho &= \phi^a\chi^\lambda ,
\end{aligned}$$

$$\begin{aligned}
|\mathbf{70}, {}^2\mathbf{8}\rangle^\lambda &= \frac{1}{\sqrt{2}}(\phi^\rho\chi^\rho - \phi^\lambda\chi^\lambda) , \\
|\mathbf{70}, {}^4\mathbf{8}\rangle^\lambda &= \phi^\lambda\chi^s , \\
|\mathbf{70}, {}^2\mathbf{10}\rangle^\lambda &= \phi^s\chi^\lambda , \\
|\mathbf{70}, {}^2\mathbf{1}\rangle^\lambda &= \phi^a\chi^\rho ,
\end{aligned}$$

$$\begin{aligned}
|\mathbf{20}, {}^2\mathbf{8}\rangle^a &= \frac{1}{\sqrt{2}}(\phi^\rho\chi^\lambda - \phi^\lambda\chi^\rho) , \\
|\mathbf{20}, {}^4\mathbf{1}\rangle^a &= \phi^a\chi^s .
\end{aligned}$$

Success of potential quark model:

Hadrons are made of quarks (antiquarks) as QCD color singlet

Typical quark model prescription (e.g. Godfrey Isgur model):

$$H\,|\Psi\rangle = (H_0 + V)\,|\Psi\rangle = E\,|\Psi\rangle$$

$$H_0 = (p^2 + m_1{}^2)^{1/2} + (p^2 + m_2{}^2)^{1/2} \quad \rightarrow \sum_{i=1}^{2}\left[m_i + \frac{p^2}{2m_i}\right]$$

$$V_{ij}(\mathbf{p},\mathbf{r}) \rightarrow H_{ij}^{\text{conf}} + H_{ij}^{\text{hyp}} + H_{ij}^{\text{so}} + H_A$$

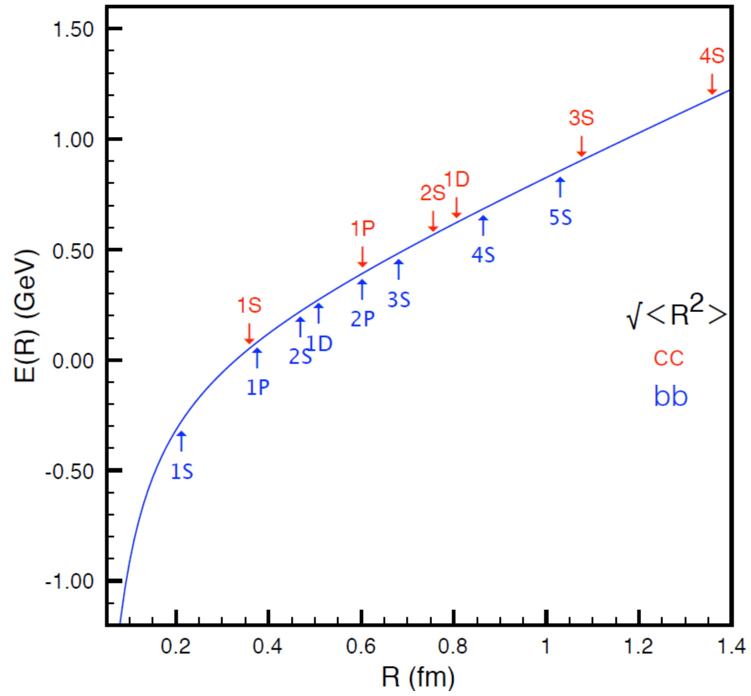
$$H_{ij}^{\text{conf}} = -\left[\tfrac{3}{4}c + \tfrac{3}{4}br - \frac{\alpha_s(r)}{r}\right]\mathbf{F}_i\cdot\mathbf{F}_j$$

$$H_{ij}^{\text{hyp}} = -\frac{\alpha_s(r)}{m_i m_j}\left[\frac{8\pi}{3}\mathbf{S}_i\cdot\mathbf{S}_j\delta^3(\mathbf{r}) + \frac{1}{r^3}\left[\frac{3\mathbf{S}_i\cdot\mathbf{r}\mathbf{S}_j\cdot\mathbf{r}}{r^2} - \mathbf{S}_i\cdot\mathbf{S}_j\right]\right]\mathbf{F}_i\cdot\mathbf{F}_j$$

$$H_{ij}^{\text{so}} = H_{ij}^{\text{so(cm)}} + H_{ij}^{\text{so(tp)}}$$

$$H_{ij}^{\text{so(cm)}} = -\frac{\alpha_s(r)}{r^3}\left[\frac{1}{m_i} + \frac{1}{m_j}\right]\left[\frac{\mathbf{S}_i}{m_i} + \frac{\mathbf{S}_j}{m_j}\right]\cdot\mathbf{L}\,(\mathbf{F}_i\cdot\mathbf{F}_j) \qquad H_{ij}^{\text{so(tp)}} = \frac{-1}{2r}\frac{\partial H_{ij}^{\text{conf}}}{\partial r}\left[\frac{\mathbf{S}_i}{m_i{}^2} + \frac{\mathbf{S}_j}{m_j{}^2}\right]\cdot\mathbf{L}$$

$$\langle\,\mathbf{F}_i\cdot\mathbf{F}_j\,\rangle = \begin{cases} -\frac{4}{3} & \text{in a meson} \\ -\frac{2}{3} & \text{in a baryon} \end{cases}$$



The spin-independent potential for quark-quark interactions

Isgur and Karl [2] first systematically studied the baryon spectroscopy by solving the Schrödinger equation for the quarks in a baryon system by expressing the Hamiltonian as

$$\hat{H} = \sum_i \left(m_i + \frac{\mathrm{p}_i^2}{2m_i} \right) + \sum_{i<j} (V_{conf}^{ij} + \hat{H}_{hyp}^{ij}) \ , \tag{2}$$

where V_{conf}^{ij} is the potential for confinement, and has the following expression:

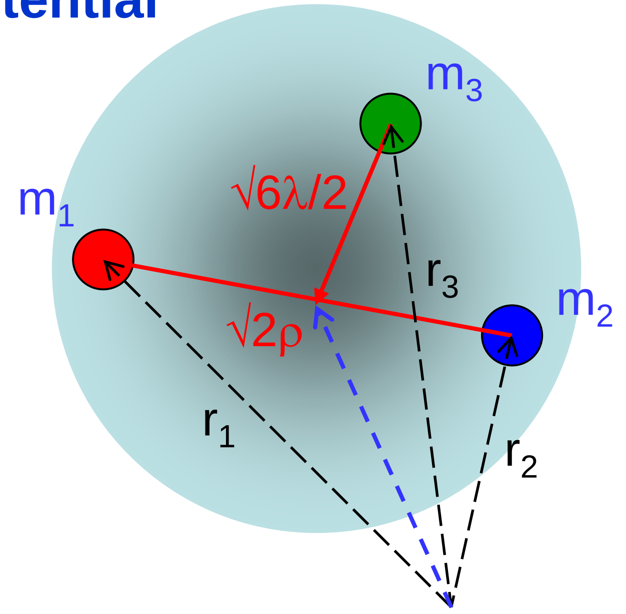
$$V_{conf}^{ij} = C_{qqq} + \frac{1}{2} b r_{ij} - \frac{2}{3} \frac{\alpha_s}{r_{ij}} \ . \tag{3}$$

In the harmonic oscillator basis, V_{conf}^{ij} can be expressed as

$$V_{conf}^{ij} \equiv \frac{1}{2} \beta r_{ij}^2 + U_{ij} \ , \tag{4}$$

where U_{ij} is treated as an anharmonic perturbation. Leaving the spin-dependent term $\hat{H}_{hyp}^{ij}$ to be discussed later, one can solve the eigenstates of the harmonic oscillator potential, and minimize the anharmonic part by choosing a proper value for β.

Spatial wavefunction in a spin-independent potential



Jacobi coordinate

Hamiltonian $H = H_{si} + H_{sd}$

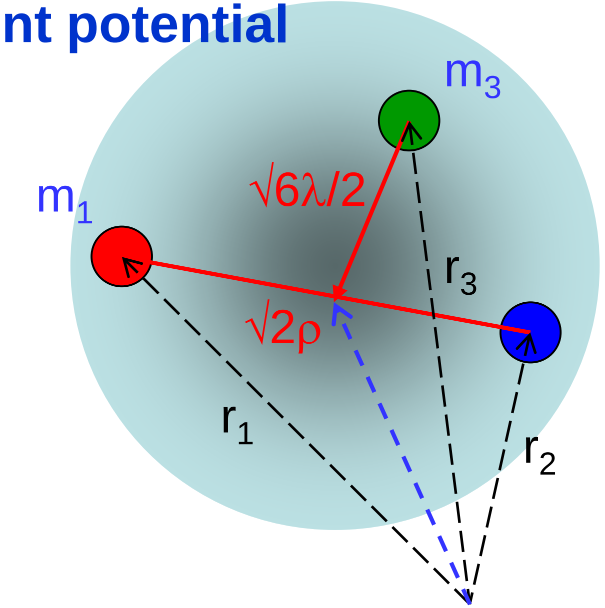
$$
\begin{aligned}
H_{si} &= \sum_i \left(m_i + \frac{\mathbf{p}_i^2}{2m_i} \right) + \sum_{i<j} \left(\frac{1}{2} b r_{ij} + c - \frac{2\alpha_s}{3r_{ij}} \right) \\
&\equiv \sum_i \left(m_i + \frac{\mathbf{p}_i^2}{2m_i} \right) + \sum_{i<j} \left(\frac{1}{2} k r_{ij}^2 + U(r_{ij}) \right) \\
&\equiv H_0 + \sum_{i<j} U(r_{ij})
\end{aligned}
$$

With an equal mass for u, d, and s quark, and $k = m_q \omega_h^2/3$, the Hamiltonian can be expressed as

$$
\hat{H} = \sum_{i=1}^{3} \frac{\mathbf{p}_i^2}{2m_q} + \frac{1}{6} m_q \omega_h^2 \sum_{i<j} (\mathbf{r}_j - \mathbf{r}_i)^2
$$

Isgur, and Karl, PLB109, 72(1977); PRD18, 4187(1978); PRD19, 2653(1979)

Spatial wavefunction in a spin-independent potential



Jacobi coordinate

$$\begin{cases} \mathbf{R} &= \frac{1}{3}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) \\ \boldsymbol{\rho} &= \frac{1}{\sqrt{2}}(\mathbf{r}_1 - \mathbf{r}_2) \\ \boldsymbol{\lambda} &= \frac{1}{\sqrt{6}}(\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3) \end{cases}$$

$$\begin{cases} \mathbf{P}_R &= \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \\ \mathbf{p}_\rho &= \frac{1}{\sqrt{2}}(\mathbf{p}_1 - \mathbf{p}_2) \\ \mathbf{p}_\lambda &= \frac{1}{\sqrt{6}}(\mathbf{p}_1 + \mathbf{p}_2 - 2\mathbf{p}_3) \end{cases}$$

The eigenstates for the above Hamitonian can be expressed as

$$\psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{(2\pi)^{3/2}} e^{i\mathbf{P}_R \cdot \mathbf{R}} \boxed{\psi^\sigma_{NLL_z}(\boldsymbol{\rho}, \boldsymbol{\lambda})}, \qquad \sigma = s, \rho, \lambda, a$$

Harmonic oscillator wavefunction

G. Karl and E. Obryk, Nucl. Phys. B 8, 609 (1968).

Taking the notation of Karl and Obryk, the harmonic oscillator wavefunction can be written as

$$\psi^{\sigma}_{NLL_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) = P^{\sigma}_{NLL_z}\frac{\alpha_h^3}{\pi^{3/2}}e^{-\alpha_h^2(\boldsymbol{\rho}^2+\boldsymbol{\lambda}^2)/2} ,$$

where $\alpha_h \equiv m\omega_h$, and $P^{\sigma}_{NLL_z}$ is a polynomial of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$. In this way, the symmetry charactor of the wavefunctions is highlighted.

Explicitly, the states with $N \leq 2$ can be obtained:

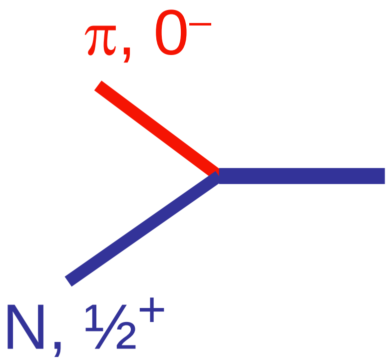
$$\begin{aligned}
N = 0,\quad L = 0,\quad & P^{s}_{000} = 1,\\
N = 1,\quad L = 1,\quad & P^{\rho}_{111} = \alpha_h\rho_+,\\
& P^{\lambda}_{111} = \alpha_h\lambda_+,\\
N = 2,\quad L = 0,\quad & P^{s}_{200} = \frac{\alpha_h^2}{\sqrt{3}}(\boldsymbol{\rho}^2+\boldsymbol{\lambda}^2-3\alpha_h^{-2}),\\
& P^{\rho}_{200} = \frac{\alpha_h^2}{\sqrt{3}}2\boldsymbol{\rho}\cdot\boldsymbol{\lambda},\\
& P^{\lambda}_{200} = \frac{\alpha_h^2}{\sqrt{3}}(\boldsymbol{\rho}^2-\boldsymbol{\lambda}^2),\\
L = 2,\quad & P^{s}_{222} = \frac{1}{2}\alpha_h^2(\rho_+^2+\lambda_+^2),\\
& P^{\rho}_{222} = \alpha_h^2\rho_+\lambda_+,
\end{aligned}$$

$$\begin{aligned}
& P^{\lambda}_{222} = \frac{1}{2}\alpha_h^2(\rho_+^2-\lambda_+^2),\\
L = 1,\quad & P^{a}_{211} = \alpha_h^2(\rho_+\lambda_z-\lambda_+\rho_z).
\end{aligned}$$

Total wavefunction of SU(6)⊗O(3) symmetry

$$\phi_c|\mathrm{SU}(6)\otimes \mathrm{O}(3)\rangle = \phi_c|\mathbf{N}_6,\ {}^{2S+1}\mathbf{N}_3,\ N, L, J\rangle$$

Naming convention for baryon:



N*, Δ*, $\mathbf{L_{2I,2J}}$

- $P_{33}(1232)$ Δ
- $P_{11}(1440)$
- $S_{11}(1535)$
- $D_{13}(1520)$
- …

i) Ground state baryons with $N=0$, $L=0$, $L^P=0^+$.

Nucleon (p and n) (****):

$$|\mathbf{56},\ {}^2\mathbf{8},\ 0,0,\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\phi^\rho\chi^\rho + \phi^\lambda\chi^\lambda)\psi^s_{000}(\boldsymbol{\rho},\boldsymbol{\lambda})\ .$$

$P_{33}(1232)$ $(\Delta^{++},\Delta^{+},\Delta^{0},\Delta^{-})$ (****):

$$|\mathbf{56},\ {}^4\mathbf{10},\ 0,0,\frac{3}{2}\rangle = \phi^s\chi^s\psi^s_{000}(\boldsymbol{\rho},\boldsymbol{\lambda})\ .$$

Isgur, and Karl, PLB109, 72(1977); PRD18, 4187(1978); PRD19, 2653(1979)

ii) The first excited states with $N=1$, $L=1$, $L^P=1^-$. Since the spatial wavefunction with $L=1$ cannot be symmetrized, it implies that only the spatial wavefunction with mixed symmetries can contribute, i.e. $\mathbf{N}_6 = \mathbf{70}$.

- $S_{11}(1535)$ (****), $D_{13}(1520)$ (****):

$$|\mathbf{70},\ {}^2\mathbf{8},\ 1,1,J\rangle = \sum_{L_z+S_z=J_z} \langle 1L_z, \frac{1}{2}S_z|JJ_z\rangle \times\frac{1}{2}[(\phi^\rho\chi^\lambda_{S_z} + \phi^\lambda\chi^\rho_{S_z})\psi^\rho_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) + (\phi^\rho\chi^\rho_{S_z} - \phi^\lambda\chi^\lambda_{S_z})\psi^\lambda_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})] \ . \qquad (21)$$

- $S_{11}(1650)$ (****), $D_{13}(1700)$ (***), $D_{15}(1675)$ (****):

$$|\mathbf{70},\ {}^4\mathbf{8},\ 1,1,J\rangle = \sum_{L_z+S_z=J_z} \langle 1L_z, \frac{3}{2}S_z|JJ_z\rangle \times\frac{1}{\sqrt{2}}[\phi^\rho\chi^s_{S_z}\psi^\rho_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) + \phi^\lambda\chi^s_{S_z}\psi^\lambda_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})] \ .$$

- $S_{31}(1620)$ (****), $D_{33}(1670)$ (****):

$$\begin{aligned}
& |\mathbf{70},\ {}^{\mathbf{2}}\mathbf{10},\ 1,1,J\rangle \\
= & \sum_{L_z+S_z=J_z} \langle 1L_z, \frac{1}{2}S_z|JJ_z\rangle \\
& \times\frac{1}{\sqrt{2}}[\phi^s\chi^\rho_{S_z}\psi^\rho_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) + \phi^s\chi^\lambda_{S_z}\psi^\lambda_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})]_J \ . \qquad (23)
\end{aligned}$$

- States with anti-symmetric flavor wavefunction (only for Λ baryons):

$$\begin{aligned}
& |\mathbf{70},\ {}^{\mathbf{2}}\mathbf{1},\ 1,1,J\rangle \\
= & \sum_{L_z+S_z=J_z} \langle 1L_z, \frac{1}{2}S_z|JJ_z\rangle \\
& \times\frac{1}{\sqrt{2}}[\phi^a\chi^\lambda_{S_z}\psi^\rho_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) - \phi^a\chi^\rho_{S_z}\psi^\lambda_{11L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})]_J \ . \qquad (24)
\end{aligned}$$

iii) Higher excited states with $N = 2$.

Since the spatial wavefunction can be symmetric, anti-symmetric or mixed symmetric, the SU(6)⊗O(3) representation can be **56**, **70**, and **20**.

A) For $\mathbf{N_6} = \mathbf{56}$:

- $L = 2$:

$P_{13}(1720)$ (****), $F_{15}(1680)$ (****)

$$|\mathbf{56},\ {}^2\mathbf{8},\ 2,2,J\rangle = \sum_{L_z+S_z=J_z} \langle 2L_z, \frac{1}{2}S_z|JJ_z\rangle \times\frac{1}{\sqrt{2}}(\phi^\rho\chi^\rho_{S_z} + \phi^\lambda\chi^\lambda_{S_z})\psi^s_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})\ . \tag{27}$$

$P_{31}(1910)$ (****), $P_{33}(1920)$ (***), $F_{35}(1905)$ (****), $F_{37}(1950)$ (****)

$$|\mathbf{56},\ {}^4\mathbf{10},\ 2,2,J\rangle = \sum_{L_z+S_z=J_z} \langle 2L_z, \frac{3}{2}S_z|JJ_z\rangle\phi^s\chi^s_{S_z}\psi^s_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})\ . \tag{28}$$

B) For $\mathbf{N}_6 = \mathbf{70}$:

- Radial excitation with $L = 0$:

  $P_{11}(1710)$ (***)

$$|\mathbf{70},\ {}^2\mathbf{8},\ 2, 0, \frac{1}{2}\rangle \ = \ \frac{1}{2}[(\phi^\rho \chi^\lambda_{S_z} + \phi^\lambda \chi^\rho_{S_z})\psi^\rho_{200}(\boldsymbol{\rho}, \boldsymbol{\lambda}) + (\phi^\rho \chi^\rho_{S_z} - \phi^\lambda \chi^\lambda_{S_z})\psi^\lambda_{200}(\boldsymbol{\rho}, \boldsymbol{\lambda})] \ .$$

  $P_{13}(1900)$ (**)

$$|\mathbf{70},\ {}^4\mathbf{8},\ 2, 0, \frac{3}{2}\rangle = \frac{1}{\sqrt{2}}[\phi^\rho \chi^s_{S_z} \psi^\rho_{200}(\boldsymbol{\rho}, \boldsymbol{\lambda}) + \phi^\lambda \chi^s_{S_z} \psi^\lambda_{200}(\boldsymbol{\rho}, \boldsymbol{\lambda})] \ .$$

  $P_{31}(1750)$ (*)

$$|\mathbf{70},\ {}^2\mathbf{10},\ 2, 0, \frac{1}{2}\rangle = \frac{1}{\sqrt{2}}[\phi^s \chi^\rho_{S_z} \psi^\rho_{200}(\boldsymbol{\rho}, \boldsymbol{\lambda}) + \phi^s \chi^\lambda_{S_z} \psi^\lambda_{200}(\boldsymbol{\rho}, \boldsymbol{\lambda})]_J \ .$$

P_{01} with anti-symmetric flavor wavefunction (only for Λ baryons)

$$
\begin{aligned}
&|\mathbf{70},\ {}^2\mathbf{1},\ 2,0,\frac{1}{2}\rangle \\
= \ & \frac{1}{\sqrt{2}}[\phi^a\chi^\lambda_{S_z}\psi^\rho_{200}(\boldsymbol{\rho},\boldsymbol{\lambda}) - \phi^a\chi^\rho_{S_z}\psi^\lambda_{200}(\boldsymbol{\rho},\boldsymbol{\lambda})]_J \ .
\end{aligned}
$$

- $L=2$

$P_{13}(1900)$ (**), $F_{15}(2000)$ (**)

$$|\mathbf{70},\ ^2\mathbf{8},\ 2,2,J\rangle = \sum_{L_z+S_z=J_z}\langle 2L_z,\frac{1}{2}S_z|JJ_z\rangle \times\frac{1}{2}[(\phi^\rho\chi^\lambda_{S_z}+\phi^\lambda\chi^\rho_{S_z})\psi^\rho_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) +(\phi^\rho\chi^\rho_{S_z}-\phi^\lambda\chi^\lambda_{S_z})\psi^\lambda_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})]\ .$$

$P_{11}(2100)$ (*), $P_{13}(1900)$ (**), $F_{15}(2000)$ (**), $F_{17}(1990)$ (**)

$$|\mathbf{70},\ ^4\mathbf{8},\ 2,2,J\rangle = \sum_{L_z+S_z=J_z}\langle 2L_z,\frac{3}{2}S_z|JJ_z\rangle \times\frac{1}{\sqrt{2}}[\phi^\rho\chi^s_{S_z}\psi^\rho_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})+\phi^\lambda\chi^s_{S_z}\psi^\lambda_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})]\ .$$

$P_{33}(1985)$ (missing), $F_{35}(2000)$ (**)

$$|\mathbf{70},\ ^2\mathbf{10},\ 2,2,J\rangle = \sum_{L_z+S_z=J_z}\langle 2L_z,\frac{1}{2}S_z|JJ_z\rangle \times\frac{1}{\sqrt{2}}[\phi^s\chi^\rho_{S_z}\psi^\rho_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})+\phi^s\chi^\lambda_{S_z}\psi^\lambda_{22L_z}(\boldsymbol{\rho},\boldsymbol{\lambda})]_J\ .$$

C) For $\mathbf{N}_6 = \mathbf{20}$, it must require $L = 1$. So far no experimental evidence for the existence of such resonances was found. For non-strange resonance, we have

$$\begin{aligned} & |\mathbf{20},\ {}^2\mathbf{8},\ 2,1,J\rangle \\ = & \sum_{L_z+S_z=J_z} \langle 1L_z, \frac{1}{2}S_z|JJ_z\rangle \\ & \times\frac{1}{\sqrt{2}}(\phi^\rho\chi^\lambda_{S_z} - \phi^\lambda\chi^\rho_{S_z})\psi^a_{21L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) \ , \end{aligned} \tag{36}$$

and for Λ baryon with anti-symmetric flavor wavefunction, we have

$$|\mathbf{20},\ {}^4\mathbf{1},\ 2,1,J\rangle = \sum_{L_z+S_z=J_z} \langle 1L_z, \frac{3}{2}S_z|JJ_z\rangle\chi^s_{S_z}\phi^a\psi^a_{21L_z}(\boldsymbol{\rho},\boldsymbol{\lambda}) \ . \tag{37}$$

Anharmonic and spin-dependent potential

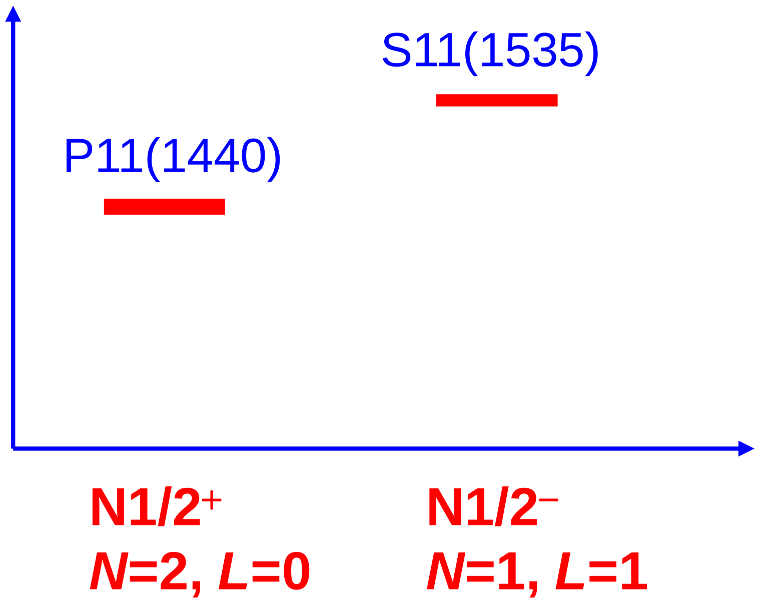
Based on the SU(6)⊗O(3) symmetry, the quark model succeeded in the classification of the baryon spectrum. But quantitative results could not be expected since more elaborate details about the dynamics were needed.

For instance, the Roper resonance P11(1440) was assigned to the radial excitation state with N = 2, and L = 0, which was found to have lower mass than the first orbital excition multiplets S11(1535) etc with N = 1 and L = 1. The inclusion of the anharmonic and spin-dependent quark potential turned to be necessary.

$$\mathbf{L} = \boldsymbol{l}_\rho + \boldsymbol{l}_\lambda$$

$$N = 2(n_\rho + n_\lambda) + l_\rho + l_\lambda$$

$$\boxed{E = (N+3)\omega}$$



n **is the radial quantum number, and** ***L*** **is the orbital angular momentum.**

The spin-dependent effects are introduced through the hyperfine interaction for the light quark system:

$$\hat{H}_{hyp} = \sum_{i<j} \frac{2\alpha_s(r_{ij})}{3m_i m_j} \left[\frac{8\pi}{3} \mathbf{S}_i \cdot \mathbf{S}_j \delta^3(\mathbf{r}_{ij}) + \frac{1}{r_{ij}^3} \left(\frac{3\mathbf{S}_i \cdot \mathbf{r}_{ij} \mathbf{S}_j \cdot \mathbf{r}_{ij}}{r_{ij}^2} - \mathbf{S}_i \cdot \mathbf{S}_j \right) \right] , \tag{38}$$

where $\mathbf{S}_i$ is the spin operator for the i-th quark.

For the ground state baryons, the hyperfine interaction could be completely isolated since both the ρ and λ type orbital angular momentum operators would annihilate the ground states. The hyperfine interaction can only contribute via the contact term of $\mathbf{S}_i \cdot \mathbf{S}_j$. Notice that

$$\langle \psi^s_{000} | \delta^3(\mathbf{r}_{ij}) | \psi^s_{000} \rangle = \frac{\alpha_h^3}{(2\pi)^{3/2}} , \tag{39}$$

and

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = \begin{cases} +1/4 & \text{if } i \text{ and } j \text{ have spin 1} \\ -3/4 & \text{if } i \text{ and } j \text{ have spin 0} \end{cases} , \tag{40}$$

the ground state diagonal splittings can be then obtained:

$$\delta \equiv \frac{4\alpha_s \alpha_h^3}{3\sqrt{2}\pi m_q^2} r_{hyp} , \tag{41}$$

where r_{hyp} is a factor introduced as a relativistic correction for the naive hyperfine interaction. Taking the approximation, $m_u = m_d$ and $x \equiv m_u/m_s \approx 2/3$, one obtains the following relation for the ground state baryons:

$$\begin{aligned}
\delta M_N &= -\frac{1}{2}\delta \,, \\
\delta M_\Delta &= +\frac{1}{2}\delta \,, \\
\delta M_\Sigma &= (\frac{1}{6} - \frac{2x}{3})\delta \,, \\
\delta M_{\Sigma^*} &= (\frac{x}{3} + \frac{1}{6})\delta \,, \\
\delta M_\Lambda &= -\frac{1}{2}\delta \,, \\
\delta M_\Xi &= (\frac{x^2}{6} - \frac{2x}{3})\delta \,, \\
\delta M_{\Xi^*} &= (\frac{x}{3} + \frac{x^2}{6})\delta \,, \\
\delta M_\Omega &= +\frac{x^2}{2}\delta \,.
\end{aligned} \tag{42}$$

An interesting feature can be immediately seen is that the hyperfine interaction leads to a heavier Δ mass than the nucleon. Nevertheless, the mass of Λ (uds, 1.115 GeV) is lowered by as much as the nucleon.

