Quantum kinetic theory
and its applications to chiral transports and spin polarizations

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中国科学技术大学

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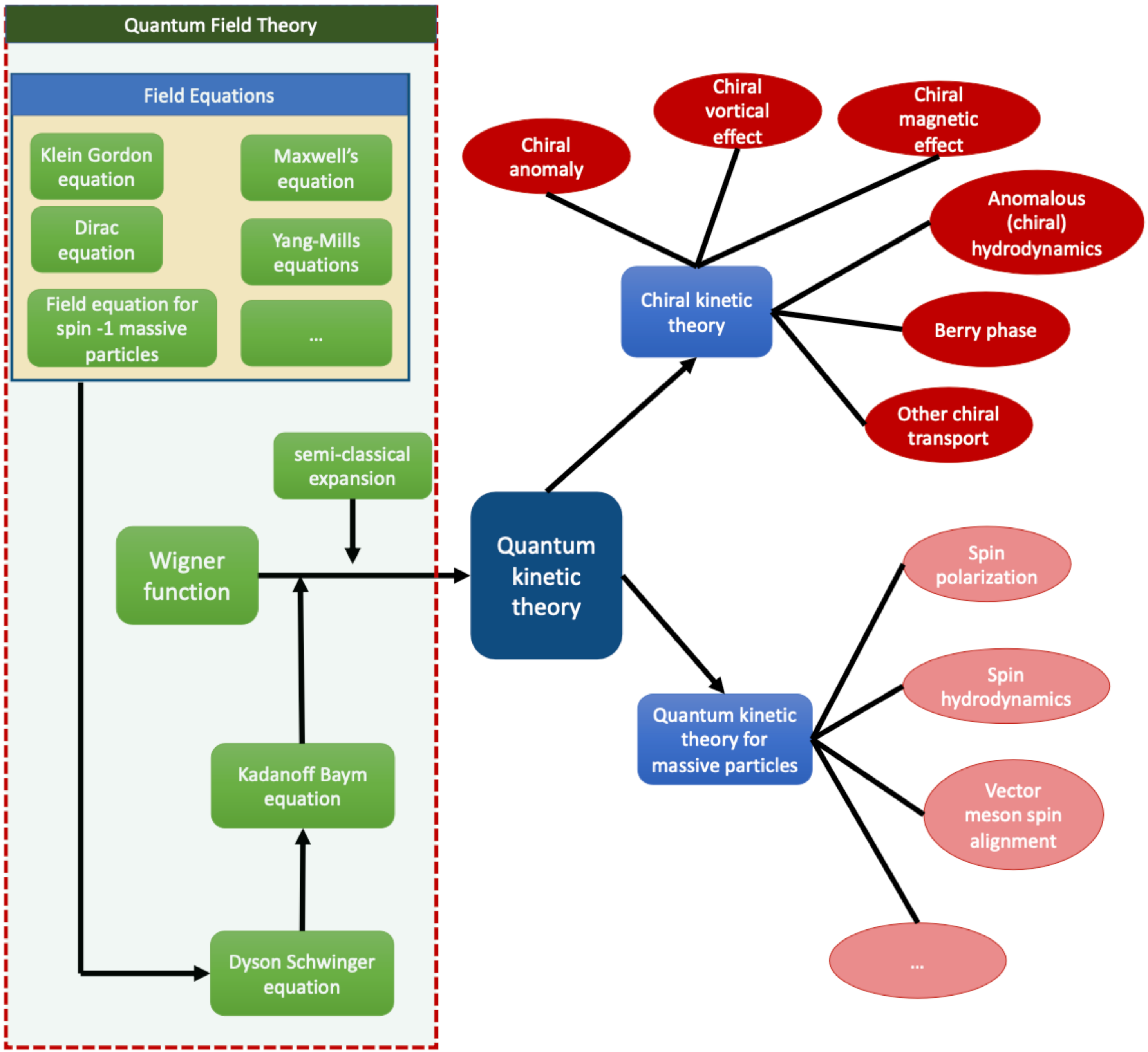
QCD与中高能核物理暑期学校
山东大学青岛校区

Recent review:

- Y. Hidaka, SP, Q. Wang, D.L. Yang, ***Foundations and Applications of Quantum Kinetic Theory***, Progress in Particle and Nuclear Physics, 127, 103989 (2022).


Quantum Field Theory
Field Equations
Klein Gordon equation
Maxwell's equation
Dirac equation
Yang-Mills equations
Field equation for spin -1 massive particles
...
semi-classical expansion
Wigner function
Kadanoff Baym equation
Dyson Schwinger equation
Quantum kinetic theory
Chiral kinetic theory
Chiral anomaly
Chiral vortical effect
Chiral magnetic effect
Anomalous (chiral) hydrodynamics
Berry phase
Other chiral transport
Quantum kinetic theory for massive particles
Spin polarization
Spin hydrodynamics
Vector meson spin alignment
...

Outline

- **Part 1:**
  Chiral magnetic effect, Berry phase and kinetic theory
- **Part 2:**
  Wigner functions and the master equations
- **Part 3:**
  Quantum kinetic theory in massless limit and collisions
- **Part 4:**
  Applications to heavy ion physics

Part 1

Chiral magnetic effect and kinetic theory

1. Chiral magnetic effect and chiral separation effect

   (1a) Strong magnetic fields in HIC and CME

   (1b) Other topics related to the CME

2. Kinetic theory and chiral kinetic theory

   (2a) Standard kinetic theory

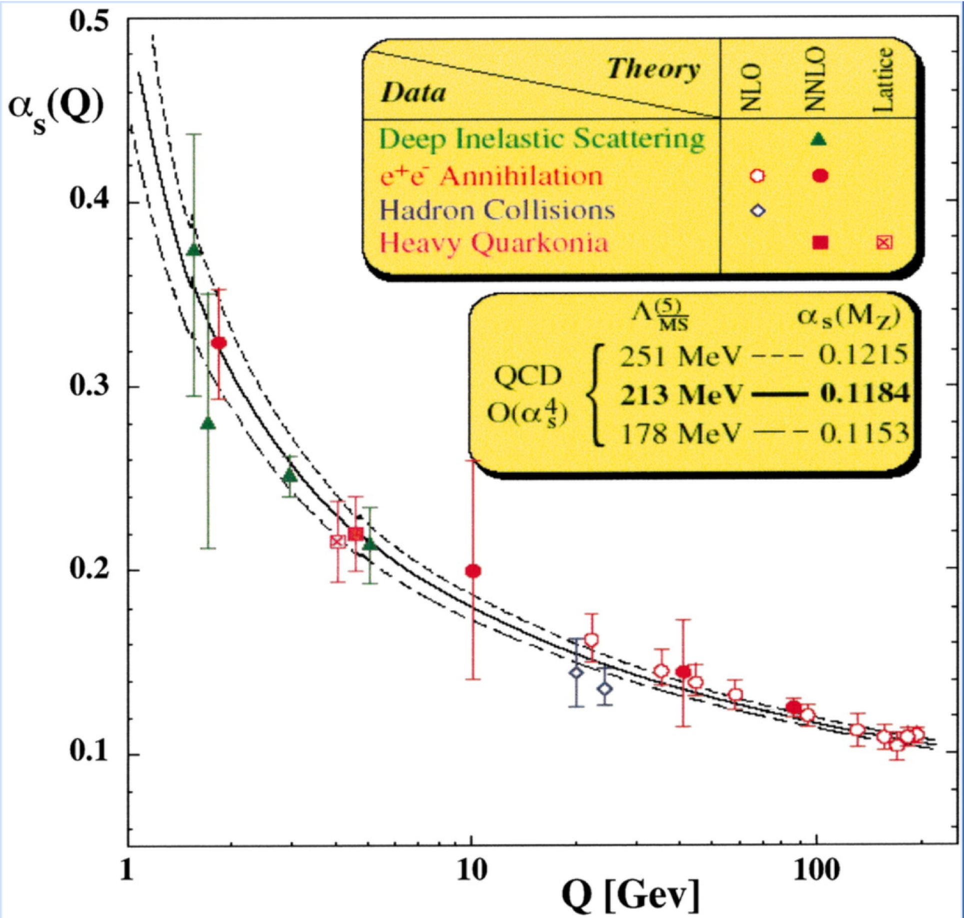
   (2b) Chiral kinetic theory: a quick look

3. Non-trivial Lorentz symmetry for chiral system

1. Chiral magnetic effect and chiral separation effect

(1a) Strong magnetic fields in HIC and CME

Asymptotic freedom of QCD



The Nobel Prize in Physics 2004

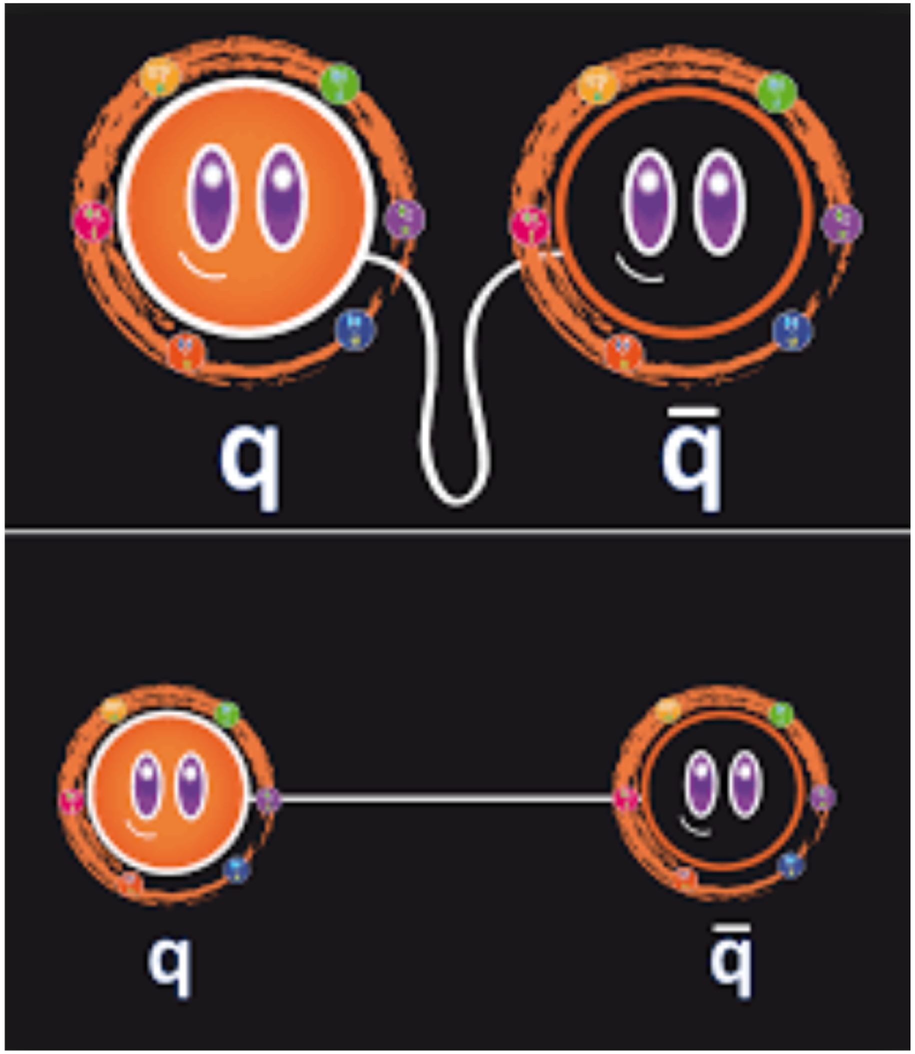
Photo from the Nobel Foundation archive.
David J. Gross
Prize share: 1/3

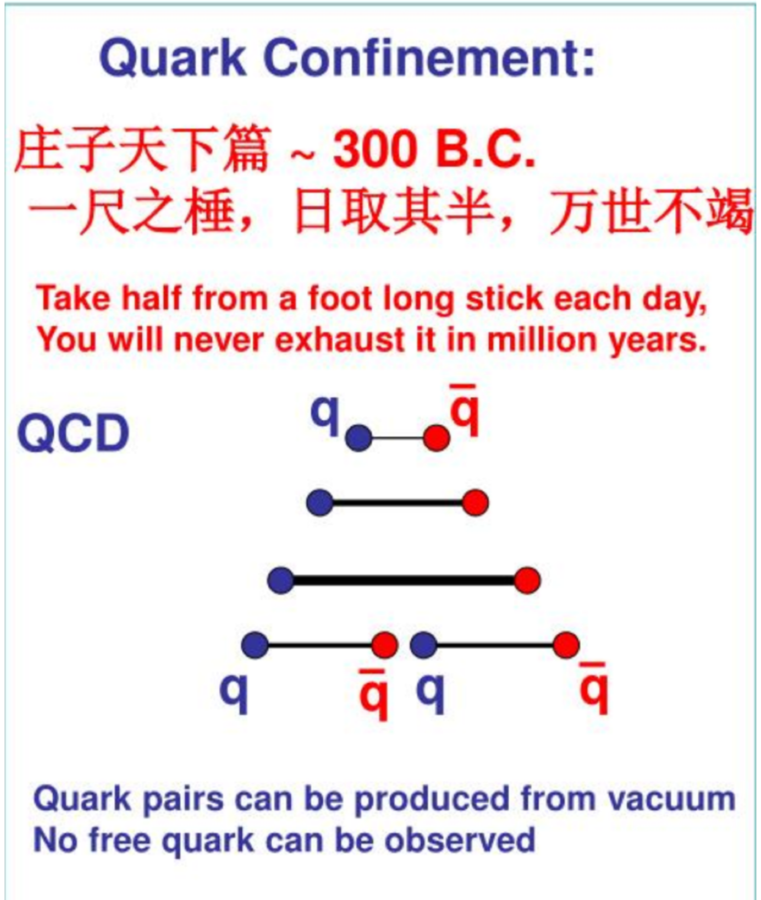
Photo from the Nobel Foundation archive.
H. David Politzer
Prize share: 1/3

Photo from the Nobel Foundation archive.
Frank Wilczek
Prize share: 1/3

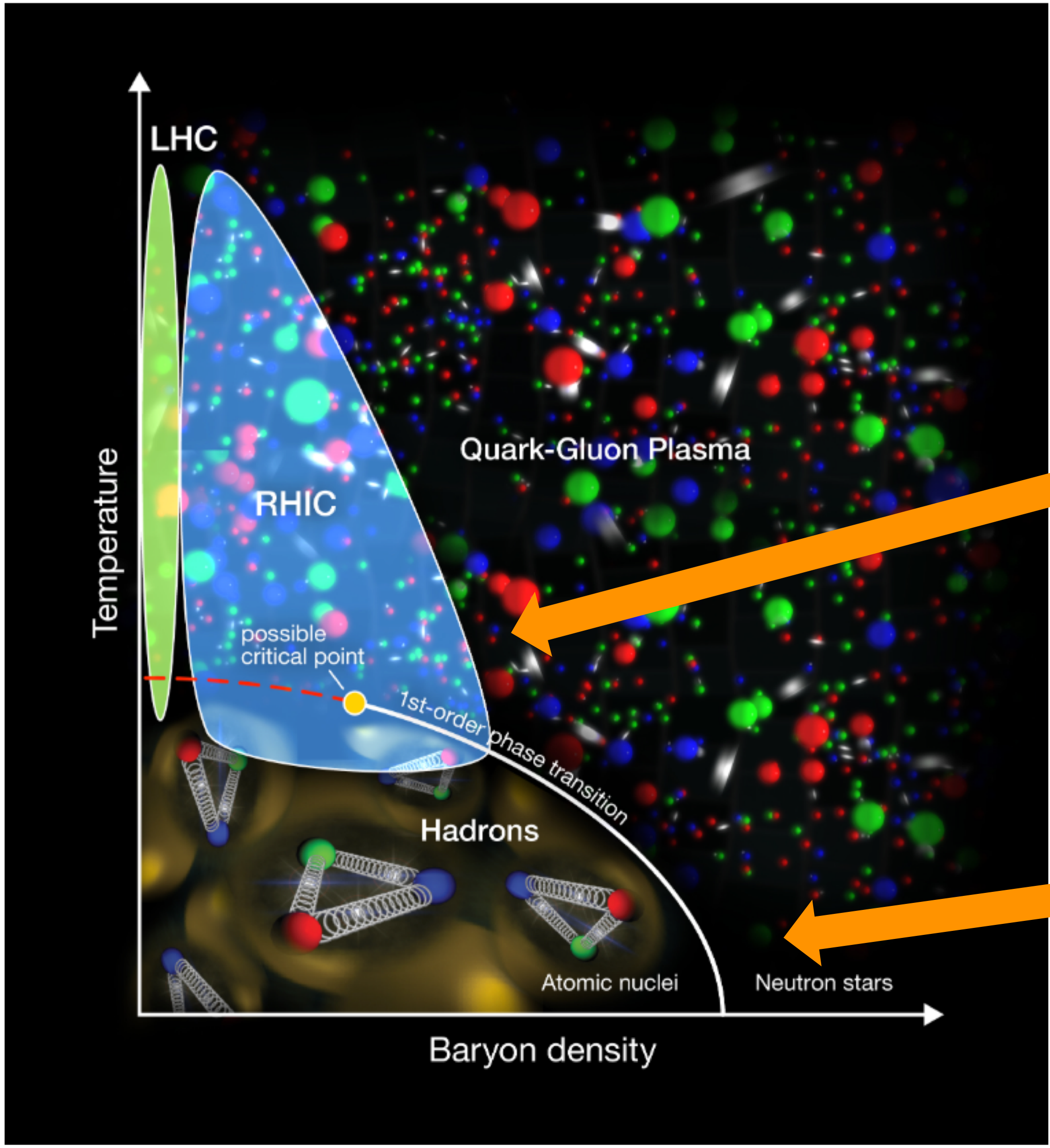
The Nobel Prize in Physics 2004 was awarded jointly to David J. Gross, H. David Politzer and Frank Wilczek "for the discovery of asymptotic freedom in the theory of the strong interaction."

