A Brief Introduction to Thermal Field Theory

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Introduction to thermal field theory ($\mu = 0$ for simplicity, $g \ll 1$)

- T much bigger than particles' bare masses: massless particles
- Quantum & statistical effects: more scales than in vacuum (screening, damping, etc)

a_0	$a_2 g^2$		$a_4 g^4$		$a_6 g^6$	$\dots$	(T, hard)
		$b_3 g^3$	$b_4 g^4$	$b_5 g^5$	$b_6 g^6$	$\dots$	(gT, soft or electric, HTL)
					$c_6 g^6$	$\dots$	($g^2 T$, ultrasoft or magnetic)
							Linde problem, nonpert.!
d_0	$d_2 g^2$	$d_3 g^3$	$d_4 g^4$	$d_5 g^5$	$d_6 g^6$	$\dots$	

- gT defines the (LO) thermal (Debye) mass
- Massless particles become massive quasiparticles
- Massive quasiparticles $\rightarrow$ short-range(!) correlations ($\sim 1/gT$)
- Where is (how to generate) LONG-RANGE correlations for QCD?

Weak-coupling expansion: massless scalar ϕ^4 theory

- Massless scalar ϕ^4 theory

$$S = \int_0^{1/T} d\tau \int d^3x \mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{g^2}{4!}\phi^4$$

- $P = (2n\pi T, \mathbf{p})$, zero Matsubara modes induce IR subtleties; thermal mass $m^2 = g^2 T^2/24$ is important here

$$\begin{aligned} \mathcal{L}_{\rm free} &= \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \delta_{P_0,0} \\ \mathcal{L}_{\rm int} &= \frac{g^2}{4!}\phi^4 - \frac{1}{2} m^2 \phi^2 \delta_{P_0,0} \end{aligned}$$

- Weak-coupling expansion (naive perturbation theory): expanding around an ideal gas of massless particles

Weak-coupling expansion: massless scalar ϕ^4 theory

- Tree-level propagator

$$\Delta(P) \;=\; \frac{1-\delta_{P_0,0}}{P^2} + \frac{\delta_{P_0,0}}{p^2+m^2}$$

- Expanding the partition function in terms of $S_{\rm int}$

$$Z \;=\; \int \mathcal{D}\phi\, e^{-(S_{\rm free}+S_{\rm int})} \;=\; \int \mathcal{D}\phi\, e^{-S_{\rm free}} \sum_{n=0}^{\infty} \frac{(-S_{\rm int})^n}{n!}$$

- Taking log for both sides

$$\log Z \;=\; \log\left[\int \mathcal{D}\phi\, e^{-S_{\rm free}}\right] + \log\left[1 + \sum_{n=1}^{\infty} \frac{\int \mathcal{D}\phi\, e^{-S_{\rm free}}(-S_{\rm int})^n}{n!\int \mathcal{D}\phi\, e^{-S_{\rm free}}}\right]$$

$$\;=\; \log Z_0 + \log Z_{\rm I}$$

Weak-coupling expansion: massless scalar ϕ^4 theory

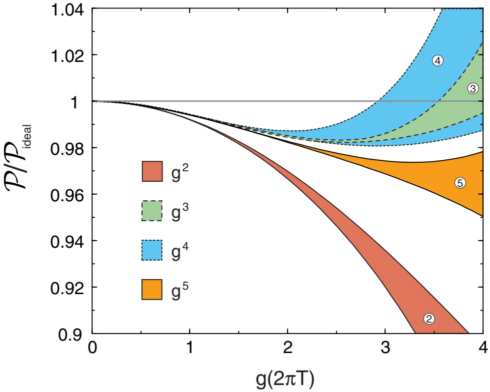
- Using $\log\det A = \mathrm{Tr}\log A$

$$P_0 \;=\; \frac{T}{V}\log Z_0 \;=\; -\frac{T}{2V}\mathrm{Tr}\log[-\partial^2] \;=\; -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \log P^2 \;=\; \frac{\pi^2 T^4}{90}$$

- Diagrams contributing to $\log Z$ up to 3-loop order, aka next-to-next-to-leading order (NNLO)

$\mathcal{F}_0$ $\quad\mathcal{F}_{1\mathrm{a}}$ $\quad\mathcal{F}_{1\mathrm{b}}$ $\quad\mathcal{F}_{2\mathrm{a}}$ $\quad\mathcal{F}_{2\mathrm{b}}$ $\quad\mathcal{F}_{2\mathrm{c}}$ $\quad\mathcal{F}_{2\mathrm{d}}$