On the other hand, the total spin and total orbital angular momentum were no longer conserved due to the tensor interaction, which led to the configuration mixings among SU(6)⊗O(3) states. One thus had to diagonise a matrix for each of the spin-parity (physical) states, which were superposition of the SU(6)⊗O(3) states. Namely

$$|\Psi_B\rangle = \sum_i C_i |\Psi^i_{SU(6)\otimes O(3)}\rangle \ . \tag{43}$$

As an example, Isgur *et al* [5] showed that the ground state nucleon $P_{11}(938)$ ($J^P = \frac{1}{2}^+$) could be expressed as

$$\begin{aligned} |\Psi_N\rangle \;\; = \;\; & +0.91|\mathbf{56},{}^2\mathbf{8},0,0,\frac{1}{2}\rangle - 0.34|\mathbf{56},{}^2\mathbf{8},2,0,\frac{1}{2}\rangle \\ & -0.27|\mathbf{70},{}^2\mathbf{8},2,0,\frac{1}{2}\rangle - 0.06|\mathbf{70},{}^2\mathbf{8},2,2,\frac{1}{2}\rangle \ . \end{aligned} \tag{44}$$

PDG 2020

p	$1/2^+$	****
n	$1/2^+$	****
$N(1440)$	$1/2^+$	****
$N(1520)$	$3/2^-$	****
$N(1535)$	$1/2^-$	****
$N(1650)$	$1/2^-$	****
$N(1675)$	$5/2^-$	****
$N(1680)$	$5/2^+$	****
$N(1700)$	$3/2^-$	***
$N(1710)$	$1/2^+$	****
$N(1720)$	$3/2^+$	****
$N(1860)$	$5/2^+$	**
$N(1875)$	$3/2^-$	***
$N(1880)$	$1/2^+$	***
$N(1895)$	$1/2^-$	****
$N(1900)$	$3/2^+$	****
$N(1990)$	$7/2^+$	**
$N(2000)$	$5/2^+$	**
$N(2040)$	$3/2^+$	*
$N(2060)$	$5/2^-$	***
$N(2100)$	$1/2^+$	***
$N(2120)$	$3/2^-$	***
$N(2190)$	$7/2^-$	****
$N(2220)$	$9/2^+$	****
$N(2250)$	$9/2^-$	****
$N(2300)$	$1/2^+$	**
$N(2570)$	$5/2^-$	**
$N(2600)$	$11/2^-$	***
$N(2700)$	$13/2^+$	**

$\Delta(1232)$	$3/2^+$	****
$\Delta(1600)$	$3/2^+$	****
$\Delta(1620)$	$1/2^-$	****
$\Delta(1700)$	$3/2^-$	****
$\Delta(1750)$	$1/2^+$	*
$\Delta(1900)$	$1/2^-$	***
$\Delta(1905)$	$5/2^+$	****
$\Delta(1910)$	$1/2^+$	****
$\Delta(1920)$	$3/2^+$	***
$\Delta(1930)$	$5/2^-$	***
$\Delta(1940)$	$3/2^-$	**
$\Delta(1950)$	$7/2^+$	****
$\Delta(2000)$	$5/2^+$	**
$\Delta(2150)$	$1/2^-$	*
$\Delta(2200)$	$7/2^-$	***
$\Delta(2300)$	$9/2^+$	**
$\Delta(2350)$	$5/2^-$	*
$\Delta(2390)$	$7/2^+$	*
$\Delta(2400)$	$9/2^-$	**
$\Delta(2420)$	$11/2^+$	****
$\Delta(2750)$	$13/2^-$	**
$\Delta(2950)$	$15/2^+$	**
Λ	$1/2^+$	****
Λ	$1/2^-$	**
$\Lambda(1405)$	$1/2^-$	****
$\Lambda(1520)$	$3/2^-$	****
$\Lambda(1600)$	$1/2^+$	****
$\Lambda(1670)$	$1/2^-$	****
$\Lambda(1690)$	$3/2^-$	****
$\Lambda(1710)$	$1/2^+$	*
$\Lambda(1800)$	$1/2^-$	***
$\Lambda(1810)$	$1/2^+$	***
$\Lambda(1820)$	$5/2^+$	****
$\Lambda(1830)$	$5/2^-$	****
$\Lambda(1890)$	$3/2^+$	****
$\Lambda(2000)$	$1/2^-$	*
$\Lambda(2050)$	$3/2^-$	*
$\Lambda(2070)$	$3/2^+$	*
$\Lambda(2080)$	$5/2^-$	*
$\Lambda(2085)$	$7/2^+$	**
$\Lambda(2100)$	$7/2^-$	****
$\Lambda(2110)$	$5/2^+$	***
$\Lambda(2325)$	$3/2^-$	*
$\Lambda(2350)$	$9/2^+$	***
$\Lambda(2585)$		**

Σ^+	$1/2^+$	****
Σ^0	$1/2^+$	****
Σ^-	$1/2^+$	****
$\Sigma(1385)$	$3/2^+$	****
$\Sigma(1580)$	$3/2^-$	*
$\Sigma(1620)$	$1/2^-$	*
$\Sigma(1660)$	$1/2^+$	***
$\Sigma(1670)$	$3/2^-$	****
$\Sigma(1750)$	$1/2^-$	***
$\Sigma(1775)$	$5/2^-$	****
$\Sigma(1780)$	$3/2^+$	*
$\Sigma(1880)$	$1/2^+$	**
$\Sigma(1900)$	$1/2^-$	**
$\Sigma(1910)$	$3/2^-$	***
$\Sigma(1915)$	$5/2^+$	****
$\Sigma(1940)$	$3/2^+$	*
$\Sigma(2010)$	$3/2^-$	*
$\Sigma(2030)$	$7/2^+$	****
$\Sigma(2070)$	$5/2^+$	*
$\Sigma(2080)$	$3/2^+$	*
$\Sigma(2100)$	$7/2^-$	*
$\Sigma(2160)$	$1/2^-$	*
$\Sigma(2230)$	$3/2^+$	*
$\Sigma(2250)$		***
$\Sigma(2455)$		**
$\Sigma(2620)$		**
$\Sigma(3000)$		*
$\Sigma(3170)$		*

Ξ^0	$1/2^+$	****
Ξ^-	$1/2^+$	****
$\Xi(1530)$	$3/2^+$	****
$\Xi(1620)$		*
$\Xi(1690)$		***
$\Xi(1820)$	$3/2^-$	***
$\Xi(1950)$		***
$\Xi(2030)$	$\geq \frac{5}{2}?$	***
$\Xi(2120)$		*
$\Xi(2250)$		**
$\Xi(2370)$		**
$\Xi(2500)$		*
Ω^-	$3/2^+$	****
$\Omega(2012)^-$	$?^-$	***
$\Omega(2250)^-$		***
$\Omega(2380)^-$		**
$\Omega(2470)^-$		**
Λ_c^+	$1/2^+$	****
$\Lambda_c(2595)^+$	$1/2^-$	***
$\Lambda_c(2625)^+$	$3/2^-$	***
$\Lambda_c(2765)^+$		*
$\Lambda_c(2860)^+$	$3/2^+$	***
$\Lambda_c(2880)^+$	$5/2^+$	***
$\Lambda_c(2940)^+$	$3/2^-$	***
$\Sigma_c(2455)$	$1/2^+$	****
$\Sigma_c(2520)$	$3/2^+$	***
$\Sigma_c(2800)$		***
Ξ_c^+	$1/2^+$	***
Ξ_c^0	$1/2^+$	****
$\Xi_c'^+$	$1/2^+$	***
$\Xi_c'^0$	$1/2^+$	***
$\Xi_c(2645)$	$3/2^+$	***
$\Xi_c(2790)$	$1/2^-$	***
$\Xi_c(2815)$	$3/2^-$	***
$\Xi_c(2930)$		**
$\Xi_c(2970)$		***
$\Xi_c(3055)$		***
$\Xi_c(3080)$		***
$\Xi_c(3123)$		*
Ω_c^0	$1/2^+$	***
$\Omega_c(2770)^0$	$3/2^+$	***
$\Omega_c(3000)^0$		***
$\Omega_c(3050)^0$		***
$\Omega_c(3065)^0$		***
$\Omega_c(3090)^0$		***
$\Omega_c(3120)^0$		***

Ξ_{cc}^{++}		***
Λ_b^0	$1/2^+$	***
$\Lambda_b(5912)^0$	$1/2^-$	***
$\Lambda_b(5920)^0$	$3/2^-$	***
$\Lambda_b(6146)^0$	$3/2^+$	***
$\Lambda_b(6152)^0$	$5/2^+$	***
Σ_b	$1/2^+$	***
Σ_b^*	$3/2^+$	***
$\Sigma_b(6097)^+$		***
$\Sigma_b(6097)^-$		***
Ξ_b^0, Ξ_b^-	$1/2^+$	***
$\Xi_b'(5935)^-$	$1/2^+$	***
$\Xi_b(5945)^0$	$3/2^+$	***
$\Xi_b(5955)^-$	$3/2^+$	***
$\Xi_b(6227)$		***
Ω_b^-	$1/2^+$	***
$P_c(4312)^+$		*
$P_c(4380)^+$		*
$P_c(4440)^+$		*
$P_c(4457)^+$		*

- **QCD exotics and the search for exotic hadrons in various processes**

1. Hadrons beyond the conventional quark model

Exotics of Type-I:

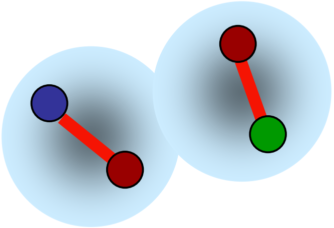
J^{PC} are not allowed by $Q\bar{Q}$ configurations, e.g. $0^{--}, 1^{-+}$ …

- Direct observation

Exotics of Type-II:

J^{PC} are the same as $Q\bar{Q}$ configurations

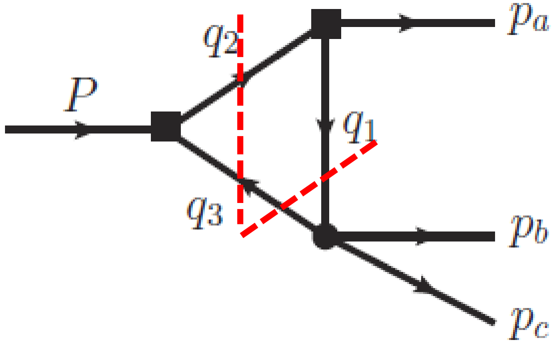
- Outnumbering of conventional QM states?
- Peculiar properties?



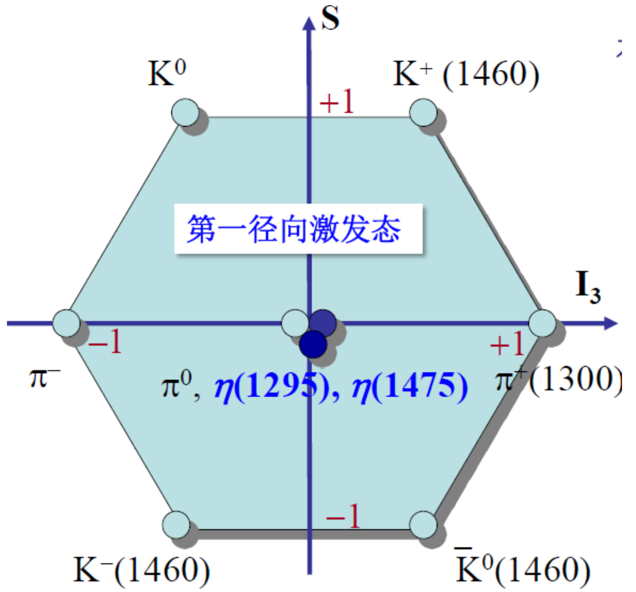
"Exotics" of Type-III:

Leading kinematic singularity can cause measurable effects, e.g. the triangle singularity.

- What's the impact?
- How to distinguish a genuine state from kinematic effects?



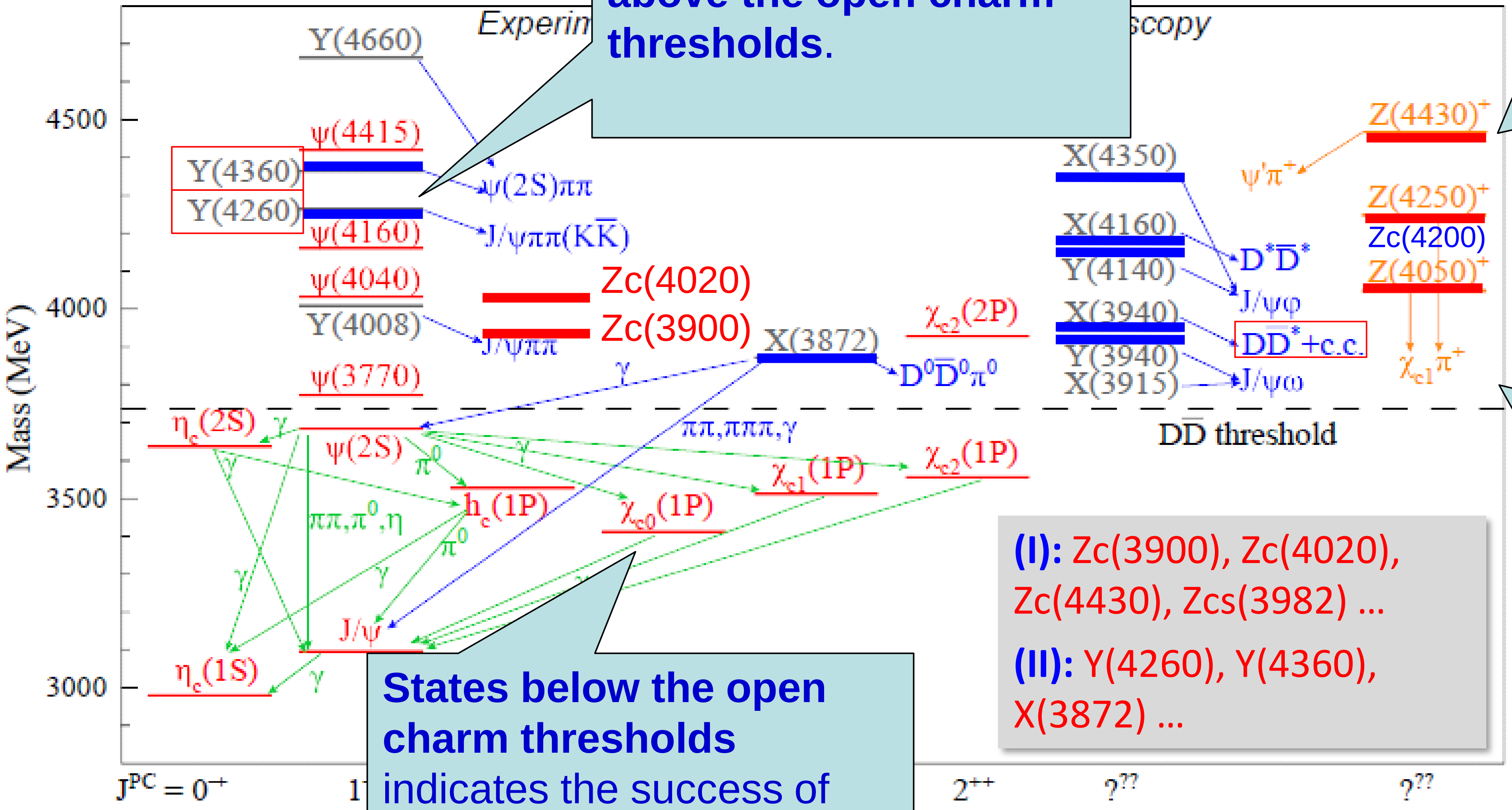
Additional states beyond the QM flavor symm. pattern imply “exotic” signals!



Light hadrons: $\bar{q}q$ SU(3) flavor nonet: $\bar{3}\otimes 3 = 1\oplus 8$

$n\,^{2s+1}\ell_J$	J^{PC}	I = 1 $u\bar{d}$, $\bar{u}d$, $\frac{1}{\sqrt{2}}(d\bar{d}-u\bar{u})$	I = $\frac{1}{2}$ $u\bar{s}$, $d\bar{s}$; $\bar{d}s$, $-\bar{u}s$	I = 0 f'	I = 0 f	$\theta_{\rm quad}$ [°]	$\theta_{\rm lin}$ [°]
$1\,^1S_0$	0^{-+}	π	K	η	$\eta'(958)$	−11.5	−24.6
$1\,^3S_1$	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	38.7	36.0
$1\,^1P_1$	1^{+-}	$b_1(1235)$	$K_{1B}{}^\dagger$	$h_1(1380)$	$h_1(1170)$		
$1\,^3P_0$	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$	$f_0(1500)$?	
$1\,^3P_1$	1^{++}	$a_1(1260)$	$K_{1A}{}^\dagger$	$f_1(1420)$	$f_1(1285)$	$a_1(1420)$?	
$2\,^1S_0$	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$	$\eta(1295)$	$\eta(1405)$?	
$2\,^3S_1$	1^{--}	$\rho(1450)$	$K^*(1410)$	$\phi(1680)$	$\omega(1420)$		

Heavy flavor states: Charmonia and charmonium-like states, i.e. X, Y, Z's.



Apparent deviations occur **above the open charm thresholds.**

Charged charmonium states definitely indicate novel phenomena beyond the potential QM.

Most of these new states are **close to some S-wave open thresholds.**

(I): Zc(3900), Zc(4020), Zc(4430), Zcs(3982) …

(II): Y(4260), Y(4360), X(3872) …

States below the open charm thresholds indicates the success of the potential quark model.

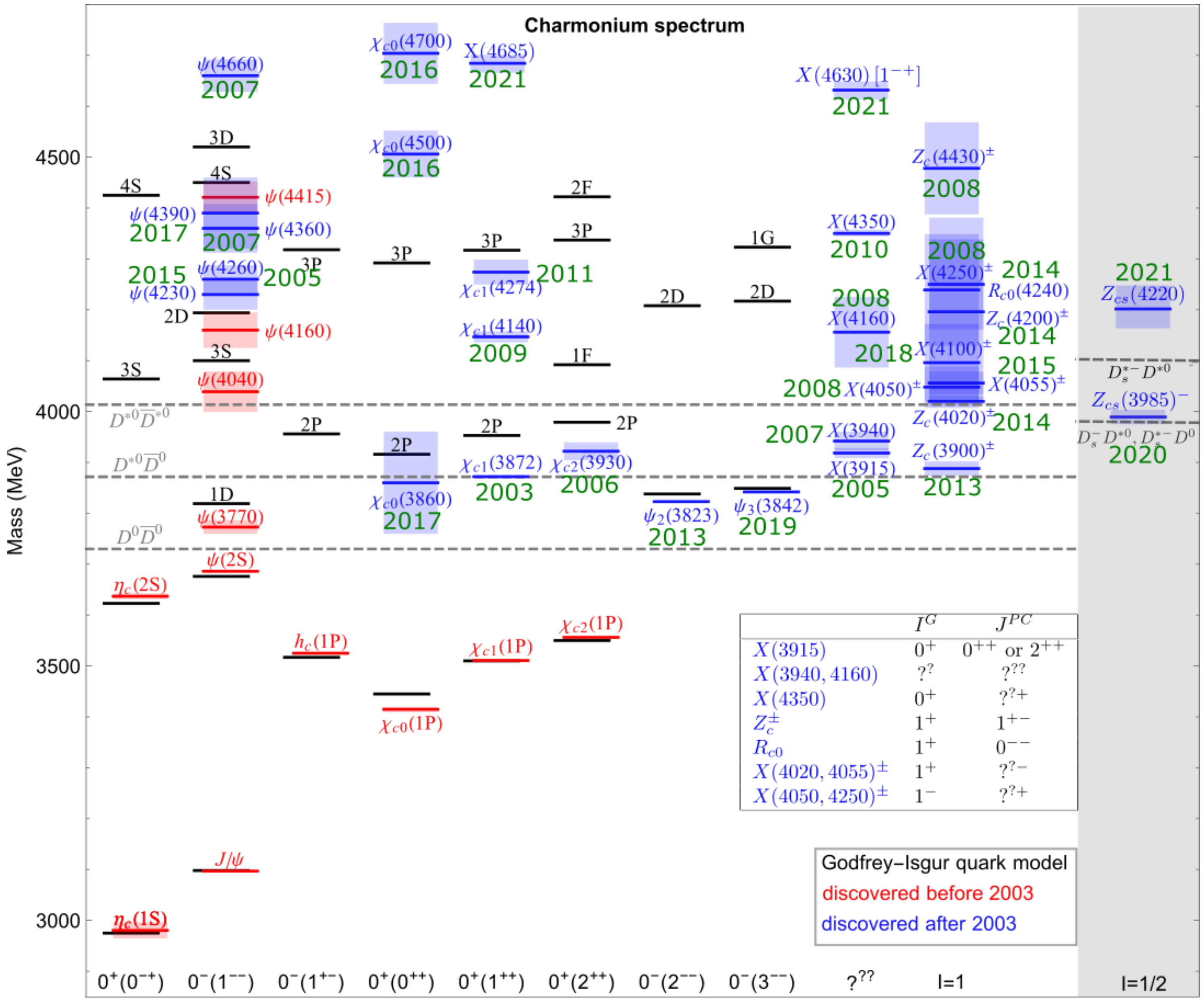
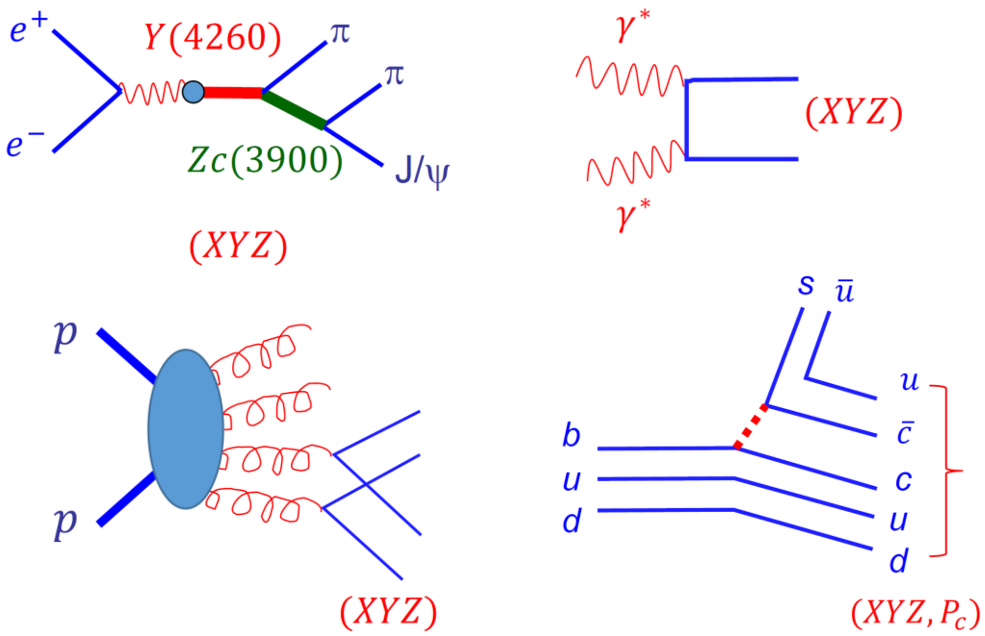
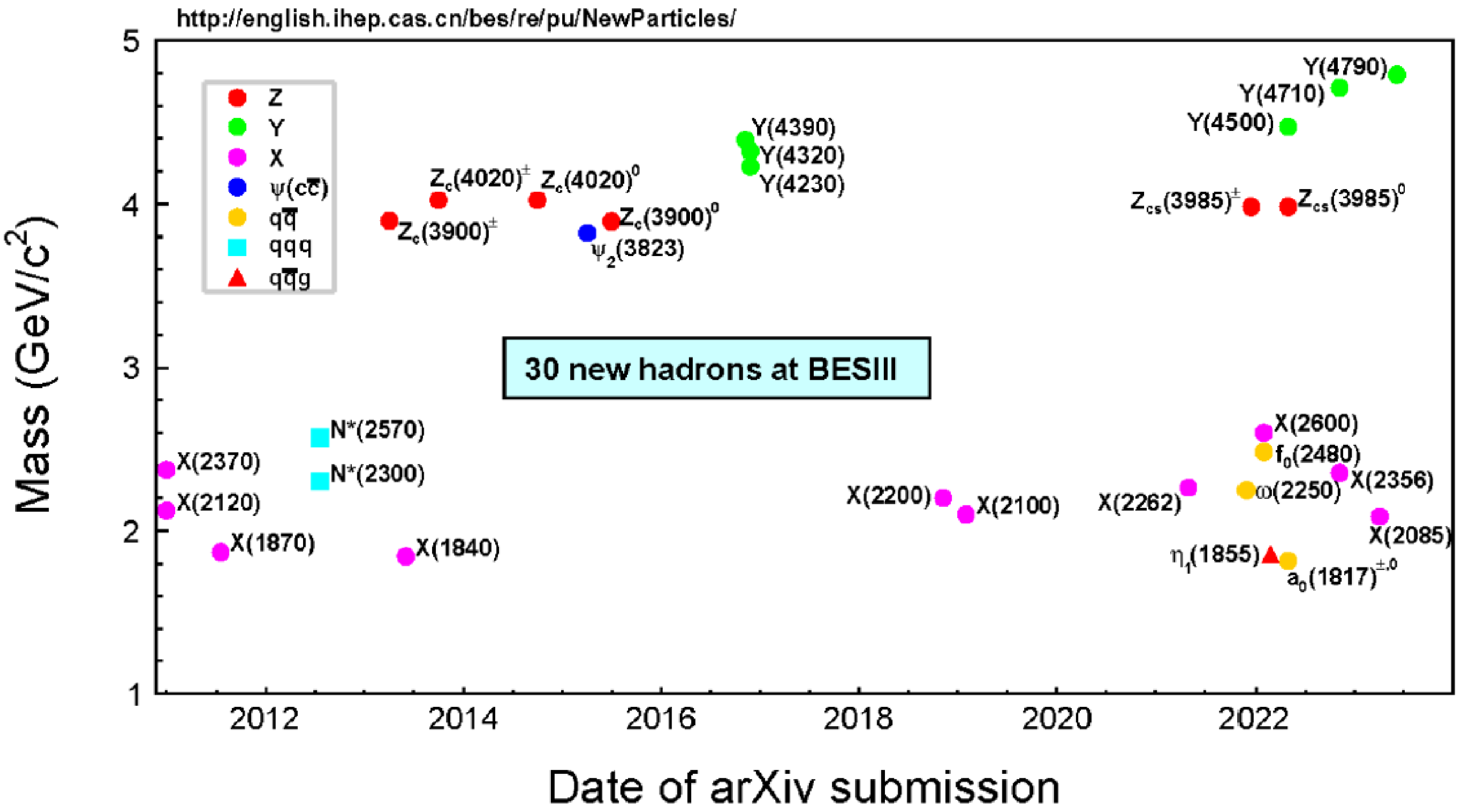