Quark Confinement





Deconfinement phase transition



High temperature

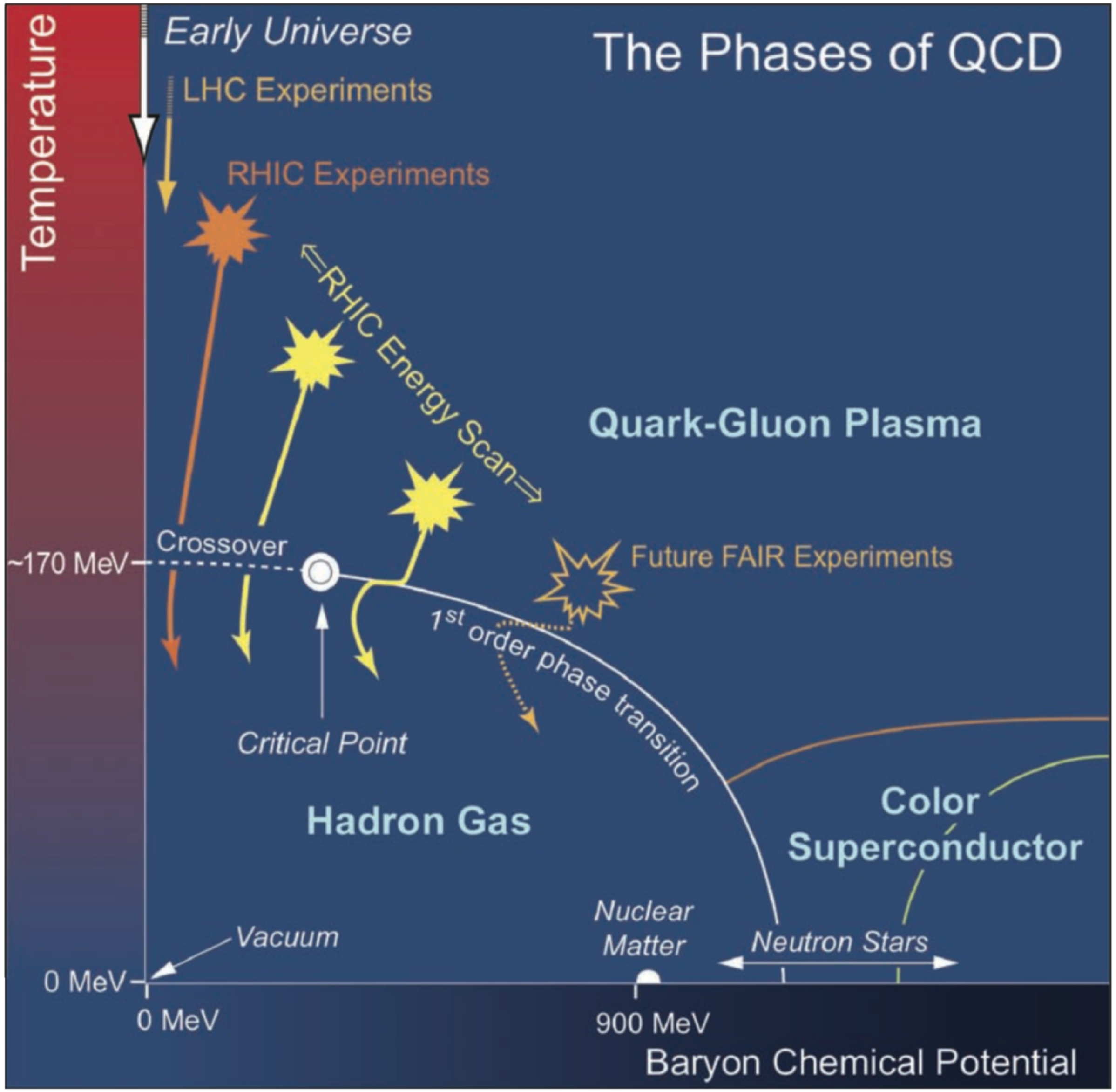
High pressure

核子重如牛 对撞生新态

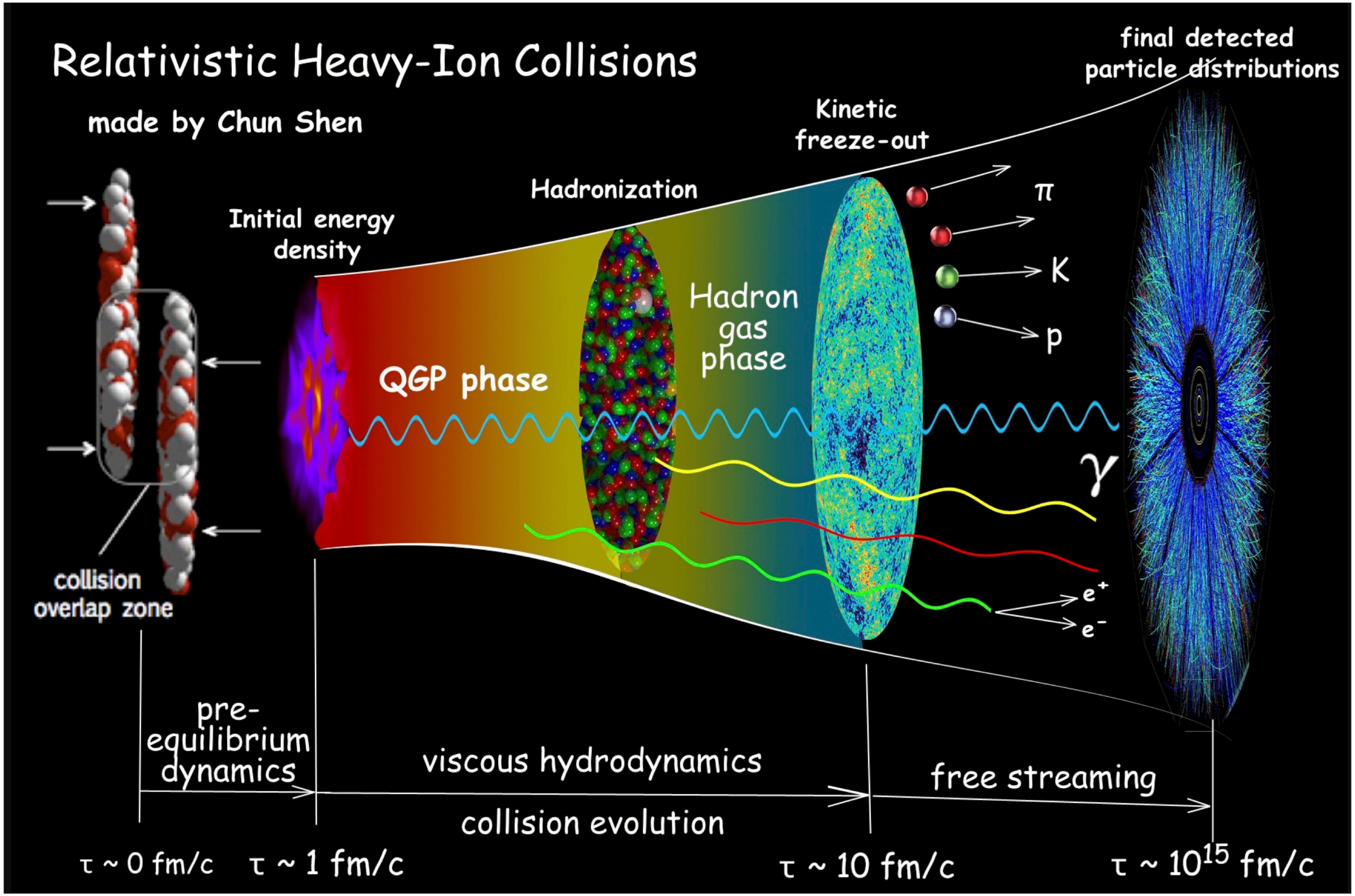
T.D. Lee (1974) and Collins (1975): Heavy ion collision to create a new form of matter!

李可染大师 1986年，
为 李政道先生作。

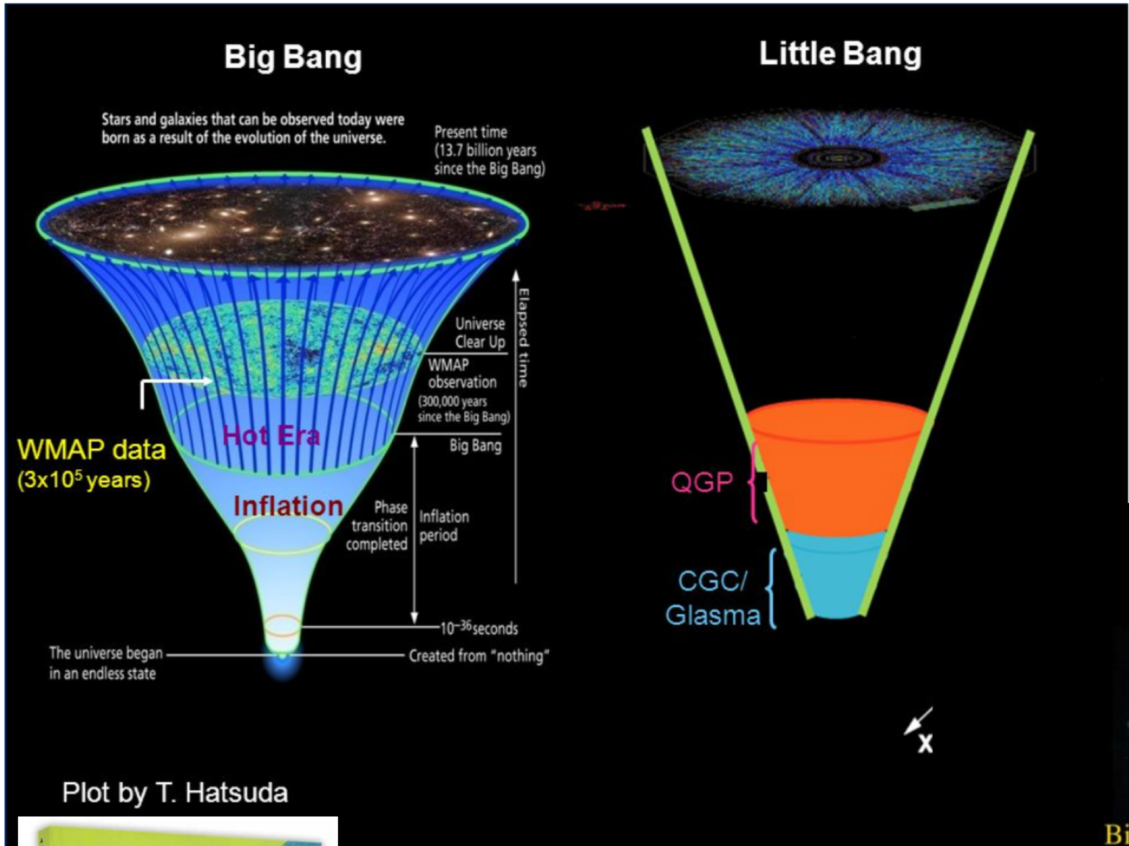
Phases of QCD

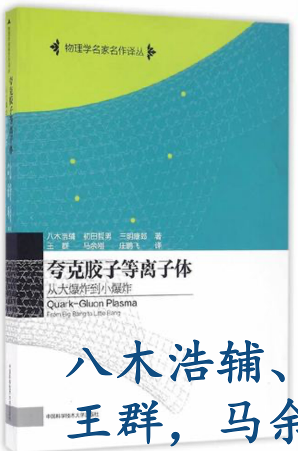


Relativistic heavy ion collisions



Little Bang VS Big Bang





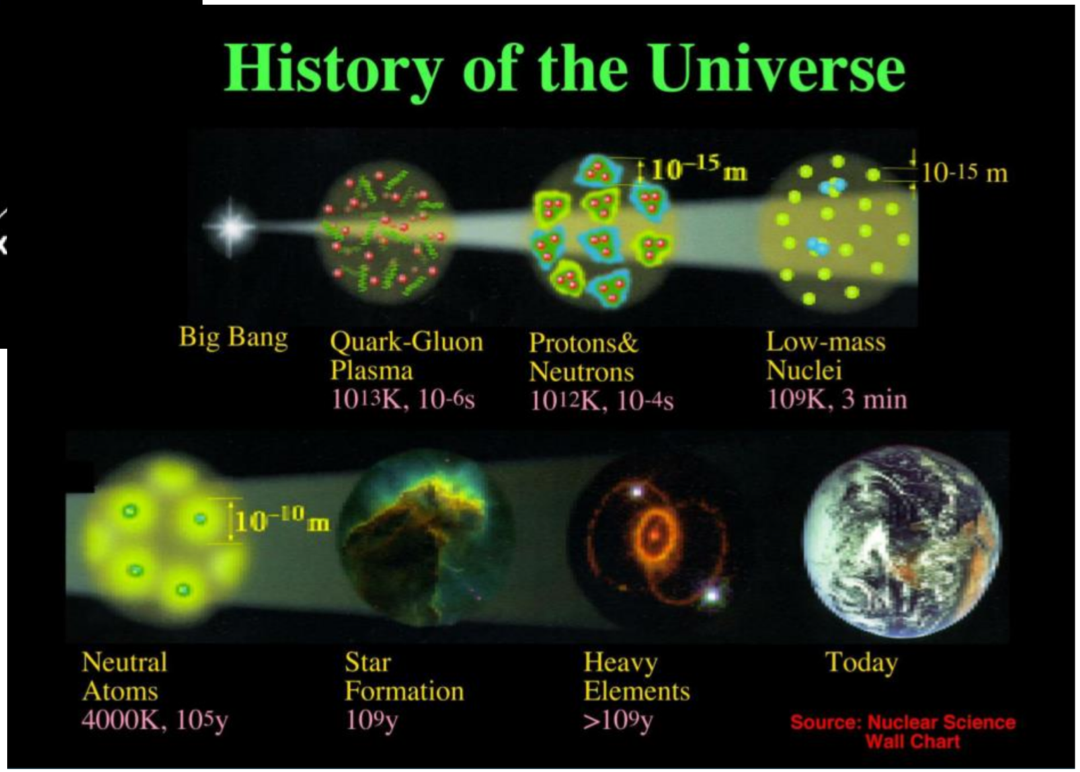
夸克胶子等离子体

从大爆炸到小爆炸

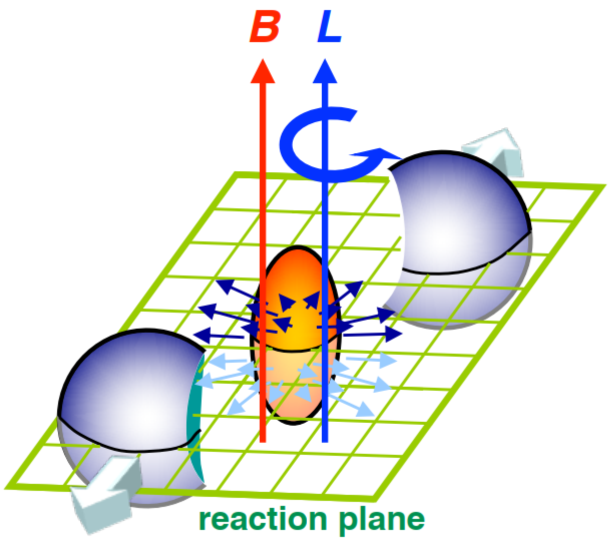
2022年8月 第2次印刷

八木浩辅、初田哲男、三明康郎 著；

王群，马余刚，庄鹏飞 译



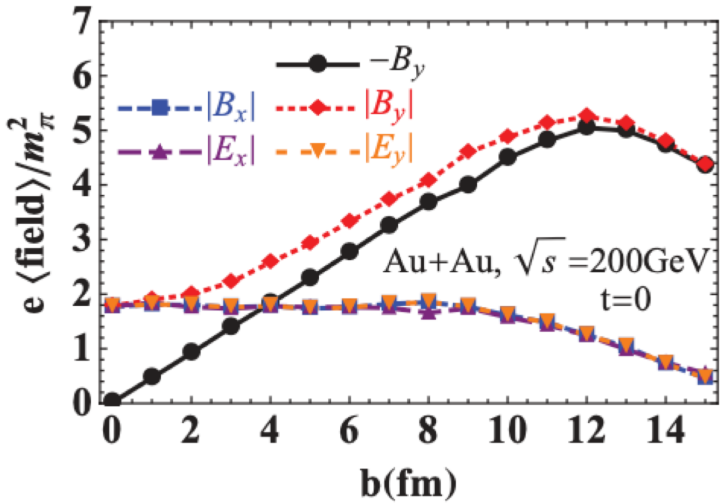
Strong electromagnetic fields in HIC

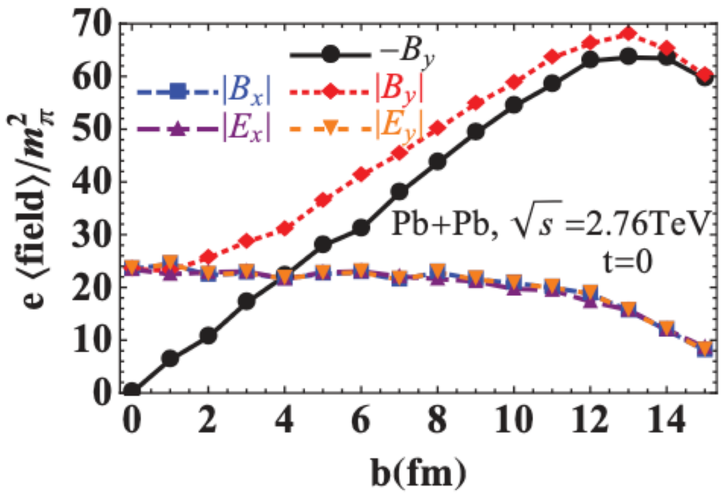


- **Two charged nuclei moving alone z direction generate the EB fields.**

- **Theoretical estimation:**

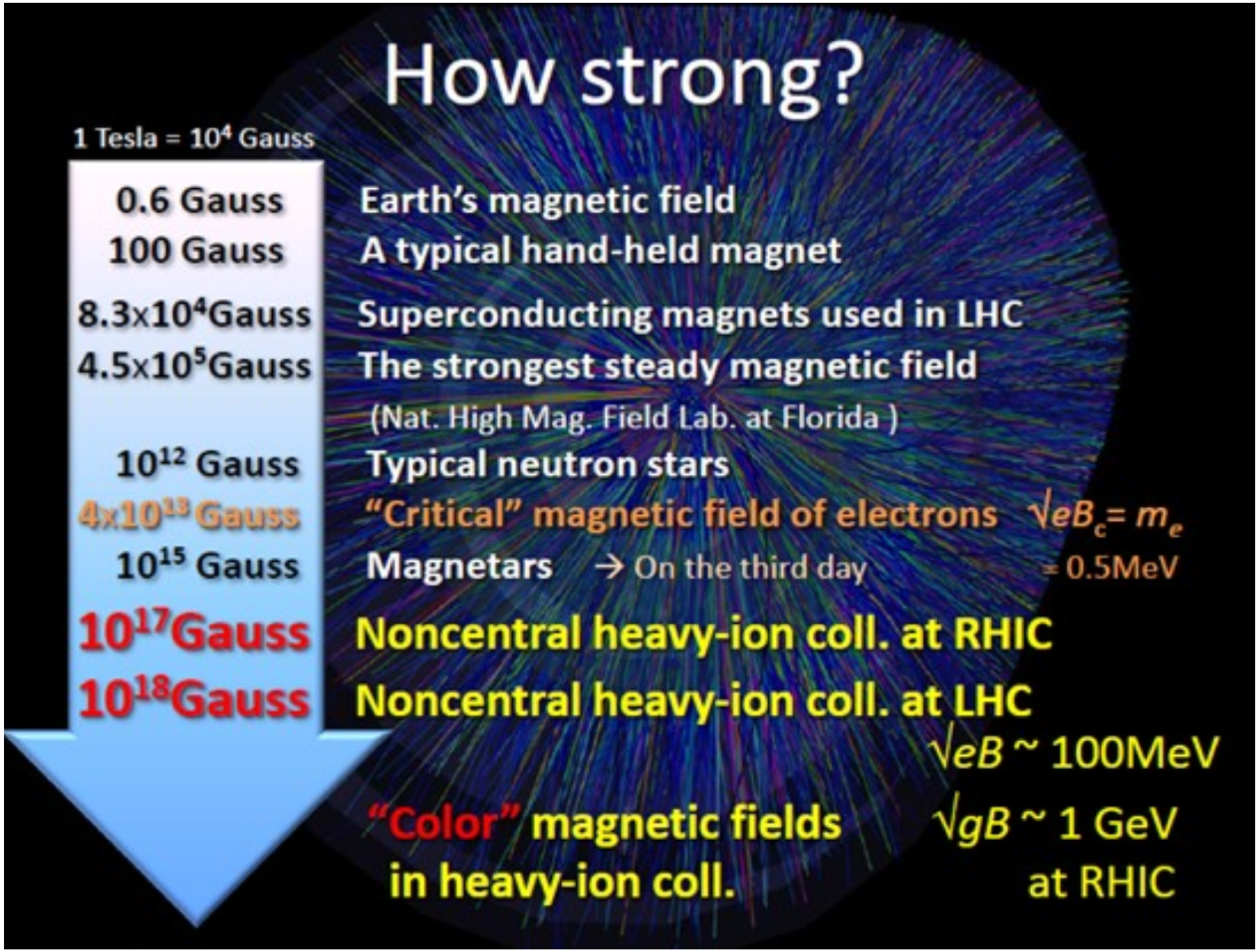
Lienard-Wiechert potential + Event-by-event



