

PRL.129, 051601,
 in collaboration with
 Marcela Carena, Henry Lamm,
 Wanqiang Liu



Ying-Ying Li

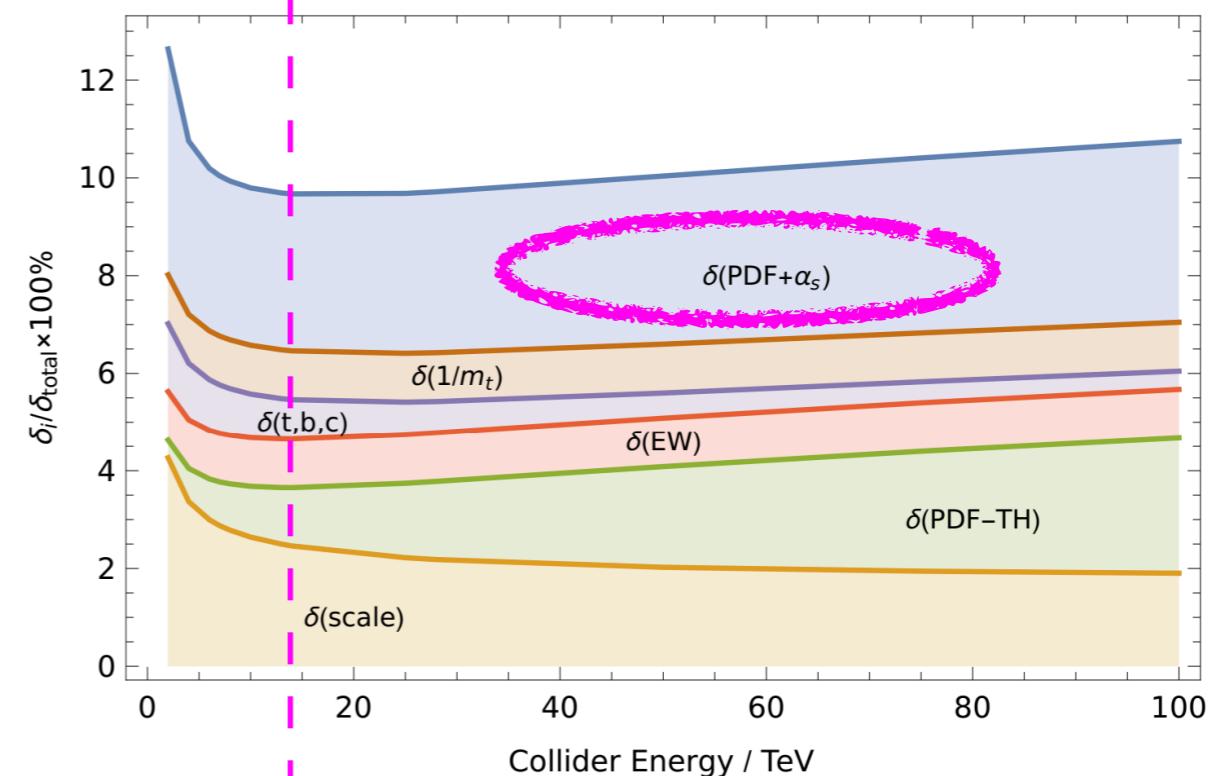
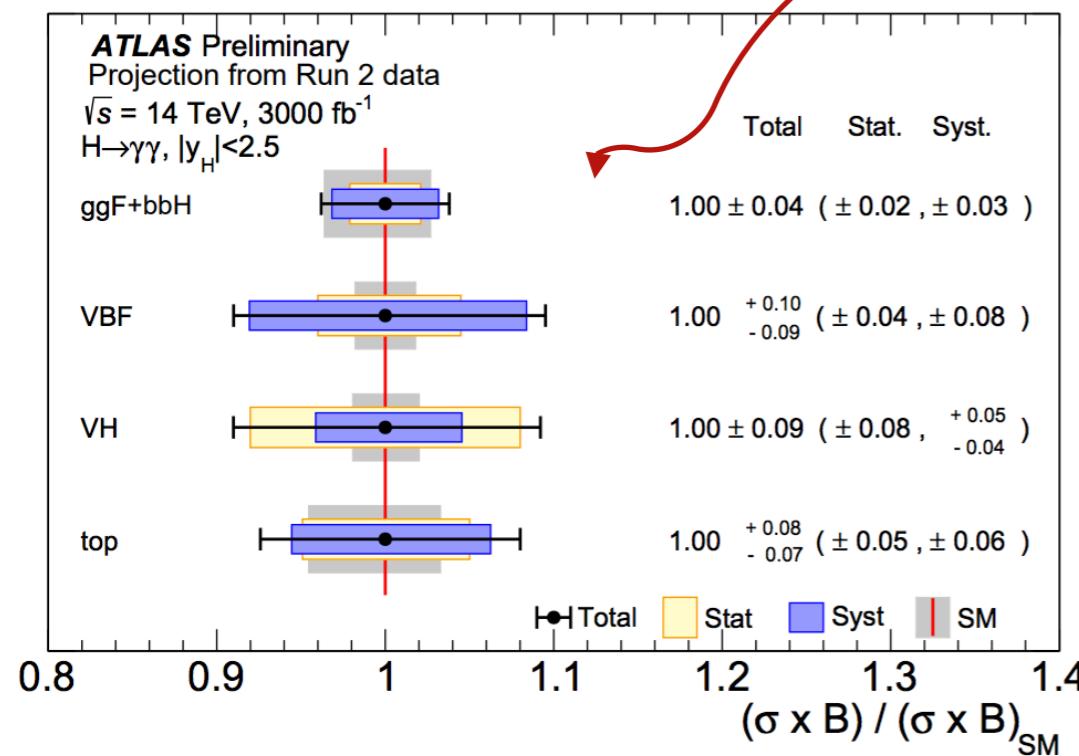
Improved Hamiltonians for Quantum Simulations of Gauge Theories