Chart plotted by Fengkun

Production processes in experiment

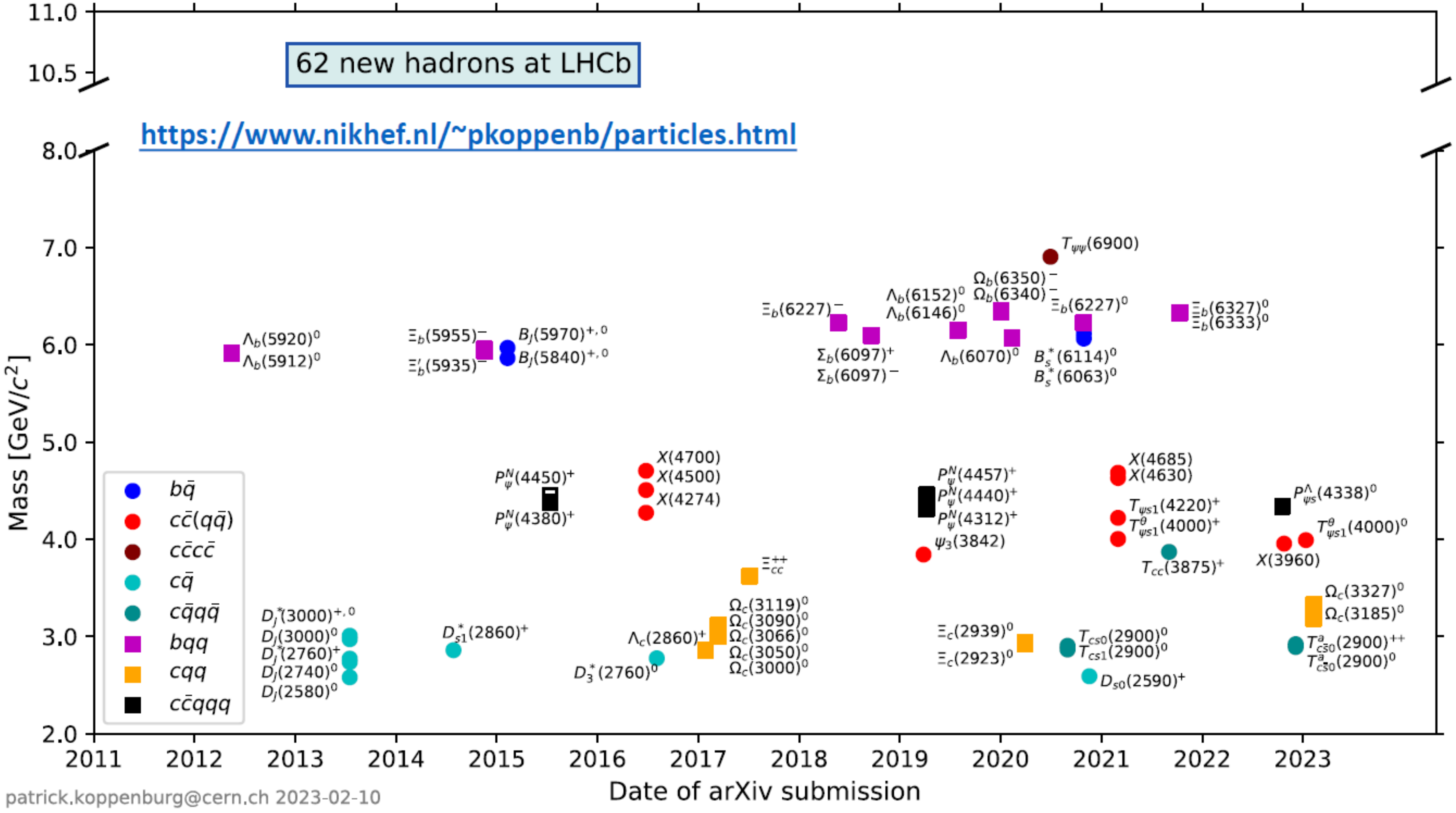


New Particles discovered at BESIII



2. Open threshold phenomena and near-threshold dynamics

Hadrons observed at LHCb

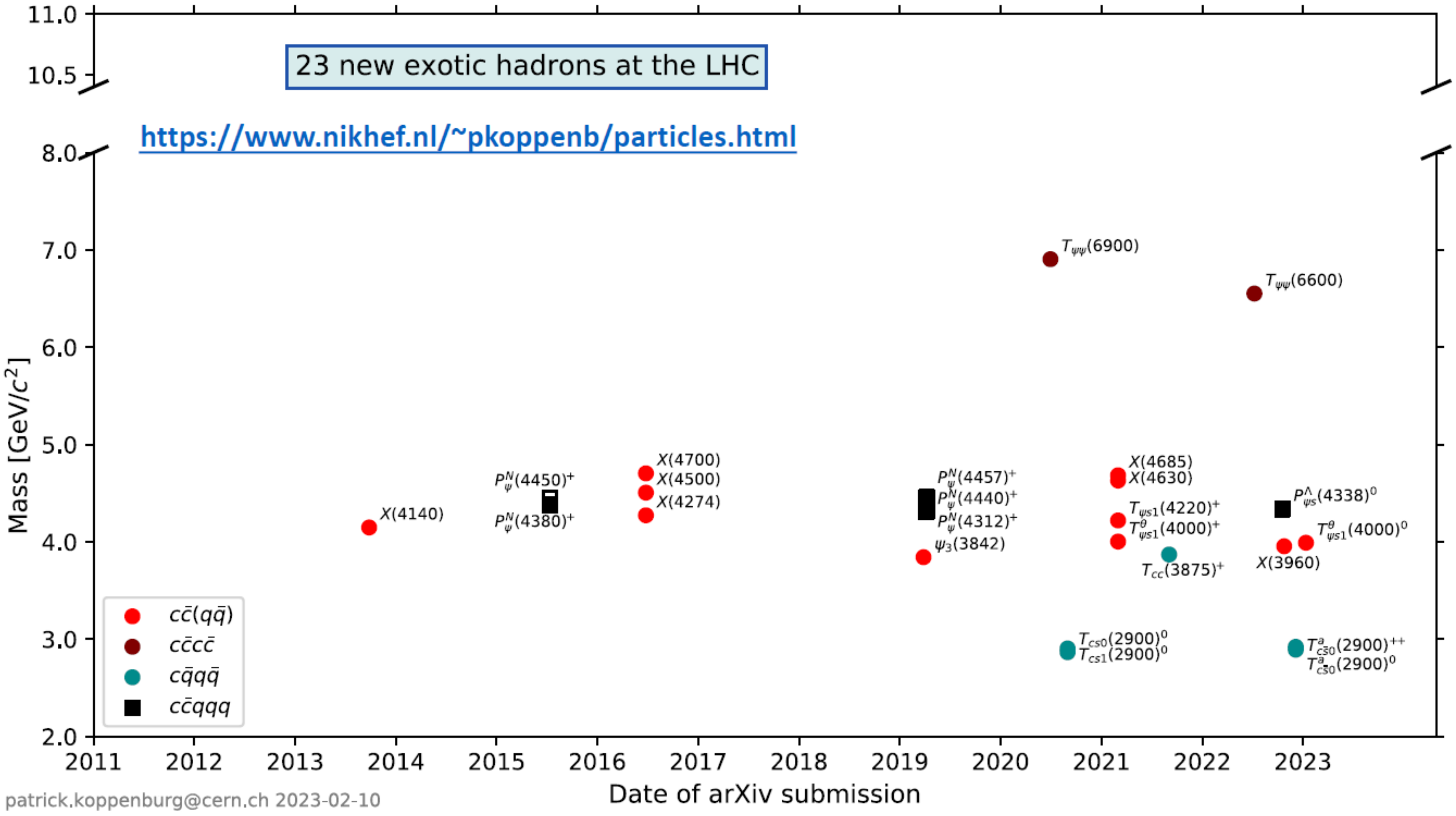


- Following "Exotic hadron naming convention" proposed by LHCb recently

[arXiv: 2206.15233]

Slide from Liu-Pan An

Exotic hadrons observed at LHC



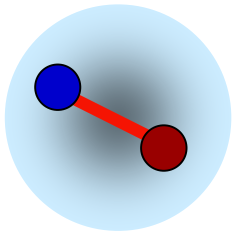
- Following "Exotic hadron naming convention" proposed by LHCb recently

[arXiv: 2206.15233]

Slide from Liu-Pan An

Success of potential quark model:

Hadrons are made of quarks (antiquarks) as QCD color singlet



Typical quark model prescription (e.g. Godfrey Isgur model):

$$H\,|\,\Psi\rangle=(H_0+V)\,|\,\Psi\rangle=E\,|\,\Psi\rangle$$

$$H_0=(p^2+m_1{}^2)^{1/2}+(p^2+m_2{}^2)^{1/2} \quad \rightarrow \sum_{i=1}^{2}\left[m_i+\frac{p^2}{2m_i}\right]$$

$$V_{ij}(\mathbf{p},\mathbf{r})\rightarrow H_{ij}^{\rm conf}+H_{ij}^{\rm hyp}+H_{ij}^{\rm so}+H_A$$

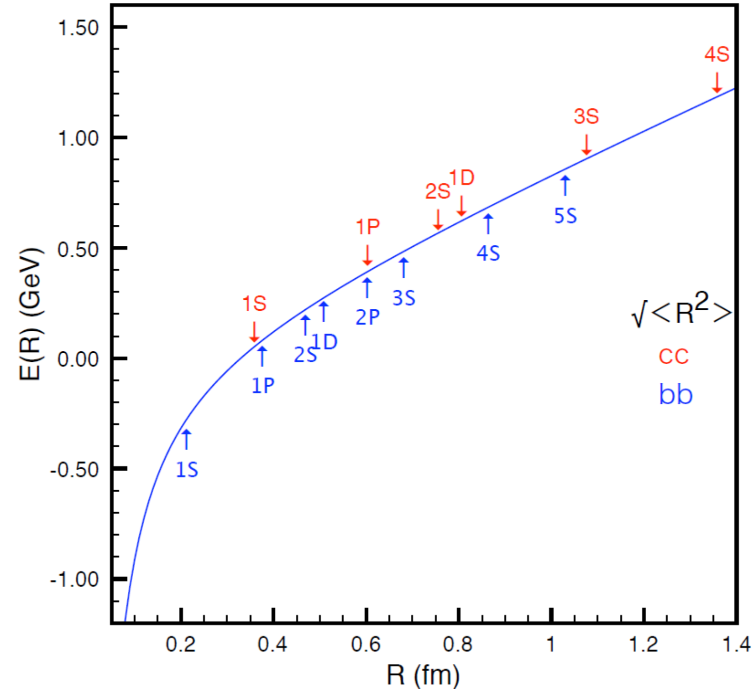
$$H_{ij}^{\rm conf}=-\left[\tfrac{3}{4}c+\tfrac{3}{4}br-\frac{\alpha_s(r)}{r}\right]\mathbf{F}_i\cdot\mathbf{F}_j$$

$$H_{ij}^{\rm hyp}=-\frac{\alpha_s(r)}{m_i m_j}\left[\frac{8\pi}{3}\mathbf{S}_i\cdot\mathbf{S}_j\delta^3(\mathbf{r})+\frac{1}{r^3}\left[\frac{3\mathbf{S}_i\cdot\mathbf{r}\mathbf{S}_j\cdot\mathbf{r}}{r^2}-\mathbf{S}_i\cdot\mathbf{S}_j\right]\right]\mathbf{F}_i\cdot\mathbf{F}_j$$

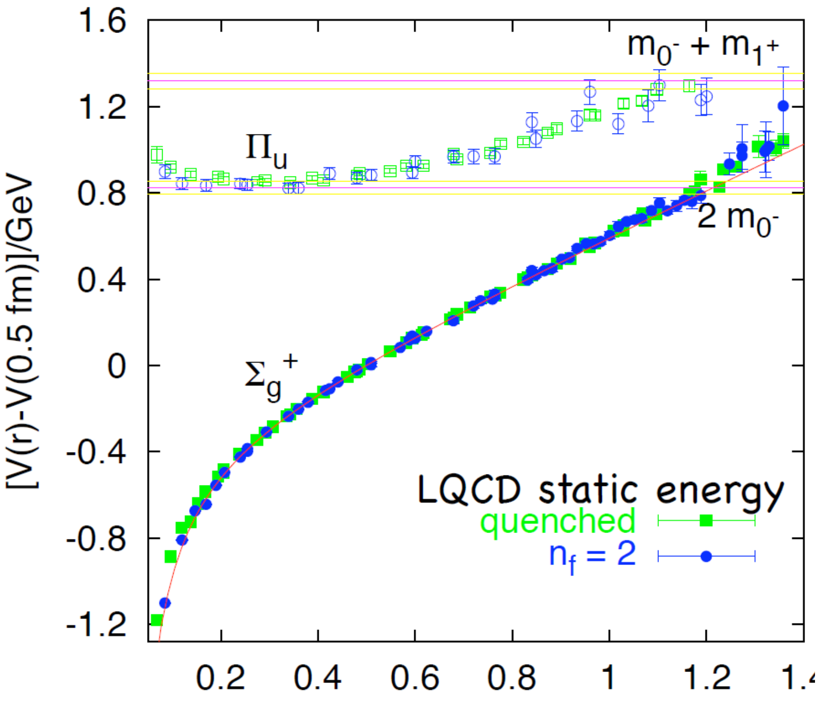
$$H_{ij}^{\rm so}=H_{ij}^{\rm so(cm)}+H_{ij}^{\rm so(tp)}$$

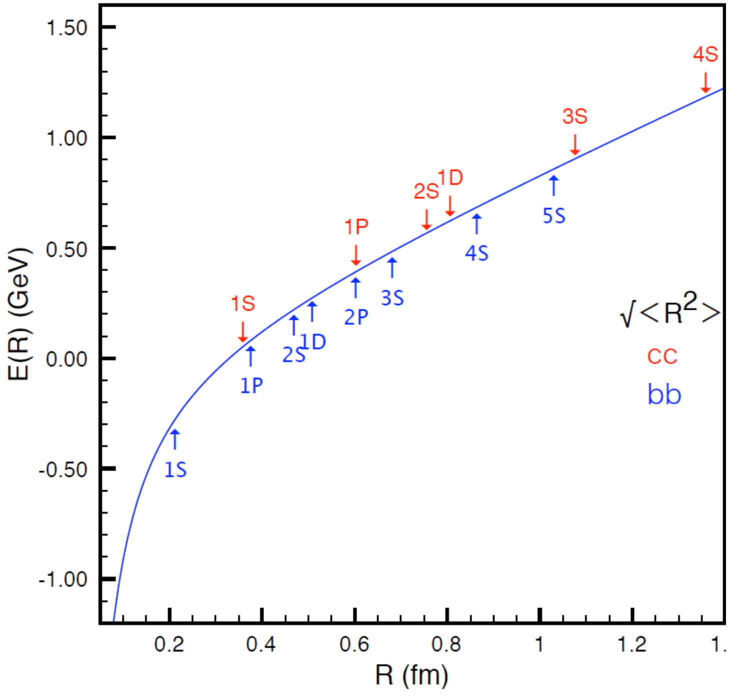
$$H_{ij}^{\rm so(cm)}=-\frac{\alpha_s(r)}{r^3}\left[\frac{1}{m_i}+\frac{1}{m_j}\right]\left[\frac{\mathbf{S}_i}{m_i}+\frac{\mathbf{S}_j}{m_j}\right]\cdot\mathbf{L}(\mathbf{F}_i\cdot\mathbf{F}_j) \qquad H_{ij}^{\rm so(tp)}=\frac{-1}{2r}\frac{\partial H_{ij}^{\rm conf}}{\partial r}\left[\frac{\mathbf{S}_i}{m_i{}^2}+\frac{\mathbf{S}_j}{m_j{}^2}\right]\cdot\mathbf{L}$$

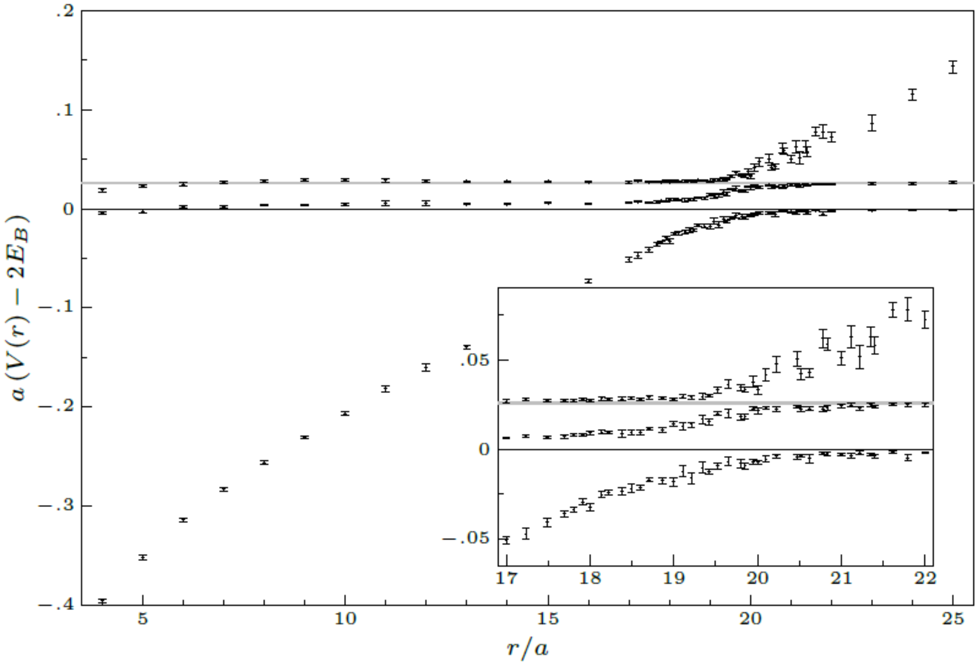
$$\langle\mathbf{F}_i\cdot\mathbf{F}_j\rangle=\begin{cases}-\frac{4}{3} & \text{in a meson}\\ -\frac{2}{3} & \text{in a baryon}\end{cases}$$



The connection between the quark model and QCD **ONLY** becomes clear in certain circumstances: in the heavy quark limit the soft QCD for quark-antiquark or quark-quark interactions can become much simpler.





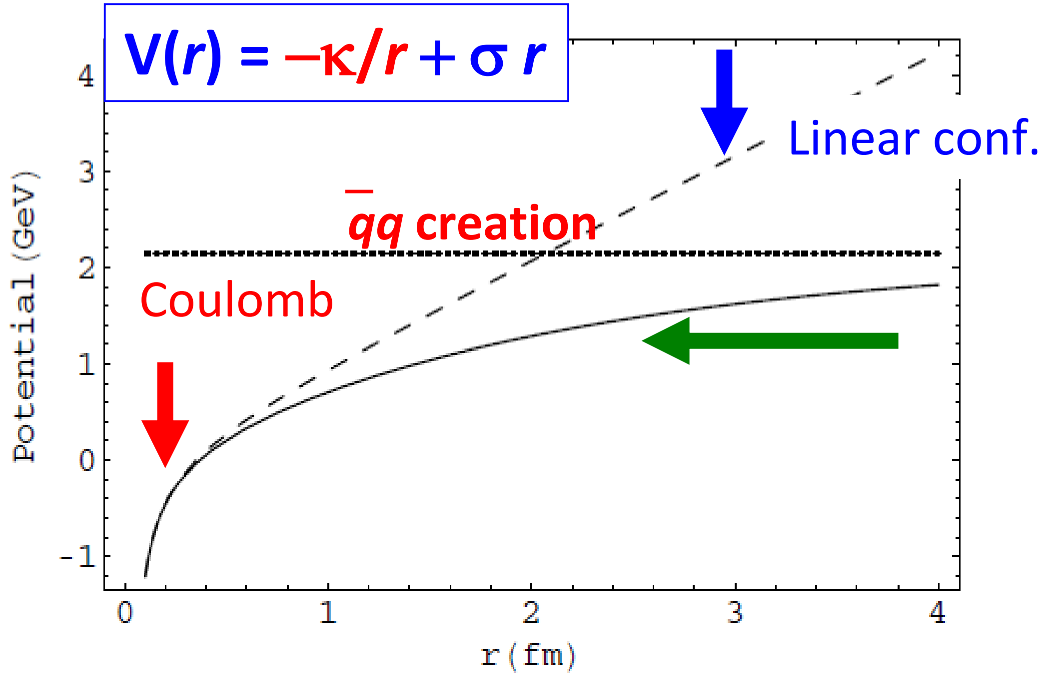


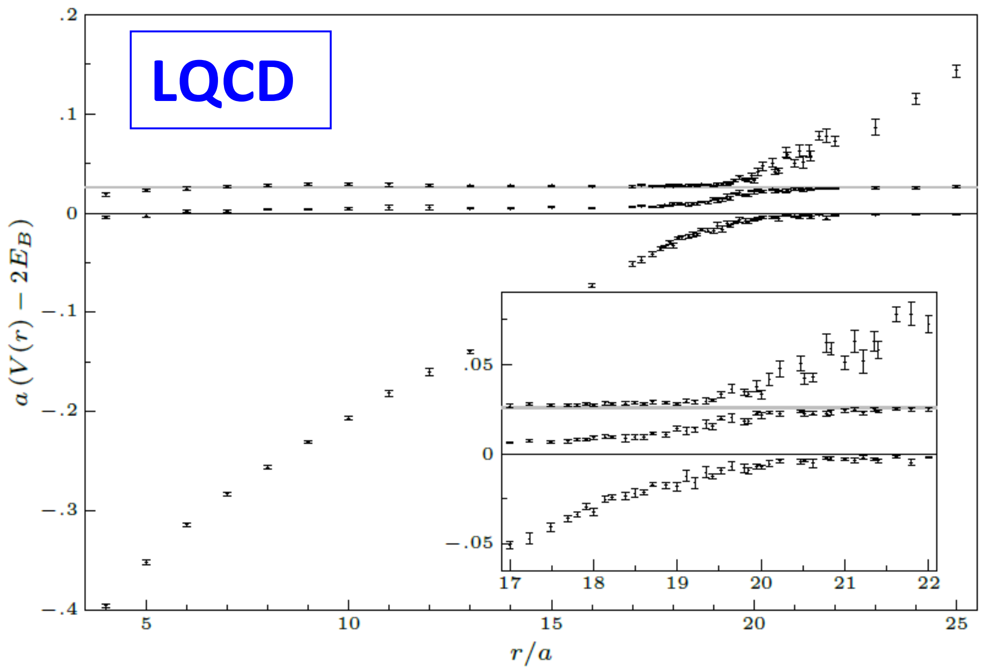
However, the effects of the open channels on the soft QCD potential is also evident!

G. S. Bali, et al., Phys. Rev. D62, 054503 (2000)
J. Bulava, et al., Phys. Lett. B793, 493 (2019),
M. Foster and C. Michael (UKQCD), Phys. Rev. D59, 094509 (1999)

Threshold phenomena and dynamics

- **Color screening effects? String breaking effects? Unquenched effects? Coupled-channel effects?**

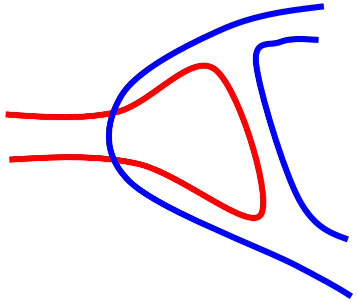




- **The effect of vacuum polarization due to dynamical quark pair creation may be manifested by the strong coupling to open thresholds and compensated by that of the hadron loops, i.e. coupled-channel effects.**

E. Eichten et al., PRD17, 3090 (1987)
E. J. Eichten, K. Lane, and C. Quigg, Phys. Rev. D 69, 094019 (2004)
B.-Q. Li and K.-T. Chao, Phys. Rev. D79, 094004 (2009);
T. Barnes and E. Swanson, Phys.Rev. C77, 055206 (2008)

Typical processes where the open threshold coupled channels can play a role

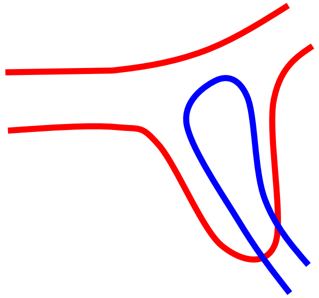


$\psi(3770) \to nonD\bar{D}$ Y.J. Zhang, G. Li, Q. Zhao, PRL(2009); X. Liu, B. Zhang, X.Q. Li, PLB(2009)

"ρπ puzzle" Q. Wang et al. PRD(2012), PLB(2012)

$\chi_{c1} \to VV,\ \chi_{c2} \to VP$ X.-H. Liu et al, PRD81, 014017(2010); X. Liu et al, PRD81, 074006(2010)

$\eta_c(\eta_c') \to VV$ Q. Wang et al, PRD2012



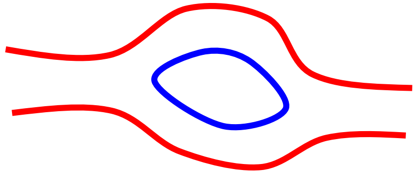
$\psi' \to J/\psi\pi^0, \psi' \to J/\psi\eta$

$\psi' \to \gamma\eta_c, J/\psi \to \gamma\eta_c$

G. Li and Q. Zhao, PRD(2011)074005

F.K. Guo, C. Hanhart, G. Li, U.-G. Meiβner and Q. Zhao, PRD82, 034025 (2010); PRD83, 034013 (2011)

F.K. Guo and Ulf-G Meiβner, PRL108(2012)112002

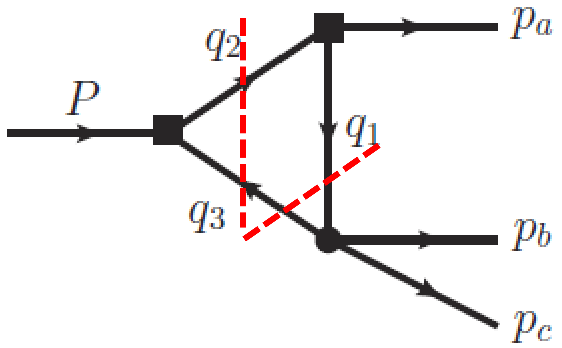


$D_{s1}(2460) - D_{s1}(2536)$

The mass shift in charmonia and charmed mesons, E.Eichten et al., PRD17(1987)3090

X.-G. Wu and Q. Zhao, PRD85, 034040 (2012)

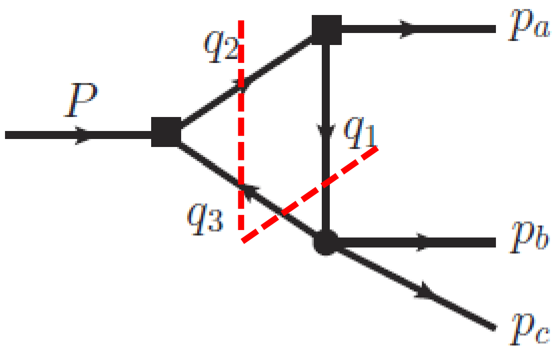
The open channel couplings introduce NOT ONLY additional dynamics (add. effective DOF) into the hadron structures, BUT ALSO novel kinematic effects, i.e. triangle singularity …



Exotics of Type-III:
Peak structures caused by kinematic effects, in particular, by triangle singularity.

$$\Gamma_3(s_1,s_2,s_3) = \frac{1}{i(2\pi)^4}\int \frac{d^4q_1}{(q_1^2-m_1^2+i\epsilon)(q_2^2-m_2^2+i\epsilon)(q_3^2-m_3^2+i\epsilon)}$$
$$= \frac{-1}{16\pi^2}\int_0^1\int_0^1\int_0^1 da_1\,da_2\,da_3\,\frac{\delta(1-a_1-a_2-a_3)}{D-i\epsilon},$$

$$D \equiv \sum_{i,j=1}^{3} a_i a_j Y_{ij},\quad Y_{ij} = \frac{1}{2}\left[m_i^2+m_j^2-(q_i-q_j)^2\right]$$



The TS occurs when all the three internal particles can approach their on-shell condition simultaneously:

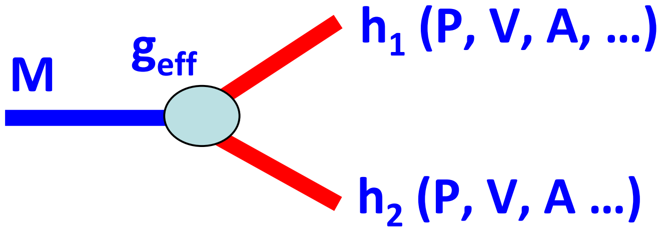
$\partial D/\partial a_j = 0$ for all j=1,2,3. $\Rightarrow$ $\det[Y_{ij}] = 0$

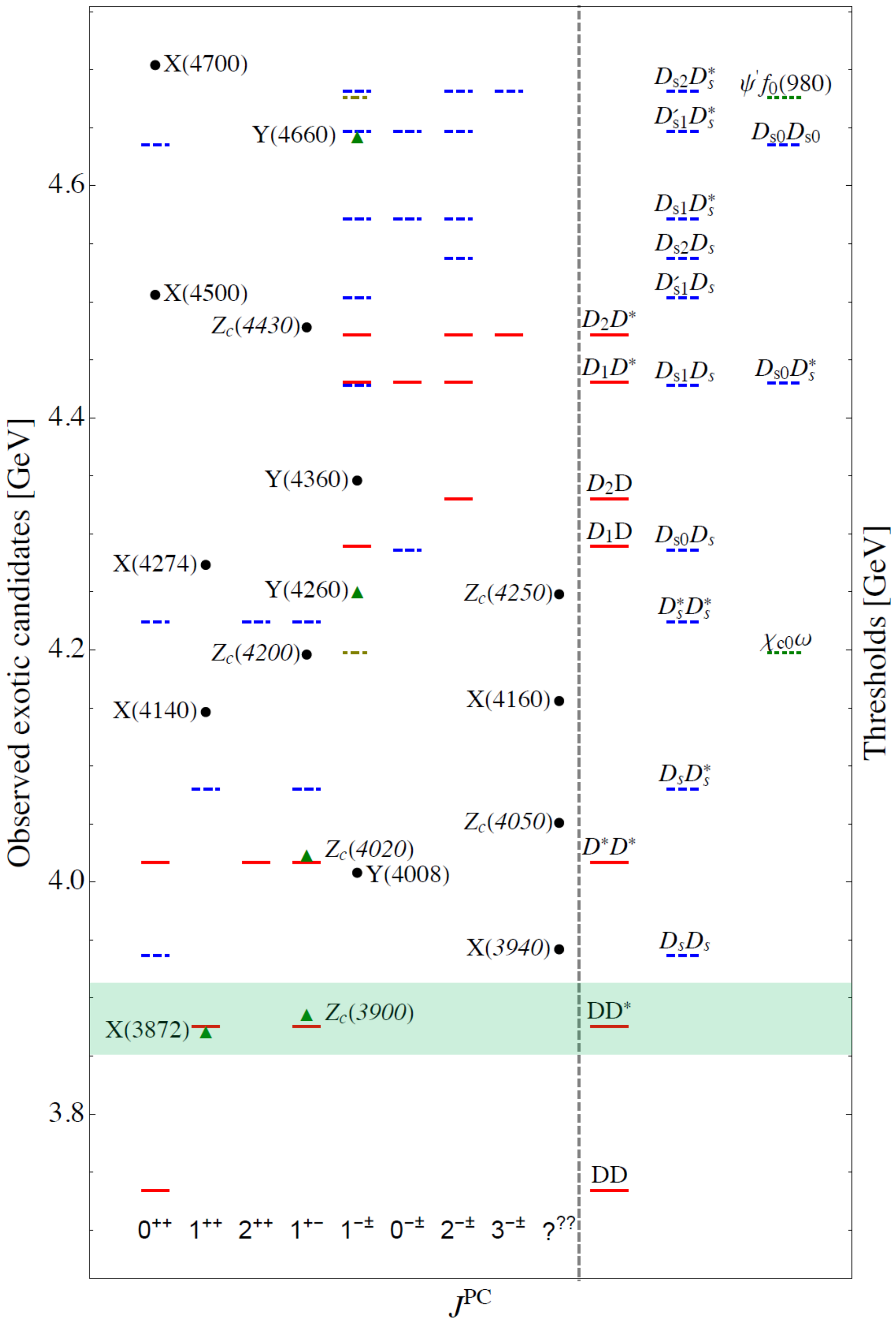
L. D. Landau, Nucl. Phys. 13, 181 (1959);
J.J. Wu, X.-H. Liu, Q. Zhao, B.-S. Zou, Phys. Rev. Lett. 108, 081003 (2012);
Q. Wang, C. Hanhart, Q. Zhao, Phys. Rev. Lett. 111, 132003 (2013); Phys. Lett. B 725, 106 (2013)
X.-H. Liu, M. Oka and Q. Zhao, PLB753, 297(2016);
F.-K. Guo, C. Hanhart, U.-G. Meissner, Q. Wang, Q. Zhao, B.-S. Zou, arXiv:1705.00141[hep-ph], Rev. Mod. Phys. 90, 015004 (2018) ; F.-K. Guo, X.-H. Liu and S. Sakai, Prog. Part. Nucl. Phys. 112, 103757 (2020)

The narrow two-body open thresholds:

Their possible impact on the spectrum should be systematically investigated.