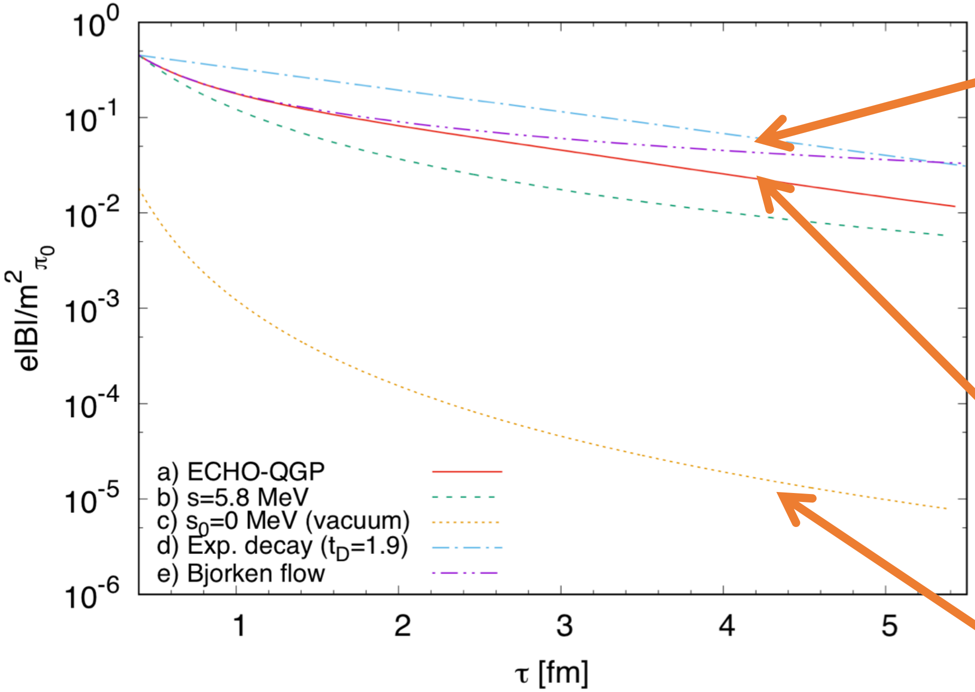


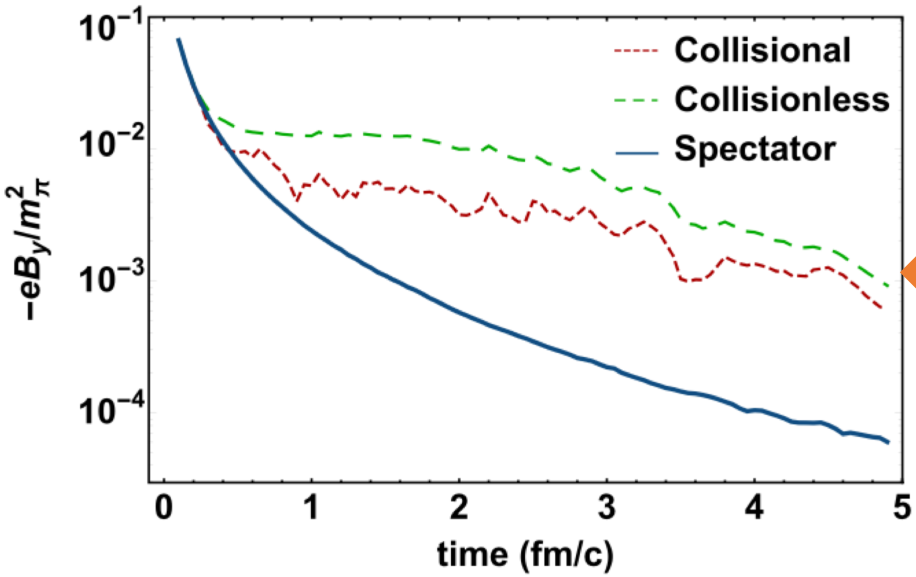
A. Bzdak, V. Skokov PRC 2012 ;W.T. Deng, X.G. Huang PRC 2012;V.Roy, SP, PRC 2015;
H. Li, X.l. Sheng, Q.Wang, 2016; etc. / review: K. Tuchin 2013

Strong magnetic fields



Evolution of electromagnetic fields





Ideal Bjorken MHD

Pu, Roy, Rezzolla, Rischke, PRD(2015)

MHD + CME + Chiral anomaly

Siddique, Wang, Pu, Wang, PRD (2019)

Peng, Wu, Wang, She, Pu, PRD(2023)

ECHO-QGP

Inghirami, Zanna, Moghaddam,Becattini, Bleicher, EPJC(2016)

Vacuum

Kharzeev, McLerran,Warringa,NPA(2008)

Relativistic kinetic theory coupled with Maxwell's equations + 2->2 Leading-Log pQCD scattering

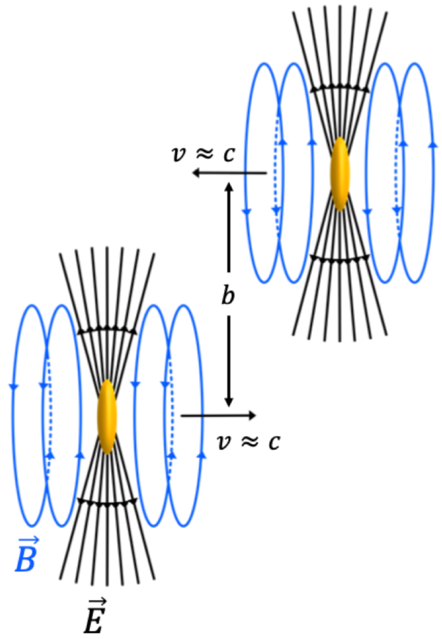
Zhang, Sheng, Pu, Chen, Peng, Wang, Wang, PRR (2022)

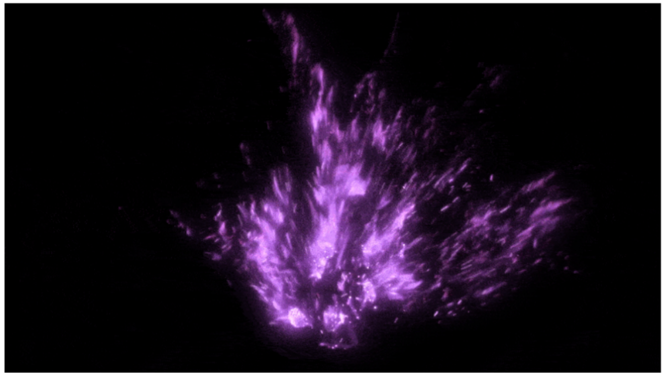
Two ways to study the effects to EM fields

- Consider EM fields as the (real) photon fields
  - Photo-photon, photon-nuclear interaction

- Consider EM fields as the background fields
  - Quantum transport phenomena
  - …

Ultra-Peripheral Collisions





Scientists Generate Matter Directly From Light – Physics Phenomena Predicted More Than 80 Years Ago

TOPICS: Antimatter Atomic Physics Brookhaven National Laboratory DOE Popular

By BROOKHAVEN NATIONAL LABORATORY JULY 30, 2021

Abstract energy concept illustration.

- Ultra-Peripheral Collisions (UPC): the impact parameter is larger than 2 times the radius of a nucleus
- Since the QCD effects are higher orders and QED effects are enhanced by the Ze, UPC provides a nice platform to study the strong EB effects.

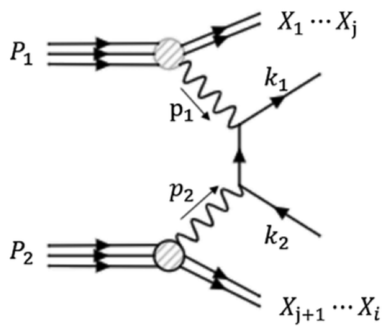
  $Z\alpha \approx 1$ → High photon density
- Because the relativity, the photon (EB fields) are almost real. Photon-photon, photon-nuclear interactions

STAR Collaboration, ”Measurement of e+e− Momentum and Angular Distributions from Linearly Polarized Photon Collisions”, Phys. Rev. Lett 127, 052302

Dilepton photoproduction in UPC

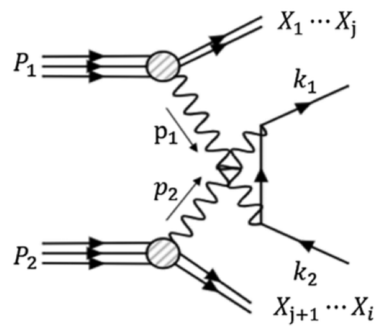
Equivalent photon approximation (EPA)

- Baltz, Gorbunov, Klein, Nystrand, PRC 80, 044902 (2009)
- Zha, Ruan, Tang, Xu, Yang, PLB 781, 182 (2018)
- Zha, Brandenburg, Tang, Xu, PLB 800 (2020) 135089



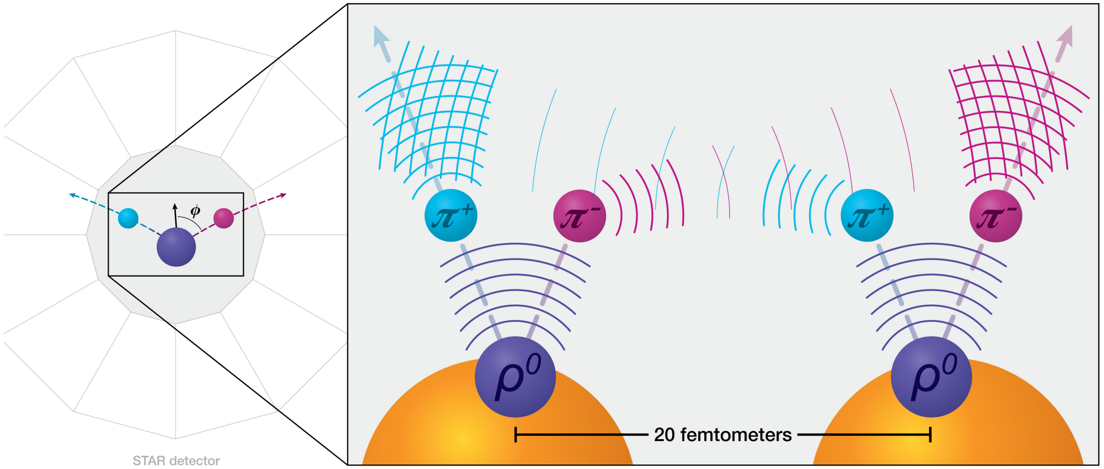
Based on QED calculations:

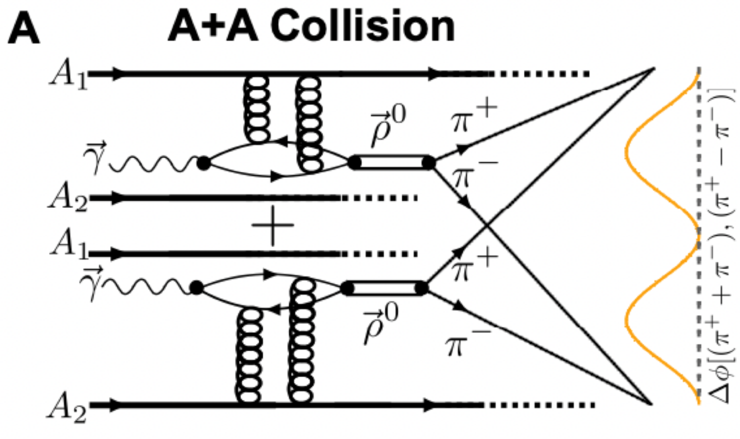
- **Transverse momentum dependent (TMD) formulism**
  - C. Li, J. Zhou and Y. J. Zhou, Phys. Lett. B 795, 576 (2019)
  - Klein, Muller, Xiao, Yuan, PRL 122 (2019) 13, 132301; PRD 102 (2020) 9, 094013
  - Xiao, Yuan, Zhou, PRL 125 (2020) 23, 232301



- **QED in classical field approximation**
  - Vidovic, Greiner, Best, Soff, PRC (1993)
  - W. Zha, J. D. Brandenburg, Z. Tang and Z. Xu, PLB 800 (2020) 135089
- **QED with wave packet description of nuclei**
  - Wang, SP, Wang, PRD (2021); Wang, SP, Zhang, Wang, PRD (2022);
  - Lin, Wang, Wang, Xu, SP, Wang, PRD (2022)

Polarization dependent vector meson production



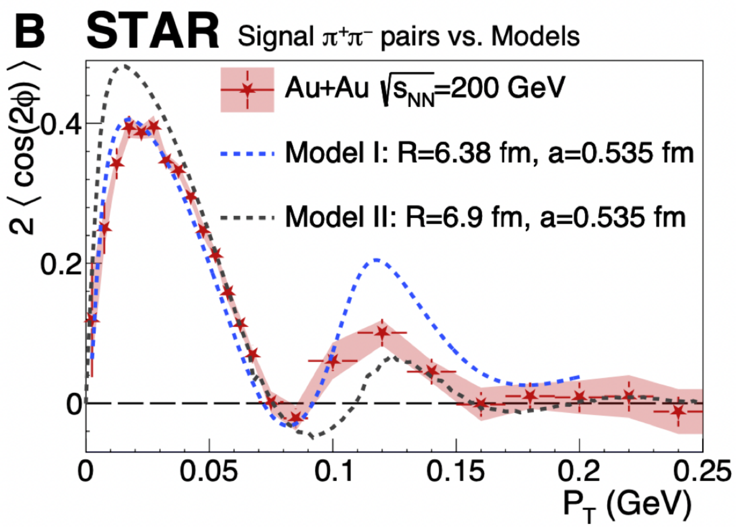


$\rho^0 \to \pi^+\pi^-$

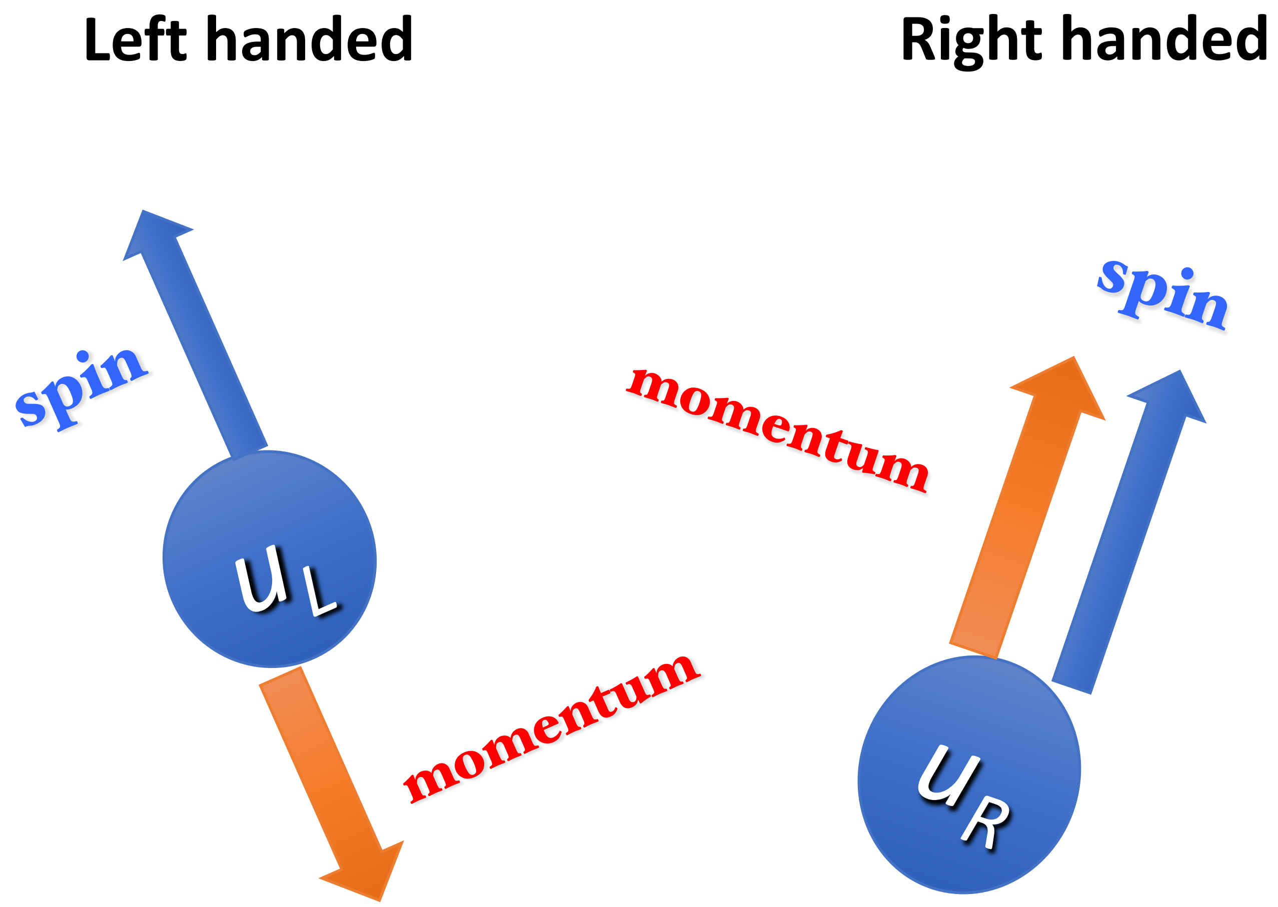
- **Azimuthal asymmetries $\cos(2\phi)$ in diffractive vector meson production in UPC**
- **STAR: Sci. Adv. 9 (2023) 1, eabq3903**
- **Theory:**
  - **Zha, Brandenburg, Ruan, Tang, Xu, PRD 2021**
  - **Xing, Zhang, Zhou, Zhou, JHEP 2020**