Weak-coupling expansion: massless scalar ϕ^4 theory

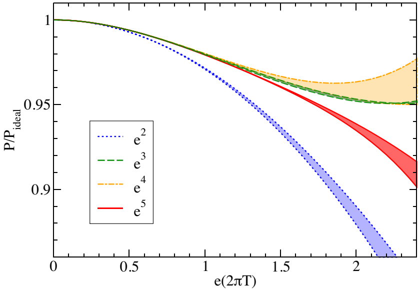


Weak-coupling expansion pressure ($\pi T \leq \mu \leq 4\pi T$)

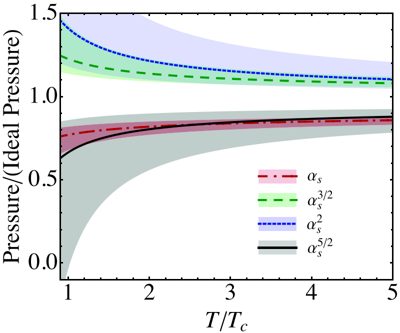
- Weak-coupling expansion pressure up to 3 loops (accurate to g^5)
- Renormalization scale μ is a free parameter: typical value $\mu \sim 2\pi T$
- μ-dependence varies at different orders
- Resulting series does not converge!

Weak-coupling expansion: QED and Yang-Mills

QED



Yang-Mills



Nonconvergence is a generic issue for weak-coupling expansion at finite T
RESUMMATION NEEDED!

Weak-coupling expansion: small coupling $\stackrel{?}{=}$ perturbative

- Consider the perturbation series for the ground state energy, E, of a simple anharmonic oscillator with potential

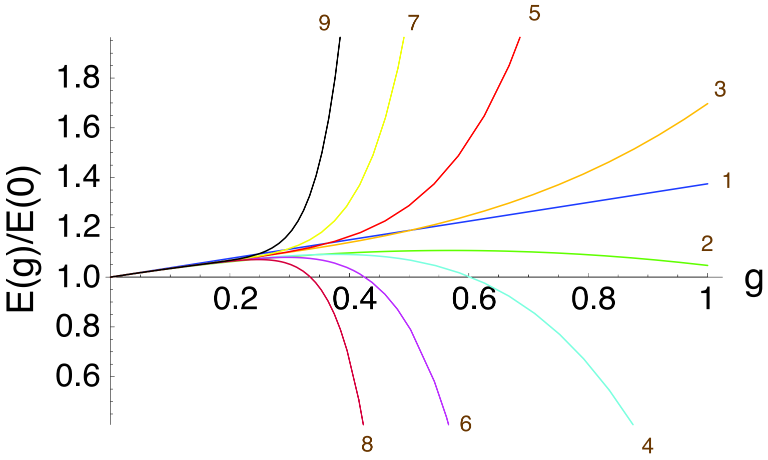
$$V(x) = \frac{1}{2}\omega^2 x^2 + \frac{g}{4}x^4 \quad (\omega^2, g > 0)$$

- Weak-coupling expansion of the ground state energy $E(g)$ is known to all orders [Bender and Wu, 69/73]

$$E(g) = \omega \sum_{n=0}^{\infty} c_n^{\mathrm{BW}} \left(\frac{g}{4\omega^3}\right)^n, \quad c_n^{\mathrm{BW}} = \left\{\frac{1}{2}, \frac{3}{4}, -\frac{21}{8}, \frac{333}{16}, -\frac{30885}{16}, \ldots\right\}$$

- $\lim_{n\to\infty} c_n^{\mathrm{BW}} = (-1)^{n+1}\sqrt{\frac{6}{\pi^3}}3^n(n-\frac{1}{2})!$
- Because of the factorial growth, the expansion is an asymptotic series with zero radius of convergence!

Weak-coupling expansion: small coupling ≠ perturbative!!