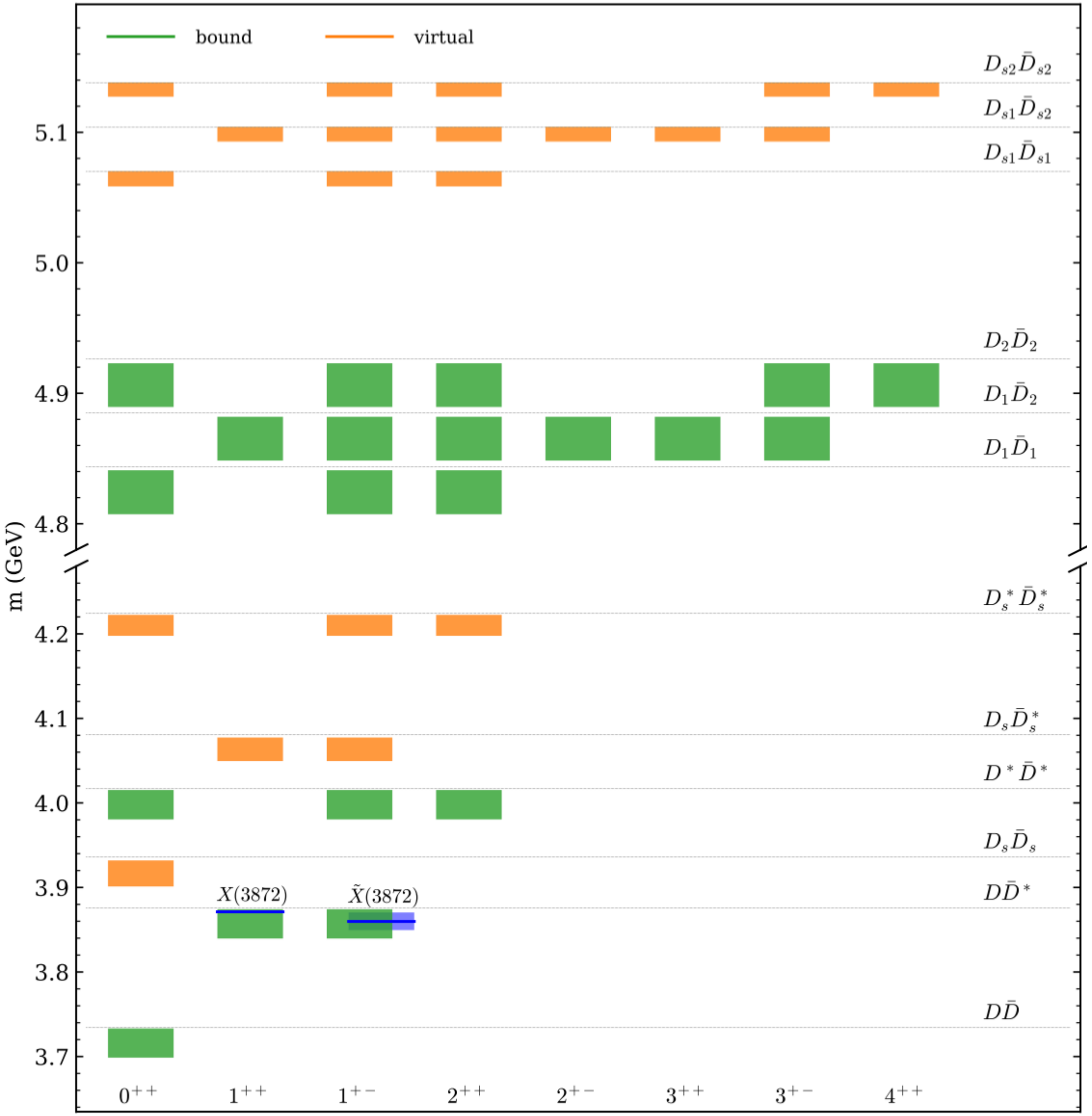
	$S-wave(L=0)$	$P-wave(L=1)$
PP	$0^{+(\pm)}$	$1^{-(\pm)}$
PV	$1^{+(\pm)}$	$0^{-(\pm)},\ 1^{-(\mp)},\ 2^{-(\pm)}$
VV	$0^{+(+)},\ 1^{+(-)},\ 2^{+(+)}$	$1^{-(+)};$ $0^{-(-)},\ 1^{-(-)},\ 2^{-(-)};$ $1^{-(+)},\ 2^{-(+)},\ 3^{-(+)}$
PA	$1^{-(-)}$	$\ldots\ldots$
VA	$0^{-(\pm)},\ 1^{-(\mp)},\ 2^{-(\pm)}$	$\ldots\ldots$

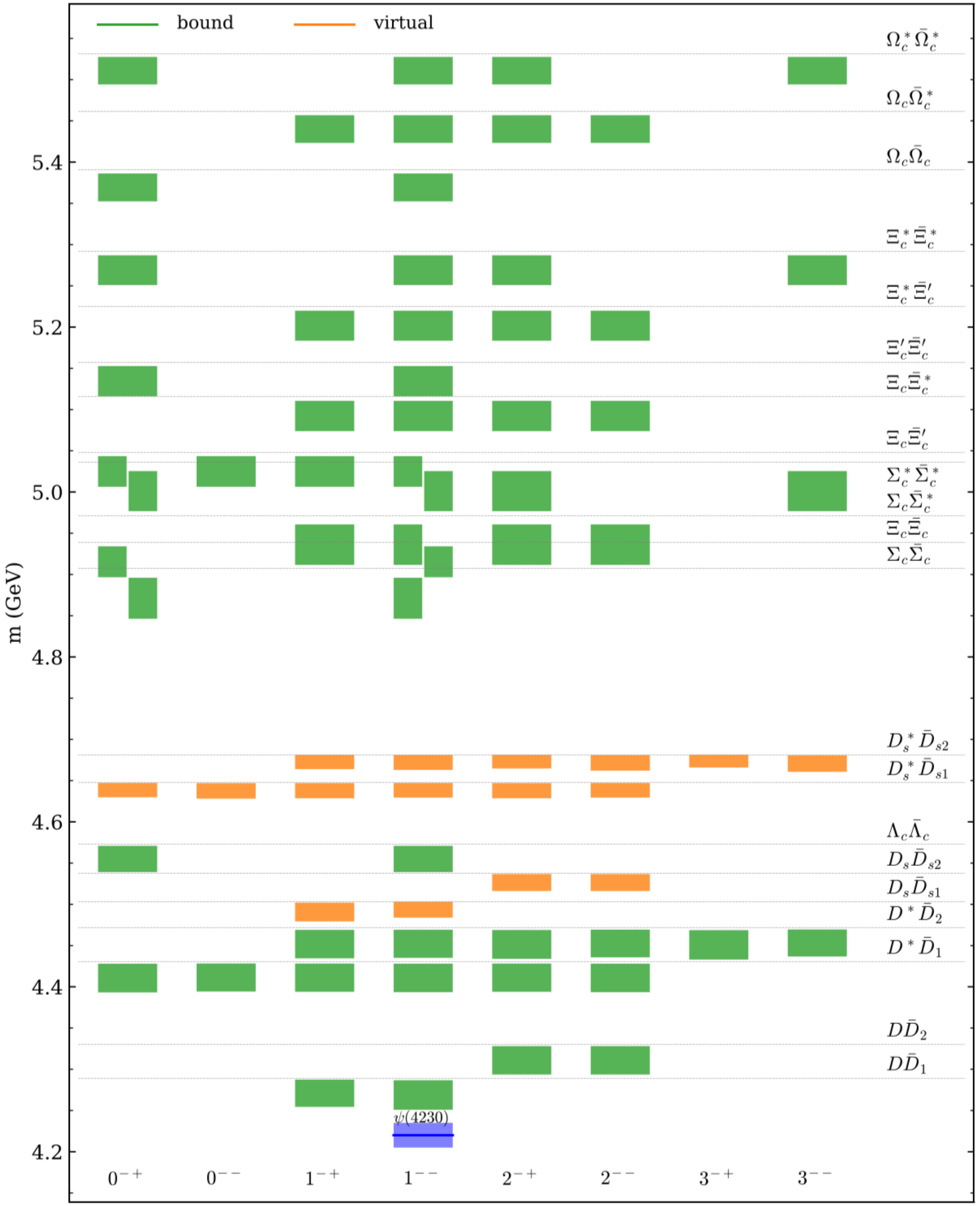




Implementation of EFT with heavy quark symmetry (HQS) and heavy quark spin symmetry (HQSS)

X.-K. Dong, F.-K. Guo, B.-S. Zou, Progr. Phys. 41 (2021) 65 [arXiv:2101.01021]

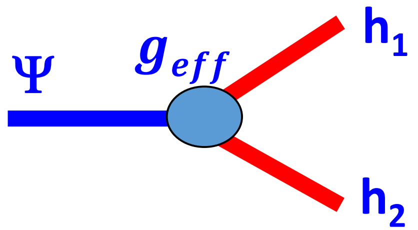




Dynamics that can contribute in additional to the potential quark model Hamiltonian

Weinberg's compositeness theorem:

$$|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi(\boldsymbol{k})|h_1 h_2\rangle \end{pmatrix}$$



$$\hat{\mathcal{H}}|\Psi\rangle = E|\Psi\rangle, \qquad \hat{\mathcal{H}} = \begin{pmatrix} \hat{H}_c & \hat{V} \\ \hat{V} & \hat{H}^0_{hh} \end{pmatrix}$$

$\hat{H}^0_{hh} = k^2/(2\mu)$: two-hadron kinetic energy.

$\mu = m_1 m_2/(m_1 + m_2)$: two-hadron reduced mass.

$\langle\psi_0|\hat{V}|h_1 h_2\rangle = f(\boldsymbol{k})$: transition amplitude between the elementary component and two-hadron state.

Wave function in momentum space: $\chi(\boldsymbol{k}) = \lambda \dfrac{f(\boldsymbol{k})}{E - k^2/(2\mu)}$

Normalization of the physical wavefunction leads to the interpretation of the states mixings:

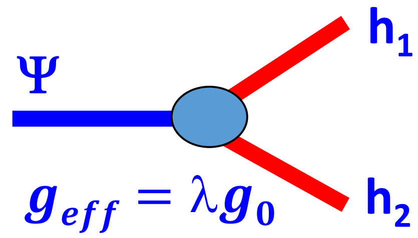
$$1 = \langle\Psi|\Psi\rangle = \lambda^2\langle\psi_0|\psi_0\rangle + \int \frac{d^3k}{(2\pi)^3}|\chi(\boldsymbol{k})|^2\langle h_1 h_2|h_1 h_2\rangle$$

$$= \lambda^2\left\{1 + \int \frac{d^3k}{(2\pi)^3}\frac{f^2(\boldsymbol{k})}{[E_B + k^2/(2\mu)]^2}\right\}.$$

$$E_B = m_1 + m_2 - M$$

$$\gamma = \sqrt{2\mu E_B}$$

Formation of hadronic molecules



Given that the inverse range of force $\beta >> \gamma$ (=$(2\mu E_B)^{1/2}$), the integral can be evaluated model independently for the case of S-wave coupling:

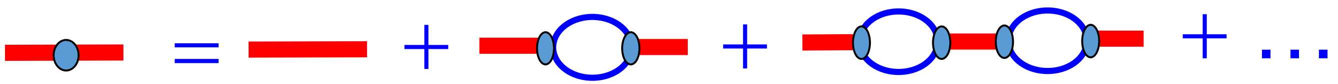
$$1 = \lambda^2 \left[1 + \frac{\mu^2 g_0^2}{2\pi\sqrt{2\mu E_B}} + \mathcal{O}\left(\frac{\gamma}{\beta}\right) \right] \qquad g_0 = f(0)$$

$$g_{eff}^2 \equiv \lambda^2 g_0^2 = \frac{2\pi\gamma}{\mu^2}(1-\lambda^2) = \frac{2\pi\gamma}{\mu^2}\chi^2$$

In the kinematic region of near threshold, a fixed E_B and sufficiently large g_0 imply a maximized rate for χ/λ. Note: Only with $E_B \to 0$, can it lead to $\lambda \to 0$. Otherwise, the physical state will always be associated with a compact component.

$$\Sigma(E) = -\int \frac{d^3k}{(2\pi)^3} \frac{f^2(\mathbf{k})}{E - k^2/(2\mu) + i\epsilon}$$

$$= \Sigma(-E_B) + i g_0^2 \frac{\mu}{2\pi}\sqrt{2\mu E + i\epsilon} + \mathcal{O}\left(\frac{\gamma}{\beta}\right)$$

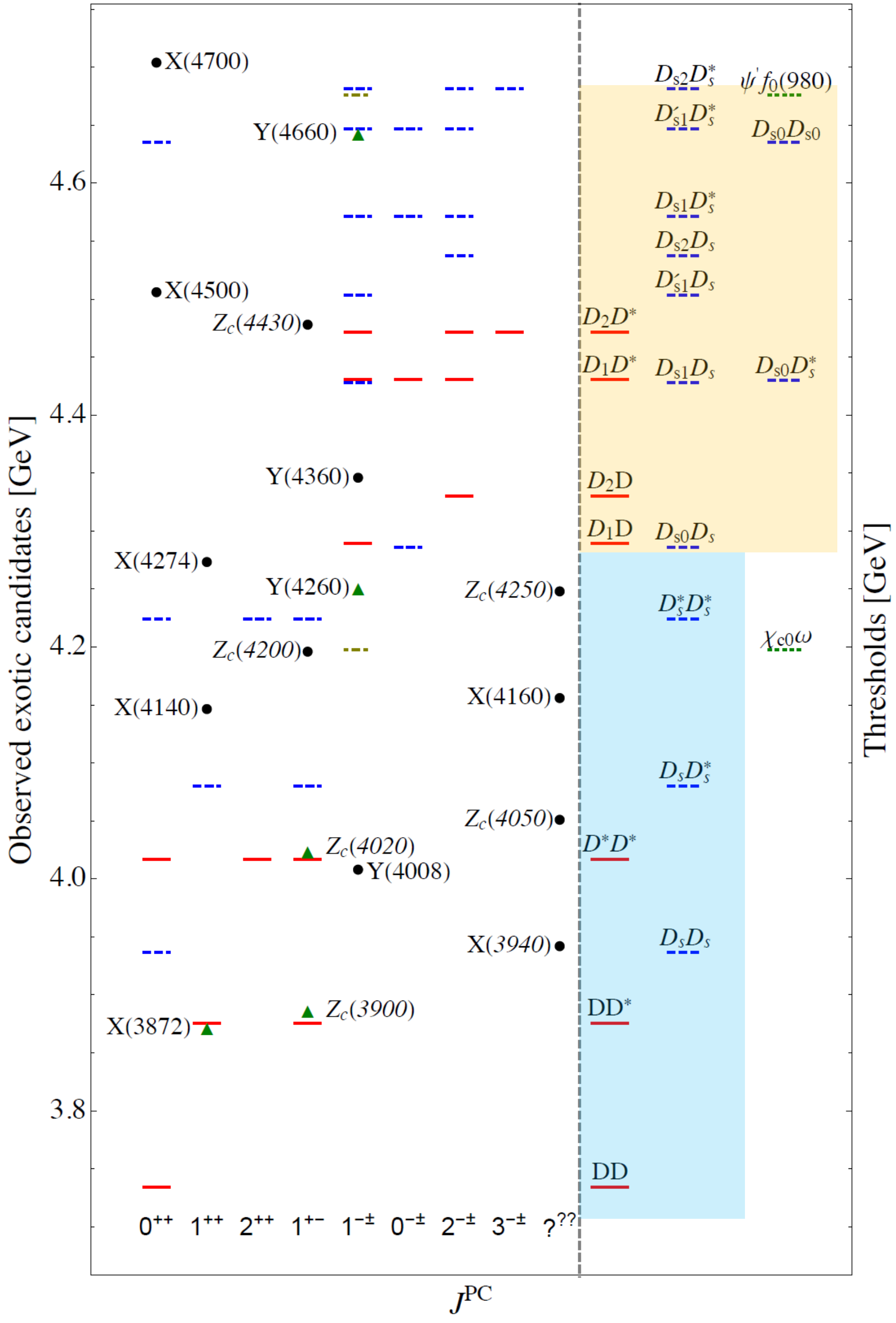


The molecular property, to some extent, needs a reasonably good understanding of the short-range component of the physical state.

T matrix for the two continuum hadron scattering:

$$T_{\rm NR}(E) = \frac{g_0^2}{E - E_0 + \Sigma(E)} + (\text{nonpole terms}) = \frac{g_0^2}{E + E_B + g_0^2\mu/(2\pi)(ik+\gamma)}$$

S wave thresholds and effects on the lineshapes



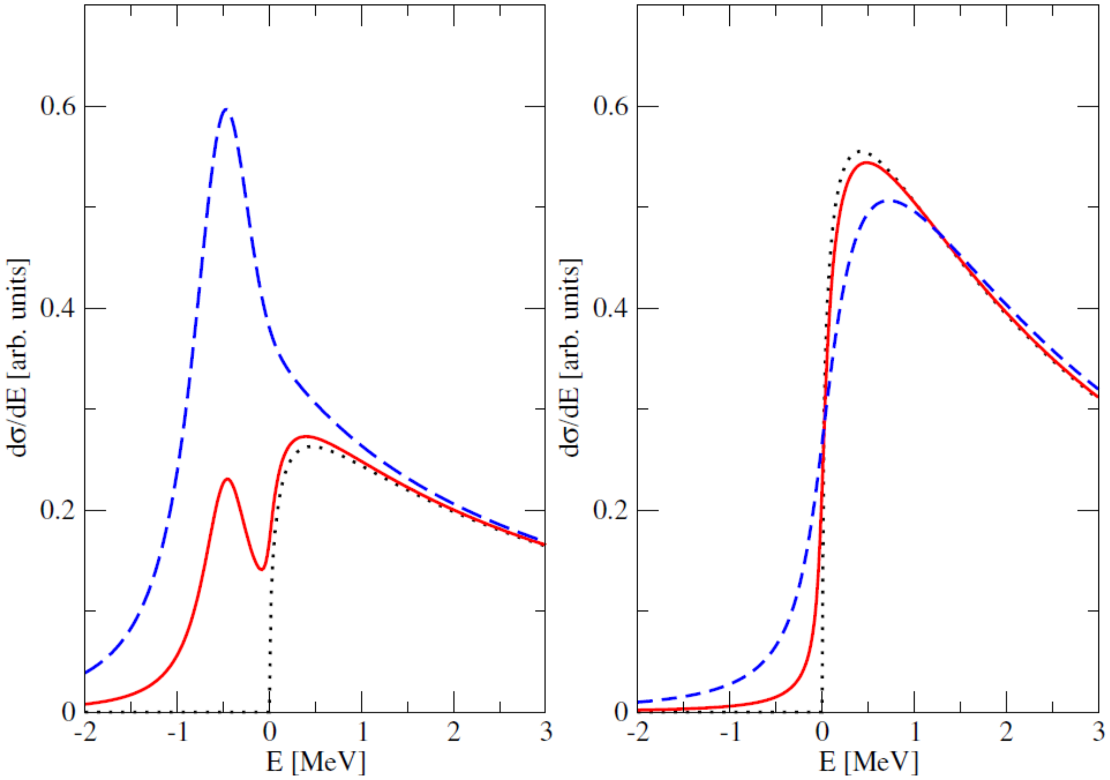
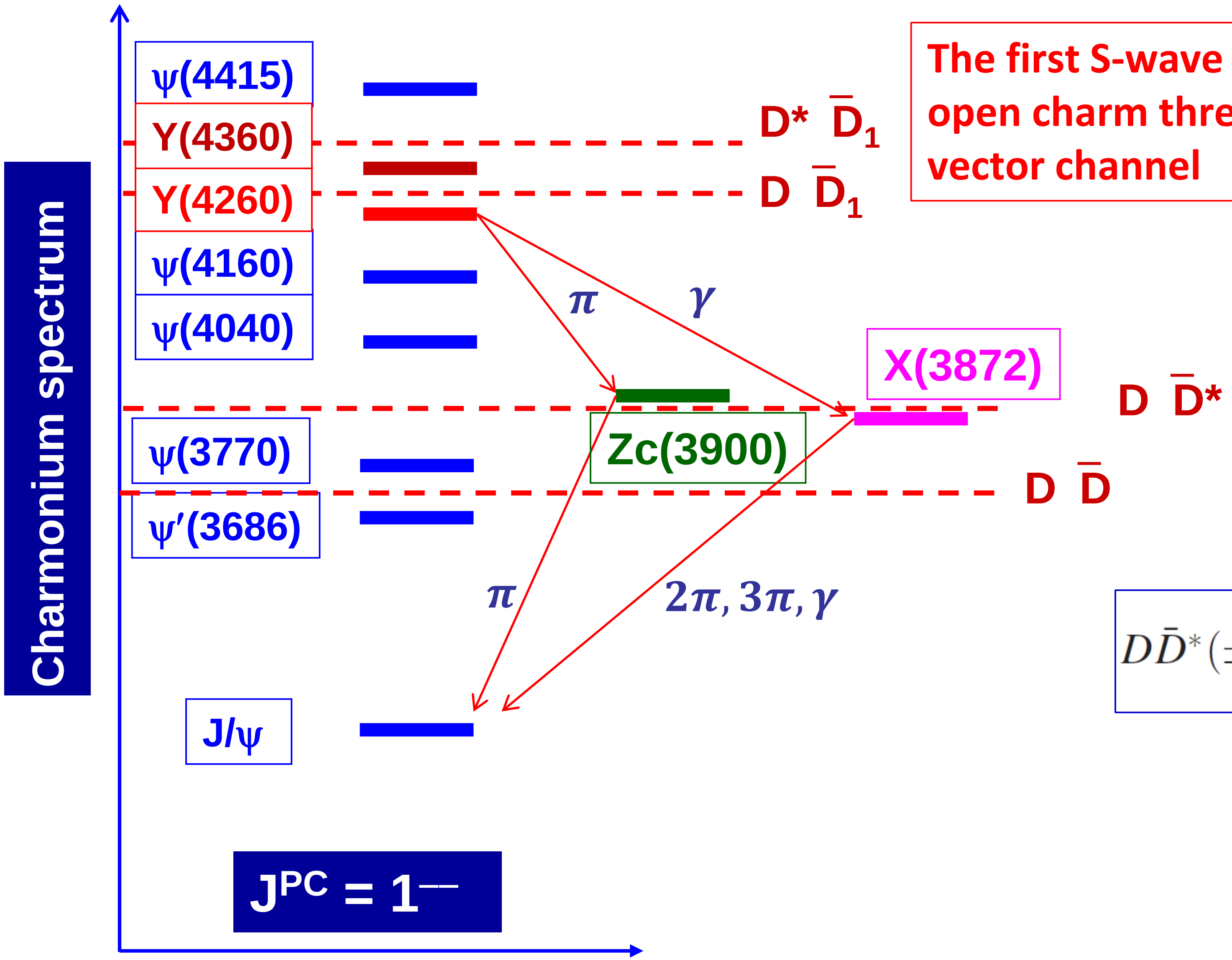


FIG. 10 Line shapes that emerge for a bound state (left panel) and for a virtual state (right panel) once one of the constituents is unstable. The dotted, solid and dashed line show the results for $\Gamma = 0, 0.1$ and 1 MeV, respectively. The other parameters of the calculation are given in Eq. (36).