For $\cos(\phi)$ and $\cos(3\phi)$ related to ρ^0, see
Hagiwara, Zhang, Zhou, Zhou, PRD 2021

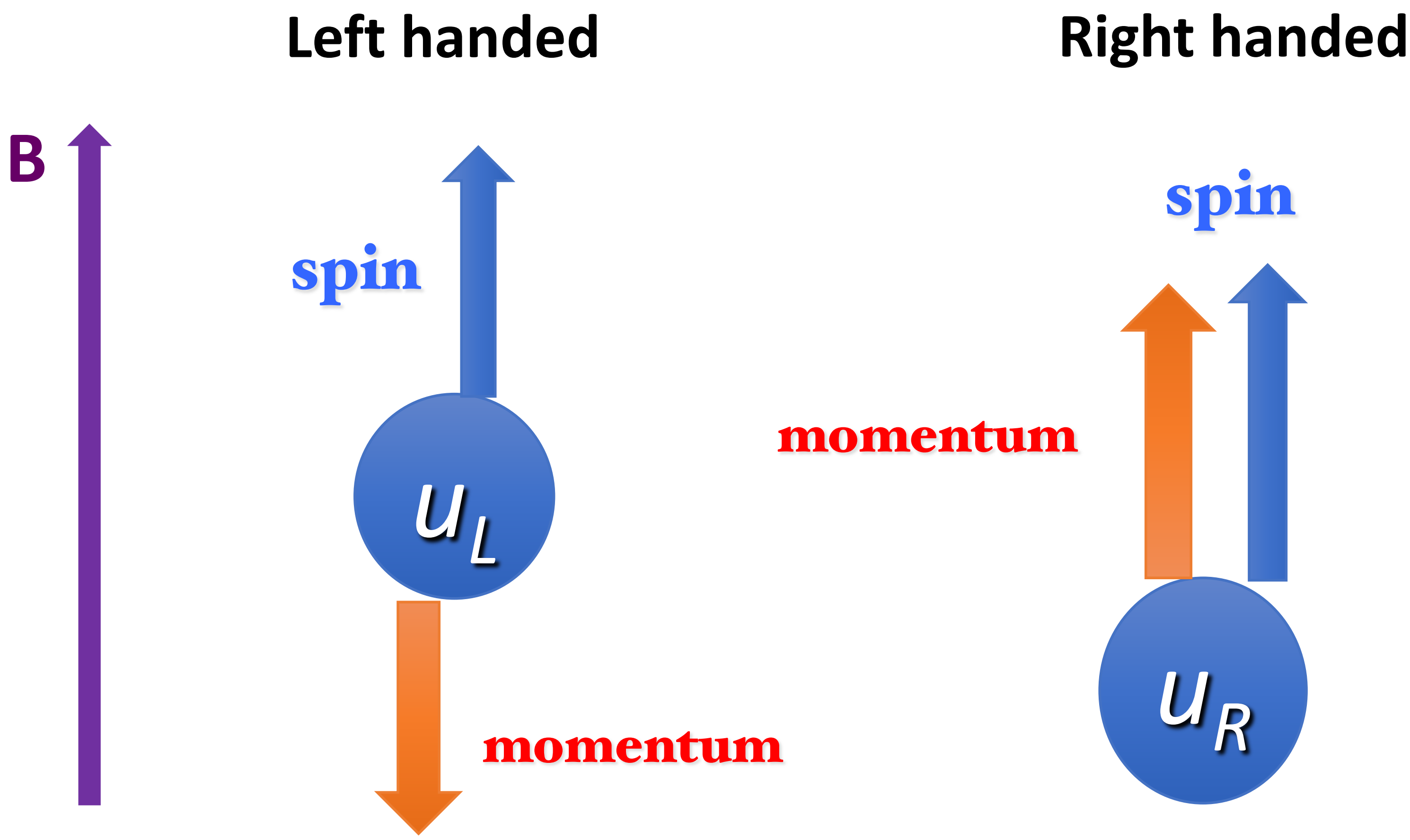
Also see studies for J/ψ:
Brandenburg, Xu, Zha, Zhang, Zhou, Zhou, PRD 2022



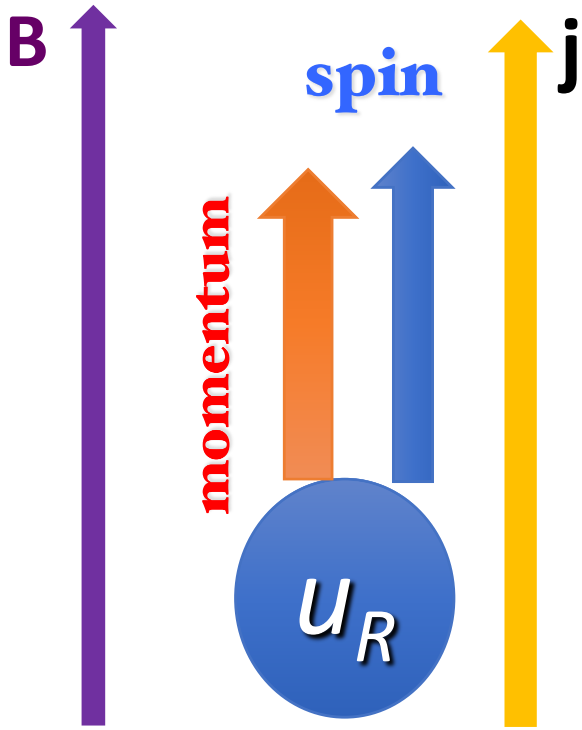
Chirality and massless fermions



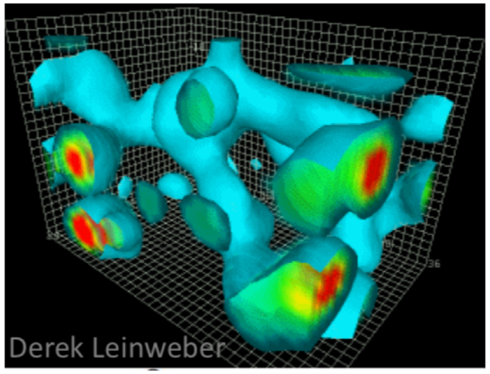
Polarization induced by magnetic fields

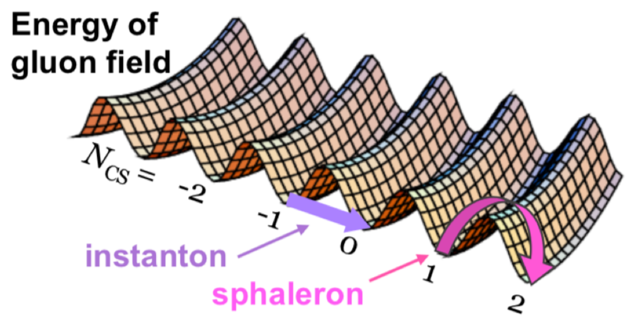


Chiral Magnetic Effect



- Magnetic fields
- Nonzero axial chemical potential





- **Charge current: charge separation**

$$\mathbf{j} = \frac{e^2}{2\pi^2}\mu_5 \mathbf{B},$$

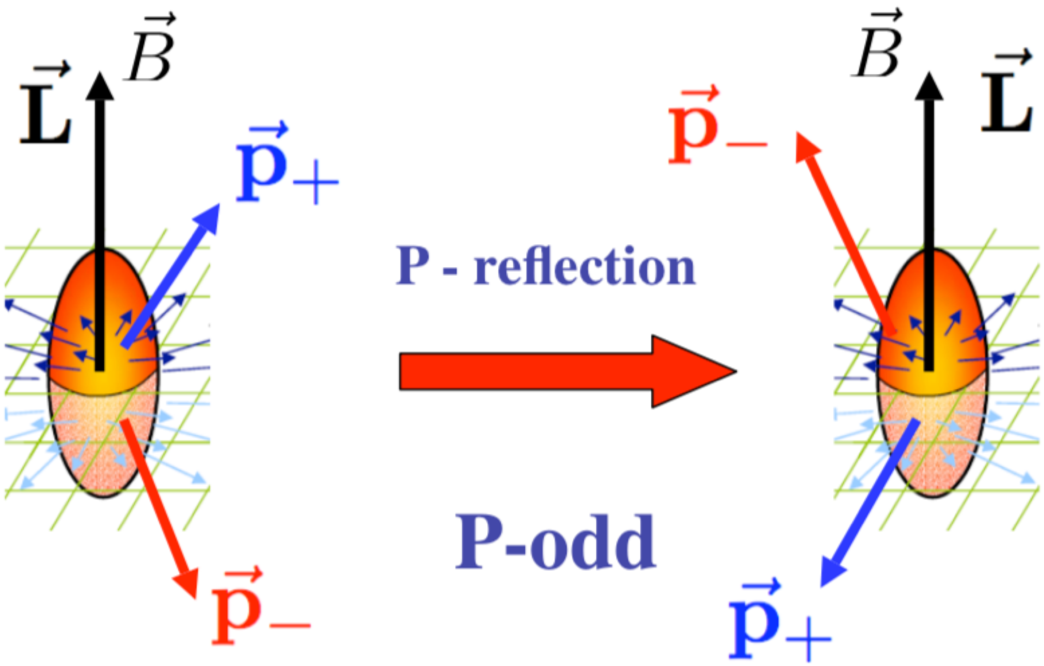
Kharzeev, Fukushima, Warrigna, (08,09), etc. ...

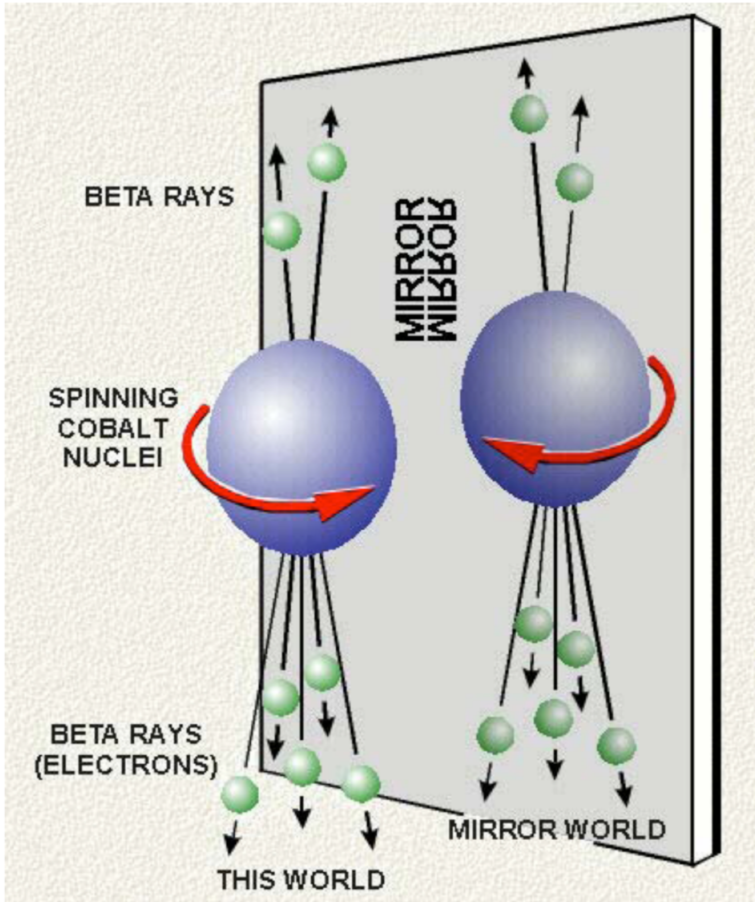
Also see the works done by the groups in PKU, Tsinghua Uni, Fudan Uni., IMP, SINAP, USTC, CCNU…

Charge separation ?= Parity Violation

Slides from Kharzeev's talk at 26th Winter Workshop on Nuclear Dynamics (2010)

Charge separation = parity violation:



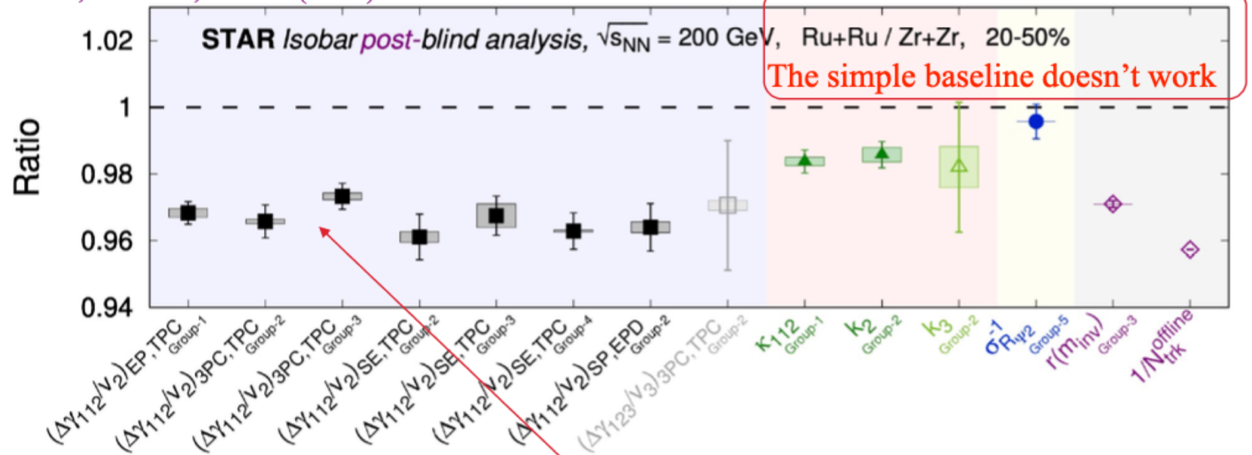


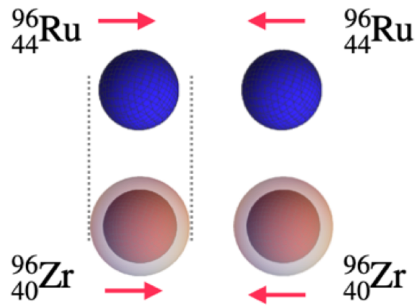
$$\mathcal{P}: \quad \vec{p} \to -\vec{p}; \quad \vec{B} \to \vec{B}; \quad \vec{L} \to \vec{L}$$

CME and isobaric collisions

To get a better understanding of the data, symmetrical studies from both experiential and theoretical sides are needed.

STAR, Isobar, PRC105, 014901(2022)





$$\Delta\gamma_{\rm bkg} = \langle \cos(\varphi_\alpha + \varphi_\beta - 2\Psi_{RP}) \rangle = \frac{N_{\rm cluster}}{N_\alpha N_\beta} \times \langle \cos(\varphi_\alpha + \varphi_\beta - 2\Psi_{\rm cluster}) \rangle \times v_{2,\rm cluster}$$

Multiplicity differences

Flow differences

Copy from Hao-jie Xu's slides

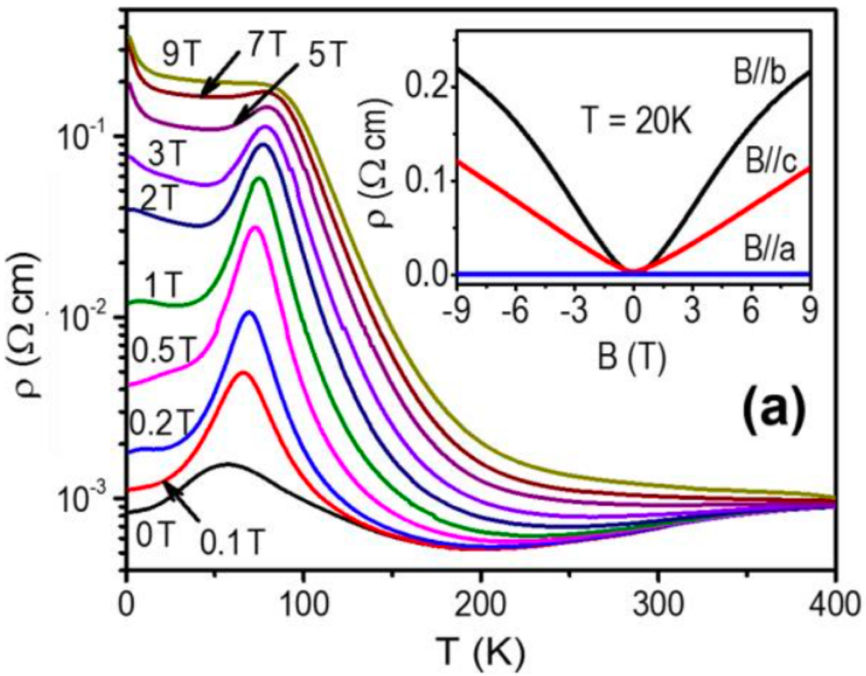
The multiplicity and v2 differences from isobar structure are crucial for the CME search in the isobar collisions at RHIC

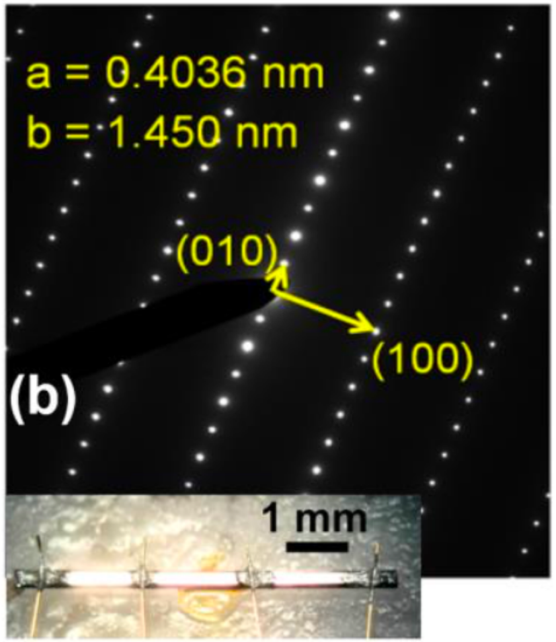
Nuclear structure plays an important role to understand the data.