Mar. 31, 2023 @CCAST

Theoretical inputs to colliders

theoretical uncertainties

real-time strong
dynamics
PDF knowledge from data



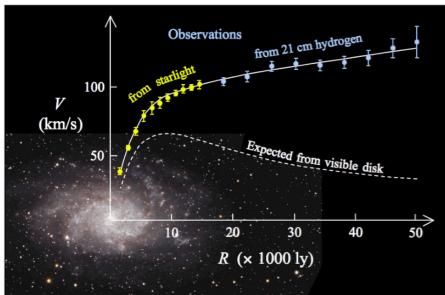
[F. Dulat, et al, arXiv:1802.00827]
[M. Cepeda, et al, arXiv:1902.00134]

non perturbative, non equilibrium dynamics of QCD in QGP,
parton shower, etc



Puzzles We Encountered

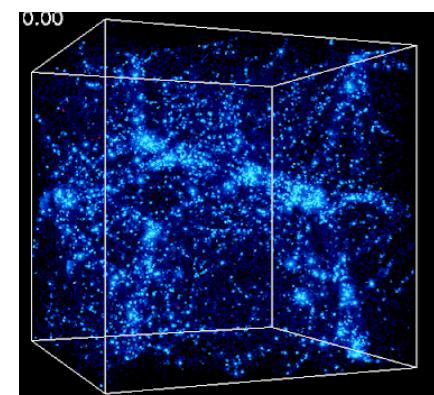
1933-Now
dark matter



rotation curves



bullet cluster



structure formation

$$m_a^2(T)f_a^2 \propto \frac{\partial^2 F(\theta, T)}{\partial \theta^2} |_{\theta=0}$$

lattice non-perturbative calculations

$$\begin{array}{ccc} F(\theta, T) & \leftarrow & \langle O \rangle = \frac{\sum_{\mathcal{C}} O(\mathcal{C}) W(\mathcal{C})}{\sum_{\mathcal{C}} W(\mathcal{C})} \\ \text{QCD free energy} & & \end{array}$$

Imaginary time problem : $W(\mathcal{C}) \sim \exp(-S(\mathcal{C}))$

complex $S(\mathcal{C})$ for non vanishing θ

Sign Problem!!!

configuration space \mathcal{C} is exponentially large in system size

out-of-equilibrium,
non-perturbative
higher order processes,
quantum interference
cannot be solved classically
due to theoretical or computational limitations:

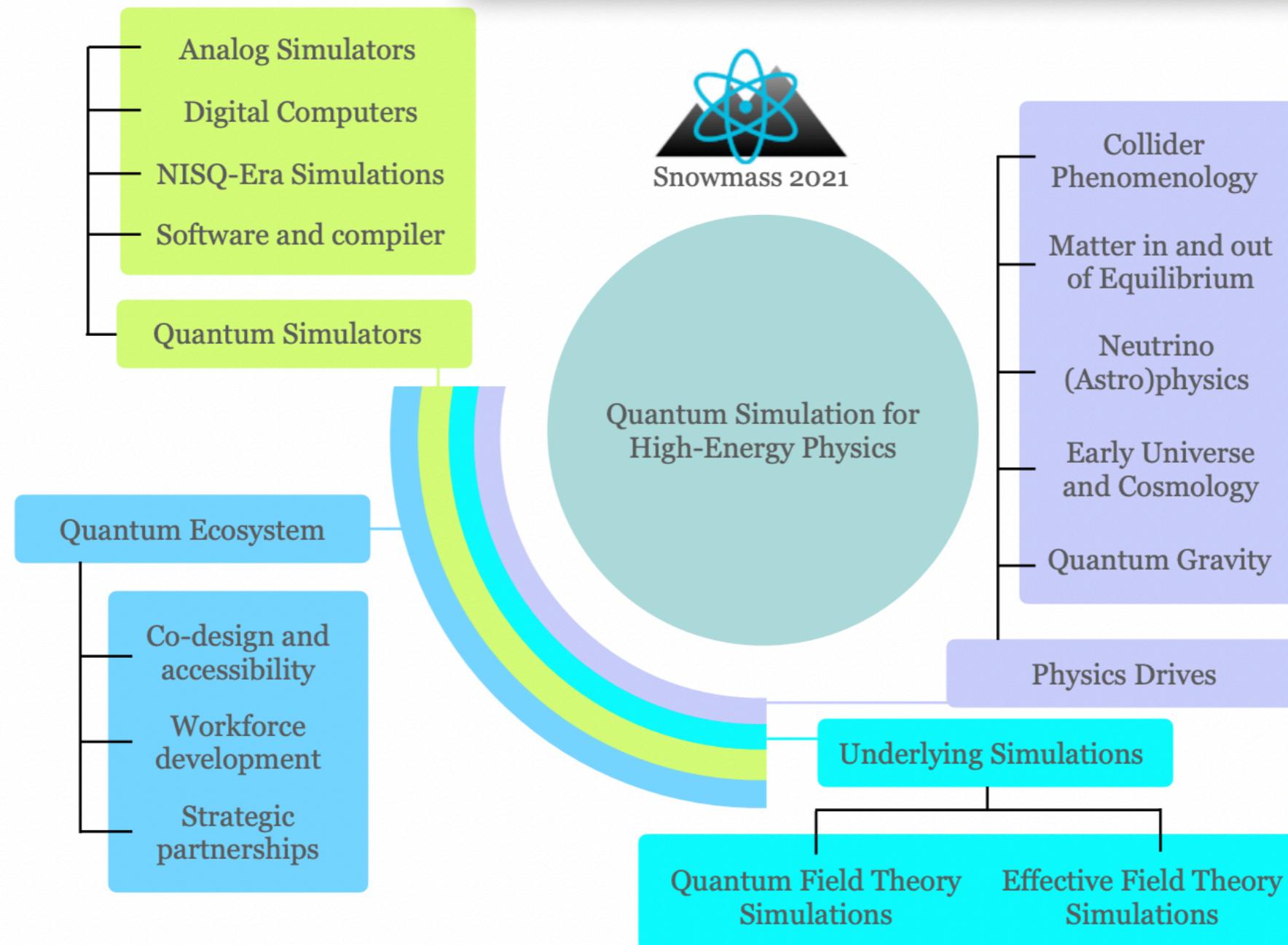
**sign problem or rare events that has
exponential-scaling of the complexity**

$$\langle x | e^{-iHt} | y \rangle = \int \mathcal{D}\phi e^{iS}$$

[arXiv:2204.03381]

Quantum Simulation for High Energy Physics

Christian W. Bauer,^{1, a} Zohreh Davoudi,^{2, b} A. Baha Balantekin,³ Tanmoy Bhattacharya,⁴ Marcela Carena,^{5, 6, 7, 8} Wibe A. de Jong,¹ Patrick Draper,⁹ Aida El-Khadra,⁹ Nate Gemelke,¹⁰ Masanori Hanada,¹¹ Dmitri Kharzeev,^{12, 13} Henry Lamm,⁵ Ying-Ying Li,⁵ Junyu Liu,^{14, 15} Mikhail Lukin,¹⁶ Yannick Meurice,¹⁷ Christopher Monroe,^{18, 19, 20, 21} Benjamin Nachman,¹ Guido Pagano,²² John Preskill,²³ Enrico Rinaldi,^{24, 25, 26} Alessandro Roggero,^{27, 28} David I. Santiago,^{29, 30} Martin J. Savage,³¹ Irfan Siddiqi,^{29, 30, 32} George Siopsis,³³ David Van Zanten,⁵ Nathan Wiebe,^{34, 35} Yukari Yamauchi,² Kübra Yeter-Aydeniz,³⁶ and Silvia Zorzetti⁵