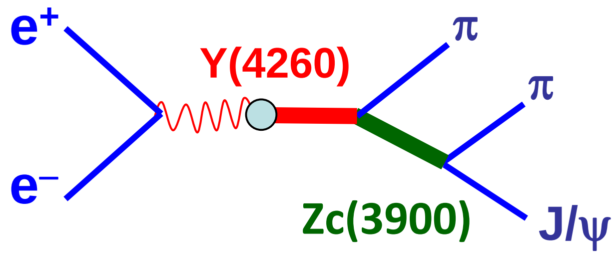
F.-K. Guo, C. Hanhart, U.-G. Meissner, Q. Wang, Q. Zhao, B.-S. Zou, Rev. Mod. Phys. 90, 015004 (2018)

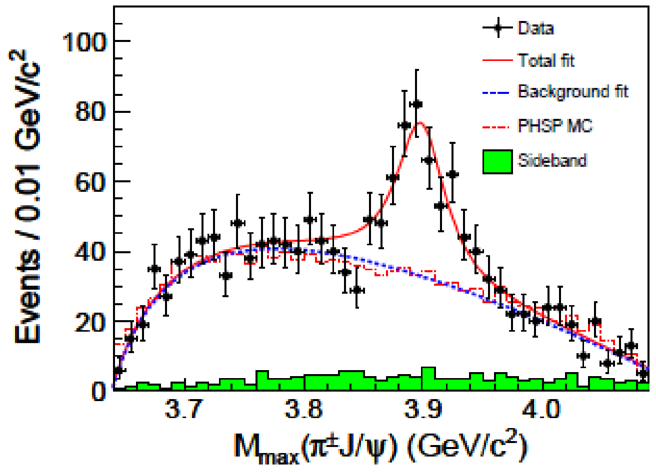
e^+e^- annihilations

3. Y(4260) and Zc(3900)/X(3872)

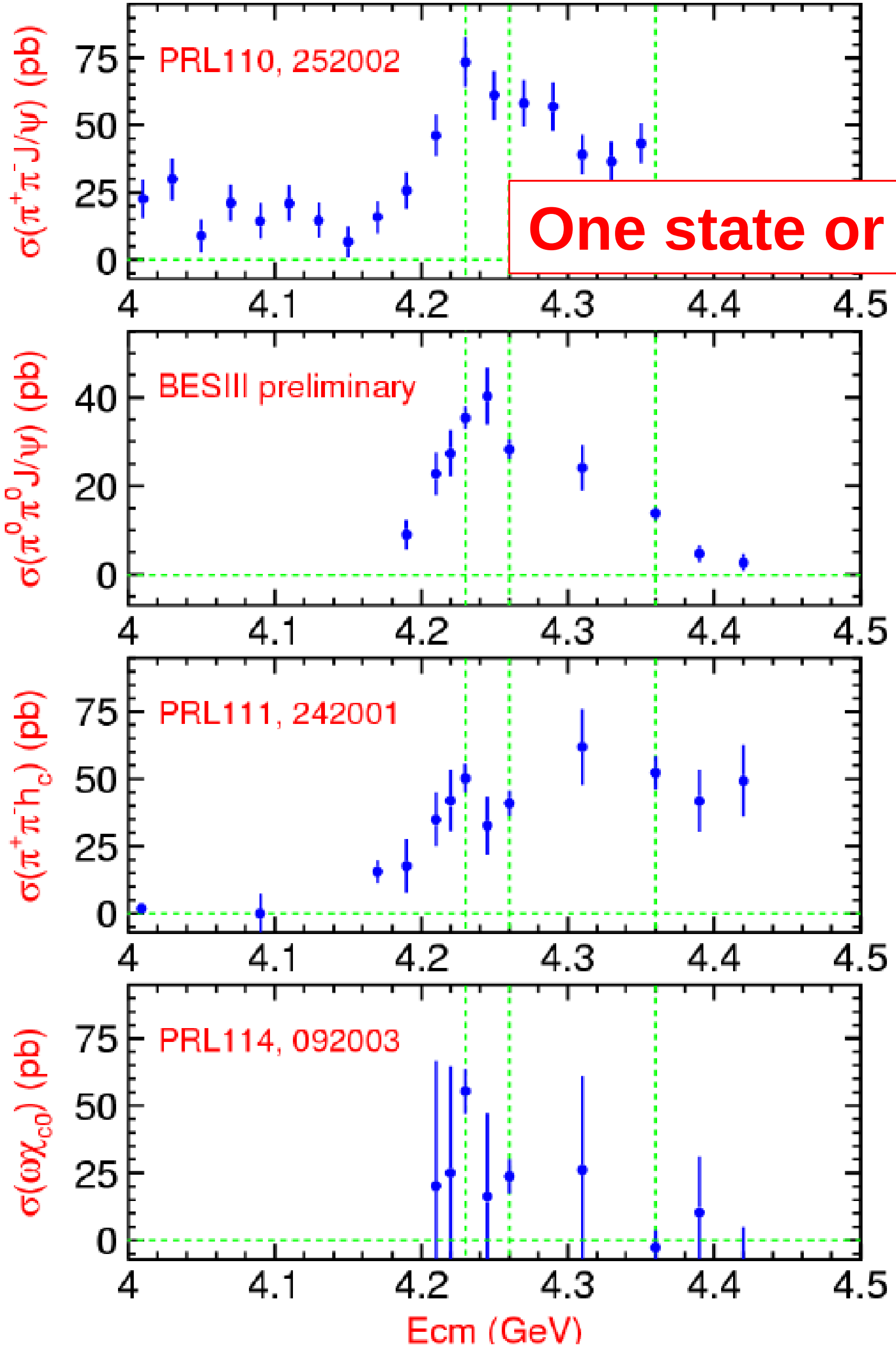


The first S-wave narrow open charm threshold in vector channel

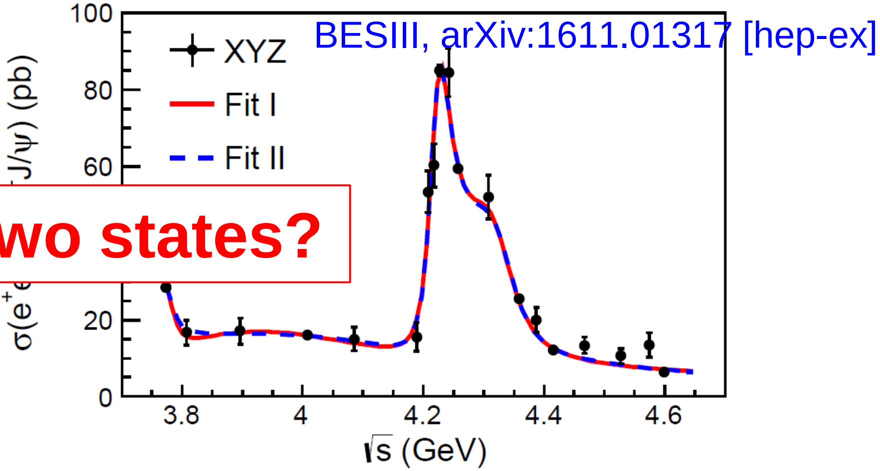


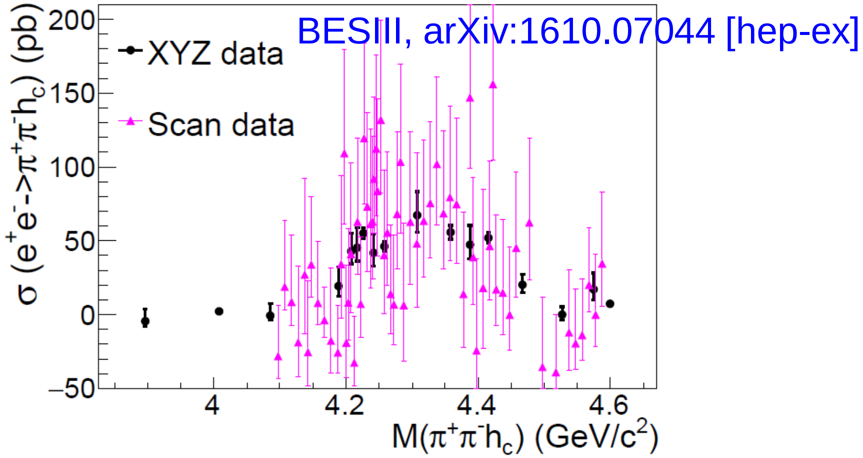


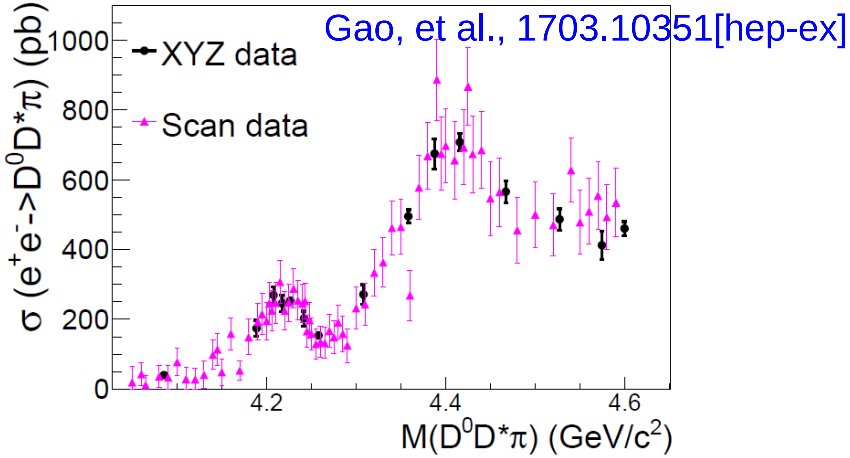
$$D\bar{D}^*(\pm) = \frac{1}{\sqrt{2}}(D\bar{D}^* \pm D^*\bar{D})$$



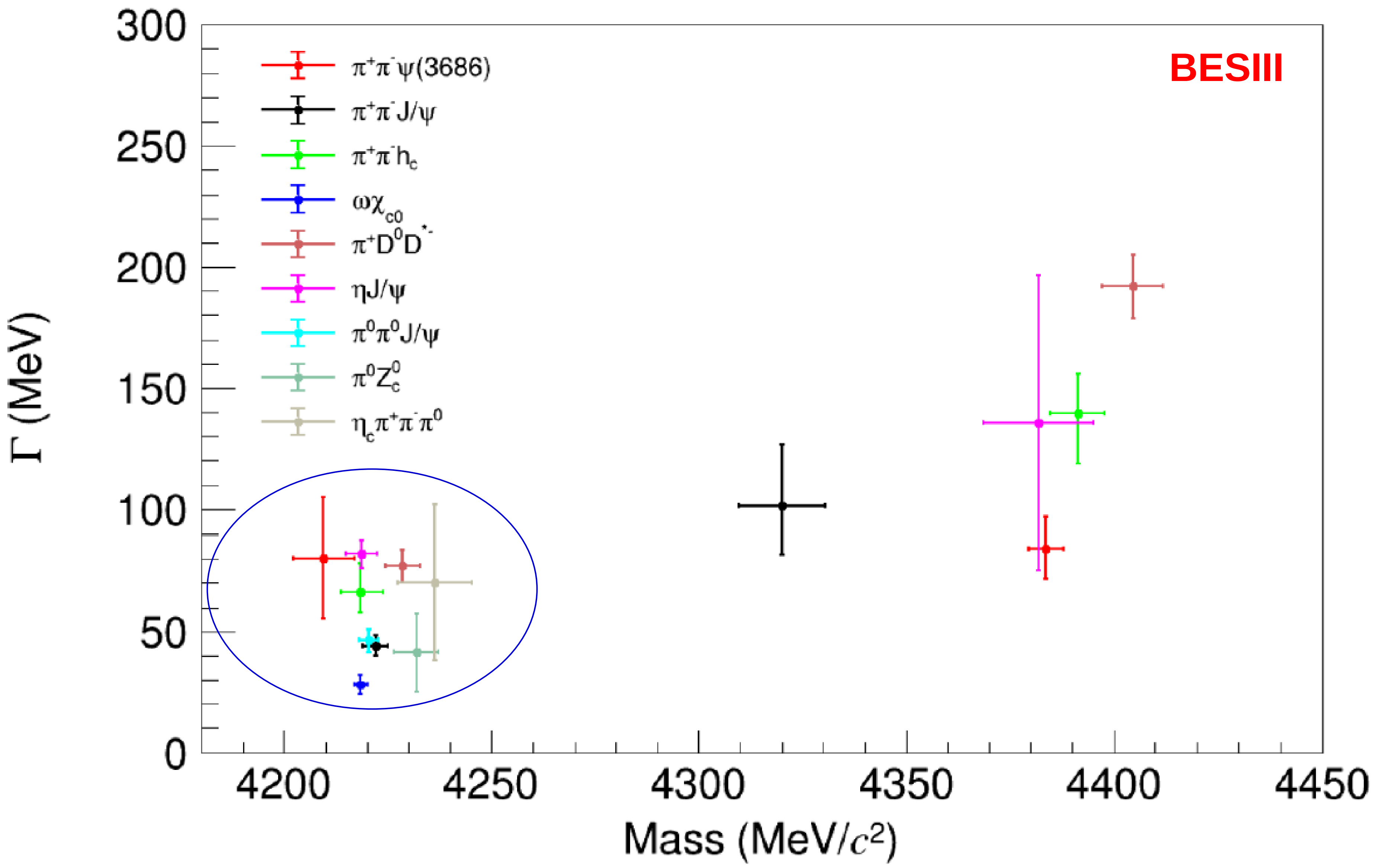
One state or two states?







Resonance parameters extracted around 4.26 GeV in different channels



Many papers on the properties of Y(4260), e.g. see recent reviews and reference therein:
H.-X. Chen, W. Chen, X. Liu and S.-L. Zhu, **Phys. Rept.** 639, 1 (2016)
F.K. Guo, C. Hanhart, U.-G. Meissner, Q. Wang, Q. Zhao, B.-S. Zou, **Rev. Mod. Phys.** 90, 015004 (2018)
A. Esposito, A. Pilloni and A.D. Polosa, **Phys. Rept.** 668, 1 (2017)

- **Why the DD*pi channel has the largest cross sections?**

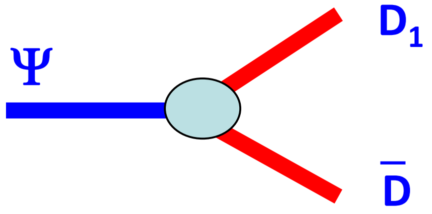
  Hadronic molecule picture can explain.

  Hybrid picture can possibly explain.

  Tetraquark picture seems to be failed.

- **What are the probes for the internal structures? How to distinguish short-ranged and long-ranged dynamics?**

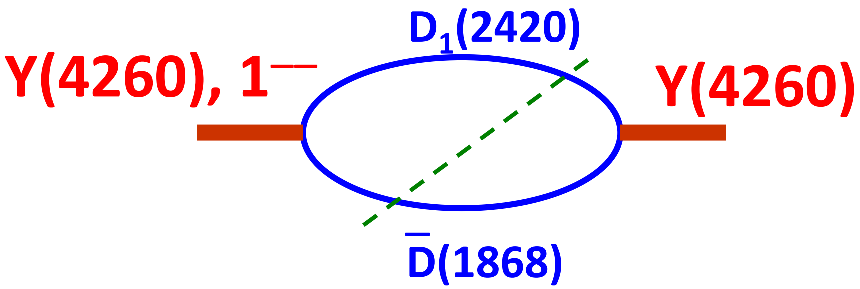
$$|\Psi\rangle = \begin{pmatrix} \lambda|\psi_0\rangle \\ \chi(\boldsymbol{k})|h_1 h_2\rangle \end{pmatrix}$$

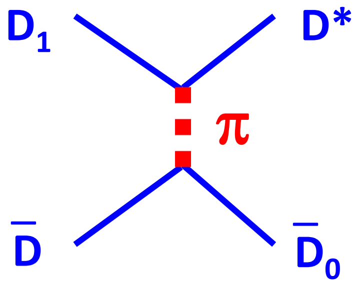


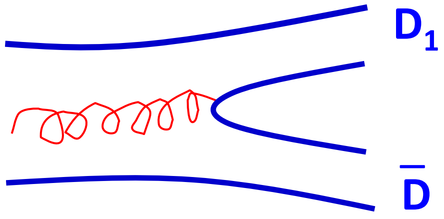
- **What is the dynamical connection with the production of Zc(3900) and X(3872)?**

- **Y(4260) could be a hadronic molecule made of $DD_1(2420)$ with coupled channel effects.**

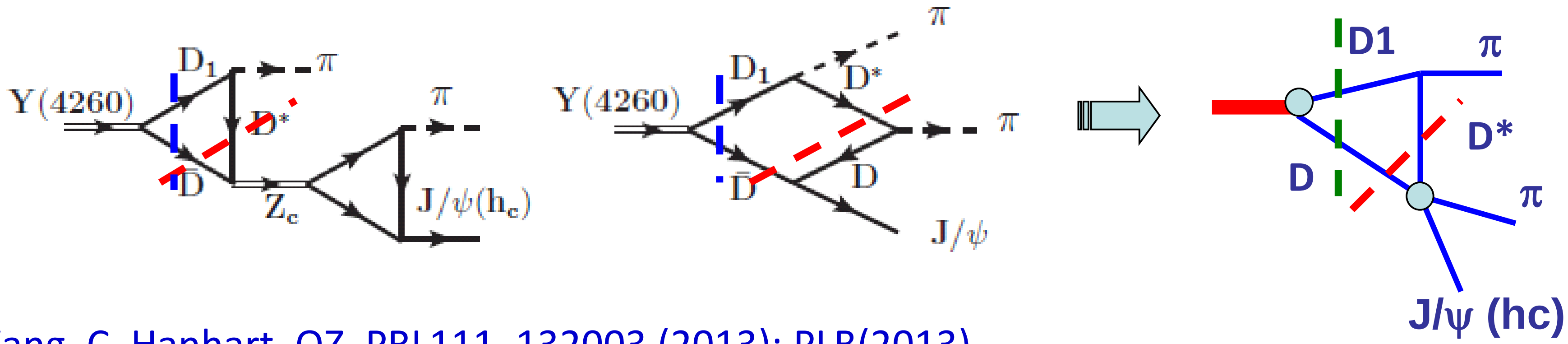
Y(4260) may have sizeable couplings to $DD_1(2420)$ if it is a hybrid. Then, how to distinguish them?







- **The production of Zc(3900) is strongly correlated with Y(4260) and enhanced by the triangle singularity kinematics.**



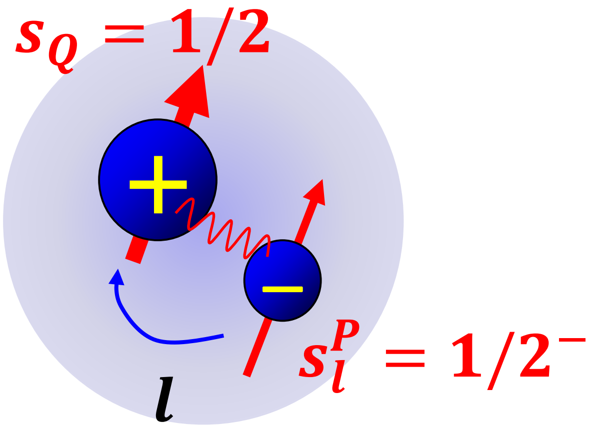
Q. Wang, C. Hanhart, QZ, PRL111, 132003 (2013); PLB(2013)
Q. Wang et al., PRD89, 034001 (2014); M. Cleven et al., PRD90, 074039 (2014);
W. Qin, S.R. Xue, QZ, PRD94, 054035 (2016)

Heavy quark spin symmetry (HQSS) and heavy quark flavor symmetry (HQFS): $SU(2)_s \otimes SU(2)_f$

1) Heavy quark spin symmetry (HQSS):

The chromo-magnetic interaction to flip the heavy quark spin: $\propto \frac{(\vec{\sigma}\cdot\vec{B})}{m_Q}$.

$\rightarrow$ Heavy quark will decouple from the light quarks with $m_Q \rightarrow \infty$.



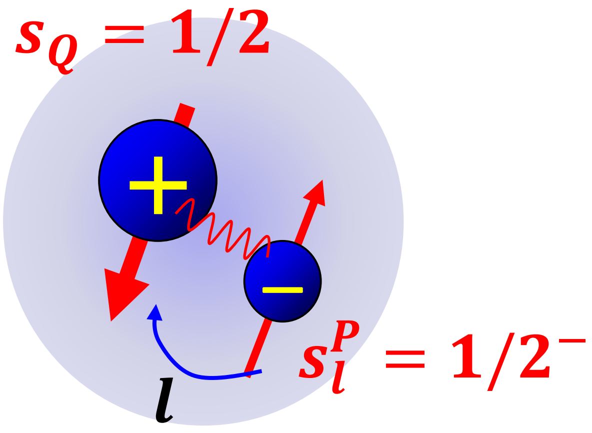
2) Heavy quark flavor symmetry (HQFS):

For any hadron containing one heavy quark, its velocity remains unchanged in the limit of $m_Q \rightarrow \infty$, because

$\Delta v = \frac{\Delta p}{m_Q} \simeq \frac{\Lambda_{QCD}}{m_Q}$. It means that the heavy quark only plays a role as a static color triplet source.

One can focus on the light quarks as the relevant D.O.F. in the description of the hadron property:

$$\boldsymbol{J} = \boldsymbol{s}_Q + \boldsymbol{s}_l, \qquad \boldsymbol{s}_l = \boldsymbol{s}_q + \boldsymbol{l}\,.$$



Flavor doublets: {D,D*}, {B, B*} → $s_l^P = 1/2^-$

Constructing super fields respecting heavy quark symmetry and chiral symmetry

For instance, the S-wave (D_a^*, D_a), $a = u, d, s$, the super field can be written as

$$H_a = \frac{1+\not{v}}{2}[D^*_{a\mu}\gamma^\mu - D_a\gamma_5]\,, \quad \bar{H}_a = \gamma_0 H_a^\dagger \gamma_0$$

Charmed meson of HQSS douplet with $s_l^P = 1/2^-$

$$\begin{cases} (D^{*0}, D^0) \\ (D^{*+}, D^+) \\ (D_s^*, D_s) \end{cases}$$

For the P-wave heavy-light mesons, two doublets can be formed, i.e. $J_{s_l}^P = (0^+, 1^+)_{1/2}$ and $J_{s_l}^P = (1^+, 2^+)_{3/2}$

$$S_a = \frac{1+\not{v}}{2}[\tilde{D}^\mu_{a1}\gamma_\mu\gamma_5 - D_{a0}],$$

$$T_a^\mu = \frac{1+\not{v}}{2}\left[D^{\mu\nu}_{a2}\gamma_\nu - \sqrt{\frac{3}{2}}D_{a1\nu}\gamma_5\left(g^{\mu\nu} - \frac{1}{3}\gamma^\nu(\gamma^\mu - v^\mu)\right)\right]$$

$\Rightarrow$

$$J_{s_l}^P = (0^+, 1^+)_{1/2} \to (D_0, \tilde{D}_1)$$

$$J_{s_l}^P = (1^+, 2^+)_{3/2} \to (D_1, D_2)$$

Lagrangians in the NREFT

- **$Y(4260)D_1D$ coupling:**

$$\mathcal{L}_Y = i\frac{y}{\sqrt{2}}\left(\bar{D}_a^\dagger Y^i D_{1a}^{i\dagger} - \bar{D}_{1a}^{i\dagger} Y^i D_a^\dagger\right) + \text{H.c.},$$

$$|y| = (3.28^{+0.25}_{-0.28} \pm 1.39) \quad \text{GeV}^{-1/2}$$

- **Zc(3900)DD* coupling:**

$$\mathcal{L}_Z = \frac{z}{\sqrt{2}}\left[\bar{V}^{\dagger i} Z^i P^\dagger - \bar{P}^\dagger Z^i V^{\dagger i}\right] + \text{H.c.} \ ,$$

$$Z_{ba}^i = \begin{pmatrix} \frac{1}{\sqrt{2}} Z^{0i} & Z^{+i} \\ Z^{-i} & -\frac{1}{\sqrt{2}} Z^{0i} \end{pmatrix}_{ba} \qquad P(V) = \left(D^{(*)0}, D^{(*)+}\right)$$

- **D_1D^*pi coupling:**

$$\mathcal{L}_{D_1} = i\frac{h'}{f_\pi}\left[3D_{1a}^i(\partial^i\partial^j\phi_{ab})D_b^{*\dagger j} - D_{1a}^i(\partial^j\partial^j\phi_{ab})D_b^{*\dagger i}\right.$$
$$\left. -3\bar{D}_a^{*\dagger i}(\partial^i\partial^j\phi_{ab})\bar{D}_{1b}^j + \bar{D}_a^{*\dagger i}(\partial^j\partial^j\phi_{ab})\bar{D}_{1b}^i\right] + \text{H.c.}, (2)$$

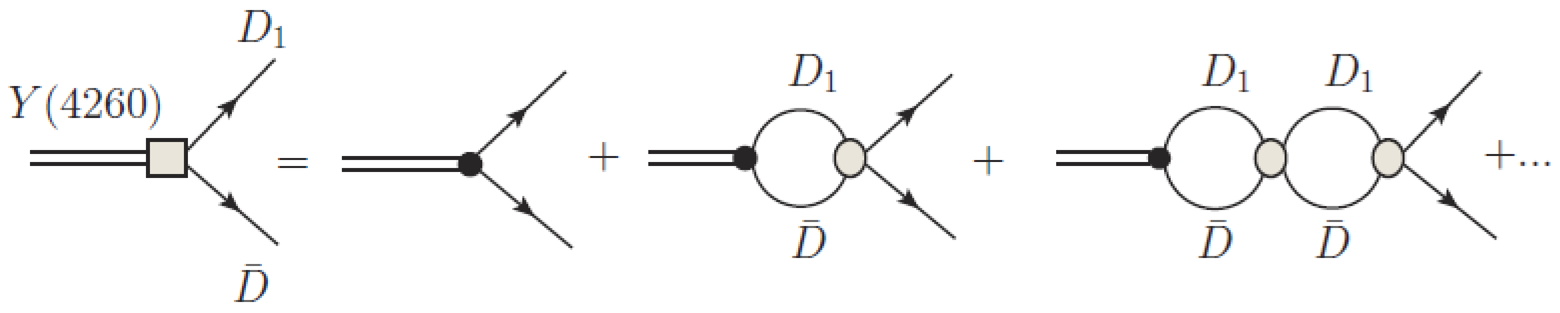
Q. Wang, C. Hanhart, QZ, PRL111, 132003 (2013); PLB(2013)
Q. Wang et al., PRD89, 034001 (2014); M. Cleven et al., PRD90, 074039 (2014);
W. Qin et al., PRD94, 054035 (2016)

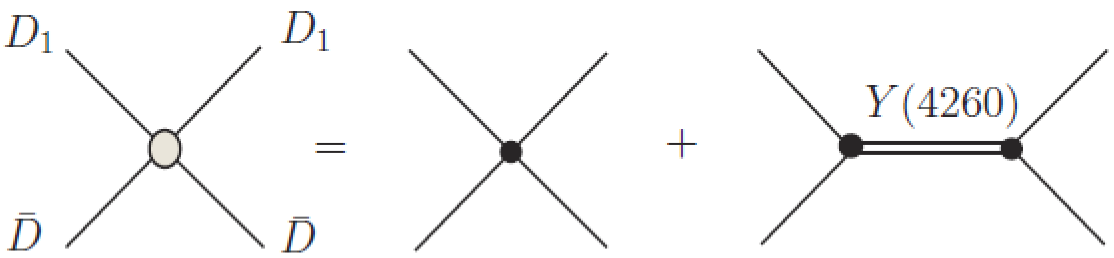
Defining the molecular component of Y(4260)

$$|Y(4260)\rangle = \alpha|c\bar{c}\rangle + \beta|D_1\bar{D} + c.c.\rangle$$

HQSS breaking is implicated

$$\mathcal{L}_Y = \frac{y^{\rm bare}}{\sqrt{2}}(\bar{D}_a^\dagger Y^i D_{1a}^{i\dagger} - \bar{D}_{1a}^{i\dagger} Y^i D_a^\dagger) + g_1\{(D_{1a}^i \bar{D}_a)^\dagger (D_{1a}^i \bar{D}_a) + (D_a \bar{D}_{1a}^i)^\dagger (D_a \bar{D}_{1a}^i)\} + H.c.$$

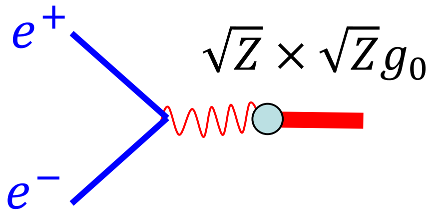




The propagator of Y(4260)

$$\mathcal{G}_Y(E) = \frac{1}{2}\frac{i}{E - m_0 + \Sigma_{D_1\bar{D}}(E) \times [i(y^{\text{bare}})^2 - 4i(E-m_0)g_1]}$$

$$\equiv \frac{1}{2}\frac{i}{E - m_0 - \Sigma_1(E)},$$



$$\Sigma_{D_1\bar{D}}(E) = \frac{-1}{4}\int \frac{d^D l}{(2\pi)^D} \frac{1}{(l^0 - \vec{l}^{\,2}/(2m_D) + i\epsilon)(E - l^0 - \vec{l}^{\,2}/(2m_{D_1}) + i\Gamma_{D_1}/2)}.$$

$$\Sigma^{\overline{\text{MS}}}_{D_1\bar{D}}(E) = \frac{\mu}{8\pi}\sqrt{2\mu(E - m_D - m_{D_1}) + i\mu\Gamma_{D_1}},$$

$$m_Y = m_0 + \text{Re}\Sigma_1(m_Y)$$

$$\tilde{\Sigma}_1(E) \equiv \Sigma_1(E) - \text{Re}(\Sigma_1(m_Y)) - (E - m_Y)\text{Re}(\partial_E \Sigma_1(m_Y))$$

$$\mathcal{G}_Y(E) = \frac{1}{2}\frac{iZ}{E - m_Y - Z\tilde{\Sigma}_1(E)}$$

$$Z \equiv 1/[1 - \text{Re}(\partial_E \Sigma_1(m_Y))]$$

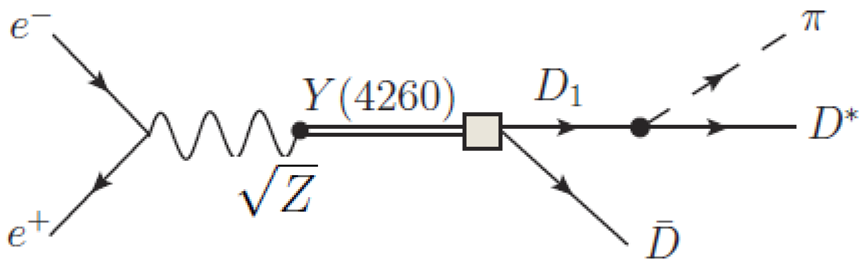
$$|Y(4260)\rangle = \alpha|c\bar{c}\rangle + \beta|D_1\bar{D} + c.c.\rangle$$

$$|Y(4260)\rangle = 0.359|c\bar{c}\rangle + 0.933|D_1\bar{D} + c.c.\rangle$$

Short-range component ← → Long-range component

$$\begin{cases} |\alpha| \simeq \sqrt{Z} \\ |\beta| = \sqrt{1-Z} \end{cases} \qquad \begin{cases} Z = \alpha^2 = 0.129 \\ 1 - Z = \beta^2 = 0.871 \end{cases}$$

$$\mathcal{G}_Y(E) = \frac{1}{2}\frac{iZ}{E - m_Y - Z\tilde{\Sigma}_1(E) + i\Gamma^{\mathrm{non}-D_1\bar{D}}/2}$$



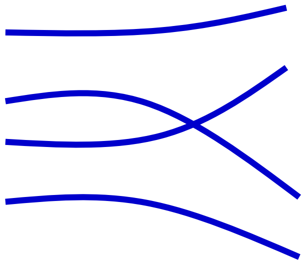
In the HQS limit, i.e. $Z \to 0$, then $\alpha \to 0$, which means that $Y(4260)$ cannot be produced in e^+e^- annihilations. Or as an alternative observation, the production of $Y(4260)$ has indicated the HQS breaking.