Variational perturbation theory [Janke and Kleinert, 95/97]

- Split the harmonic term into two pieces (with $r \equiv \frac{2}{g}\left(\omega^2 - \Omega^2\right)$)

$$\omega^2 \to \Omega^2 + \left(\omega^2 - \Omega^2\right) \implies V_{\rm int}(x) = \frac{g}{4}\left(rx^2 + x^4\right)$$

- Weak-coupling expansion in g

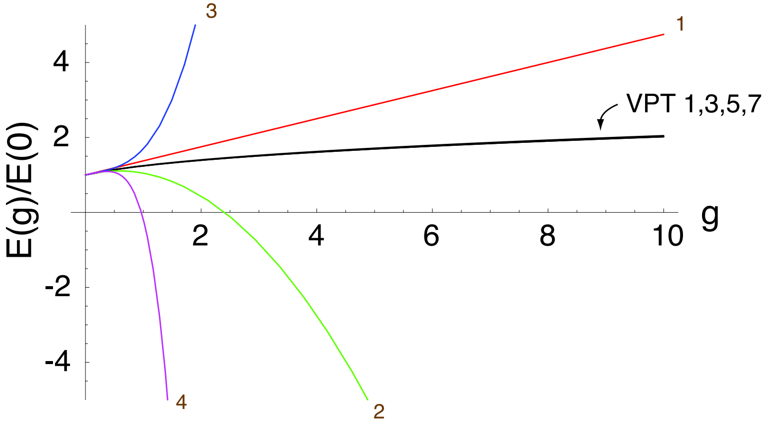
$$E_N(g, r) = \Omega \sum_{n=0}^{N} c_n(r) \left(\frac{g}{4\Omega^3}\right)^n$$

- New coefficients c_n can be obtained by

$$c_n(r) = \sum_{j=0}^{n} c_j^{\rm BW} \left(\begin{array}{c} (1-3j)/2 \\ n-j \end{array} \right) (2r\Omega)^{n-j}$$

- Fix Ω_N by requiring that at each order N: $\frac{\partial E_N}{\partial \Omega}\Big|_{\Omega=\Omega_N} = 0$

VPT: perturbation theory can tackle strong coupling!!



Screened perturbation theory

- SPT: generalization of VPT to scalar field theory
- A mass parameter is added and then subtracted (at higher orders) to the Lagrangian

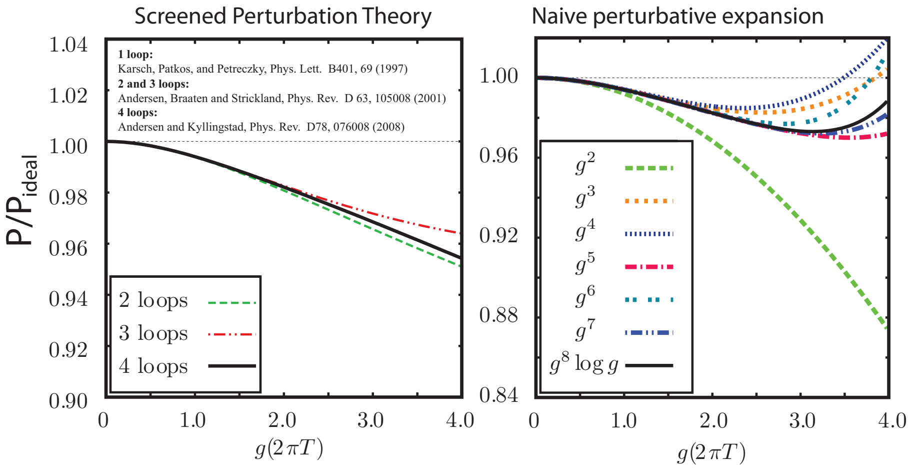
$$\mathcal{L}_{\rm SPT} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2(1-\delta)\phi^2 - \frac{\delta}{4!}g^2\phi^4$$

- $\mathcal{L}_{\rm SPT}$ can be split into free and interaction parts

$$\begin{aligned}\mathcal{L}_{\rm free} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \\ \mathcal{L}_{\rm int} &= \delta(-\frac{1}{4!}g^2\phi^4 + \frac{1}{2}m^2\phi^2)\end{aligned}$$

- An expansion around ideal gas of massive quasi-particles: more appropriate d.o.f.s at high T than massless ones

Screened perturbation theory



Improved convergence over weak-coupling expansion even at large g!