Universal Quantum Computing

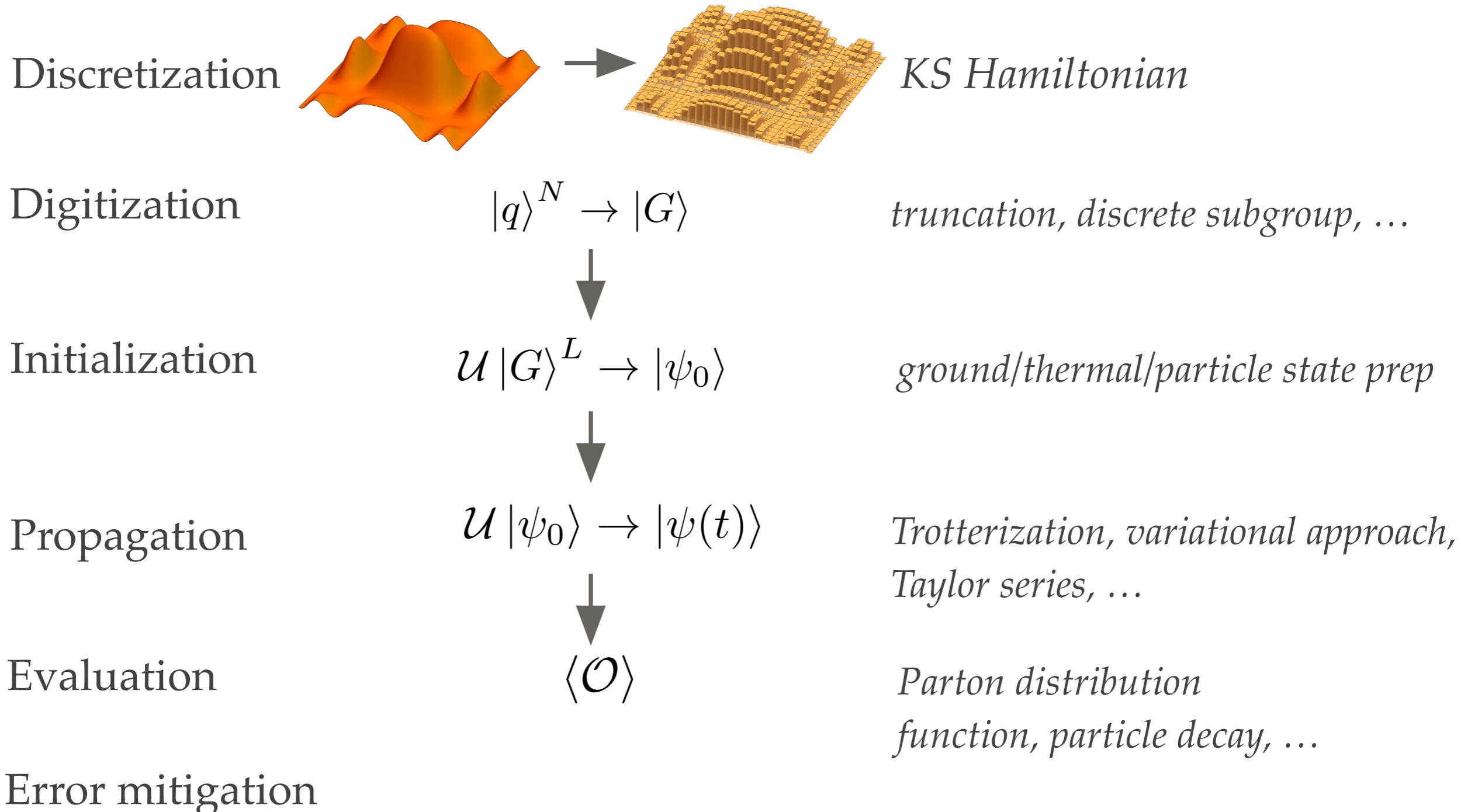
State of the art devices: $\mathcal{O}(10)$ **physical** qubits with $\mathcal{O}(10)$ gates
Error correction: Noiseless, **logical** qubits from set of physical ones

Noise limits fidelity of primitive gates to 95 – 99% today

entering Noisy Intermediate-Scale Quantum (NISQ) era:
more than 50 well controlled qubits, not error-corrected yet.

Quantum Computing

galactic algorithm to simulate quantum field theory



See references in [M. Carena, H. Lamm, YYL, W. Liu, PRD. 104, 094519]

Discretization of space

J. Kogut and L. Susskind [Phys. Rev. D **11**, 395]

For pure SU(N) gauge theory,

$$H_{\text{co}} = \frac{1}{2} \int d^d x \text{Tr} [\mathbf{E}^2(\mathbf{x}) + \mathbf{B}^2(\mathbf{x})]$$