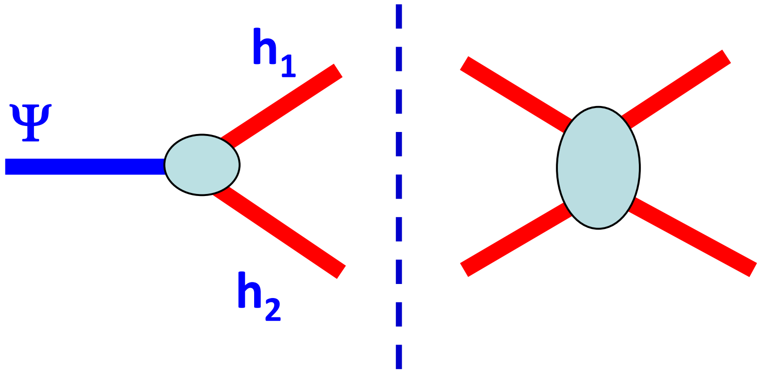
Y(4260) is dominated by molecular component but also contains a component (charmonium? Or something else?) as a compact structure.

The general features with the tetraquark or hadronic molecule scenarios:

- Very rich spectra are expected.
- The form of diquark DoF can lead to very different results for the tetraquark production and decay.
- A strong coupling for a tetraquark state into a nearby S-wave threshold may need unitarization which will still introduce a molecular component into the wavefunction.

- Limited number of states close to threshold.
- Possible mixing with the kinematic singularity, but can be clarified by energy dependence of the lineshape measurement.
- Unitarization is crucial and EFT can be implemented. However, whether or not a molecular state can be formed would depend on the detailed dynamics.





We presumably have understand the main feature of hadronic molecules which are originated from the threshold dynamics. But their relation with the elementary quark model states is still obscure.

Clues for the following issues are also essentially important:

1) What drives higher states to be unstable?

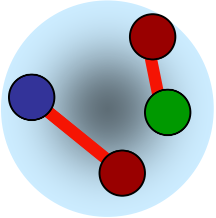
2) To what extent the genuine color-singlet multiquark states are allowed?

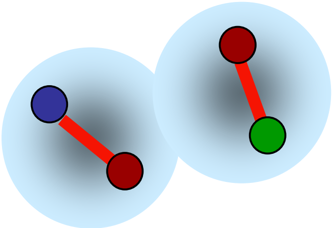
3) Do we have a reasonable scheme to combine the genuine color-singlet multiquark states with the long-ranged molecular states arising from the threshold dynamics?

3. Back to the quark model: where are the genuine color-singlet multiquark states?

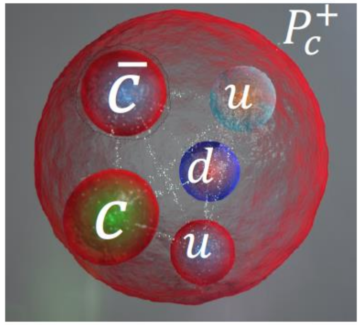
Genuine color-singlet multiquark states *vs.* hadron molecules

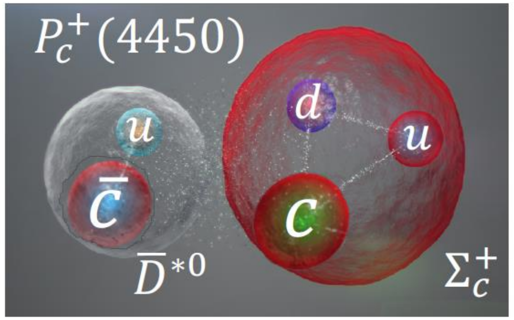
Tetraquark vs. hadronic molecule





Pentaquark vs. hadronic molecule





Status rating	
****	$X(3872), Z_c(3900)$
***	$Y(4260)/Y(4230)$, $P_c(4440), P_c(4457), P_c(4312), T_{cc}(3876)$, $X(6900)$
**	$\psi_2(3823), X(4140), X(4274)$, $X(4500), X(4700), Z_c(4360), Z_c(4430)$, $Z_{c1}(4050), Z_{c2}(4250), Z_c(4200), Z_c(4020)$, $Z_b(10610), Z_b(10650), Y(4660)$, $X(6200), X(7200)$ …
*	$Y(4008), Z_{cs}(3985), Z_{cs}(4000)$ ...

Why we do not see rich spectra arising from genuine color-singlet multiquark states?

Heavy quarkonium spectrum *vs.* fully-heavy genuine color-singlet tetraquark states in the quark model

Hamiltonian in a non-relativistic quark model :

$$H = \left(\sum_{i=1}^{4} m_i + T_i \right) - T_G + \sum_{i<j} V_{ij}(r_{ij})$$

$$T_i = \frac{p_i^2}{2m_i}, \qquad V_{ij}(r_{ij}) = V_{ij}^{\rm OGE}(r_{ij}) + V_{ij}^{\rm Conf}(r_{ij}),$$

$$V_{ij}^{\rm Conf}(r_{ij}) = -\frac{3}{16}(\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j) \cdot b r_{ij},$$

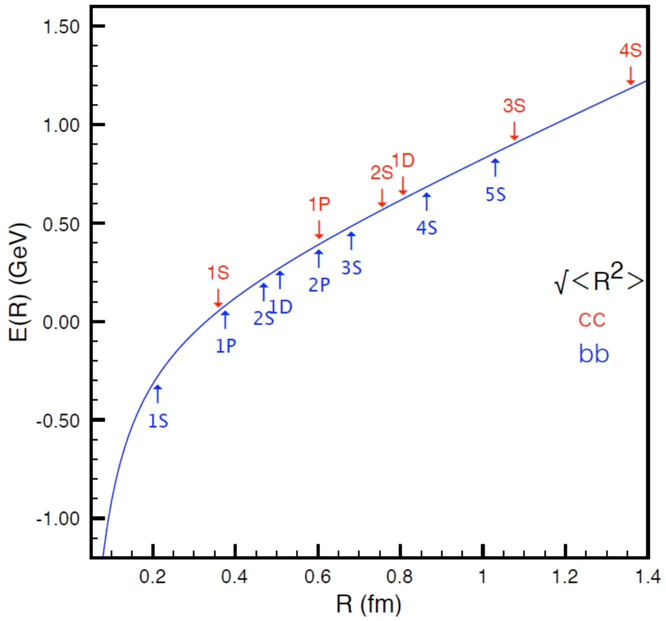
Potential smearing factor

$$V_{ij}^{\rm OGE} = \frac{\alpha_{ij}}{4}(\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j)\left\{ \frac{1}{r_{ij}} - \frac{\pi}{2} \cdot \frac{\sigma_{ij}^3 e^{-\sigma_{ij}^2 r_{ij}^2}}{\pi^{3/2}} \cdot \frac{4}{3m_i m_j}(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) \right\}$$

Coulomb **Spin-spin correl.**

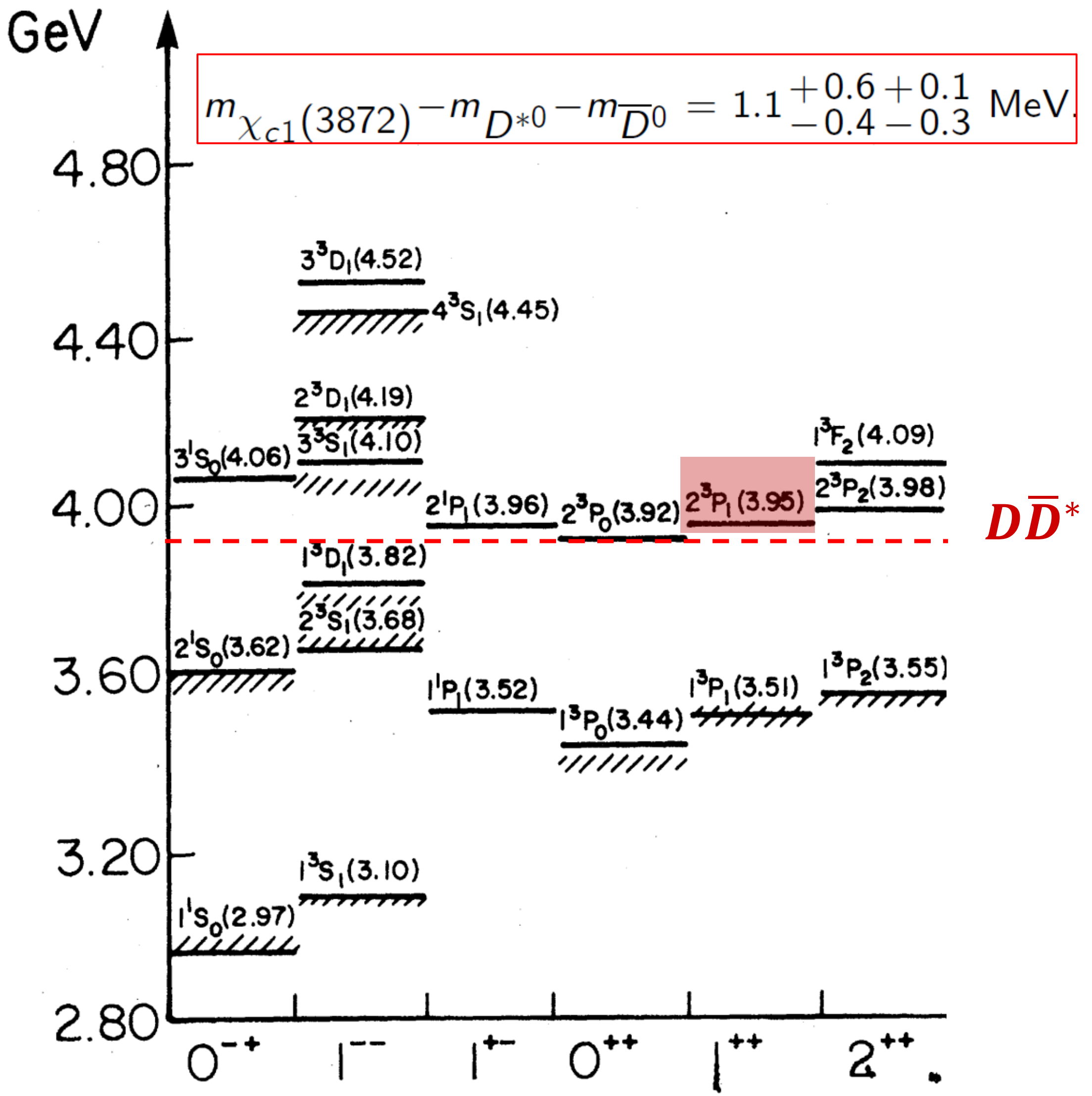
$$V_{ij}^{LS} = -\frac{\alpha_{ij}}{16}\frac{\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j}{r_{ij}^3}\left(\frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{4}{m_i m_j} \right)\left\{ \mathbf{L}_{ij} \cdot (\mathbf{S}_i + \mathbf{S}_j) \right\} - \frac{\alpha_{ij}}{16}\frac{\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j}{r_{ij}^3}\left(\frac{1}{m_i^2} - \frac{1}{m_j^2} \right)\left\{ \mathbf{L}_{ij} \cdot (\mathbf{S}_i - \mathbf{S}_j) \right\},$$

$$V_{ij}^{T} = -\frac{\alpha_{ij}}{4}(\boldsymbol{\lambda}_i \cdot \boldsymbol{\lambda}_j)\frac{1}{m_i m_j r_{ij}^3}\left\{ \frac{3(\mathbf{S}_i \cdot \mathbf{r}_{ij})(\mathbf{S}_j \cdot \mathbf{r}_{ij})}{r_{ij}^2} - \mathbf{S}_i \cdot \mathbf{S}_j \right\}$$



The QM state $\chi_{c1}(2P)$ is about 60 MeV higher than the physical state $X(3872)$.

$n^{2S+1}L_J$	Name	J^{PC}	Exp. [6]	[8]	[11]	LP	SP
1^3S_1	J/ψ	1^{--}	3097[a]	3090	3097	3097	3097
1^1S_0	$\eta_c(1S)$	0^{-+}	2984[a]	2982	2979	2983	2984
2^3S_1	$\psi(2S)$	1^{--}	3686[a]	3672	3673	3679	3679
2^1S_0	$\eta_c(2S)$	0^{-+}	3639[a]	3630	3623	3635	3637
3^3S_1	$\psi(3S)$	1^{--}	4040[a]	4072	4022	4078	4030
3^1S_0	$\eta_c(3S)$	0^{-+}	...	4043	3991	4048	4004
4^3S_1	$\psi(4S)$	1^{--}	4415?	4406	4273	4412	4281
4^1S_0	$\eta_c(4S)$	0^{-+}	...	4384	4250	4388	4264
5^3S_1	$\psi(5S)$	1^{--}	...	...	4463	4711	4472
5^1S_0	$\eta_c(5S)$	0^{-+}	...	...	4446	4690	4459
1^3P_2	$\chi_{c2}(1P)$	2^{++}	3556[a]	3556	3554	3552	3553
1^3P_1	$\chi_{c1}(1P)$	1^{++}	3511[a]	3505	3510	3516	3521
1^3P_0	$\chi_{c0}(1P)$	0^{++}	3415[a]	3424	3433	3415	3415
1^1P_1	$h_c(1P)$	1^{+-}	3525[a]	3516	3519	3522	3526
2^3P_2	$\chi_{c2}(2P)$	2^{++}	3927[a]	3972	3937	3967	3937
2^3P_1	$\chi_{c1}(2P)$	1^{++}	...	3925	3901	3937	3914
2^3P_0	$\chi_{c0}(2P)$	0^{++}	3918?	3852	3842	3869	3848
2^1P_1	$h_c(2P)$	1^{+-}	...	3934	3908	3940	3916



W.J. Deng et al., PRD95, 034026 (2017)

Godfrey and Isgur, PRD32, 189 (1985)

[8] T. Barnes, S. Godfrey, and E. S. Swanson, PRD 72, 054026 (2005).
[11] B. Q. Li and K. T. Chao, PRD 79, 094004 (2009).

Energy decomposition of the low-lying charmonium states

	State	Mass	$\langle T\rangle$	$\langle V^{Lin}\rangle$	$\langle V^{Coul}\rangle$	$\langle V^{SS}\rangle$	$\langle V^{T}\rangle$	$\langle V^{LS}\rangle$
	1^1S_0	2984	523	232	-638	-100	...	...
J/ψ	1^3S_1	3097	379	267	-539	24	...	...
	2^1S_0	3635	451	554	-303	-33	...	...
$\psi(2S)$	2^3S_1	3679	426	572	-297	11	...	...
	1^3P_0	3417	555	383	-329	6	-115	-48
	1^1P_1	3522	375	456	-268	-7	...	...
	1^3P_1	3516	387	449	-272	3	-31	14
	1^3P_2	3552	328	485	-247	1	21	-2

For J/ψ, $\left|\frac{\langle V^{Lin}\rangle}{\langle V^{Coul}\rangle}\right| \simeq 0.49$

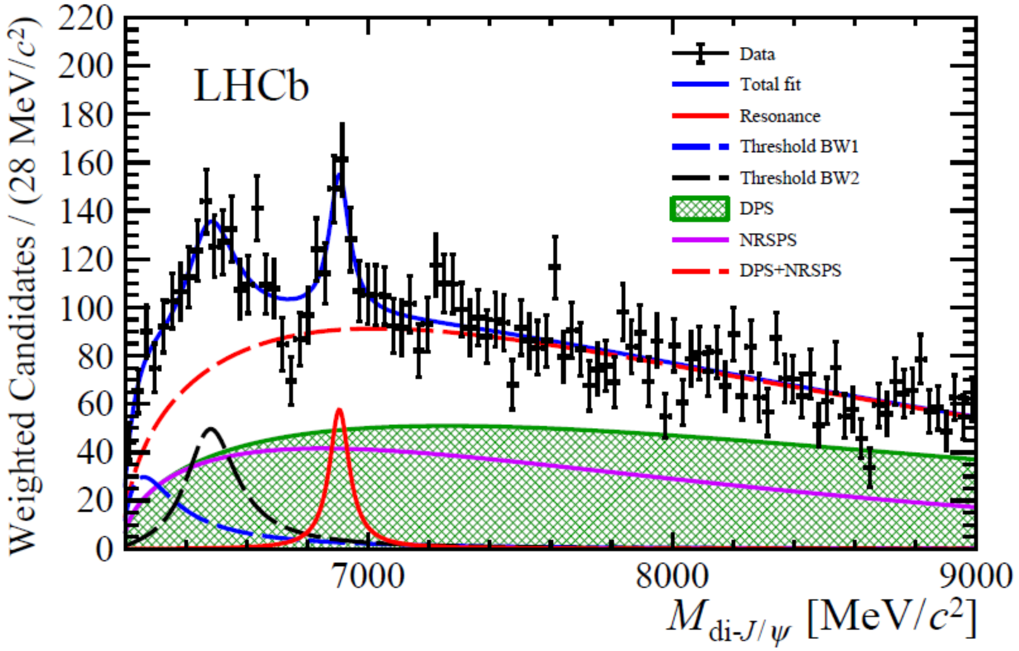
For $\psi(2S)$, $\left|\frac{\langle V^{Lin}\rangle}{\langle V^{Coul}\rangle}\right| \simeq 1.92$

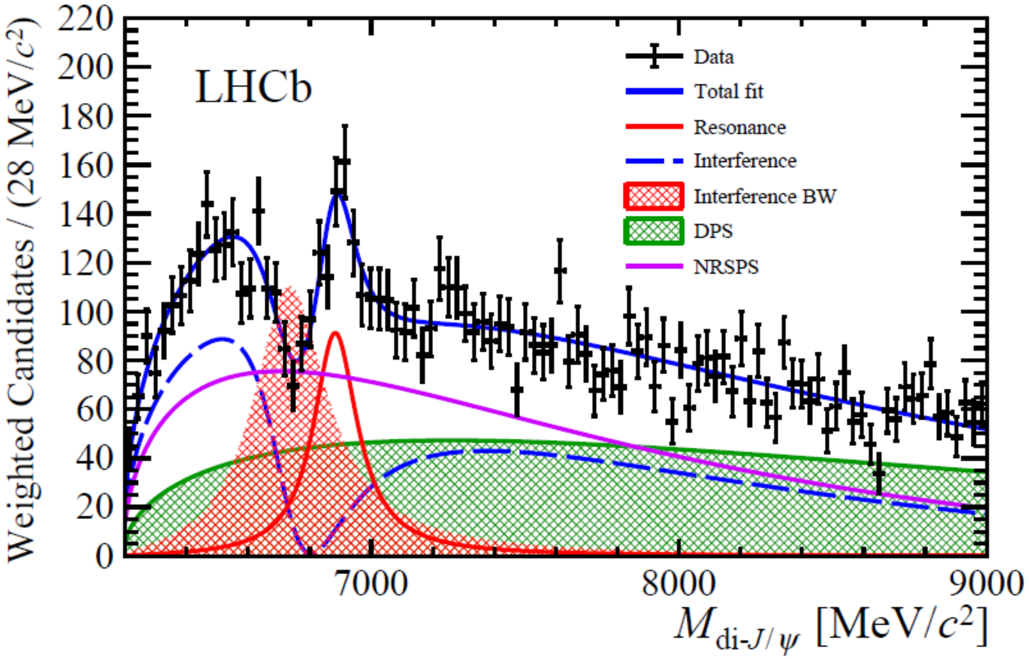
The ratio increases quickly for excited states.

- The linear confinement potential increases with the excitation quantum numbers, while the Coulomb potential decreases. Their combined contribution drives the instability of the excited states.
- How do these energy terms evolve with the increased constituent degrees of freedom?

W.J. Deng, H. Liu, L.C. Gui, X.H. Zhong, PRD95, 034026 (2017)
M. S. Liu, Q. F. Lu, X. H. Zhong and Q. Zhao, PRD 100, 016006 (2019)

Fully-heavy tetraquarks as ideal exotic candidates





$$m[X(6900)] = 6905 \pm 11 \pm 7\,\text{MeV}/c^2$$

$$\Gamma[X(6900)] = 80 \pm 19 \pm 33\,\text{MeV},$$

$$m[X(6900)] = 6886 \pm 11 \pm 11\,\text{MeV}/c^2$$

$$\Gamma[X(6900)] = 168 \pm 33 \pm 69\,\text{MeV}.$$

LHCb Collaboration, arXiv:2006.16957v1 [hep-ex]

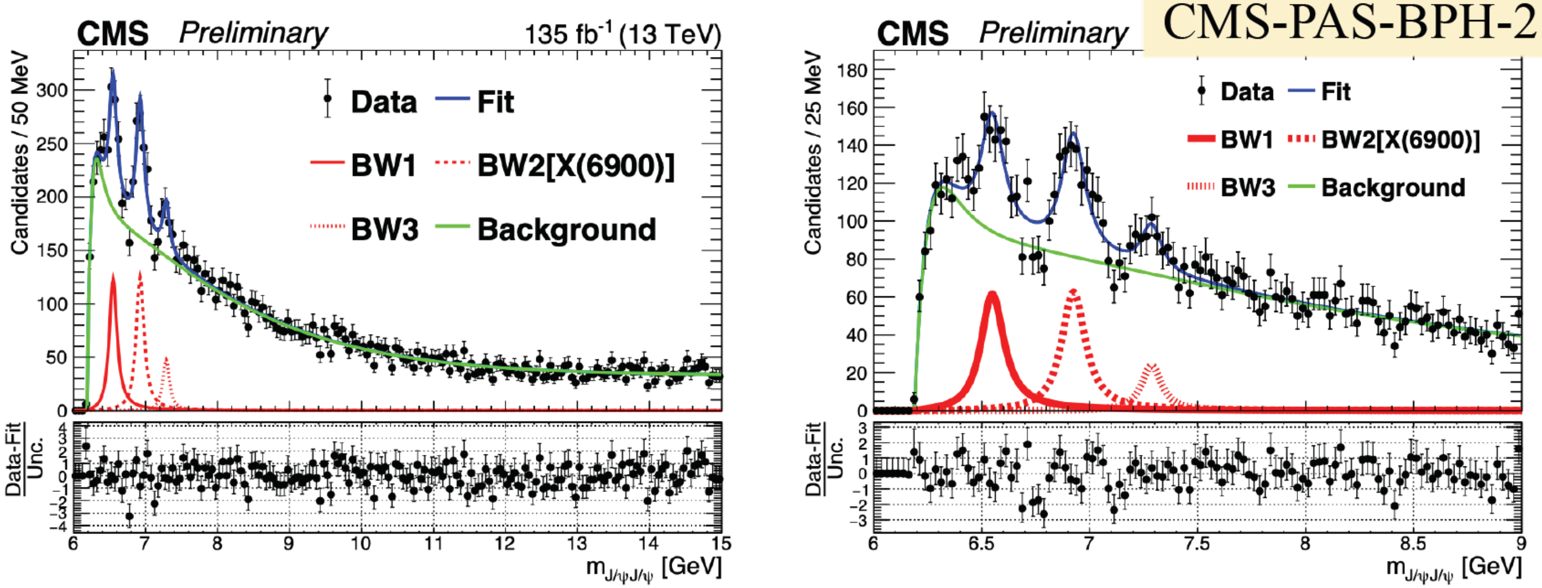
Unbound fully-heavy tetraquarks:

1. J.~P.~Ader, J.~M.~Richard and P.~Taxil, Phys.\ Rev.\ D {\bf 25}, 2370 (1982)
2. J.~M.~Richard, A.~Valcarce and J.~Vijande, Phys.\ Rev.\ D {\bf 95}, 054019 (2017); Phys.\ Rev.\ C {\bf 97}, 035211 (2018)
3. M.~Karliner, S.~Nussinov and J.~L.~Rosner, Phys.\ Rev.\ D {\bf 95}, 034011 (2017)
4. J.~Wu, Y.~R.~Liu, K.~Chen, X.~Liu, and S.~L.~Zhu, Phys.\ Rev.\ D {\bf 97}, 094015 (2018)
5. M. S. Liu, Q. F. Lu, X. H. Zhong and Q. Zhao, Phys. Rev. D 100, 016006 (2019).
6. M.S. Liu, F.X. Liu, X.H. Zhong and Q. Zhao, 2006.11952 [hep-ph].
7. X.Z. Weng, X.L. Chen, W.Z. Deng, and S.L. Zhu, arXiv:2010.05163v1 [hep-ph]

Bound fully-heavy tetraquarks:

1. L.~Heller and J.~A.~Tjon, Phys.\ Rev.\ D {\bf 32}, 755 (1985)
2. R.~J.~Lloyd and J.~P.~Vary, Phys.\ Rev.\ D {\bf 70}, 014009 (2004)
3. N.~Barnea, J.~Vijande and A.~Valcarce, Phys.\ Rev.\ D {\bf 73}, 054004 (2006);
4. M.~N.~Anwar, J.~Ferretti, F.~K.~Guo, E.~Santopinto and B.~S.~Zou, Eur.\ Phys.\ J.\ C {\bf 78}, 647 (2018)
5. Y.~Bai, S.~Lu and J.~Osborne, Phys.Lett.B 798 (2019) 134930

CMS-PAS-BPH-21-003



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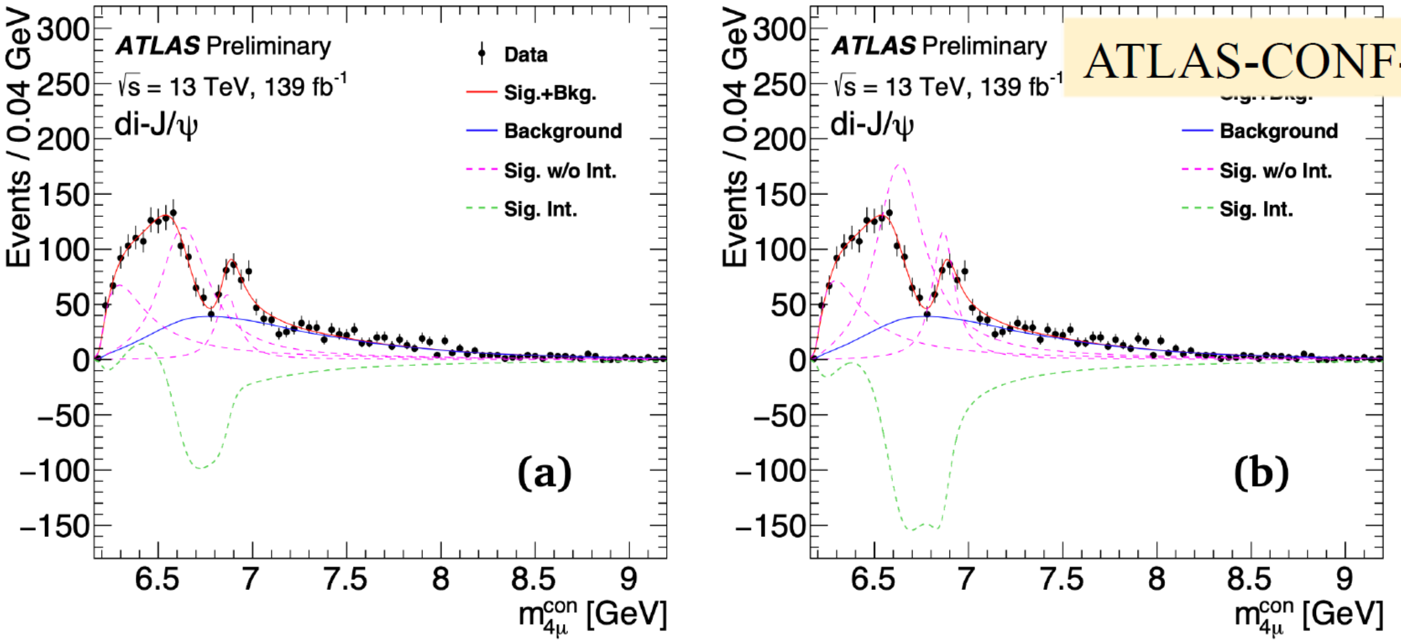