Hard-thermal-loop perturbation theory

- HTLpt: generalization of SPT to gauge theory
- Non-trivial since adding a naive mass term like A^2 would violate gauge invariance
- Hard-thermal-loop effective action is gauge invariant

$$\mathcal{L}_{\rm HTLpt} = \left(\mathcal{L}_{\rm QED} + \mathcal{L}_{\rm HTL} - \delta\mathcal{L}_{\rm HTL}\right)\Big|_{g\to\sqrt{\delta}g}$$

with

$$\mathcal{L}_{\rm HTL} = -\frac{1}{4}m_D^2 F_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot\partial)^2}\right\rangle_{\hat{\mathbf{y}}} F^\mu{}_\beta + i m_f^2 \bar{\psi}\gamma^\mu \left\langle \frac{y_\mu}{y\cdot D}\right\rangle_{\hat{\mathbf{y}}}\psi$$

HTLpt: 1-loop free energy for photons

- Separation into hard and soft contributions ($d = 3 - 2\epsilon$)

$$\mathcal{F}_1 = -\frac{1}{2}\sum\!\!\!\!\!\!\!\int_P \{(d-1)\log[-\Delta_T(P)] + \log \Delta_L(P)\}$$

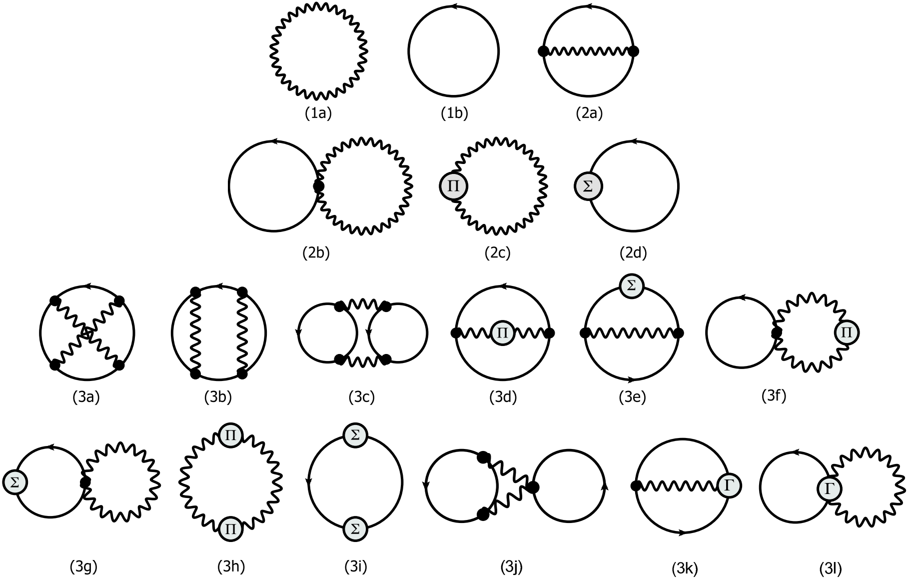
- Hard momenta ($\omega, \mathbf{p} \sim T$): expansion in $m_D/T \sim g$

$$\mathcal{F}_1^{(h)} = \frac{d-1}{2}\sum\!\!\!\!\!\!\!\int_P \log(P^2) + \frac{1}{2}m_D^2 \sum\!\!\!\!\!\!\!\int_P \frac{1}{P^2} - \frac{1}{4(d-1)} m_D^4 \sum\!\!\!\!\!\!\!\int_P \left[\frac{1}{(P^2)^2} - 2\frac{1}{p^2 P^2} - 2d\frac{1}{p^4}\mathcal{T}_P + 2\frac{1}{p^2 P^2}\mathcal{T}_P + d\frac{1}{p^4}(\mathcal{T}_P)^2\right] + \mathcal{O}(m_D^6)$$

- Soft momenta ($\omega, \mathbf{p} \sim gT$)