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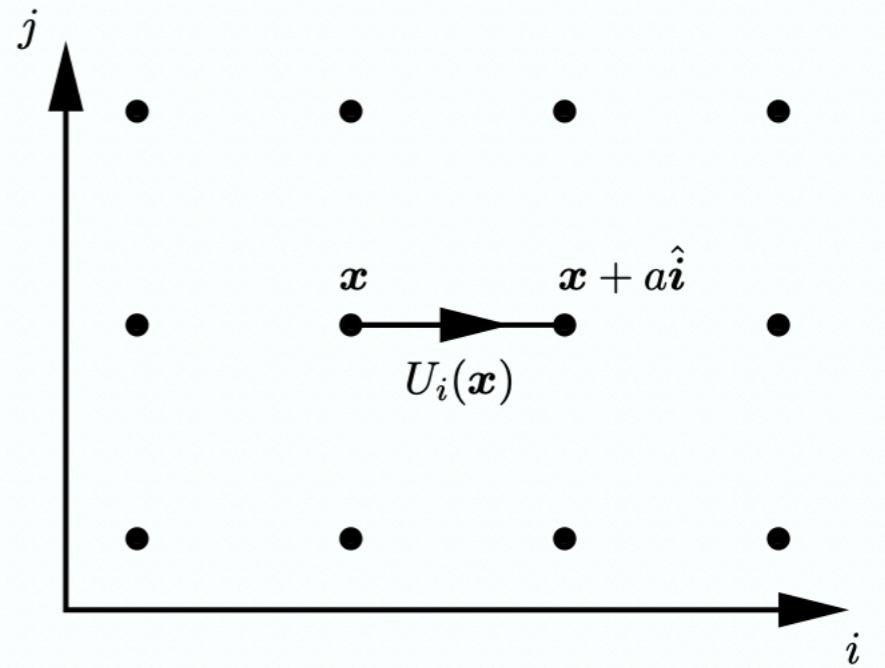
$$H_{\text{co}} = \frac{1}{2} \int d^d x \text{Tr} [\mathbf{E}^2(\mathbf{x}) + \mathbf{B}^2(\mathbf{x})]$$

On a lattice, to build a discrete theory
with exact gauge invariance

Wilson loop

$$P_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \left\{ \begin{array}{c} \square \\ \rightarrow \\ i \\ \downarrow \\ j \end{array} \right\}$$

$$P_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \exp \left\{ ig \oint_{\square} A \cdot dx \right\}$$



$$U_i(x) = e^{ig \int_a^0 dt A_i(x + t\hat{i})}$$

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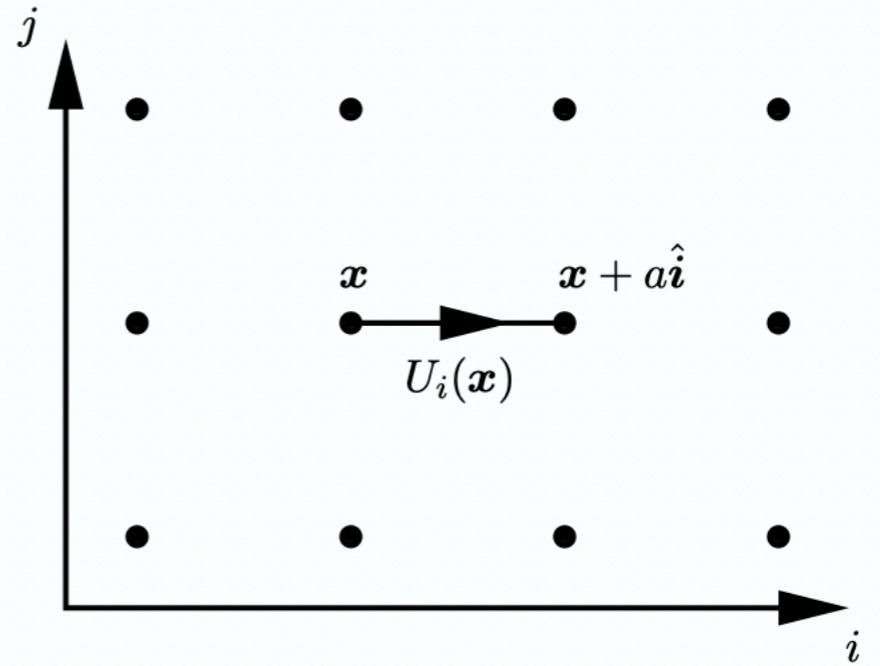
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$$E_i^L \rightarrow L_i$$



$$U_i(x) = e^{ig \int_a^0 dt A_i(x + t\hat{i})}$$

$$H_{KS} = K_{KS} + V_{KS},$$

$$K_{KS} = \sum_{\mathbf{x}, i} \frac{g_t^2}{a} \text{Tr} L_i^2(\mathbf{x})$$

$$V_{KS} = - \sum_{\mathbf{x}, i < j} \frac{2}{g_s^2 a} \text{Re Tr} P_{ij}(\mathbf{x})$$

Discretization of space

$$U_{\mathbf{n},k}^{(r)} = e^{i\theta_{\mathbf{n},k}^i T_i^{(r)}} \quad L_{\mathbf{n},k} = -i\dot{U}_{\mathbf{n},k} U_{\mathbf{n},k}^\dagger \quad ; \quad R_{\mathbf{n},k} = -iU_{\mathbf{n},k}^\dagger \dot{U}_{\mathbf{n},k}$$