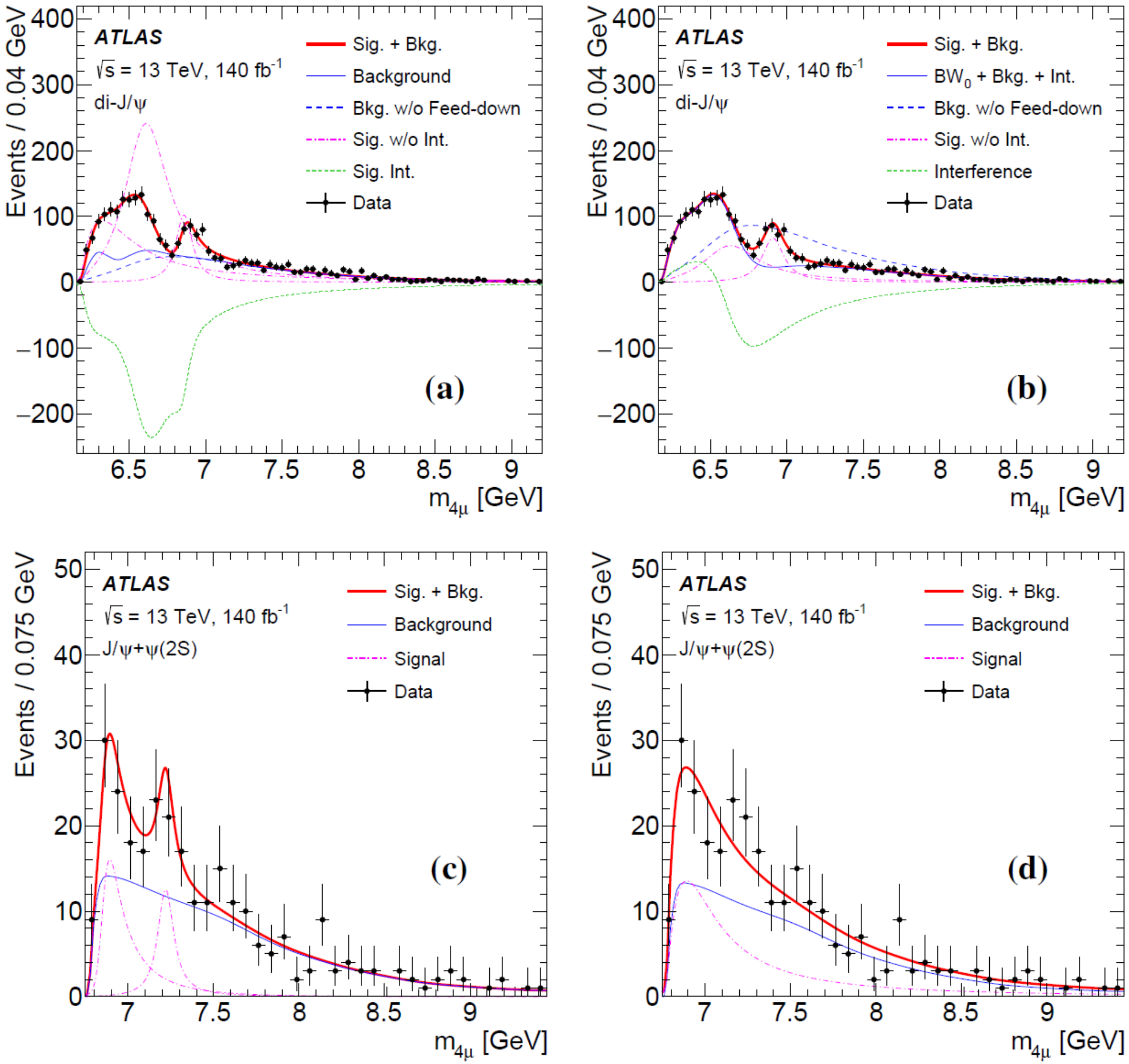

di-J/ψ

J/ψ-$\psi(2S)$

ATLAS Colla., arXiv:2304.08962v1 [hep-ex]

Constituent quark models:

X. Jin et al., Eur.Phys.J.C 80 (2020) 11, 1083
G. Yang, J.L. Ping, J. Segovia, Symmetry 12 (2020) 11, 1869
X.Z. Wen et al., Phys.Rev.D 103 (2021) 3, 034001 [only the color-magnetic interaction included]
Q.F. Lyu et al., Eur.Phys.J.C 80 (2020) 9, 871 [Extended relativized quark model]
M.C. Gordillo, F. De. Soto, J. Segovia, Phys.Rev.D 102 (2020) 11, 114007 [Difussion Monto Carlo mechod]
M.S. Liu, F.X. Liu, X.H. Zhong, Q. Zhao, 2006.11952 [hep-ph]
G.J. Wang, L. Meng, M. Oka, S.L. Zhu, PRD 104, 036016 (2021)

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X.K. Dong, V. Baru, C. Hanhart, F.K. Guo, A. Nefediev, Phys.Rev.Lett. 126 (2021) 13, 132001
C. Gong, M.C. Du, Q. Zhao, X.H. Zhong, B. Zhou, Phys.Lett.B 824 (2022) 136794
Z. Zhang et al., Phys.Rev.D 105 (2022) 5, 054026
Z.H. Guo, J.A. Oller, Phys.Rev.D 103 (2021) 3, 034024
C. Gong, M.C. Du, Q. Zhao, Phys.Rev.D 106 (2022), 054011

Diquark model:

M.A. Bedolla et al., Eur.Phys.J.C 80 (2020) 11, 1004
J.F. Giron, R.F. Lebed, Phys.Rev.D 102 (2020) 7, 074003
M. Karliner, J.L. Rosner, Phys.Rev.D 102 (2020) 11, 114039
R.N. Faustov, et al., Universe 7 (2021) 4, 94
H.W. Ke, X.H. Liu, Y.L. Shi, Eur.Phys.J.C 81 (2021) 5, 427

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L.B. Wang and C.F. Qiao, Phys.Lett.B 817 (2021) 136339

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Z.G. Wang, Int.J.Mod.Phys.A 36 (2021) 02, 2150014
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R.H. Wu et al., JHEP 11 (2022) 023

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C. Becchi, A. Giachino, L. Maiani, E. Santopinto, Phys.Lett.B 811 (2020) 135952
J.Z. Wang, D.Y. Chen, X. Liu, T. Matsuki, Phys.Rev.D 103 (2021) 7, 071503
R.L. Zhu, Nucl.Phys.B 966 (2021) 115393
Rafał Maciuła et al., Phys.Lett.B 812 (2021) 136010
Y.Q. Ma and H.F. Zhang, 2009.08376 [hep-ph]
F. Feng et al., Phys.Rev.D 106 (2022) 11, 114029
V. P. Gonçalves, Bruno D. Moreira, Phys.Lett.B 816 (2021) 136249

Now, all the calculated masses are above the di-charmonium threshold!

Typical fully-heavy tetraquark system with color-spin-flavor symmetry

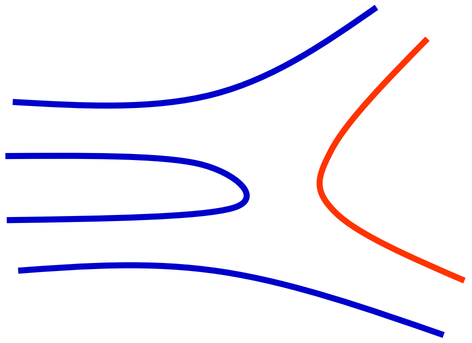
$T = Q_1 Q_2 \bar{Q}_3 \bar{Q}_4$ $(cc\bar{c}\bar{c}, bb\bar{b}\bar{b}, cc\bar{b}\bar{b}, bc\bar{b}\bar{c}, bc\bar{c}\bar{c}, bc\bar{b}\bar{b}, \ldots)$

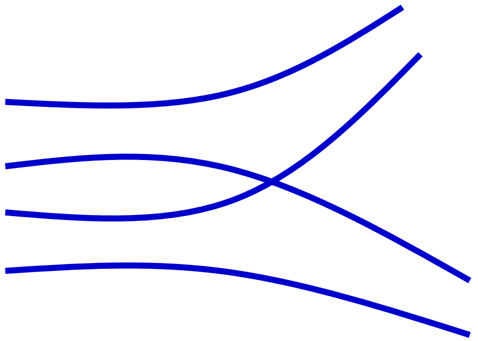
– ideal candidates for genuine tetraquark states

Stability of the four-body system:

$M_T < 4m_Q$ → stable

$M_T > 4m_Q$ → unstable.





Typical fully-heavy tetraquark system with color-spin-flavor symmetry

$T = Q_1 Q_2 \bar{Q}_3 \bar{Q}_4 \; (cc\bar{c}\bar{c}, bb\bar{b}\bar{b}, cc\bar{b}\bar{b}, bc\bar{b}\bar{c}, bc\bar{c}\bar{c}, bc\bar{b}\bar{b}, \ldots)$

Diquark (anti-diquark) configurations:

- Spin: $2 \otimes 2 = 1 + 5$
- Color: $3 \otimes 3 = \bar{3} + 6\,;\; \bar{3} \otimes \bar{3} = 3 + \bar{6}$
- Flavor: $\{Q_1 Q_2\} = (Q_1 Q_2 + Q_2 Q_1)/\sqrt{2}\,;\, [Q_1 Q_2] = (Q_1 Q_2 - Q_2 Q_1)/\sqrt{2}$

$$
\begin{aligned}
|1\rangle &= |[Q_1 Q_2]_1^6 [\bar{Q}_3 \bar{Q}_4]_1^{\bar{6}}\rangle_0^0, & |2\rangle &= |\{Q_1 Q_2\}_0^6 \{\bar{Q}_3 \bar{Q}_4\}_0^{\bar{6}}\rangle_0^0, \\
|3\rangle &= |\{Q_1 Q_2\}_1^{\bar{3}} \{\bar{Q}_3 \bar{Q}_4\}_1^{3}\rangle_0^0, & |4\rangle &= |[Q_1 Q_2]_0^{\bar{3}} [\bar{Q}_3 \bar{Q}_4]_0^{3}\rangle_0^0, \\
|5\rangle &= |[Q_1 Q_2]_1^6 [\bar{Q}_3 \bar{Q}_4]_1^{\bar{6}}\rangle_1^0, & |6\rangle &= |[Q_1 Q_2]_1^6 \{\bar{Q}_3 \bar{Q}_4\}_0^{\bar{6}}\rangle_1^0, \\
|7\rangle &= |\{Q_1 Q_2\}_0^6 [\bar{Q}_3 \bar{Q}_4]_1^{\bar{6}}\rangle_1^0, & |8\rangle &= |\{Q_1 Q_2\}_1^{\bar{3}} \{\bar{Q}_3 \bar{Q}_4\}_1^{3}\rangle_1^0, \\
|9\rangle &= |\{Q_1 Q_2\}_1^{\bar{3}} [\bar{Q}_3 \bar{Q}_4]_0^{3}\rangle_1^0, & |10\rangle &= |[Q_1 Q_2]_0^{\bar{3}} \{\bar{Q}_3 \bar{Q}_4\}_1^{3}\rangle_1^0, \\
|11\rangle &= |[Q_1 Q_2]_1^6 [\bar{Q}_3 \bar{Q}_4]_1^{\bar{6}}\rangle_2^0, & |12\rangle &= |\{Q_1 Q_2\}_1^{\bar{3}} \{\bar{Q}_3 \bar{Q}_4\}_1^{3}\rangle_2^0,
\end{aligned}
$$

TABLE I. Configurations of all-heavy tetraquarks.

System	$J^{P(C)}$	Configuration		
$cc\bar{c}\,\bar{c}$	0^{++}	$\vert\{cc\}^{6}_{0}\{\bar{c}\,\bar{c}\}^{\bar{6}}_{0}\rangle^{0}_{0}$	$\vert\{cc\}^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{0}$	$\cdots$
	1^{+-}	$\vert\{cc\}^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{1}$	$\cdots$	$\cdots$
	2^{++}	$\vert\{cc\}^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{2}$	$\cdots$	$\cdots$
$bb\bar{b}\,\bar{b}$	0^{++}	$\vert\{bb\}^{6}_{0}\{\bar{b}\,\bar{b}\}^{\bar{6}}_{0}\rangle^{0}_{0}$	$\vert\{bb\}^{\bar{3}}_{1}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{0}$	$\cdots$
	1^{+-}	$\vert\{bb\}^{\bar{3}}_{1}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{1}$	$\cdots$	$\cdots$
	2^{++}	$\vert\{bb\}^{\bar{3}}_{1}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{2}$	$\cdots$	$\cdots$
$bb\bar{c}\,\bar{c}$	0^{+}	$\vert\{bb\}^{6}_{0}\{\bar{c}\,\bar{c}\}^{\bar{6}}_{0}\rangle^{0}_{0}$	$\vert\{bb\}^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{0}$	$\cdots$
	1^{+}	$\vert\{bb\}^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{1}$	$\cdots$	$\cdots$
	2^{+}	$\vert\{bb\}^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{2}$	$\cdots$	$\cdots$
$bc\bar{c}\,\bar{c}$	0^{+}	$\vert(bc)^{6}_{0}\{\bar{c}\,\bar{c}\}^{\bar{6}}_{0}\rangle^{0}_{0}$	$\vert(bc)^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{0}$	$\cdots$
	1^{+}	$\vert(bc)^{6}_{1}\{\bar{c}\,\bar{c}\}^{\bar{6}}_{0}\rangle^{0}_{1}$	$\vert(bc)^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{1}$	$\vert(bc)^{\bar{3}}_{0}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{1}$
	2^{+}	$\vert(bc)^{\bar{3}}_{1}\{\bar{c}\,\bar{c}\}^{3}_{1}\rangle^{0}_{2}$	$\cdots$	$\cdots$
$bc\bar{b}\,\bar{b}$	0^{+}	$\vert(bc)^{6}_{0}\{\bar{b}\,\bar{b}\}^{\bar{6}}_{0}\rangle^{0}_{0}$	$\vert(bc)^{\bar{3}}_{1}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{0}$	$\cdots$
	1^{+}	$\vert(bc)^{6}_{1}\{\bar{b}\,\bar{b}\}^{\bar{6}}_{0}\rangle^{0}_{1}$	$\vert(bc)^{\bar{3}}_{1}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{1}$	$\vert(bc)^{\bar{3}}_{0}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{1}$
	2^{+}	$\vert(bc)^{\bar{3}}_{1}\{\bar{b}\,\bar{b}\}^{3}_{1}\rangle^{0}_{2}$	$\cdots$	$\cdots$
$bc\bar{b}\,\bar{c}$	0^{++}	$\vert(bc)^{6}_{1}(\bar{b}\,\bar{c})^{\bar{6}}_{1}\rangle^{0}_{0}$	$\vert(bc)^{6}_{0}(\bar{b}\,\bar{c})^{\bar{6}}_{0}\rangle^{0}_{0}$	$\cdots$
		$\vert(bc)^{\bar{3}}_{1}(\bar{b}\,\bar{c})^{3}_{1}\rangle^{0}_{0}$	$\vert(bc)^{\bar{3}}_{0}(\bar{b}\,\bar{c})^{3}_{0}\rangle^{0}_{0}$	$\cdots$
	1^{+-}	$\vert(bc)^{6}_{1}(\bar{b}\,\bar{c})^{\bar{6}}_{1}\rangle^{0}_{1}$	$\frac{1}{\sqrt{2}}\vert(bc)^{6}_{1}(\bar{b}\,\bar{c})^{\bar{6}}_{0}\rangle^{0}_{1} - \vert(bc)^{6}_{0}(\bar{b}\,\bar{c})^{\bar{6}}_{1}\rangle^{0}_{1}$	$\cdots$
		$\vert(bc)^{\bar{3}}_{1}(\bar{b}\,\bar{c})^{3}_{1}\rangle^{0}_{1}$	$\frac{1}{\sqrt{2}}\vert(bc)^{\bar{3}}_{1}(\bar{b}\,\bar{c})^{3}_{0}\rangle^{0}_{1} - \vert(bc)^{\bar{3}}_{0}(\bar{b}\,\bar{c})^{3}_{1}\rangle^{0}_{1}$	$\cdots$
	1^{++}	$\frac{1}{\sqrt{2}}\vert(bc)^{6}_{1}(\bar{b}\,\bar{c})^{\bar{6}}_{0}\rangle^{0}_{1} + \vert(bc)^{6}_{0}(\bar{b}\,\bar{c})^{\bar{6}}_{1}\rangle^{0}_{1}$	$\frac{1}{\sqrt{2}}\vert(bc)^{\bar{3}}_{1}(\bar{b}\,\bar{c})^{3}_{0}\rangle^{0}_{1} + \vert(bc)^{\bar{3}}_{0}(\bar{b}\,\bar{c})^{3}_{1}\rangle^{0}_{1}$	$\cdots$
	2^{++}	$\vert(bc)^{6}_{1}(\bar{b}\,\bar{c})^{\bar{6}}_{1}\rangle^{0}_{2}$	$\vert(bc)^{\bar{3}}_{1}(\bar{b}\,\bar{c})^{3}_{1}\rangle^{0}_{2}$	$\cdots$

Reminder of the Hamiltonian adopted

Hamiltonian in a non-relativistic quark model:

$$H=\left(\sum_{i=1}^{4} m_i+T_i\right)-T_G+\sum_{i<j} V_{ij}(r_{ij})$$

$$T_i=\frac{p_i^2}{2m_i}$$

$$V_{ij}(r_{ij})=V_{ij}^{\mathrm{OGE}}(r_{ij})+V_{ij}^{\mathrm{Conf}}(r_{ij}),$$

$$V_{ij}^{\mathrm{Conf}}(r_{ij})=-\frac{3}{16}(\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j)\cdot b r_{ij},$$

$$V_{ij}^{\mathrm{OGE}}=\frac{\alpha_{ij}}{4}(\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j)\left\{\frac{1}{r_{ij}}-\frac{\pi}{2}\cdot\frac{\sigma_{ij}^3 e^{-\sigma_{ij}^2 r_{ij}^2}}{\pi^{3/2}}\cdot\frac{4}{3m_i m_j}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\right\}$$

$$\begin{aligned}V_{ij}^{LS}=&-\frac{\alpha_{ij}}{16}\frac{\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j}{r_{ij}^3}\left(\frac{1}{m_i^2}+\frac{1}{m_j^2}+\frac{4}{m_i m_j}\right)\left\{\mathbf{L}_{ij}\cdot(\mathbf{S}_i+\mathbf{S}_j)\right\}\\&-\frac{\alpha_{ij}}{16}\frac{\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j}{r_{ij}^3}\left(\frac{1}{m_i^2}-\frac{1}{m_j^2}\right)\left\{\mathbf{L}_{ij}\cdot(\mathbf{S}_i-\mathbf{S}_j)\right\},\end{aligned}$$

$$V_{ij}^{T}=-\frac{\alpha_{ij}}{4}(\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j)\frac{1}{m_i m_j r_{ij}^3}\left\{\frac{3(\mathbf{S}_i\cdot\mathbf{r}_{ij})(\mathbf{S}_j\cdot\mathbf{r}_{ij})}{r_{ij}^2}-\mathbf{S}_i\cdot\mathbf{S}_j\right\}$$

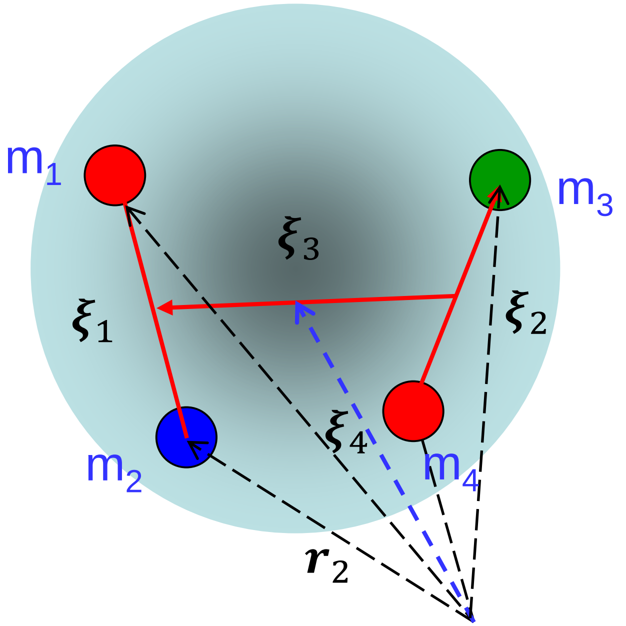
Color matrix elements:

	$\langle\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2\rangle$	$\langle\boldsymbol{\lambda}_3\cdot\boldsymbol{\lambda}_4\rangle$	$\langle\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_3\rangle$	$\langle\boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}_4\rangle$	$\langle\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_4\rangle$	$\langle\boldsymbol{\lambda}_2\cdot\boldsymbol{\lambda}_3\rangle$
$\langle\zeta_1\|\hat{O}\|\zeta_1\rangle$	$4/3$	$4/3$	$-10/3$	$-10/3$	$-10/3$	$-10/3$
$\langle\zeta_2\|\hat{O}\|\zeta_2\rangle$	$-8/3$	$-8/3$	$-4/3$	$-4/3$	$-4/3$	$-4/3$
$\langle\zeta_1\|\hat{O}\|\zeta_2\rangle$	0	0	$-2\sqrt{2}$	$-2\sqrt{2}$	$2\sqrt{2}$	$2\sqrt{2}$

Dynamic feature for fully-heavy tetraquark system

Jacobi coordinates:

$$\begin{cases} \boldsymbol{\xi}_1 \equiv \mathbf{r_1} - \mathbf{r_2}, \\ \boldsymbol{\xi}_2 \equiv \mathbf{r_3} - \mathbf{r_4}, \\ \boldsymbol{\xi}_3 \equiv \dfrac{m_1\mathbf{r_1} + m_2\mathbf{r_2}}{m_1 + m_2} - \dfrac{m_3\mathbf{r_3} + m_4\mathbf{r_4}}{m_3 + m_4}, \\ \boldsymbol{\xi}_4 \equiv \dfrac{m_1\mathbf{r_1} + m_2\mathbf{r_2} + m_3\mathbf{r_3} + m_4\mathbf{r_4}}{m_1 + m_2 + m_3 + m_4}, \end{cases}$$



Jacobi coordinate

Trial wavefunction for the ground states expanded by a series of Gaussian functions:

$$\psi(\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}, \mathbf{r_4}) = \sum_{\ell}^{n} C_\ell \prod_{i=1}^{4} \left(\frac{m_i \omega_\ell}{\pi}\right)^{3/4} \exp\left[-\frac{m_i \omega_\ell}{2} r_i^2\right]$$

$$\Rightarrow \quad \psi(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3, \boldsymbol{\xi}_4) = \sum_{\ell}^{n} C_\ell \prod_{i=1}^{4} \left(\frac{\mu_i \omega_\ell}{\pi}\right)^{3/4} \exp\left[-\frac{\mu_i \omega_\ell}{2} \xi_i^2\right]$$

E. Hiyama, Y. Kino, and M. Kamimura, Gaussian expansion method for few-body systems, Prog. Part. Nucl. Phys. 51, 223 (2003).

Liu, Lyu, Zhong, and Zhao, Phys. Rev. D 100, 016006 (2019) ; M. S. Liu, Q. F. Lu, X. H. Zhong and Q. Zhao, Phys. Rev. D 100, 016006 (2019). M.S. Liu, F.X. Liu, X.H. Zhong and Q. Zhao, 2006.11952 [hep-ph].

The contributions from each part of the Hamiltonian of the $cc\bar{c}\bar{c}$ and $bb\bar{b}\bar{b}$ systems in units of MeV with $L = 0$.

J^{PC}	Configuration	M	$\langle T \rangle$	$\langle V^{\mathrm{Conf}} \rangle$	$\langle V^{\mathrm{OGE}}_{\mathrm{coul}} \rangle$	$\langle V^{\mathrm{OGE}}_{CM} \rangle$
0^{++}	$\vert\{cc\}^6_0\{\bar{c}\bar{c}\}^{\bar{6}}_0\rangle^0_0$	6518	715	664	−811	18
	$\vert\{cc\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_0$	6487	756	646	−834	−13
1^{+-}	$\vert\{cc\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_1$	6500	739	653	−825	0
2^{++}	$\vert\{cc\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_2$	6524	708	667	−806	23
0^{++}	$\vert\{bb\}^6_0\{\bar{b}\bar{b}\}^{\bar{6}}_0\rangle^0_0$	19338	768	356	−1203	9
	$\vert\{bb\}^{\bar{3}}_1\{\bar{b}\bar{b}\}^3_1\rangle^0_0$	19322	796	350	−1225	−6
1^{+-}	$\vert\{bb\}^{\bar{3}}_1\{\bar{b}\bar{b}\}^3_1\rangle^0_1$	19329	785	353	−1216	0
2^{++}	$\vert\{bb\}^{\bar{3}}_1\{\bar{b}\bar{b}\}^3_1\rangle^0_2$	19341	763	357	−1199	12

- The confining potential contributes a positive energy and cannot be neglected.
- This implies that the four-quark systems are located above the two-heavy meson thresholds.
- The treatment of the confinement potential seems to distinguish models in the literature.

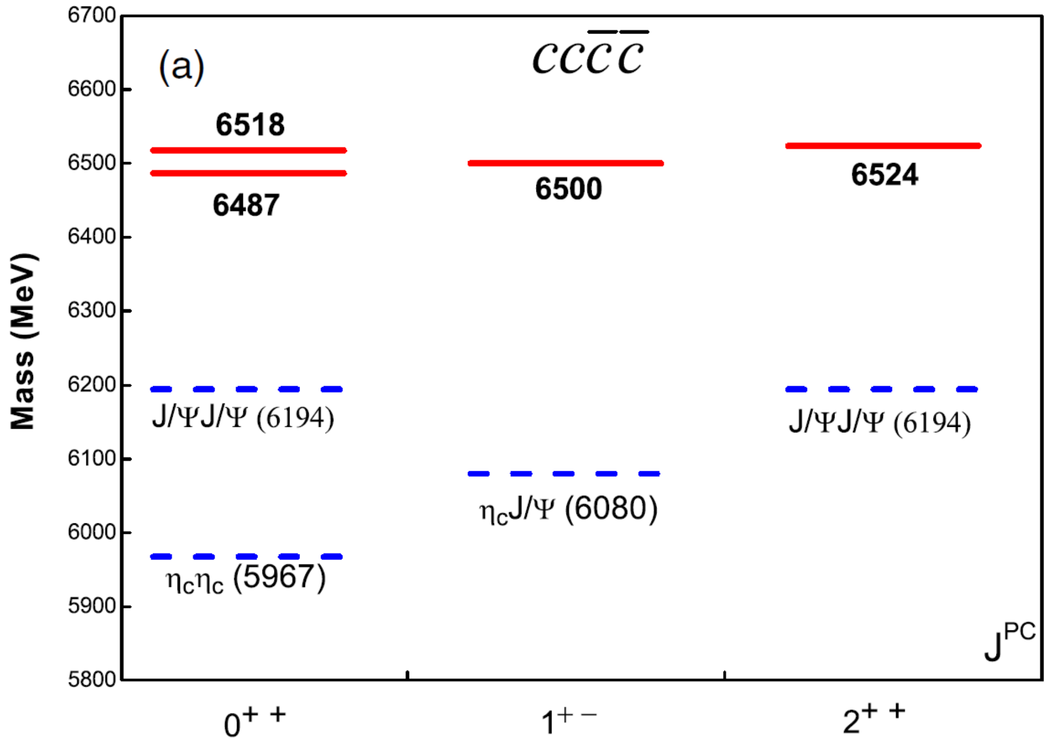
(See M. S. Liu, Q. F. Lu, X. H. Zhong and Q. Zhao, Phys. Rev. D 100, 016006 (2019), and references therein.)

Predicted masses (MeV) for the $cc\bar{c}\bar{c}$ system compared with other models.

State	Ours	Ref. [29]	Ref. [16]	Ref. [11]	Ref. [12]	Ref. [13]	Ref. [22]	Ref. [17]	Refs. [25,26]	Ref. [27]	Ref. [24]	Ref. [21]
0^{++}	6487	6797	6477	6460–6470	6437	6200	6192	6038–6115	5990	5969	5966	<6140
0^{++}	6518	7016	6695	6440–6820	6383	...	...	...	...	...	...	
1^{+-}	6500	6899	6528	6370–6510	6437	...	...	6101–6176	6050	6021	6051	
2^{++}	6524	6956	6573	6370–6510	6437	...	...	6172–6216	6090	6115	6223	

Predicted mass spectra for the $cc\bar{c}\bar{c}$, $bb\bar{b}\bar{b}$ and $bb\bar{c}\bar{c}$ systems.

$J^{P(C)}$	Configuration	$\langle H \rangle$ (MeV)
0^{++}	$\lvert\{cc\}^6_0\{\bar{c}\bar{c}\}^{\bar{6}}_0\rangle^0_0$ $\lvert\{cc\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_0$	$\begin{pmatrix} 6518 & -0.2371 \\ -0.2371 & 6487 \end{pmatrix}$
1^{+-}	$\lvert\{cc\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_1$	(6500)
2^{++}	$\lvert\{cc\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_2$	(6524)
0^{++}	$\lvert\{bb\}^6_0\{\bar{b}\bar{b}\}^{\bar{6}}_0\rangle^0_0$ $\lvert\{bb\}^{\bar{3}}_1\{\bar{b}\bar{b}\}^3_1\rangle^0_0$	$\begin{pmatrix} 19338 & -0.1102 \\ -0.1102 & 19322 \end{pmatrix}$
1^{+-}	$\lvert\{bb\}^{\bar{3}}_1\{\bar{b}\bar{b}\}^3_1\rangle^0_1$	(19329)
2^{++}	$\lvert\{bb\}^{\bar{3}}_1\{\bar{b}\bar{b}\}^3_1\rangle^0_2$	(19341)
0^{+}	$\lvert\{bb\}^6_0\{\bar{c}\bar{c}\}^{\bar{6}}_0\rangle^0_0$ $\lvert\{bb\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_0$	$\begin{pmatrix} 13032 & -0.1105 \\ -0.1105 & 12953 \end{pmatrix}$
1^{+}	$\lvert\{bb\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_1$	(12960)
2^{+}	$\lvert\{bb\}^{\bar{3}}_1\{\bar{c}\bar{c}\}^3_1\rangle^0_2$	(12972)



The transition between color representations $\mathbf{6} \otimes \bar{\mathbf{6}}$ and $\mathbf{3} \otimes \bar{\mathbf{3}}$ seems to be small.