Xu, et al., PRL 2018; Li, et al, PRC 2018; Zhang, Jia, PRL 2022; Deng, Huang, Ma, Wang PRC 2016; etc. …

CME in condense matter

- **Weyl Semi-metal: new transport effects**





ZrTe$_5$: Nature Physics, 12 , 550–554, (2016)

2. kinetic theory and chiral kinetic theory

(2a) Standard kinetic theory

Kinetic theory

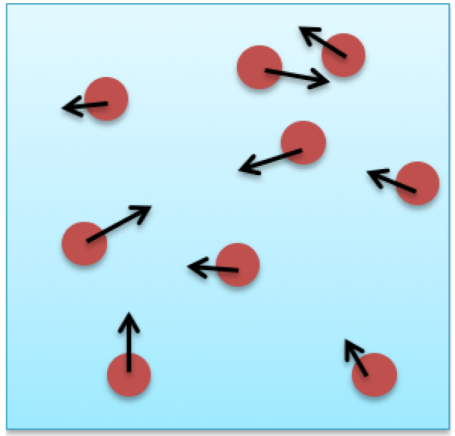
- **Assumptions:**

Mean free path >> collision length scaling

- **“distribution function” f(x,p,t)**

how many particles in a small volume of phase space (x+dx, p+dp)

e.g. Fermi-Dirac distribution function



- **Ordinary kinetic theory: Boltzmann equation**

Dynamical evolution equation for f(x,p,t)

Conventional Boltzmann equation

Particle's velocity:

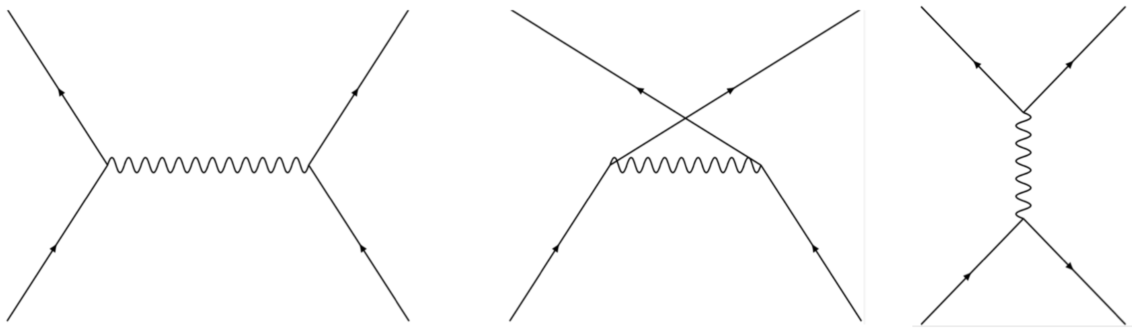
$$\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}},$$

ε : Particle's energy

Lorentz force:

$$\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B},$$

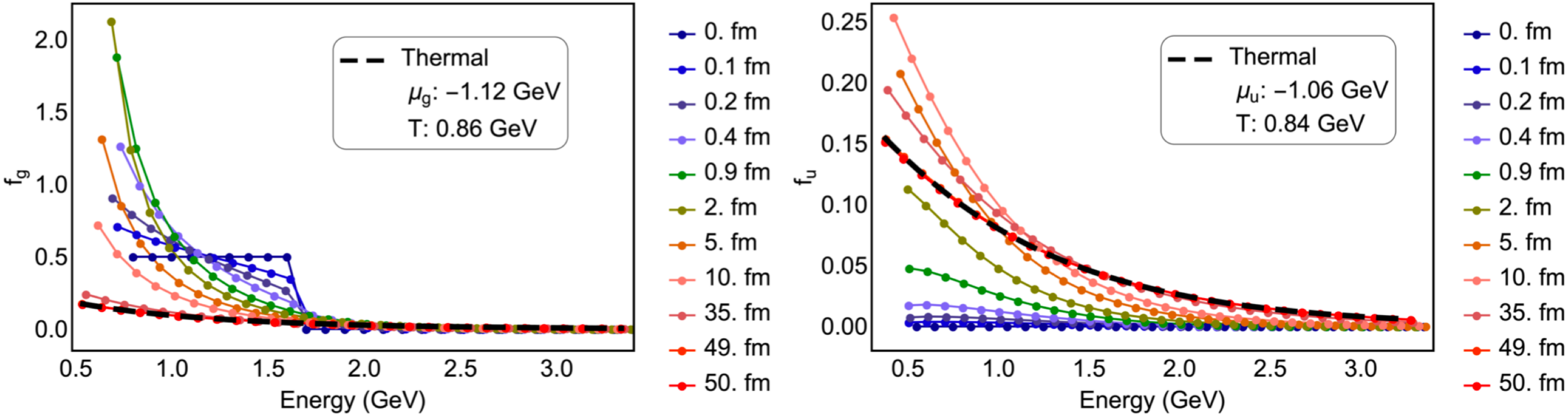
$$\partial_t f + \dot{\mathbf{x}} \cdot \nabla_x f + \dot{\mathbf{p}} \cdot \nabla_p f = C[f],$$



Collison term:

An example: evolution of a quark-gluon system

Gluons + quarks with Leading-Log order QCD scatterings



Phase space box is of size $[-3\text{fm}, 3\text{fm}]^3 \times [-2\text{GeV}, 2\text{GeV}]^3$.

- **Grids:** space: 1 grid; momentum: 30x30x30=27,000
- **Phase space size:** $[-3\text{fm},3\text{fm}]^3$ x $[-2\text{GeV},2\text{GeV}]^3$
- **Time step:** dt=0.0005fm ; 100,000 steps
- **Time cost:** around 50 hours on a Nvidia Tesla V100 card

Jun-jie Zhang, Hong-zhong Wu, SP, Guang-you Qin, Qun Wang, PRD 2020

2. kinetic theory and chiral kinetic theory

(2b) chiral kinetic theory: a quick look

What is Chiral kinetic theory?

Chiral Kinetic Theory = **Boltzmann equation** + **Massless fermions** + **Quantum Corrections**

Boltzmann equation ⟷

$$\partial_t f + \dot{\mathbf{x}} \cdot \nabla_x f + \dot{\mathbf{p}} \cdot \nabla_p f = C[f]$$

+

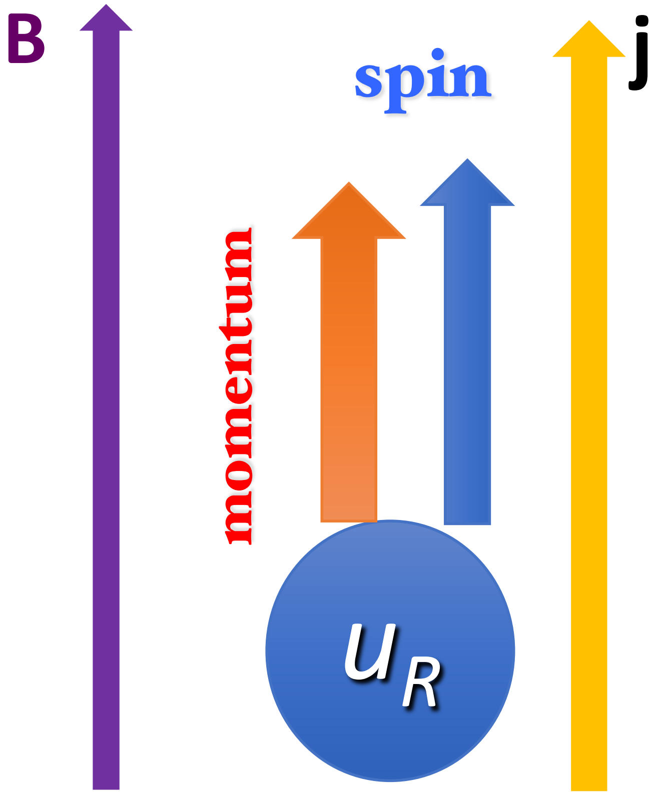
Quantum Corrections ⟷

$$\varepsilon = |\mathbf{p}| + \hbar \ldots$$

$$\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}} + \hbar \ldots$$

$$\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar \ldots$$

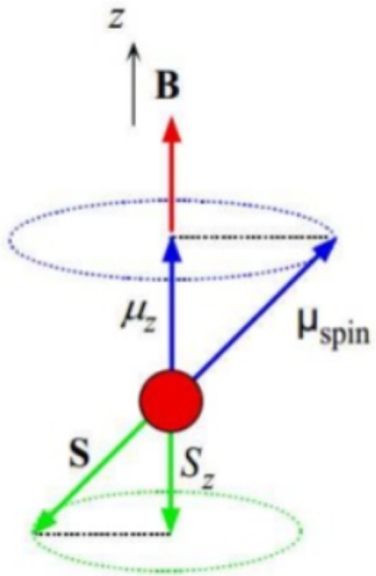
Let us "Guess" what the corrections are



Chiral Magnetic Effect

Spin coupled with magnetic fields

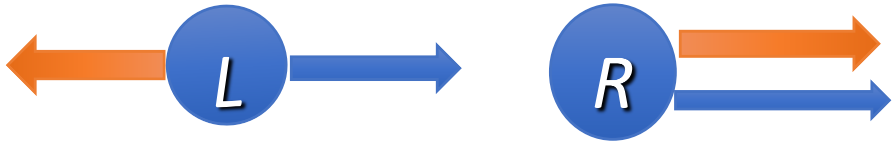
Quantum correction (I)



Spin magnetic moment:

$$\mu_S = -g_S \frac{e}{2m_e}\mathbf{S} \textcolor{red}{\rightarrow} -\frac{e}{|\mathbf{p}|}\mathbf{S} \textcolor{red}{\rightarrow} \mp \frac{e}{|\mathbf{p}|}\frac{\mathbf{p}}{2|\mathbf{p}|}$$

Massless Chirality

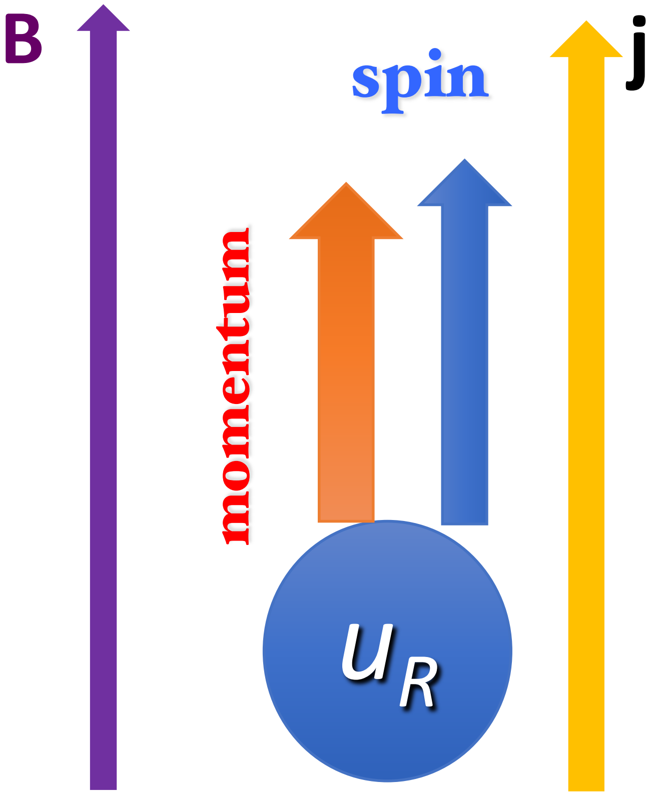


Zeeman effect:

$$\Delta\varepsilon = -\hbar\mu_S \cdot \mathbf{B} = \mp\hbar\frac{|e|}{|\mathbf{p}|}\frac{\mathbf{p}\cdot\mathbf{B}}{2|\mathbf{p}|}$$

Quantum correction (II)

- **Correction to effective velocity/w.o. E fields**



Particles move parallel or anti-parallel to B

$$\Delta \dot{\mathbf{x}} \propto \mathbf{B}$$

Dimension analysis

$$\Delta \dot{\mathbf{x}} \propto \frac{\mathbf{B}}{|\mathbf{p}|^2}$$

Final results:

$$\Delta \dot{\mathbf{x}} = \hbar \frac{\mathbf{B}}{2|\mathbf{p}|^2}$$

Quantum correction (II)

- **Correction to effective velocity with E fields**

$$\Delta\dot{\mathbf{x}} = \hbar\frac{\mathbf{B}}{2|\mathbf{p}|^2}$$

For moving particles, they feel like:

$$\mathbf{B} \to \mathbf{B} + \mathbf{E} \times \mathbf{v}$$

$$\Rightarrow \quad \Delta\dot{\mathbf{x}} = \hbar\frac{\mathbf{B}}{2|\mathbf{p}|^2} + \hbar\frac{1}{2|\mathbf{p}|^2}\mathbf{E} \times \mathbf{v}$$

Quantum correction (III)

- **Are there corrections to effective force?**

$$\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar \ldots$$

- **History: in condensate matter physics:**

  D. Xiao, M.C. Chang, Q. Niu, Rev. Mod. Phys. 82, 1959 (2010)

- **QFT: Chiral anomaly!**

  Son, Yamamoto, PRL, (2012); PRD (2013)

  Stephanov, Yin, PRL (2012);

  J.W. Chen, SP, Q. Wang, X.N. Wang, PRL (2013);

Chiral kinetic equation

$$\sqrt{G}\partial_t f + \sqrt{G}\dot{\mathbf{x}} \cdot \nabla_x f + \sqrt{G}\dot{\mathbf{p}} \cdot \nabla_p f = C[f].$$

- **Particle's effective velocity:**

$$\sqrt{G}\dot{\mathbf{x}} = \frac{\partial \varepsilon}{\partial \mathbf{p}} + \hbar \left(\frac{\partial \varepsilon}{\partial \mathbf{p}} \cdot \mathbf{\Omega} \right) \mathbf{B} + \hbar \mathbf{E} \times \mathbf{\Omega},$$

- **Effective force:**

$$\sqrt{G}\dot{\mathbf{p}} = \mathbf{E} + \frac{\partial \varepsilon}{\partial \mathbf{p}} \times \mathbf{B} + \hbar (\mathbf{E} \cdot \mathbf{B}) \mathbf{\Omega},$$

- **Berry curvature**

$$\sqrt{G} = 1 + \hbar \mathbf{B} \cdot \mathbf{\Omega}, \qquad \mathbf{\Omega} = \frac{\mathbf{p}}{2|\mathbf{p}|^3},$$

Chiral kinetic theory (massless fermions)

- **Hamiltonian formulism, effective theory**

  Son, Yamamoto, PRL, (2012); PRD (2013)

- **Path integration**

  Stephanov, Yin, PRL (2012);

  Chen, Son, Stephanov, Yee, Yin, PRL, (2014);

  J.W. Chen, J.Y. Pang, SP, Q. Wang, PRD (2014)

- **Wigner function (Quantum field theory)**
  - hydrodynamics, equilibrium

    J.W. Chen, SP, Q. Wang, X.N. Wang, PRL (2013);
  - out-of-equilibrium, quantum field theory

    Y. Hidaka, SP, D.L. Yang, PRD(RC) (2017)
  - Other studies

    A.P. Huang, S.Z. Su, Y. Jiang, J.F. Liao, P.F. Zhuang, PRD (2018)

- **World-line formulism**

  N. Muller, R, Venugopalan PRD 2017

Also see recent review:

Gao, Liang, Wang, Int.J.Mod.Phys A 36 (2021), 2130001

Hidaka, SP, D.L. Yang, Q. Wang, Particle and Nuclear Physics, 127, 103989 (2022).