Fourier transformation

$$|x\rangle = \sum_k |k\rangle \langle k|x\rangle$$

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k) e^{\frac{2\pi i k x}{T}}$$

$$|U\rangle = \sum_{\rho,i,j} |\rho, i, j\rangle \langle \rho, i, j| U \rangle$$

$$\sum_{g \in G} f(g) |g\rangle = \sum_{\rho \in \hat{G}} \hat{f}(\rho)_{ij} |\rho, i, j\rangle$$

ρ : irreducible rep

Discretization of space

[J. Carlsson, et al, hep-lat/0105018]

$$P_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \exp \left\{ ig \oint_{\square} A \cdot dx \right\} \approx \frac{g^2 a^4}{2N} \text{Tr} \{ F_{ij}(x) F_{ij}(x) \} + \frac{g^2 a^6}{12N} \text{Tr} \{ F_{ij}(x) (D_i^2 + D_j^2) F_{ij}(x) \} + \dots$$

deviations from the continuum, starts
from a^2 error, classical computational
resources proportional to a^{-k}
for Wilson action

$$R_{ij}(x) = 1 - \frac{1}{N} \text{ReTr} \left\{ \begin{array}{c} \text{square loop with arrows} \\ i \quad j \end{array} \right\} = \frac{4g^2 a^4}{2N} \text{Tr} \{ F_{ij}(x) F_{ij}(x) \} + \frac{4g^2 a^6}{24N} \text{Tr} \{ F_{ij}(x) (4D_i^2 + D_j^2) F_{ij}(x) \} + \dots$$

deviations from the continuum
starts from $a^2 g^2$
at quantum level

Discretization of space

$$H_I = K_I + V_I$$

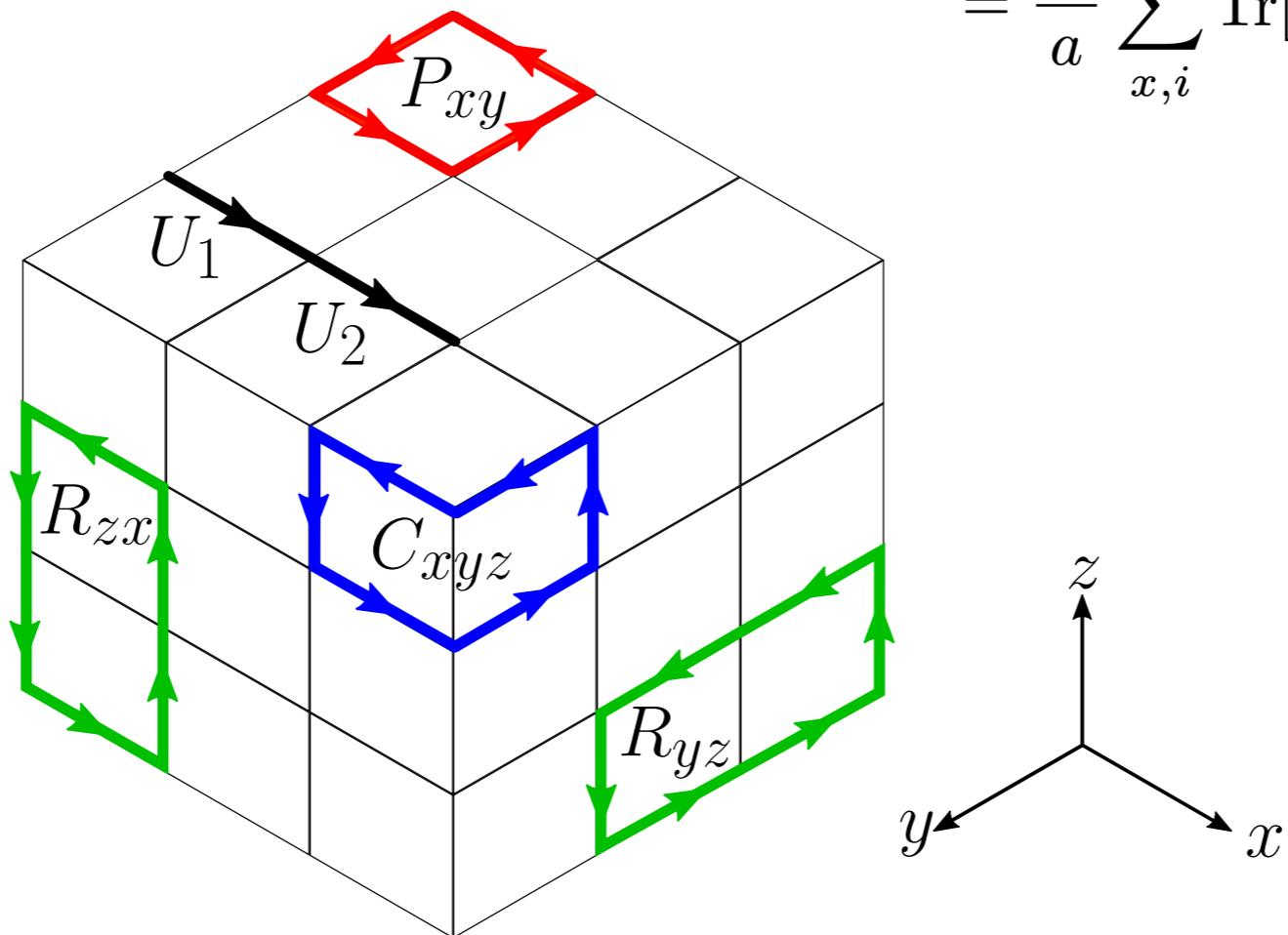
$$V_I = \beta_{V0} V_{KS} + \beta_{V1} V_{\text{rect}}$$