G.J. Wang, L. Meng, M. Oka, S.L. Zhu, PRD 104, 036016 (2021)

First orbital excitation states

The radial part of the wavefunction is expanded with a series of harmonic oscillator functions:

$$R_{n_\xi l_\xi}(\xi) = \sum_{\ell=1}^{n} C_{\xi\ell}\, \phi_{n_\xi l_\xi}(d_{\xi\ell}, \xi),$$

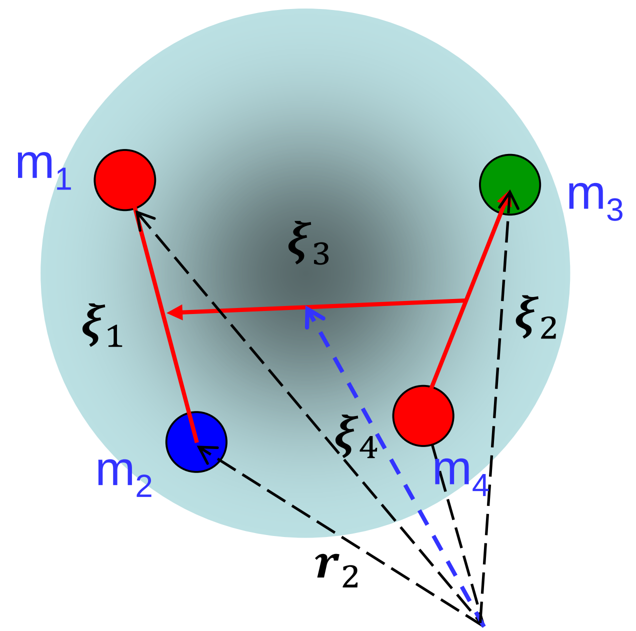
where

$$\phi_{n_\xi l_\xi}(d_{\xi\ell}, \xi) = \left(\frac{1}{d_{\xi\ell}}\right)^{\frac{3}{2}} \left[\frac{2^{l_\xi+2-n_\xi}(2l_\xi+2n_\xi+1)!!}{\sqrt{\pi}n_\xi![(2l_\xi+1)!!]^2}\right]^{\frac{1}{2}} \left(\frac{\xi}{d_{\xi\ell}}\right)^{l_\xi} \times e^{-\frac{1}{2}\left(\frac{\xi}{d_{\xi\ell}}\right)^2} F\left(-n_\xi, l_\xi+\frac{3}{2}, \left(\frac{\xi}{d_{\xi\ell}}\right)^2\right),$$

Parameter $d_{\xi l}$ is oscillator length and is related to the harmonic oscillator frequency $\omega_{\xi l}$:

$$\frac{1}{d_{\xi l}^2} = M_\xi \omega_{\xi l}$$

The number of harmonic oscillator functions n is fitted together with $d_{\xi l}$ with $l = 1, 2, \dots n$, via relation: $d_{\xi l} = d_{\xi 1} a^{l-1}$, with a the ratio coefficient.



Jacobi coordinate

With $\boldsymbol{S} = 0, 1, 2$, and $\boldsymbol{L} = 1$, the total spin of a tetraquark state can be: $\boldsymbol{J} = \boldsymbol{S} + \boldsymbol{L} = 0, 1, 2, 3$.

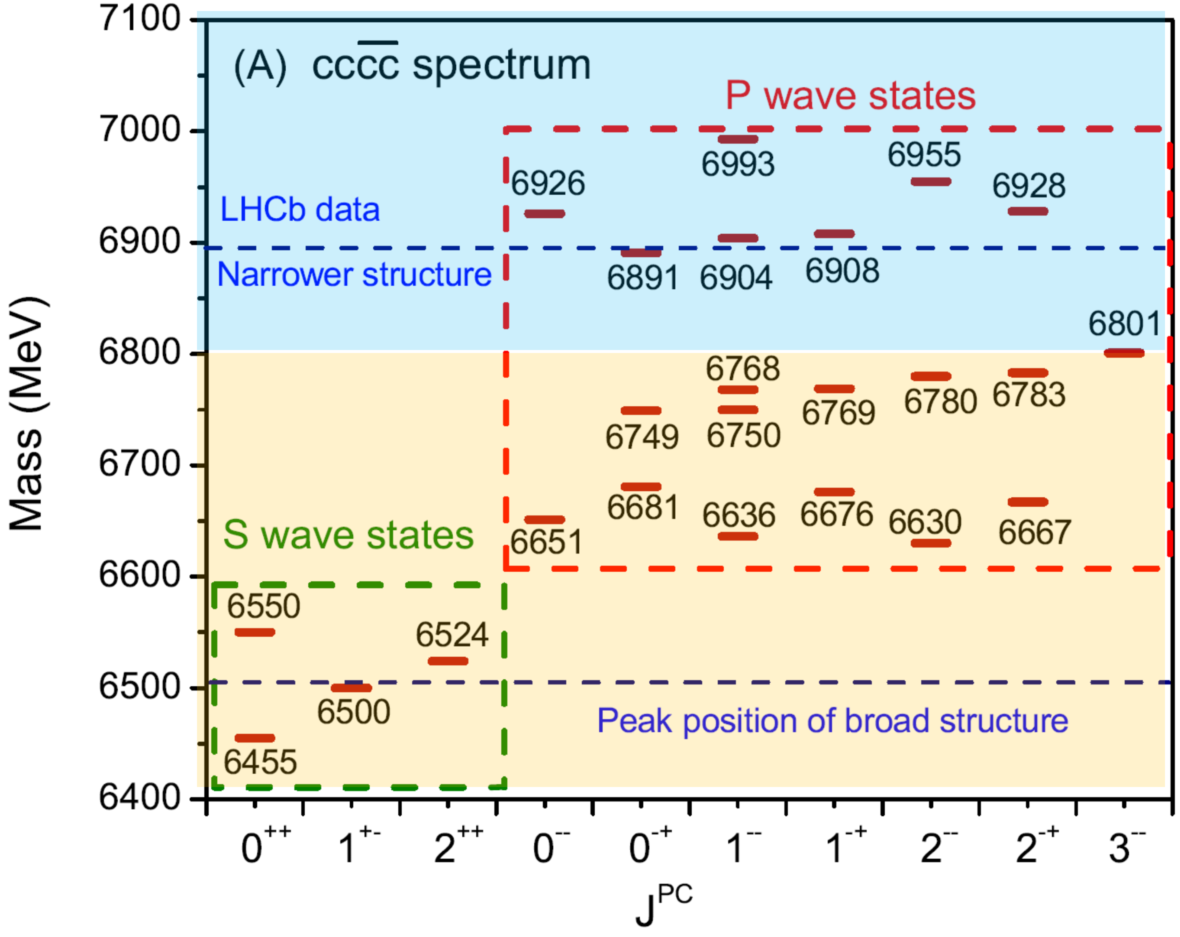
Predicted masses for the *P*-wave $cc\bar{c}\bar{c}$ states. (ξ_1, ξ_2) stands for a configuration containing both ξ_1 and ξ_2-mode of orbital excitations, while (ξ_3) stands for a configuration containing ξ_3-mode of orbital excitation.

$J^{P(C)}$	Configuration	$\langle H \rangle$ (MeV)	Mass (MeV)	Eigenvector
0^{--}	${}^3P_{0^{--}(6\bar{6})_c(\xi_1,\xi_2)}$ ${}^3P_{0^{--}(\bar{3}3)_c(\xi_1,\xi_2)}$	$\begin{pmatrix} 6751 & -132 \\ -132 & 6827 \end{pmatrix}$	$\begin{pmatrix} 6651 \\ 6926 \end{pmatrix}$	$\begin{pmatrix} (-0.7985 & -0.6020) \\ (-0.6020 & 0.7985) \end{pmatrix}$
0^{-+}	${}^3P_{0^{-+}(6\bar{6})_c(\xi_1,\xi_2)}$ ${}^3P_{0^{-+}(\bar{3}3)_c(\xi_1,\xi_2)}$ ${}^3P_{0^{-+}(\bar{3}3)_c(\xi_3)}$	$\begin{pmatrix} 6746 & 88 & 37 \\ 88 & 6825 & 18 \\ 37 & 18 & 6750 \end{pmatrix}$	$\begin{pmatrix} 6681 \\ 6749 \\ 6891 \end{pmatrix}$	$\begin{pmatrix} (0.82 & -0.47 & -0.32) \\ (-0.14 & 0.38 & 0.91) \\ (0.55 & 0.80 & 0.25) \end{pmatrix}$
1^{--}	${}^3P_{1^{--}(6\bar{6})_c(\xi_1,\xi_2)}$ ${}^3P_{1^{--}(\bar{3}3)_c(\xi_1,\xi_2)}$ ${}^5P_{1^{--}(\bar{3}3)_c(\xi_3)}$ ${}^1P_{1^{--}(\bar{3}3)_c(\xi_3)}$ ${}^1P_{1^{--}(6\bar{6})_c(\xi_3)}$	$\begin{pmatrix} 6733 & 132 & -29 & -16 & 31 \\ 132 & 6827 & -14 & -7 & 26 \\ -29 & -14 & 6754 & -3 & 10 \\ -16 & -7 & -3 & 6770 & -19 \\ 31 & 26 & 10 & -19 & 6968 \end{pmatrix}$	$\begin{pmatrix} 6636 \\ 6750 \\ 6768 \\ 6904 \\ 6993 \end{pmatrix}$	$\begin{pmatrix} (0.82 & -0.55 & 0.12 & 0.06 & -0.03) \\ (0.02 & -0.24 & -0.96 & -0.16 & 0.06) \\ (-0.01 & 0.05 & -0.17 & 0.98 & 0.10) \\ (-0.48 & -0.69 & 0.19 & 0.02 & 0.50) \\ (0.31 & 0.39 & -0.02 & -0.11 & 0.86) \end{pmatrix}$
1^{-+}	${}^3P_{1^{-+}(6\bar{6})_c(\xi_1,\xi_2)}$ ${}^3P_{1^{-+}(\bar{3}3)_c(\xi_1,\xi_2)}$ ${}^3P_{1^{-+}(\bar{3}3)_c(\xi_3)}$	$\begin{pmatrix} 6751 & -108 & 9 \\ -108 & 6834 & -4 \\ 9 & -4 & 6769 \end{pmatrix}$	$\begin{pmatrix} 6676 \\ 6769 \\ 6908 \end{pmatrix}$	$\begin{pmatrix} (0.82 & 0.56 & -0.05) \\ (-0.01 & -0.08 & -1.00) \\ (-0.57 & 0.82 & -0.06) \end{pmatrix}$
2^{--}	${}^3P_{2^{--}(6\bar{6})_c(\xi_1,\xi_2)}$ ${}^3P_{2^{--}(\bar{3}3)_c(\xi_1,\xi_2)}$ ${}^5P_{2^{--}(\bar{3}3)_c(\xi_3)}$	$\begin{pmatrix} 6746 & -155 & -18 \\ -155 & 6837 & 9 \\ -18 & 9 & 6781 \end{pmatrix}$	$\begin{pmatrix} 6630 \\ 6780 \\ 6955 \end{pmatrix}$	$\begin{pmatrix} (0.80 & 0.59 & 0.06) \\ (-0.01 & 0.12 & -1.00) \\ (-0.60 & 0.80 & 0.10) \end{pmatrix}$
2^{-+}	${}^3P_{2^{-+}(6\bar{6})_c(\xi_1,\xi_2)}$ ${}^3P_{2^{-+}(\bar{3}3)_c(\xi_1,\xi_2)}$ ${}^3P_{2^{-+}(\bar{3}3)_c(\xi_3)}$	$\begin{pmatrix} 6754 & 123 & 12 \\ 123 & 6841 & 6 \\ 12 & 6 & 6783 \end{pmatrix}$	$\begin{pmatrix} 6667 \\ 6783 \\ 6928 \end{pmatrix}$	$\begin{pmatrix} (0.82 & -0.57 & -0.06) \\ (0.00 & 0.10 & -1.00) \\ (0.58 & 0.81 & 0.08) \end{pmatrix}$
3^{--}	${}^5P_{3^{--}(\bar{3}3)_c(\xi_3)}$	(6801)	6801	1

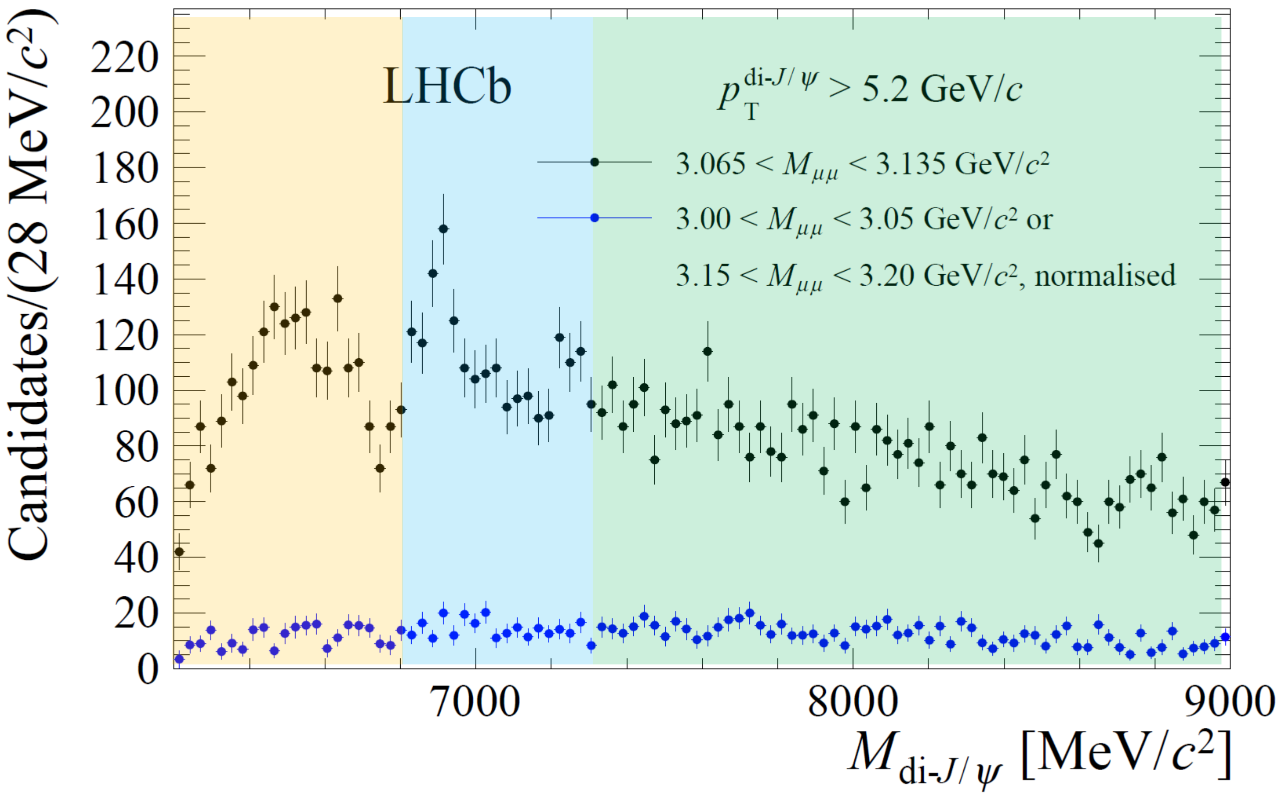
The contributions from each part of the Hamiltonian of the $cc\bar{c}\bar{c}$ systems in units of MeV with $L=1$.

$J^{P(C)}$	Configuration	Mass	$\langle T\rangle$	$\langle V^{Conf}\rangle$	$\langle V^{OGE}_{coul}\rangle$	$\langle V^{OGE}_{SS}\rangle$	$\langle V^{OGE}_{tensor}\rangle$	$\langle V^{OGE}_{LS}\rangle$
0^{--}	$^3P_{0^{--}(6\bar{6})_c(\xi_1,\xi_2)}$	6751	717	778	−686	1.62	12.92	−4.31
	$^3P_{0^{--}(\bar{3}3)_c(\xi_1,\xi_2)}$	6827	741	810	−651	0.62	3.04	−9.11
0^{-+}	$^3P_{0^{-+}(6\bar{6})_c(\xi_1,\xi_2)}$	6746	727	773	−691	11.59	−1.48	−4.43
	$^3P_{0^{-+}(\bar{3}3)_c(\xi_1,\xi_2)}$	6825	745	808	−653	4.70	−3.07	−9.21
	$^3P_{0^{-+}(\bar{3}3)_c(\xi_3)}$	6750	765	769	−694	4.22	−6.45	−19.35
1^{--}	$^3P_{1^{--}(6\bar{6})_c(\xi_1,\xi_2)}$	6733	743	765	−699	1.73	−6.89	−2.30
	$^3P_{1^{--}(\bar{3}3)_c(\xi_1,\xi_2)}$	6827	741	810	−651	0.62	−1.52	−4.55
	$^5P_{1^{--}(\bar{3}3)_c(\xi_3)}$	6754	761	771	−692	15.62	−4.47	−28.76
	$^1P_{1^{--}(\bar{3}3)_c(\xi_3)}$	6770	734	784	−679	−1.38	0	0
	$^1P_{1^{--}(6\bar{6})_c(\xi_3)}$	6968	714	885	−578	13.51	0	0
1^{-+}	$^3P_{1^{-+}(6\bar{6})_c(\xi_1,\xi_2)}$	6751	720	776	−688	11.45	0.73	−2.18
	$^3P_{1^{-+}(\bar{3}3)_c(\xi_1,\xi_2)}$	6834	732	815	−647	4.63	1.49	−4.46
	$^3P_{1^{-+}(\bar{3}3)_c(\xi_3)}$	6769	736	783	−680	4.03	3.01	−9.03
2^{--}	$^3P_{2^{--}(6\bar{6})_c(\xi_1,\xi_2)}$	6746	724	774	−690	1.65	1.32	2.19
	$^3P_{2^{--}(\bar{3}3)_c(\xi_1,\xi_2)}$	6837	725	819	−644	0.64	0.29	4.38
	$^5P_{2^{--}(\bar{3}3)_c(\xi_3)}$	6781	720	791	−672	14.38	4.05	−8.69
2^{-+}	$^3P_{2^{-+}(6\bar{6})_c(\xi_1,\xi_2)}$	6754	715	779	−685	11.35	−0.14	2.15
	$^3P_{2^{-+}(\bar{3}3)_c(\xi_1,\xi_2)}$	6841	722	821	−642	4.57	−0.29	4.35
	$^3P_{2^{-+}(\bar{3}3)_c(\xi_3)}$	6783	715	794	−670	3.89	−0.57	8.59
3^{--}	$^5P_{3^{--}(\bar{3}3)_c(\xi_3)}$	6801	692	807	−658	13.55	−1.08	16.20

- One still sees significant contributions from the confinement potential which push the energy level further beyond the two-heavy meson thresholds.



(A) cc$\overline{cc}$ spectrum
P wave states
LHCb data
Narrower structure
S wave states
Peak position of broad structure
Mass (MeV)
J^{PC}
6926
6993
6955
6928
6891
6904
6908
6801
6768
6749
6750
6769
6780
6783
6651
6681
6636
6676
6630
6667
6550
6524
6500
6455
0^{++} 1^{+-} 2^{++} 0^{--} 0^{-+} 1^{--} 1^{-+} 2^{--} 2^{-+} 3^{--}



LHCb
$p_T^{\text{di-}J/\psi} > 5.2$ GeV/c
$3.065 < M_{\mu\mu} < 3.135$ GeV/c^2
$3.00 < M_{\mu\mu} < 3.05$ GeV/c^2 or
$3.15 < M_{\mu\mu} < 3.20$ GeV/c^2, normalised
Candidates/(28 MeV/c^2)
$M_{\text{di-}J/\psi}$ [MeV/c^2]

Energy decomposition for the 0^{++} states in the ground state, and first and second radial excitations

J^{PC}	Configuration	Mass	$\langle T\rangle$	$\langle V^{Lin}\rangle$	$\langle V^{Coul}\rangle$	$\langle V^{SS}\rangle$	$\langle V^{T}\rangle$	$\langle V^{LS}\rangle$
0^{++}	$1^1S_{0^{++}(6\bar{6})_c}$	6518	715	664	-811	17.82		
	$1^1S_{0^{++}(\bar{3}3)_c}$	6487	756	646	-834	-12.79		
0^{++}	$2^1S_{0^{++}(6\bar{6})_c(\xi_1,\xi_2)}$	6954	725	883	-598	11.39		
	$2^1S_{0^{++}(\bar{3}3)_c(\xi_1,\xi_2)}$	7000	774	919	-622	-3.790		
	$2^1S_{0^{++}(6\bar{6})_c(\xi_3)}$	7183	757	1010	-522	6.520		
	$2^1S_{0^{++}(\bar{3}3)_c(\xi_3)}$	6930	761	876	-642	2.560		
0^{++}	$3^1S_{0^{++}(6\bar{6})_c(\xi_1,\xi_2)}$	7347	764	1097	-455	9.02		
	$3^1S_{0^{++}(\bar{3}3)_c(\xi_1,\xi_2)}$	7403	838	1150	-518	0.53		
	$3^1S_{0^{++}(6\bar{6})_c(\xi_3)}$	7720	855	1326	-397	4.59		
	$3^1S_{0^{++}(\bar{3}3)_c(\xi_3)}$	7241	815	1064	-575	4.46		

For 0^{++},

$$\left|\frac{\langle V^{Lin}\rangle}{\langle V^{Coul}\rangle}\right|_{1S} \simeq 0.77\sim0.81$$

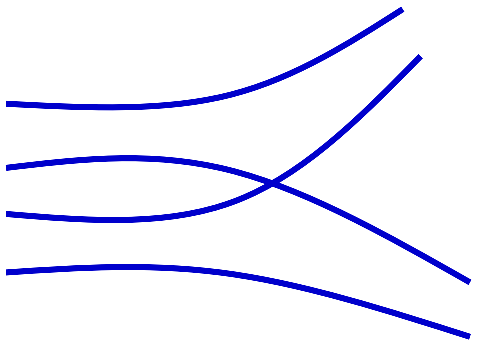
$$\left|\frac{\langle V^{Lin}\rangle}{\langle V^{Coul}\rangle}\right|_{2S} \simeq 1.36\sim1.93$$

$$\left|\frac{\langle V^{Lin}\rangle}{\langle V^{Coul}\rangle}\right|_{3S} \simeq 1.85\sim3.34$$

Fast growth of the linear confinement potential with the excitation quantum numbers

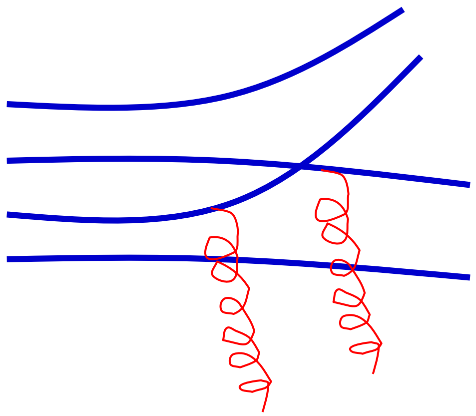
Sufficient energies for non-annihilation decays:

- Rearrangement decays



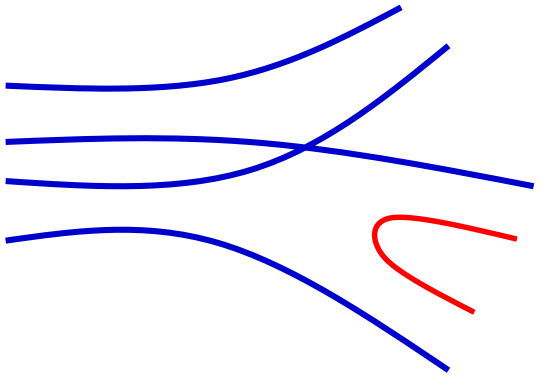
$$M_T - 4m_Q > 2M_{c\bar{c}}$$

- Two pion production



$$M_T - 4m_Q > 2M_{c\bar{c}} + 2M_\pi$$

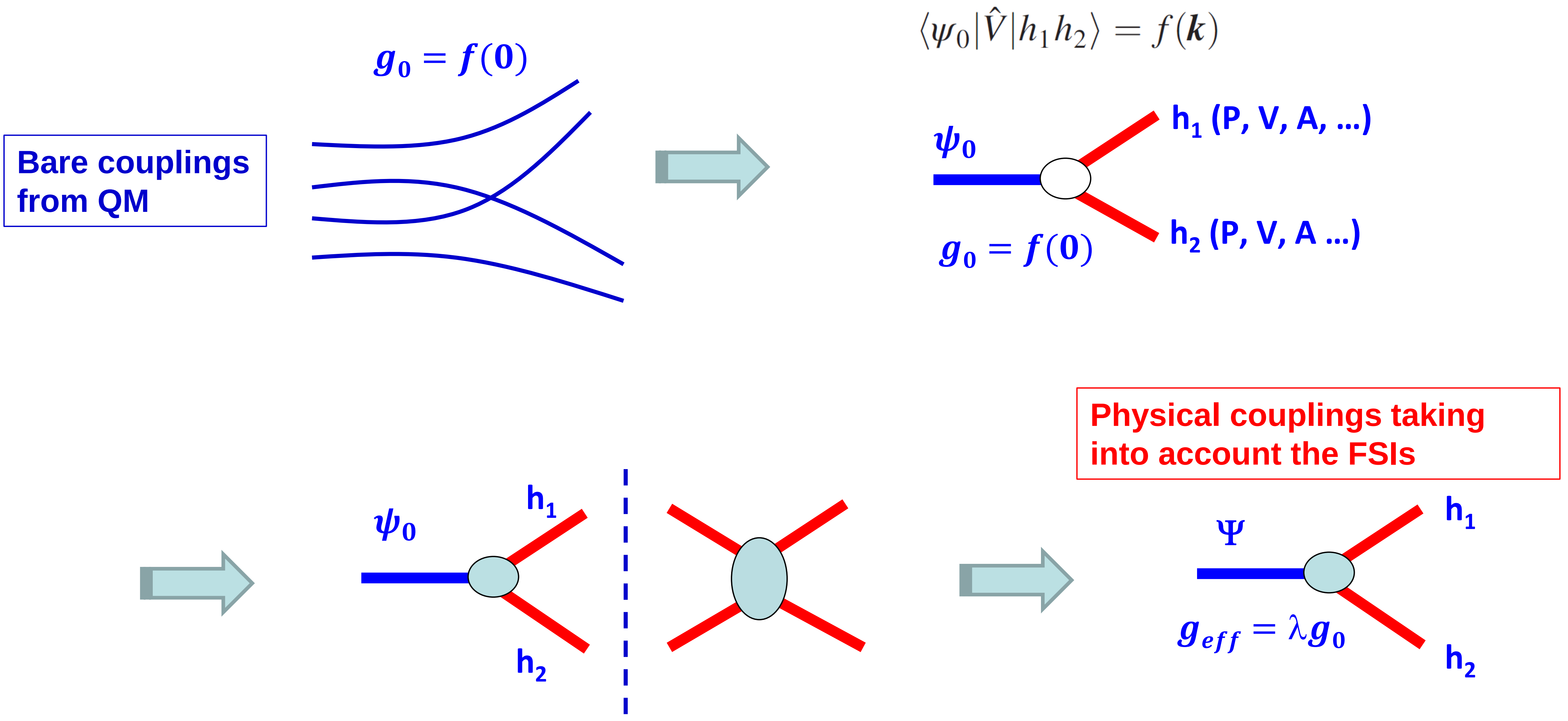
Open channel decays



$$M_T - 4m_Q > M_{c\bar{c}} + 2M_D$$

Broad widths for higher excitation multiquark states.

Quark-rearrangement couplings and final-state interactions



4. Some crucial issues to be noted

- A genuine color-singlet multiquark state may not exist as a stable physical state. However, it should exist as the short-distance component in the physical state.
- With the increase of the constituents within the genuine color-singlet multiquark states they will become unstable due to the confinement potential.
- The nearby threshold may bring down the energy to stabilize the system which bridges it to the hadronic molecule scenario.
- The role played by the triangle singularity mechanism should be included in the analysis.
- LQCD can tell more…
- … …

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