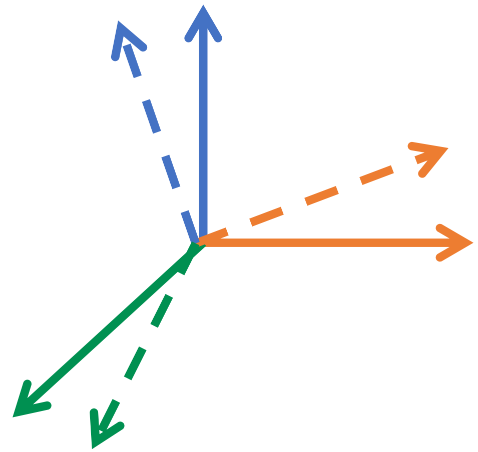
4. Non-trivial Lorentz symmetry

Subgroup of Lorentz symmetry

- **Massive particles: Rest frame**

$$p^\mu = (m, 0, 0, 0)$$

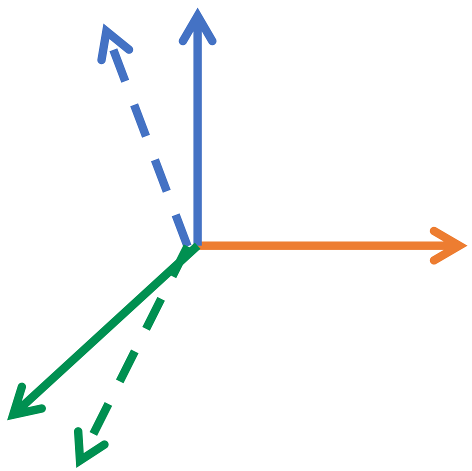
Subgroup: SO(3)



- **Massless particles: No rest frame**

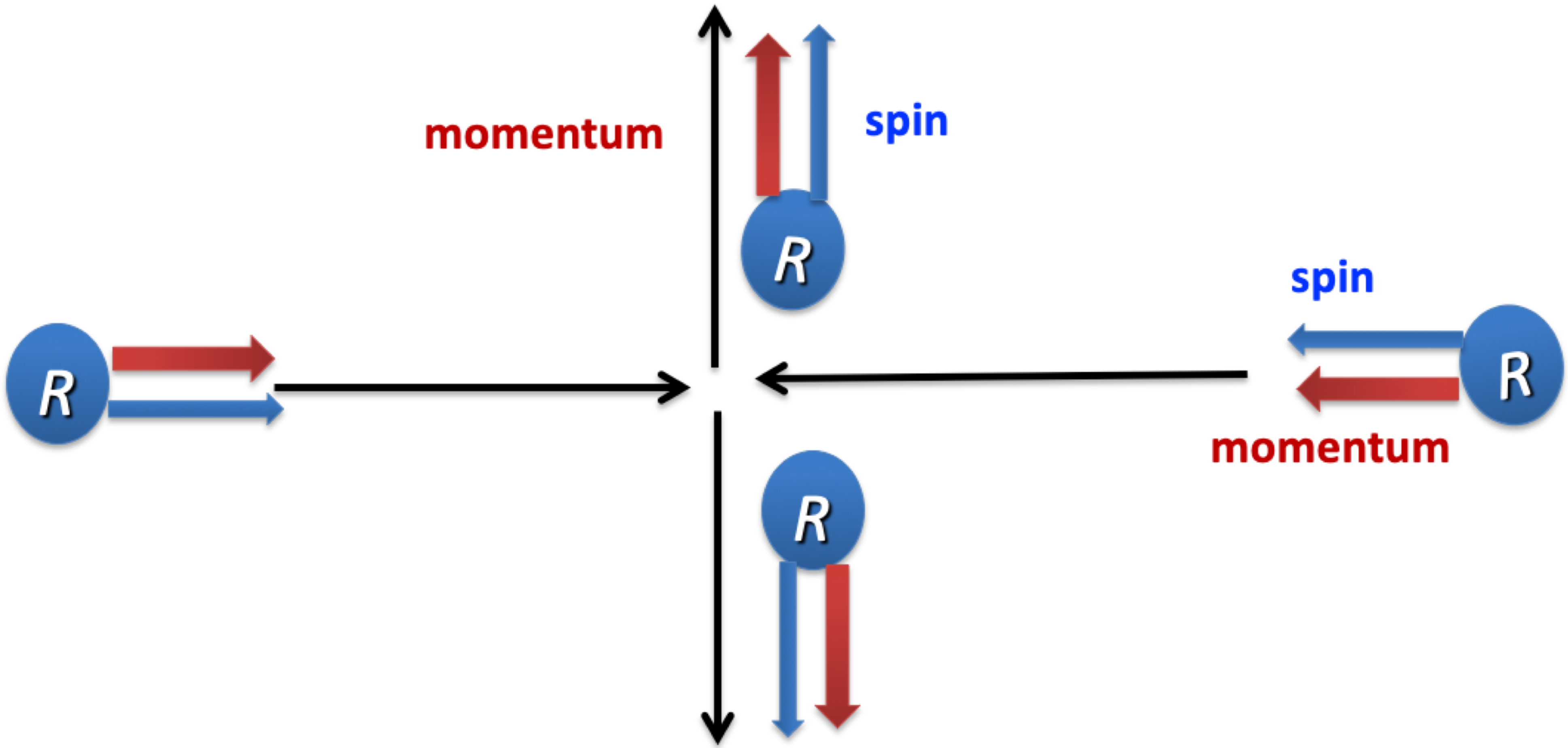
$$p^\mu = (|p_z|, 0, 0, p_z)$$

Subgroup: ISO(2)



Side-jump (I)

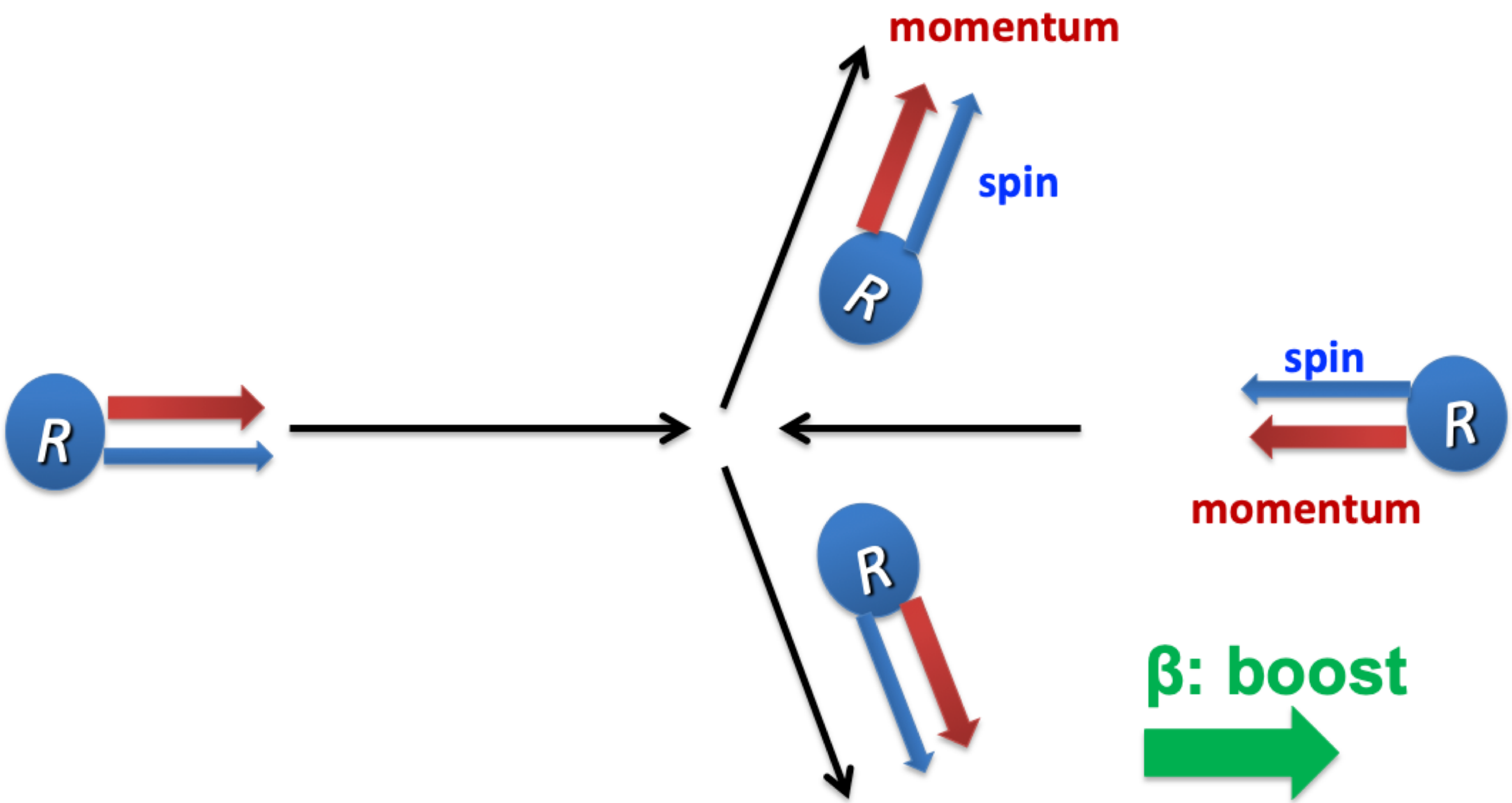
Orbital angular momentum and spin are conserved separately



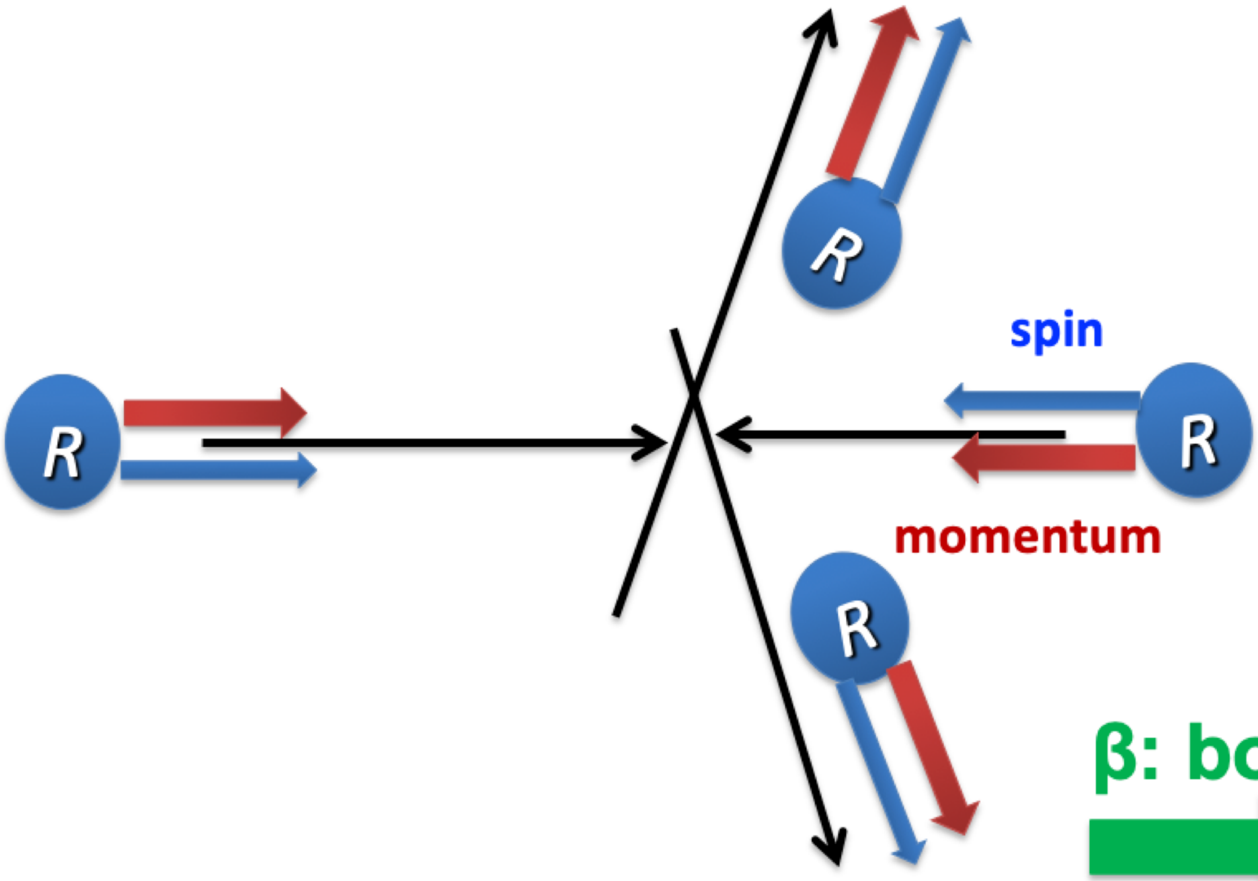
Chen, Son, Stephanov, Yee, Yin, PRL, (2014)

Side-jump (II)

Orbital angular momentum seems to be conserved separately.
Spin is NOT conserved!



Side-jump (III)



x has a shift!!!

"Side-jump"：

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \boldsymbol{\beta} t + \delta \mathbf{x}, \\ \mathbf{p}' &= \mathbf{p} + \boldsymbol{\beta} \varepsilon + \delta \mathbf{p}, \end{aligned}$$

$$\begin{aligned} \delta \mathbf{x} &= \hbar \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}}{2|\mathbf{p}|}, \\ \delta \mathbf{p} &= \hbar \frac{\boldsymbol{\beta} \times \hat{\mathbf{p}}}{2|\mathbf{p}|} \times \mathbf{B} \end{aligned}$$

β: boost

Chen, Son, Stephanov, PRL, (2015);
Y. Hidaka, SP, D.L. Yang, PRD (2016)

Non-trivial Lorentz symmetry

- **Quantum field theory**

$$j^\mu = \overline{\psi}\sigma^\mu\psi \rightarrow \Lambda^\mu_\nu j^\nu$$

- **Lorentz transformation**

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu, \; p^{\mu\prime} = \Lambda^\mu_\nu p^\nu,$$

$$\left\{ f'(x',p',t') = f(x',p',t') + \hbar N^\mu(\partial^x_\mu + F_{\nu\mu}\partial^\nu_p)f, \right\}$$

Infinitesimal Lorentz Transform

$$\left\{ \begin{array}{lcl} \delta\mathbf{x} & = & \hbar\dfrac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}, \\ \delta\mathbf{p} & = & \hbar\dfrac{\boldsymbol{\beta}\times\hat{\mathbf{p}}}{2|\mathbf{p}|}\times\mathbf{B} \end{array} \right\}$$

Chen, Son, Stephanov, PRL, (2015);
Y. Hidaka, SP, D.L. Yang, PRD (2016)

Part 2

Wigner functions and the master equations

1. Definition of gauge invariant covariant Wigner function
   (1a) Gauge invariant covariant Wigner function
   (1b) Vector, chiral currents
   (1c) The choice of gauge link
2. Master equation for covariant Wigner function
   (2a) From Dirac equation to master equations for chiral fermions
   (2c) Master equations in general case (massive case)
3. Equal-time formulism for Wigner function

1. Definition of gauge invariant covariant Wigner function

(1a) Gauge invariant covariant Wigner function

How to define “position” and “momentum”?

- In a ”classical” many body system, we use distribution function f(x,p).

- In Quantum mechanics, we know,

$$[x, p_x] = i\hbar.$$

- If we still want to use the distribution function f(t,x,p) for a quantum many body system, how can we measure the x and p at the same time t?

（For simplicity, we use the natural unit $\hbar = c = k_B = 1$.)

"position" and "momentum"

- Assuming we have two points, x_1^μ and x_2^μ with $\mu = 0, 1, 2, 3$. (For simplicity, we neglect the upper index μ here)
- We introduce another two variables,

$x = \frac{x_1+x_2}{2}$, and $y = x_1 - x_2$

$$\partial_x = \partial_{x_1} + \partial_{x_2}$$

$$\partial_y = \frac{1}{2}(\partial_{x_1} - \partial_{x_2}) \rightarrow p = -i\partial_y$$

$$\rightarrow [x, p] = [\frac{x_1 + x_2}{2}, -i\frac{1}{2}(\partial_{x_1} - \partial_{x_2})] = 0$$

Wigner-Weyl transformation

$$O(x_1, x_2) = \psi^*(x_1)\psi(x_2) \rightarrow O(x, p) = \int d^3y e^{-ip\cdot y}\psi^*\left(x + \frac{y}{2}\right)\psi\left(x - \frac{y}{2}\right)$$

Definition of Wigner function in Quantum Mechanics

$$W(\mathbf{x},\mathbf{p}) = \int d^3\mathbf{y} \exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{y}\right)\left\langle \mathbf{x}-\frac{\mathbf{y}}{2}\right|\widehat{\rho}\left|\mathbf{x}+\frac{\mathbf{y}}{2}\right\rangle,$$

where ρ is the density matrix in QM.

$$\int d^3\mathbf{p} W(\mathbf{x},\mathbf{p}) = |\psi(\mathbf{x})|^2$$

$$\int d^3\mathbf{x} W(\mathbf{x},\mathbf{p}) = |\psi(\mathbf{p})|^2$$

$$\int d^3\mathbf{x}\int d^3\mathbf{p} W(\mathbf{x},\mathbf{p}) = \mathrm{Tr}\ \hat{\rho} = 1$$

Wigner function plays a role as the “probability” in phase space (x,p).

Note that, Wigner function itself could be negative sometimes.

Covariant Wigner function in quantum field theory

- (Covariant) Wigner operator

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}_\beta(x_1) U(x_1, x_2) \psi_\alpha(x_2)$$

with gauge link

$$U(x_1, x_2) = \exp\left[-iQ \int_{x_2}^{x_1} dz \cdot A(z)\right]$$

- Wigner function:

$$W(x,p) = \left\langle : \hat{W}(x,p) : \right\rangle$$

W operator in thermal ensemble average and normal ordering of the operators.

Vasak, Gyulassy, Elze, Ann. Phys. (N.Y.) 173, 462 (1987);

Elze, Heinz,Phys.Rep.183,81(1989).

1. Definition of gauge invariant covariant Wigner function

(1b) Vector, chiral currents

Decomposition of Wigner function

- **For massive spinors, Wigner function is a 4x4 matrix.**

$$W = \mathcal{F} + i\mathcal{P}\gamma^5 + \mathcal{V}^\mu \gamma_\mu + \mathcal{A}^\mu \gamma^5 \gamma_\mu + \frac{1}{2}\mathcal{S}^{\mu\nu}\sigma_{\mu\nu},$$

$$\sigma_{\mu\nu} = i[\gamma_\mu, \gamma_\nu]/2$$

$$\mathcal{F} = \mathrm{Tr}\ W(x,p) \sim \langle \overline{\psi}\psi \rangle$$

$$\mathcal{P} = \mathrm{Tr}\ \gamma^5 W(x,p) \sim \langle \overline{\psi}\gamma^5\psi \rangle$$

$$\mathcal{V}^\mu = \mathrm{Tr}\ \gamma^\mu W(x,p) \sim \langle \overline{\psi}\gamma^\mu\psi \rangle$$

Vector current

$$\mathcal{A}^\mu = \mathrm{Tr}\ \gamma^\mu\gamma^5 W(x,p) \sim \langle \overline{\psi}\gamma^5\gamma^\mu\psi \rangle$$

Chiral current (Axial vector current)

$$\mathcal{S}^{\mu\nu} = \mathrm{Tr}\ \sigma^{\mu\nu} W(x,p) \sim \langle \overline{\psi}\sigma^{\mu\nu}\psi \rangle$$

$$W^\dagger = \gamma^0 W \gamma^0$$

All the coefficients, F, P, V, A, S are real!

$$\mathcal{V}^{\mu\dagger} = \mathrm{Tr}\ W^\dagger(\gamma^\mu)^\dagger = \mathrm{Tr}\ [(\gamma^0 W\gamma^0)(\gamma^0\gamma^\mu\gamma^0)] = \mathcal{V}^\mu$$

Vector and chiral currents

$$\mathcal{V}^{\mu} = \mathrm{Tr}\,\gamma^{\mu} W(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}\left(x+\frac{y}{2}\right)\gamma^{\mu} \exp\left[-i\int_{x-y/2}^{x+y/2} dz\cdot A(z)\right]\psi\left(x-\frac{y}{2}\right)$$

$$\mathcal{A}^{\mu} = \mathrm{Tr}\,\gamma^{\mu}\gamma^5 W(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}\left(x+\frac{y}{2}\right)\gamma^{\mu}\gamma^5 U \exp\left[-i\int_{x-y/2}^{x+y/2} dz\cdot A(z)\right]\psi\left(x-\frac{y}{2}\right)$$

- **Our definition means the vector and chiral currents are regularized. If we integrate over momentum,**

$$j_5^{\mu} = \int d^4p \mathcal{A}^{\mu}(x,p) \sim \lim_{y\to 0} \overline{\psi}\left(x+\frac{y}{2}\right)\gamma^{\mu}\gamma^5 U \exp\left[-i\int_{x-y/2}^{x+y/2} dz\cdot A(z)\right]\psi\left(x-\frac{y}{2}\right)$$

which is the chiral current in position space in Peskin's textbook QFT Sec. 19,

$$j^{\mu 5} = \operatorname*{symm\,lim}_{\epsilon\to 0}\left\{\overline{\psi}(x+\tfrac{\epsilon}{2})\gamma^{\mu}\gamma^5 \exp\left[-ie\int_{x-\epsilon/2}^{x+\epsilon/2} dz\cdot A(z)\right]\psi(x-\tfrac{\epsilon}{2})\right\}. \quad (19.22)$$

That regularization and gauge link are the key to get chiral anomaly!