$$K_I = \beta_{K0} K_{KS} + \beta_{K1} K_{2L}$$

$$V_{\text{rect}} = \frac{2}{a g_s^2} \sum_{\mathbf{x}, i < j} \text{Re} \operatorname{Tr} [R_{ij}(\mathbf{x}) + R_{ji}(\mathbf{x})]$$

$$K_{2L} = \frac{g_t^2}{a} \sum_{\mathbf{x}, i} \operatorname{Tr} [L_i(\mathbf{x}) U_i(\mathbf{x}) L_i(\mathbf{x} + a\mathbf{i}) U_i^\dagger(\mathbf{x})]$$

$$= \frac{g_t^2}{a} \sum_{x, i} \operatorname{Tr} [R_i(\mathbf{x}) L_i(\mathbf{x} + a\mathbf{i})]$$



[M. Carena, H. Lamm, YYL, W. Liu, PRL. 129, 051601]

Discretization of space

[M. Alford, et al, hep-lat/9507010]
[..., ...]

With improved action,
for Euclidean spacetime
at the same error level,
simulations can be done
with a lattice spacing of
at least 2 larger

Qubits required

$$N_q \sim \left(\frac{L}{a}\right)^d$$

Only count qubits:
saving us at least 3 years for 3+1d,
assuming number of qubits increases by
a factor of 2 each year on hardware

circuits for improved Hamiltonian need to be designed
classical field to quantum operator

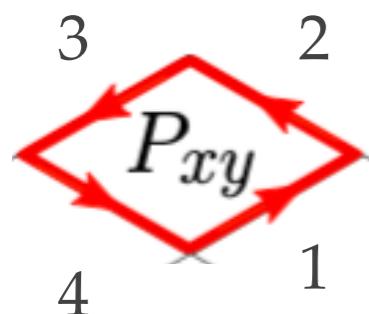
[M. Carena, H. Lamm, YYL, W. Liu, PRL. 129, 051601]

Propagation

$$K_{KS} = \sum_{\mathbf{x}, i} \frac{g_t^2}{a} \operatorname{Tr} L_i^2(\mathbf{x})$$

$$V_{KS} = - \sum_{\mathbf{x}, i < j} \frac{2}{g_s^2 a} \operatorname{Re} \operatorname{Tr} P_{ij}(\mathbf{x})$$

$$P_{ij}(x) = 1 - \frac{1}{N} \operatorname{Re} \operatorname{Tr} \left\{ \begin{array}{c} \text{square with arrows} \\ i \quad j \end{array} \right\}$$



$$\operatorname{Tr}\{U_1 U_2 U_3^\dagger U_4^\dagger\}$$

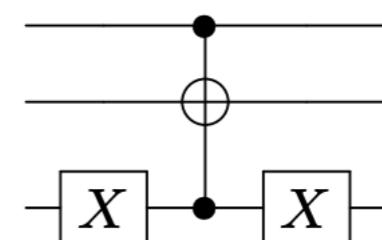
General Method

[H. Lamm, et al, arXiv:1903.08807]

G -register : $ g\rangle$
$\mathfrak{U}_x g\rangle h\rangle = g\rangle gh\rangle$
$\mathfrak{U}_{-1} g\rangle = g^{-1}\rangle$
$\mathfrak{U}_{\operatorname{Tr}}(\theta) g\rangle = e^{i\theta \operatorname{Re} \operatorname{Tr} g} g\rangle$

$$\mathfrak{U}_F \sum_{g \in G} f(g) |g\rangle = \sum_{\rho \in \hat{G}} \hat{f}(\rho)_{ij} |\rho, i, j\rangle$$