$$\mathcal{F}_1^{(s)} = \frac{1}{2}T\int_{\mathbf{p}} \log(p^2 + m_D^2)$$

HTLpt: diagrams for QED free energy up to 3 loops



HTLpt: 3-loop free energy (negative pressure) for QED

- The NNLO thermodynamic potential reads

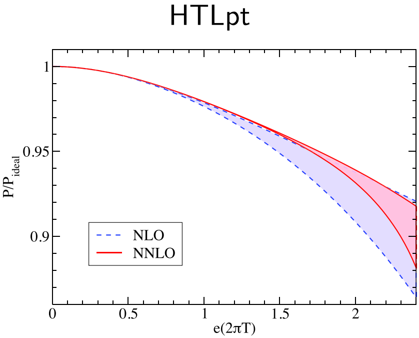
$$\begin{aligned}
\Omega_{\rm NNLO} \;=\; -\frac{\pi^2 T^4}{45}\Bigg\{ & 1+\frac{7}{4}N_f-\frac{15}{4}\hat{m}_D^3 \\
& +N_f\frac{\alpha}{\pi}\left[-\frac{25}{8}+\frac{15}{2}\hat{m}_D+15\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\hat{m}_D^3-90\hat{m}_D\hat{m}_f^2\right] \\
& +N_f\left(\frac{\alpha}{\pi}\right)^2\left[\frac{15}{64}(35-32\log 2)-\frac{45}{2}\hat{m}_D\right] \\
& +N_f^2\left(\frac{\alpha}{\pi}\right)^2\Bigg[\frac{25}{12}\left(\log\frac{\hat{\mu}}{2}+\frac{1}{20}+\frac{3}{5}\gamma-\frac{66}{25}\log 2+\frac{4}{5}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{2}{5}\frac{\zeta'(-3)}{\zeta(-3)}\right) \\
& \qquad +\frac{5}{4}\frac{1}{\hat{m}_D}-15\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\hat{m}_D+30\frac{\hat{m}_f^2}{\hat{m}_D}\Bigg]\Bigg\}
\end{aligned}$$

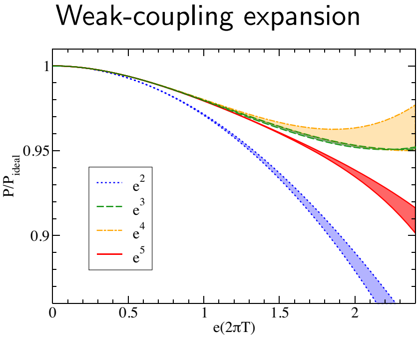
PURELY ANALYTIC!

- To eliminate the m_D and m_f dependence, the gap equations are imposed

$$\begin{aligned}
\frac{\partial}{\partial m_D}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0 \\
\frac{\partial}{\partial m_f}\Omega(T,\alpha,m_D,m_f,\delta=1) &= 0
\end{aligned}$$

HTLpt: 3-loop pressure for QED

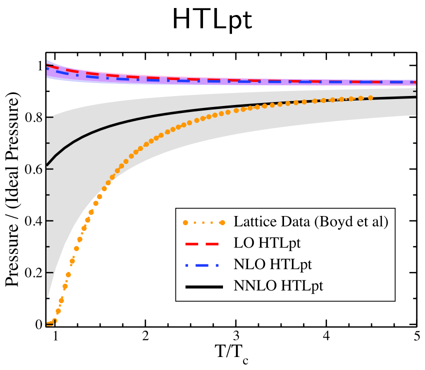


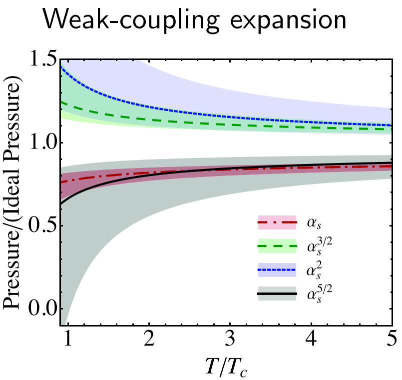


Andersen, Strickland, NS, PRD 80, 085015 (2009)

Improved convergence over weak-coupling expansion even at large g!

HTLpt: 3-loop pressure for Yang-Mills



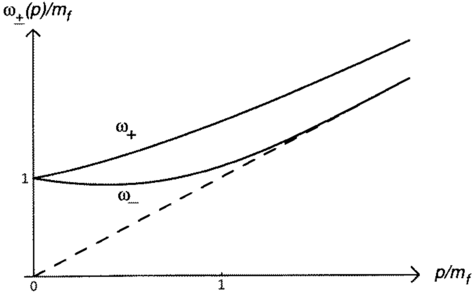


Andersen, Strickland, NS, PRL 104, 122003 (2010)

- Improved convergence, but not as good as the QED and scalar cases
- YM/QCD is much more challenging, more scales due to the non-Abelian symmetry, only resummation for gT is not enough

Collective excitations of hot plasmas – challenges!