An example: 1+1 dim chiral anomaly in operator level

$$\partial_\mu j^{\mu 5} = \text{symm} \lim_{\epsilon\to 0} \left\{ \bar\psi(x+\tfrac{\epsilon}{2}) \left[-ie\gamma^\mu \epsilon^\nu (\partial_\mu A_\nu - \partial_\nu A_\mu) \right] \gamma^5 \psi(x-\tfrac{\epsilon}{2}) \right\}.$$

Coming from gauge link

- In 1+1 dim, we can have

$$\overline{\psi}(x+\tfrac{\epsilon}{2})\Gamma\psi(x-\tfrac{\epsilon}{2}) = \frac{-i}{2\pi} \text{tr}\left[\frac{\gamma^\alpha \epsilon_\alpha}{\epsilon^2}\Gamma\right]$$

$$\psi(y)\bar\psi(z) = \int \frac{d^2k}{(2\pi)^2} e^{-ik\cdot(y-z)} \frac{i\not k}{k^2} = -\not\partial\left(\frac{i}{4\pi}\log(y-z)^2\right) = \frac{-i}{2\pi}\frac{\gamma^\alpha (y-z)_\alpha}{(y-z)^2}.$$

Regularization also plays a crucial ro

$$\partial_\mu j^{\mu 5} = \text{symm} \lim_{\epsilon\to 0} \left\{ \frac{-i}{2\pi} \text{tr}\left[\frac{\gamma^\alpha \epsilon_\alpha}{\epsilon^2}\gamma^\mu\gamma^5\right] \cdot (-ie\epsilon^\nu F_{\mu\nu}) \right\}.$$

$$= \frac{e}{2\pi} \text{symm} \lim_{\epsilon\to 0} \left\{ 2\frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2} \right\} \epsilon^{\mu\alpha} F_{\nu\alpha}$$

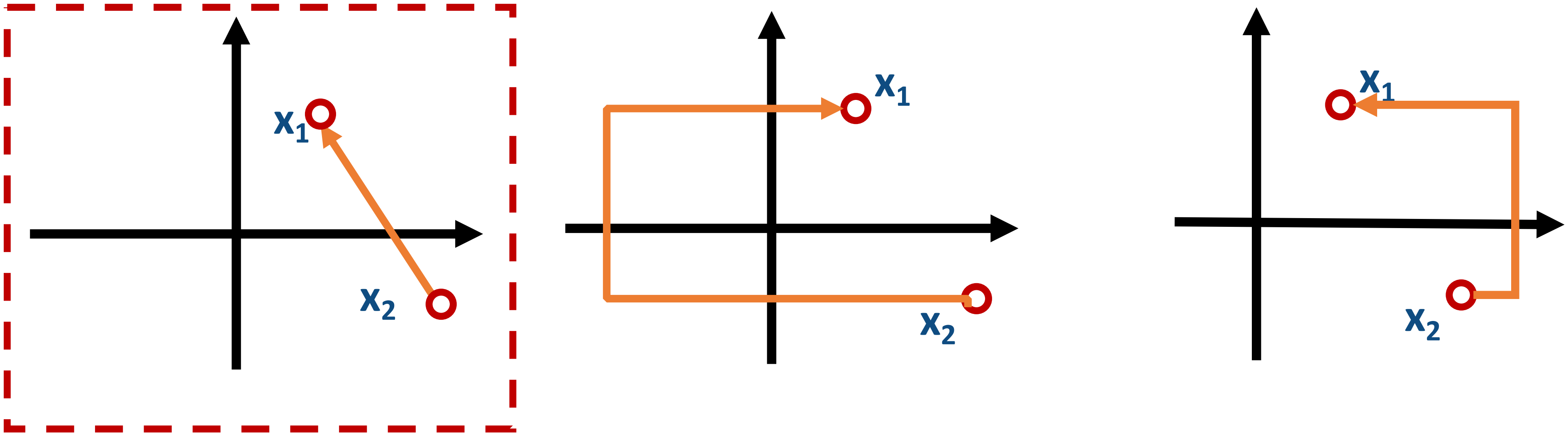
$$= \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$

chiral anomaly in 1+1 dim

1. Definition of gauge invariant covariant Wigner function

(1c) The choice of gauge link

The choice of gauge link



A compact form for Wigner function (I)

- For a function A(a+x) with a small x, we can expand it as

$$A(a+x) = A(a) + A'(a)x + \frac{1}{2}A''(a)x^2 + \ldots \frac{1}{n!}\frac{d^{(n)}A}{dx^n}x^n = e^{x\cdot\partial}A(a)$$

- Similarly, we have

$$\overline{\psi}_\beta\left(x+\frac{y}{2}\right)\psi_\alpha\left(x-\frac{y}{2}\right) \to \overline{\psi}_\beta(x)e^{+\frac{1}{2}y\cdot\overleftarrow{\partial}} \times e^{-\frac{1}{2}y\cdot\partial}\psi_\alpha(x)$$

$$= \overline{\psi}_\beta(x)\exp\left[-\frac{1}{2}y\cdot(\partial-\overleftarrow{\partial})\right]\psi_\alpha(x)$$

- Elze, Gyulassy, Vasak, Nucl.Phys.B 276 (1980), 706

A compact form for Wigner function (II)

- **With the straight line type gauge link, we find**

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\overline{\psi}_\beta\left(x+\frac{y}{2}\right)\exp\left[-i\int_{x_2}^{x_1} dz\cdot A(z)\right]\psi_\alpha\left(x-\frac{y}{2}\right)$$

$$\to \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\overline{\psi}_\beta(x)\exp\left[-\frac{1}{2}y\cdot(D-\overleftarrow{D})\right]\psi_\alpha(x)$$

$$\sim \overline{\psi}_\beta(x)\delta^{(4)}(p^\mu-\hat{p}^\mu)\psi_\alpha(x) \qquad D_\mu = \partial_\mu + iA_\mu$$

$$\hat{p}^\mu = \frac{i}{2}(D^\mu - \overleftarrow{D}) = \hat{p}_c^\mu - A^\mu$$

Kinetic momentum

canonical momentum

The straight line type gauge link corresponds the momentum in W(x,p) is the kinetic one.

Elze, Gyulassy, Vasak, Nucl.Phys.B 276 (1980), 706

2. Master equation for covariant Wigner function

(2a) From Dirac equation to master equations for chiral fermions

Dirac equation for chiral fermions

- **Let us start from the Lagrangian for massless fermions in a background electromagnetic field,**

$$\mathcal{L} = \overline{\psi} i\gamma \cdot D\psi = \chi_R^\dagger i\sigma \cdot D\chi_R + \chi_L^\dagger i\overline{\sigma} \cdot D\chi_L,$$

$$\psi = (\chi_L, \chi_R)^T$$

L, R: left or right handed Pauli spinors

$$\sigma^\mu = (1, \boldsymbol{\sigma}),\ \overline{\sigma}^\mu = (1, -\boldsymbol{\sigma}),\ \text{and}\ D_\mu = \hbar\partial_\mu + iA_\mu$$

- **We concentrate on right handed fermions and suppress the subscript R**

$$i\sigma \cdot D\chi_R = 0, \qquad -\chi_R^\dagger i\sigma \cdot \overleftarrow{D}^\dagger = 0,$$

Blaizot, Iancu, Phys. Rept. 359 (2002) 355–528

Equation for $S^<$

- We define $S^>$ and $S^<$ for right-handed fermions

$$
\begin{aligned}
S^>_{ab}(x_1, x_2) &= \left\langle \chi_a(x_1)\chi^\dagger_b(x_2) \right\rangle, \\
S^<_{ab}(x_1, x_2) &= \left\langle \chi^\dagger_b(x_2)\chi_a(x_1) \right\rangle,
\end{aligned}
$$

$$\boxed{S = S^<}$$

$$
\begin{aligned}
i\sigma \cdot D\chi_R = 0, \\
-\chi^\dagger_R i\sigma \cdot \overleftarrow{D}^\dagger = 0,
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
i\sigma \cdot D_{x_1} S(x_1, x_2) &= 0, \\
-S(x_1, x_2) i\sigma \cdot \overleftarrow{D}^\dagger_{x_2} &= 0,
\end{aligned}
$$

Gauge invariant S

- We can define the gauge invariant two-point function by including gauge links,

$$\widetilde{S}(x,y) = U(x,x_1)\, S(x_1,x_2) U(x_2,x), \qquad x = \frac{1}{2}(x_1+x_2),$$

$$U(x_1,x_2) = \mathcal{P}\exp\left[-i\frac{1}{\hbar}\int_{x_2}^{x_1} dz\cdot A(z)\right], \qquad y = x_1 - x_2,$$

- If electromagnetic fields are classical background ones, $U(x_1, x_2)$ is just a phase factor

$$\widetilde{S}(x_1,x_2) = S(x_1,x_2)U(x_2,x_1) = U(x_2,x_1)S(x_1,x_2).$$

$$\Rightarrow \quad S(x_1,x_2) = U(x_2,x_1)^{\dagger}\tilde{S}(x_1,x_2) = U(x_1,x_2)\tilde{S}(x_1,x_2)$$

Equation for gauge invariant S (I)

$$
\begin{aligned}
i\sigma \cdot D_{x_1} S(x_1, x_2) &= i\sigma \cdot [\hbar \partial_{x_1} + ieA(x_1)] S(x_1, x_2) \\
&= i\sigma \cdot D_{x_1} U(x_1, x_2) \widetilde{S}(x_1, x_2) \\
&= i\sigma \cdot [D_{x_1} U(x_1, x_2)] \widetilde{S}(x_1, x_2) + U(x_1, x_2) i\hbar\sigma \cdot \partial_{x_1} \widetilde{S}(x_1, x_2),
\end{aligned}
$$

$$
\begin{aligned}
D_{x_1,\mu} U(x_1, x_2) &= -U(x_1, x_2) i(x_1^\nu - x_2^\nu) \int_0^1 ds\, s F_{\mu\nu}[z(s)] \\
&= -U(x_1, x_2) i y^\nu \int_0^1 ds\, s F_{\mu\nu}[z(s)],
\end{aligned}
$$

See next page for Details.

$$
z(s) = x + (s - 1/2) y \text{ with } z(0) = x_2 \text{ and } z(1) = x_1.
$$

- **Also see Eq. (3.16) of Elze, Gyulassy, Vasak, Nucl. Phys. B276 (1986) 706**

Get rid of $\partial_{x_1} U(x_1, x_2)$

$$\int_{x_2}^{x_1} dz \cdot A(z) = y^\nu \int_0^1 dsA_\nu \left(x + (s - \frac{1}{2})y\right)$$

$x = \frac{x_1+x_2}{2}$,
$y = x_1 - x_2$

$$\partial_\mu^{x_1} \int_{x_2}^{x_1} dz \cdot A(z) = \left(\frac{1}{2}\partial_\mu^x + \partial_y\right) \left[y^\nu \int_0^1 dsA_\nu \left(x + (s - \frac{1}{2})y\right)\right]$$

$$= \int_0^1 dsA_\mu(z(s)) + \frac{1}{2}y^\nu \int_0^1 ds\partial_\mu^x A_\nu(z(s)) + y^\nu \int_0^1 ds\partial_\mu^y A_\nu(z(s))$$

$$= \int_0^1 dsA_\mu(z(s)) + y^\nu \int_0^1 dss[\partial_\mu A_\nu(z(s))]$$

$$= \int_0^1 dsA_\mu(z(s)) + y^\nu \int_0^1 dss[F_{\mu\nu}(z(s)) + \partial_\nu A_\mu(z(s))]$$

$$= \int_0^1 dsA_\mu(z(s)) + y^\nu \int_0^1 dss[F_{\mu\nu}(z(s))] + \int_0^1 dss\frac{d}{ds}A_\mu(z(s))$$

$$= y^\nu \int_0^1 dss[F_{\mu\nu}(z(s))] + A_\mu(x_1)$$

$$y^\nu \partial_\nu^z A_\mu(z(s)) = (s - \frac{1}{2})y^\nu \partial_\nu^y A_\mu(z) = \frac{d}{ds}A_\mu(z(s))$$

$$\frac{d}{ds}A_\mu(z(s)) = \frac{\partial z^\alpha}{\partial s}\partial_\alpha^z A_\mu = y^\alpha \partial_\alpha^z A_\mu$$

Equation for gauge invariant S (II)

$$\sigma^{\mu}\left\{\frac{1}{2}i\hbar\frac{\partial}{\partial x^{\mu}}+i\hbar\frac{\partial}{\partial y^{\mu}}+y^{\nu}\int_{0}^{1}ds\,sF_{\mu\nu}[z(s)]\right\}\widetilde{S}(x,y)=0.$$

- **We can expand $F_{\mu\nu}$ in powers of y,**

$$\begin{aligned}\int_{0}^{1}ds\,sF_{\mu\nu}[z(s)] &= \sum_{n=0}\frac{1}{n!}\int_{0}^{1}ds\,s\left(s-\frac{1}{2}\right)^{n}(y\cdot\partial_{x})^{n}F_{\mu\nu}(x)\\ &= \sum_{n=0}\frac{[(-1)^{n}+3+2n]}{4(n+2)!}\left(\frac{1}{2}y\cdot\partial_{x}\right)^{n}F_{\mu\nu}(x).\end{aligned}$$

Wigner transformation to gauge invariant S

- **Wigner transformation and its inverse transformation**

$$\widetilde{S}(x,p) = \int d^4y \exp\left(\frac{i}{\hbar}p\cdot y\right)\widetilde{S}(x,y),$$

$$\widetilde{S}(x,y) = \int \frac{d^4p}{(2\pi\hbar)^4}\exp\left(-\frac{i}{\hbar}p\cdot y\right)\widetilde{S}(x,p).$$

For simplicity, we define $S(x,p) \equiv S^<(x,p) \equiv \widetilde{S}(x,p).$

- **Then, we get**

$$\left\{\begin{aligned} &\sigma\cdot\left(\frac{1}{2}i\hbar\nabla + \Pi\right)S(x,p) = 0. \\ &\left(-\frac{1}{2}i\hbar\nabla + \Pi\right)S(x,p)\cdot\sigma = 0. \end{aligned}\right\}$$

$$\nabla_\mu = \partial_\mu^x - j_0(\Delta)F_{\mu\nu}(x)\partial_p^\nu,$$

$$\Pi_\mu = p_\mu - \frac{1}{2}\hbar j_1(\Delta)F_{\mu\nu}(x)\partial_p^\nu,$$

$$\Delta = (1/2)\hbar\partial_x\cdot\partial_p$$

$$j_0(z) = \sin z/z \text{ and } j_1(z) = (\sin z - z\cos z)/z^2$$

Constant EM fields limits

$$\nabla^\mu = \sum_{n=0}^{\infty} \hbar^{2n} \nabla^\mu_{(2n)},$$

$$\Pi^\mu = \sum_{n=0}^{\infty} \hbar^{2n} \Pi^\mu_{(2n)}.$$

$$\nabla^{(0)}_\mu = \partial^x_\mu - F_{\mu\nu}\partial^\nu_p, \quad \Pi^{(0)}_\mu = p_\mu.$$

$$\nabla^\mu_{(2n)} = (-1)^{n+1} \frac{1}{(2n+1)!} \left(\frac{1}{2}\hbar\partial_x \cdot \partial_p \right)^{2n} F_{\mu\nu}(x)\partial^\nu_p,$$

$$\Pi^\mu_{(2n)} = (-1)^n \frac{n}{(2n+1)!} \left(\frac{1}{2}\hbar\partial_x \cdot \partial_p \right)^{2n-1} F_{\mu\nu}(x)\partial^\nu_p.$$

- **In constant background fields, we find**

$$\sigma^\mu \left(\frac{1}{2} i\hbar \nabla^{(0)}_\mu + p_\mu \right) S(x,p) = 0,$$

$$\left(-\frac{1}{2} i\hbar \nabla^{(0)}_\mu + p_\mu \right) S(x,p)\sigma^\mu = 0.$$

Decomposition of S

Since S(x,p) is a 2x2 matrix, we can use the 1 and Pauli matrix to decompose it,

$$\begin{cases} S_R(x,p) = \bar{\sigma}^\mu \mathscr{J}_\mu^+, \\ S_L(x,p) = \sigma^\mu \mathscr{J}_\mu^-, \end{cases} \quad \xrightarrow{\mathrm{Tr}\,(\sigma^\mu \bar{\sigma}^\nu) = 2\eta^{\mu\nu}} \quad \begin{cases} \mathscr{J}_\mu^+ = \frac{1}{2}\mathrm{Tr}\,(\sigma_\mu S_R), \\ \mathscr{J}_\mu^- = \frac{1}{2}\mathrm{Tr}\,(\bar{\sigma}_\mu S_L), \end{cases}$$

$$\sigma^\mu = (1, \boldsymbol{\sigma}),\ \bar{\sigma}^\mu = (1, -\boldsymbol{\sigma}),$$

± represent right/left-handed components of Wigner functions

which are connected to the V and A in Wigner functions

$$\left\{ \mathcal{V}_\mu = \mathscr{J}_\mu^+ + \mathscr{J}_\mu^-,\ \mathcal{A}_\mu = \mathscr{J}_\mu^+ - \mathscr{J}_\mu^-. \right\}$$

Master eq. for massless Wigner function

By using identities

$$\begin{aligned}\sigma^\mu\overline{\sigma}^\nu &= \eta^{\mu\nu}-\frac{1}{2}i\epsilon^{\mu\nu\lambda\rho}\sigma_\lambda\overline{\sigma}_\rho,\\ \overline{\sigma}^\nu\sigma^\mu &= \eta^{\mu\nu}+\frac{1}{2}i\epsilon^{\mu\nu\lambda\rho}\sigma_\lambda\overline{\sigma}_\rho,\end{aligned}$$

and insert the decomposition of S into the eqs.

$$\begin{aligned}\sigma^\mu\left(\frac{1}{2}i\hbar\nabla_\mu^{(0)}+p_\mu\right)S(x,p) &= 0,\\ \left(-\frac{1}{2}i\hbar\nabla_\mu^{(0)}+p_\mu\right)S(x,p)\sigma^\mu &= 0.\end{aligned}$$

We can get the master equation for massless Wigner function

$$\begin{aligned}\Pi^\mu\mathscr{J}_\mu^s(x,p) &= 0,\\ \nabla^\mu\mathscr{J}_\mu^s(x,p) &= 0,\\ 2s\left(\Pi^\mu\mathscr{J}_s^\nu-\Pi^\nu\mathscr{J}_s^\mu\right) &= -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_\rho\mathscr{J}_\sigma^s,\end{aligned}$$

s = ± is the chirality index for right-handed (+) and left-handed (−) fermions

Summary for Master eq.

$$
\begin{aligned}
\Pi^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0, \\
\nabla^{\mu}\mathscr{J}^{s}_{\mu}(x,p) &= 0, \\
2s\,(\Pi^{\mu}\mathscr{J}^{\nu}_{s} - \Pi^{\nu}\mathscr{J}^{\mu}_{s}) &= -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}^{s}_{\sigma},
\end{aligned}
$$

s = ± is the chirality index for right-handed (+) and left-handed (−) fermions

- **Assumption:**
  - **Constant homogenous EM fields** $\partial_\alpha F^{\mu\nu} = 0$
  - **No-interactions between particles**

2. Master equation for covariant Wigner function

(2c) Master equations in general case (massive case)

General cases

- If we start from a 4x4 Wigner function, by implementing the Dirac equations, we can get, in a classical EM fields,

$$(\gamma \cdot K - m)W = 0$$

$$K^\mu = \Pi^\mu + \frac{1}{2} i \nabla^\mu$$

$$\begin{aligned} \nabla_\mu &= \partial_\mu^x - j_0(\Delta) F_{\mu\nu}(x) \partial_p^\nu, \\ \Pi_\mu &= p_\mu - \frac{1}{2} \hbar j_1(\Delta) F_{\mu\nu}(x) \partial_p^\nu, \end{aligned}$$

- Next, we need to insert the decomposition of W into the main equations

$$W = \mathcal{F} + i\mathcal{P}\gamma^5 + \mathcal{V}^\mu \gamma_\mu + \mathcal{A}^\mu \gamma^5 \gamma_\mu + \frac{1}{2} \mathcal{S}^{\mu\nu} \sigma_{\mu\nu},$$

Vasak, Gyulassy, Elze, Ann. Phys. (N.Y.) 173, 462 (1987);
Elze, Heinz,Phys.Rep.183,81(1989).

Master equations in massive case (I)

- **Inserting the decomposition of W into the main equations, yields**

$$K^\mu = \Pi^\mu + \frac{1}{2} i \nabla^\mu$$

$$\begin{aligned}
K \cdot \mathcal{V} - m\mathcal{F} &= 0, \\
iK \cdot \mathcal{A} + m\mathcal{P} &= 0, \\
K_\mu \mathcal{F} - iK^\nu \mathcal{S}_{\mu\nu} - m\mathcal{V}_\mu &= 0, \\
iK_\mu \mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} K^\nu \mathcal{S}^{\rho\sigma} - m\mathcal{A}_\mu &= 0, \\
i(K_\mu \mathcal{V}_\nu - K_\nu \mathcal{V}_\mu) - \epsilon_{\mu\nu\rho\sigma} K^\rho \mathcal{A}^\sigma + m\mathcal{S}_{\mu\nu} &= 0.
\end{aligned}$$

Master equations in massive case (II)

- **Since the coefficients V,A,F,P,S are real, we can get**

Real part of main eq. gives

$$\left\{\begin{aligned} \Pi\cdot\mathcal{V} &= m\mathcal{F}, \\ \nabla\cdot\mathcal{A} &= 2m\mathcal{P}, \\ \Pi_\mu\mathcal{F} - iK^\nu\mathcal{S}_{\mu\nu} &= m\mathcal{V}_\mu, \\ -\nabla_\mu\mathcal{P} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathcal{S}^{\rho\sigma} &= 2m\mathcal{A}_\mu, \\ \frac{1}{2}(\nabla_\mu\mathcal{V}_\nu - \nabla_\nu\mathcal{V}_\mu) + \epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathcal{A}^\sigma &= m\mathcal{S}_{\mu\nu}. \end{aligned}\right\}$$

Imaginary part of main eq. gives

$$\left\{\begin{aligned} \nabla\cdot\mathcal{V} &= 0, \\ \Pi\cdot\mathcal{A} &= 0, \\ \frac{1}{2}\nabla_\mu\mathcal{F} - \Pi^\nu\mathcal{S}_{\mu\nu} &= 0, \\ \Pi_\mu\mathcal{P} + \frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu\mathcal{S}^{\rho\sigma} &= 0, \\ (\Pi_\mu\mathcal{V}_\nu - \Pi_\nu\mathcal{V}_\mu) - \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathcal{A}^\sigma &= 0. \end{aligned}\right\}$$

Master eqs. in massless limit (I)

- **In massless limit, V, A are decoupled with others.**

$$\nabla\cdot\mathcal{V}=0,$$
$$\Pi\cdot\mathcal{A}=0,$$
$$\Pi\cdot\mathcal{V}=0,$$
$$\nabla\cdot\mathcal{A}=0,$$
$$(\Pi_\mu\mathcal{V}_\nu-\Pi_\nu\mathcal{V}_\mu)-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathcal{A}^\sigma=0.$$

$\epsilon^{\mu\nu\alpha\beta}\times$ →
$$\frac{1}{2}(\nabla_\mu\mathcal{V}_\nu-\nabla_\nu\mathcal{V}_\mu)+\epsilon_{\mu\nu\rho\sigma}\Pi^\rho\mathcal{A}^\sigma=0.$$

$$\frac{1}{2}\nabla_\mu\mathcal{F}-\Pi^\nu\mathcal{S}_{\mu\nu}=0,$$
$$\Pi_\mu\mathcal{P}+\frac{1}{4}\epsilon_{\mu\nu\rho\sigma}\nabla^\nu\mathcal{S}^{\rho\sigma}=0,$$
$$\Pi_\mu\mathcal{F}-iK^\nu\mathcal{S}_{\mu\nu}=0,$$
$$-\nabla_\mu\mathcal{P}+\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Pi^\nu\mathcal{S}^{\rho\sigma}=0.$$

Master eqs. in massless limit (II)

- **Eqs. for V and A are similar**

$$\left\{\begin{aligned} \nabla\cdot\mathcal{V}&=0,\\ \Pi\cdot\mathcal{A}&=0,\\ \Pi\cdot\mathcal{V}&=0,\\ \nabla\cdot\mathcal{A}&=0,\\ (\Pi_\mu\mathcal{V}_\nu-\Pi_\nu\mathcal{V}_\mu)-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathcal{A}^\sigma&=0.\\ (\Pi_\mu\mathcal{A}_\nu-\Pi_\nu\mathcal{A}_\mu)-\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\nabla^\rho\mathcal{V}^\sigma&=0. \end{aligned}\right\}$$

Master equations for chiral fermions

- **Naturally, we introduce the Wigner function for right and left handed fermions,**

$$\mathscr{J}^s_\mu = \mathcal{V}_\mu + s\mathcal{A}_\mu, \ \ s = \pm$$

+: right handed
-: left handed

$$\left\{ \begin{aligned} \Pi^\mu \mathscr{J}^s_\mu(x,p) &= 0, \\ \nabla^\mu \mathscr{J}^s_\mu(x,p) &= 0, \\ 2s\left(\Pi^\mu \mathscr{J}^\nu_s - \Pi^\nu \mathscr{J}^\mu_s\right) &= -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_\rho \mathscr{J}^s_\sigma, \end{aligned} \right\}$$

3. Equal-time formulism for Wigner function

Equal-time Wigner function

- **Equal-time Wigner function**

I. Bialynicki-Birula, P. Gornicki and J. Rafelski, Phys. Rev. D44, 1825 (1991).

C. Best, P. Gornicki and W. Greiner, Ann. Phys. (N.Y.) 225, 169(1993)

Zhuang, Heinz, Phys. Rev. D 57 (1998) 6525

Ochs, Heinz, Annals Phys. 266 (1998) 351

- **Application to Schwinger pair production:**

Hebenstreit, Alkofer, Gies, Phys.Rev.D82:105026,2010

Sheng, Fang, Wang, Rischke, Phys.Rev.D 99 (2019) 5, 056004

- **Application to QKT:**

Chen, Wang, Zhuang, arXiv:2101.07596.

Definition of equal time Wigner function

- **Equal time Wigner function is integration of covariant Wigner function over p^0,**

$$W_{\alpha\beta}(t,\mathbf{x},\mathbf{p}) = \int dp^0 W_{\alpha\beta}(x,p)$$

$$= \int \frac{d^3y}{(2\pi)^3} e^{i\mathbf{p}\cdot\boldsymbol{y}} \left\langle \overline{\psi}_\beta \left(t, \mathbf{x}+\frac{\mathbf{y}}{2}\right) \exp\left[iQ\int_{\mathbf{x}-\mathbf{y}/2}^{\mathbf{x}+\mathbf{y}/2} d\mathbf{s}\cdot\mathbf{A}(t,\mathbf{s})\right] \psi_\alpha\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right\rangle$$

- **Then, the two spinors are set to be at equal time.**

$$\hat{W}_{\alpha\beta} = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \overline{\psi}_\beta(x_1) U(x_1,x_2) \psi_\alpha(x_2)$$

Equal time formalism Vs covariant formalism

- **Covariant Wigner function:**
  - Lorentz covariance
  - Might be hard to be simulated directly.

- **Equal time Wigner function:**
  - Easy to set up as an initial problem
  - No Lorentz covariance
  - e.g. Schwinger pair production

Zhuang, Heinz, Phys. Rev. D 57 (1998) 6525

Dirac equations and gauge fixing

- Next, we will derive the master eq. for equal time Wigner function.
- We start from the Dirac equation,

$$(i\gamma \cdot D - m + \mu\gamma^0)\psi = 0,$$
$$\overline{\psi}[-i\gamma \cdot (\overleftarrow{\partial} - ieA) - m + \mu\gamma^0] = 0.$$

$$i(\partial_t + ieA_t)\psi = [-i\gamma^0\boldsymbol{\gamma} \cdot (\boldsymbol{\partial} - ie\mathbf{A}) + m\gamma^0 - \mu]\psi,$$
$$-i(\partial_t - ieA_t)\overline{\psi} = \overline{\psi}[i\boldsymbol{\gamma} \cdot (\overleftarrow{\boldsymbol{\partial}} + ie\mathbf{A})\gamma^0 + m\gamma^0 - \mu].$$

- For simplicity, we choose the gauge fixing condition

$$A^t = 0.$$

$$\mathbf{E} = F^{i0} = \partial^i A^0 - \partial^0 A^i = -\partial_t \mathbf{A},$$
$$\mathbf{B} = -\frac{1}{2}\epsilon^{ijk}F^{jk} = -\epsilon^{ijk}\partial^j A^k = \nabla \times \mathbf{A}.$$

Equations for equal time Wigner function (I)

- For simplicity, we set **EM fields are homogenous in space**.

$$
\begin{aligned}
D_t W_{\alpha\beta} &= [\partial_t + e\mathbf{E}(t)\cdot\nabla_{\mathbf{p}}]W_{\alpha\beta} \\
&= \int \frac{d^3y}{(2\pi)^3}\exp\left[i\boldsymbol{y}\cdot\mathbf{p} + ie\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{s}\cdot\mathbf{A}(t,\mathbf{s})\right]\left\langle \partial_t\left[\overline{\psi}_\beta\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)\psi_\alpha\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right)\right]\right\rangle \\
&= -im\left[W,\gamma^0\right]_{\alpha\beta} \\
&\quad - \int \frac{d^3y}{(2\pi)^3}\exp\left[i\boldsymbol{y}\cdot\mathbf{p} + ie\int_{\mathbf{x}_2}^{\mathbf{x}_1} d\mathbf{s}\cdot\mathbf{A}(t,\mathbf{s})\right] \\
&\quad \times \left\langle [\overline{\psi}(\boldsymbol{\gamma}\cdot(\overleftarrow{\boldsymbol{\partial}} + ie\mathbf{A})\gamma^0]_\beta\psi_\alpha + \overline{\psi}_\beta[\gamma^0\boldsymbol{\gamma}\cdot(\boldsymbol{\partial}_X - ie\mathbf{A})\psi]_\alpha\right\rangle
\end{aligned}
$$

- Integration by part and after a long calculation, we get,

$$
[\partial_t + e\mathbf{E}(t)\cdot\nabla_{\mathbf{p}}]W = \frac{1}{2}\left(\nabla_x + e\mathbf{B}\times\nabla_{\mathbf{p}}\right)[W,\gamma^0\boldsymbol{\gamma}] - i\mathbf{p}\cdot\{W,\gamma^0\boldsymbol{\gamma}\} + im\left[W,\gamma^0\right].
$$

Equation for equal time Wigner function (II)

- We summary the master eqs. here,

$$D_t W = \frac{1}{2}\mathbf{D}_x \cdot [W, \gamma^0 \boldsymbol{\gamma}] - i\boldsymbol{\Pi} \cdot \left\{W, \gamma^0 \boldsymbol{\gamma}\right\} + im\left[W, \gamma^0\right],$$

$$D_t = \partial_t + e\mathbf{E}(t) \cdot \nabla_{\mathbf{p}},$$

$$\mathbf{D}_x = \nabla_x + e\mathbf{B} \times \nabla_{\mathbf{p}},$$

$$\boldsymbol{\Pi} = \mathbf{p}.$$

- Similarly, we decompose the Wigner function as

$$W = \mathcal{F} + i\mathcal{P}\gamma^5 + \mathcal{V}^\mu \gamma_\mu + \mathcal{A}^\mu \gamma^5 \gamma_\mu + \frac{1}{2}\mathcal{S}^{\mu\nu}\sigma_{\mu\nu},$$

$$\mathbf{T} = (\mathcal{S}^{10}, \mathcal{S}^{20}, \mathcal{S}^{30}),$$

$$\mathbf{S} = (\mathcal{S}^{23}, \mathcal{S}^{31}, \mathcal{S}^{12}).$$

Master equation for equal time Wigner function

$$
\begin{aligned}
D_t\mathcal{F} &= 2\mathbf{\Pi}\cdot\mathbf{T}, \\
D_t\mathcal{P} &= -2\mathbf{\Pi}\cdot\mathbf{S} + 2m\mathcal{A}^0, \\
D_t\mathcal{V}^0 &= -\mathbf{D}_x\cdot V, \\
D_t\mathbf{V} &= -\mathbf{D}_x V^0 + 2\mathbf{\Pi}\times\mathbf{A} - 2m\mathbf{T}, \\
D_t\mathcal{A}^0 &= -\mathbf{D}_x\cdot\mathbf{A} - 2m\mathcal{P}, \\
D_t\mathbf{A} &= -\mathbf{D}_x\mathcal{A}^0 + 2\mathbf{\Pi}\times\mathbf{V}, \\
\mathbf{D}_t\mathbf{T} &= \mathbf{D}_x\times\mathbf{S} - 2\mathbf{\Pi}\mathcal{F} + 2m\mathbf{V}, \\
\mathbf{D}_t\mathbf{S} &= \mathbf{D}_x\times\mathbf{T} + 2\mathbf{\Pi}\mathcal{P}.
\end{aligned}
$$

Part 3

Quantum kinetic theory in massless limit and collisions

1. Solve quantum kinetic theory in gradient expansion
   (1a) Gradient expansion
   (1b) Leading order results and constrains from QKT
   (1c) $\hbar$ order results
   (1d) $\hbar^2$ order results
2. Discussions on the solution of Wigner function
   (2a) CME, CVE, energy-momentum tensor and chiral anomaly
   (2b) Chiral kinetic theory
   (2c) Lorentz transformation and side jump
3. Collision effects
   (3a) Kadanoff-Baym equation
   (3b) General solution of Wigner function with collisions
   (3c) Collision term for QED in HTL approximation

1. Solve quantum kinetic theory in gradient expansion

(1a) Gradient expansion

Summary for master equations of chiral fermions

- The master equations read,

$$\left\{ \begin{aligned} \Pi^{\mu} \mathscr{J}_{\mu}^{s}(x,p) &= 0, \\ \nabla^{\mu} \mathscr{J}_{\mu}^{s}(x,p) &= 0, \\ 2s\left(\Pi^{\mu} \mathscr{J}_{s}^{\nu} - \Pi^{\nu} \mathscr{J}_{s}^{\mu}\right) &= -\hbar\epsilon^{\mu\nu\rho\sigma}\nabla_{\rho}\mathscr{J}_{\sigma}^{s}, \end{aligned} \right\}$$

$$\mathscr{J}_{\mu}^{s} = \mathcal{V}_{\mu} + s\mathcal{A}_{\mu},\ s = \pm$$

+: right handed

-: left handed

- We assume the EM fields are constant

$$\nabla_{\mu}^{(0)} = \partial_{\mu}^{x} - F_{\mu\nu}\partial_{p}^{\nu},\ \ \Pi_{\mu}^{(0)} = p_{\mu}.$$

Gradient expansion

- A common way to handle the equations in many body system is gradient expansion.

- Gradient expansion: We assume that a field A(x) changes very slowly, i.e.

$$\frac{|\partial A|}{AL^{-1}} \ll 1$$

where L is one characteristic length (or time) of the system.

- A rough way to get the power of a term is to count the number of ∂.

Gradient expansion in relativistic hydrodynamics

- In relativistic hydrodynamics, L is the mean free path and $K \sim L_{mfp}\partial$ is the Knudsen number.
- The energy-momentum tensor in the gradient expansion is usually written as,

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)}$$

- Leading order is

$$T^{\mu\nu}_{(0)} = (\epsilon + p)u^\mu u^\nu - p g^{\mu\nu}$$

- Next leading order is

$$T^{\mu\nu}_{(1)} = -\Pi(g^{\mu\nu} - u^\mu u^\nu) + \pi^{\mu\nu}$$

$$\Pi = \zeta(\partial \cdot u),$$

$$\pi^{\mu\nu} = 2\eta\partial^{<\mu}u^{\nu>}$$

Bulk pressure
Shear viscous tensor