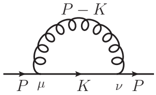
- Collective behaviors of quarks/gluons are crucial in studying QGP
  - Debye screening → massive quasiparticles $m_D \sim gT$ (electric scale)
  - Confining magnetic scale g^2T not included due to Linde problem
  - QED and QCD (collective modes) are INDISTINGUISHABLE(!) in conventional thermal-field setups → weakly coupled
  - HOW TO DISTINGUISH?
    Strongly coupled QCD formalisms NEED long-range correlations



Quark thermal self-energy

- Crucial measure of collective behaviors: screening masses, dispersion relations, spectral functions

$$\Sigma(P) = (ig)^2 C_F \sum\!\!\!\!\!\!\!\int_{\{K\}} \gamma^\mu S(K) \gamma^\nu D^{\mu\nu}(P-K)$$



- At finite T, Gribov-Zwanziger action [Gribov 78; Zwanziger 89] provides a renormalizable framework to study confinement effects at g^2T
  - Improvement over Faddeev-Popov action in IR

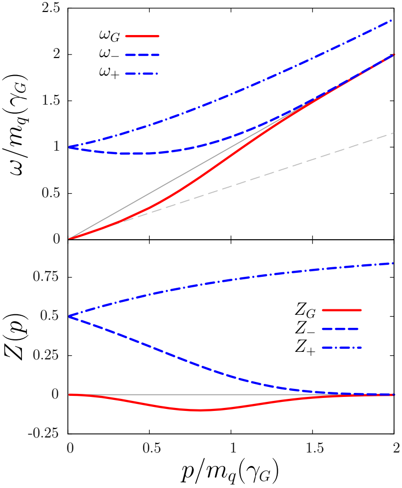
    $D^{\mu\nu}(P) = \frac{P^2}{P^4+\gamma_G^4}\left(\delta_{\mu\nu} - \frac{P_\mu P_\nu}{P^2}\right)$ with $\gamma_G = \frac{d}{d+1}\frac{N_c}{4\sqrt{2}\pi} g^2 T$

    [Zwanziger, PRD 76, 125014 (2007); Fukushima, NS, PRD 88, 076008 (2013)]

- Hard-thermal-loop (HTL) systematics [Braaten, Pisarski, 90]: contributions from $P \ll K$ on the same order as tree-level ones, gauge invariant

Dispersion relations & residues of poles

NS, Tywoniuk, PRL 114, 161601 (2015)

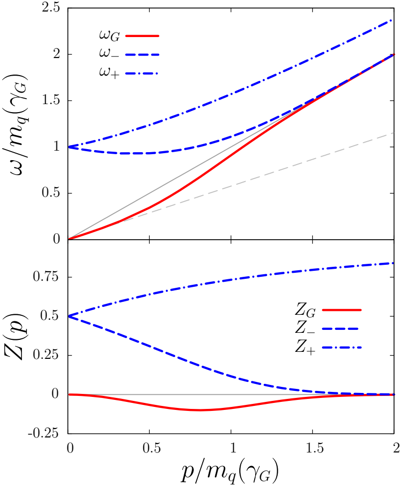


- Massive modes – time-like
- A new massless mode – space-like
- Sound wave at small p, $\omega \approx v_s p$ ($v_s = 1/\sqrt{3}$)
- Hydrodynamics-like behaviors
- Long-range correlations from $g^2 T$
- Viscosities: $\frac{\zeta}{\eta} \propto \left(\frac{1}{3} - c_s^2\right)$ $\rightarrow$ strongly coupled liquid!

  [Florkowski, Ryblewski, NS, Tywoniuk, PRC 94, 044904 (2016)]

Dispersion relations & residues of poles

NS, Tywoniuk, PRL 114, 161601 (2015)



- Non-positive residue: 1st positivity violation for thermal quarks
- Genuine feature of non-Abelian theories, distinctive from QED
- Confinement effects ($g^2 T$) persists in hot QCD even at $T \to \infty$(!)
- GZ: strongly coupled formalism
- Spectral sum rule

$$\oint \frac{d\tilde{\omega}}{2\pi} \rho_f^{\rm QCD}(\tilde{\omega}, p) = 1$$

[Du, NS, Tywoniuk, forthcoming]

Yang-Mills and QCD

More scales due to the non-Abelian symmetry: T, gT, g^2T

- QCD deconfinement transition
- QGP and heavy-ion collisions
- QCD at finite density
- QCD in strong magnetic fields
- And more ...