\mathfrak{U}_{-1} gate for D4 group



Implementation with quantum logic gate

Propagation

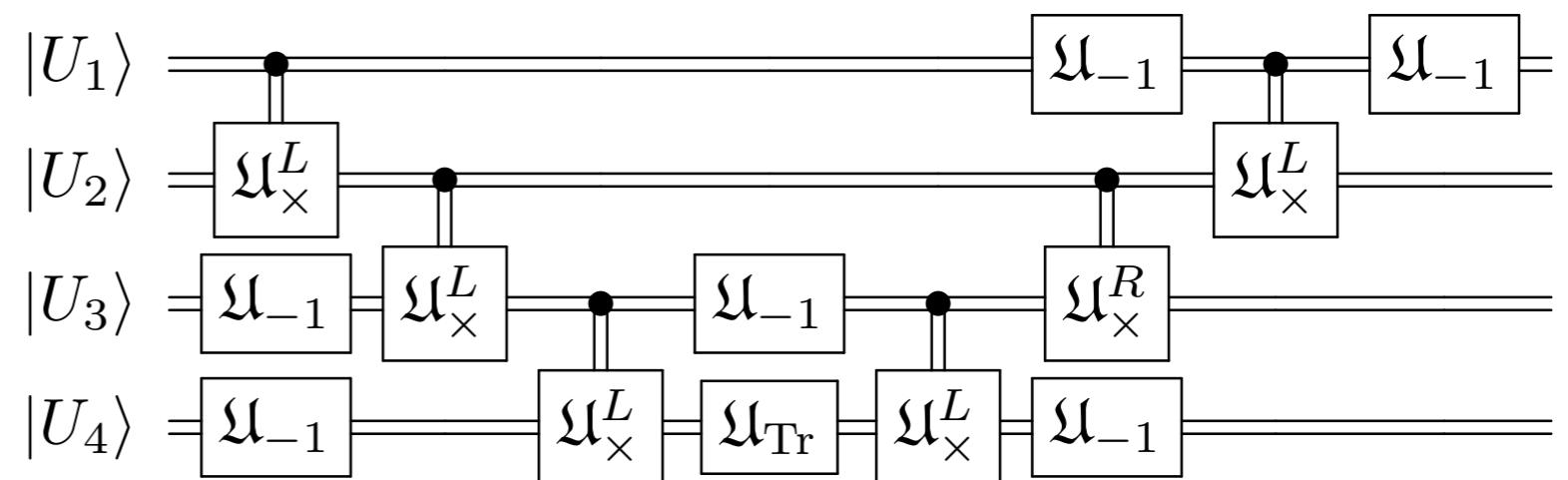
$$\mathcal{U}(t) = e^{-iH_{KS}t}$$

$$\approx [e^{-i\delta t K_{KS}} e^{-i\delta t V_{KS}}]^{t/\delta t}$$

$$\frac{g_t^2}{a} \operatorname{Tr} L_i^2(\mathbf{x}) \rightarrow$$

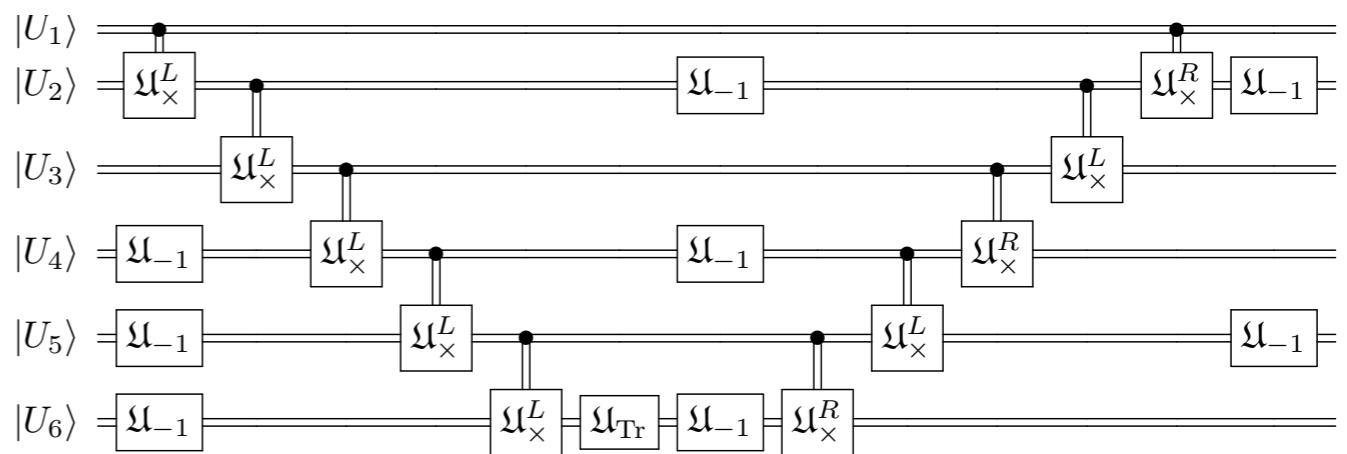
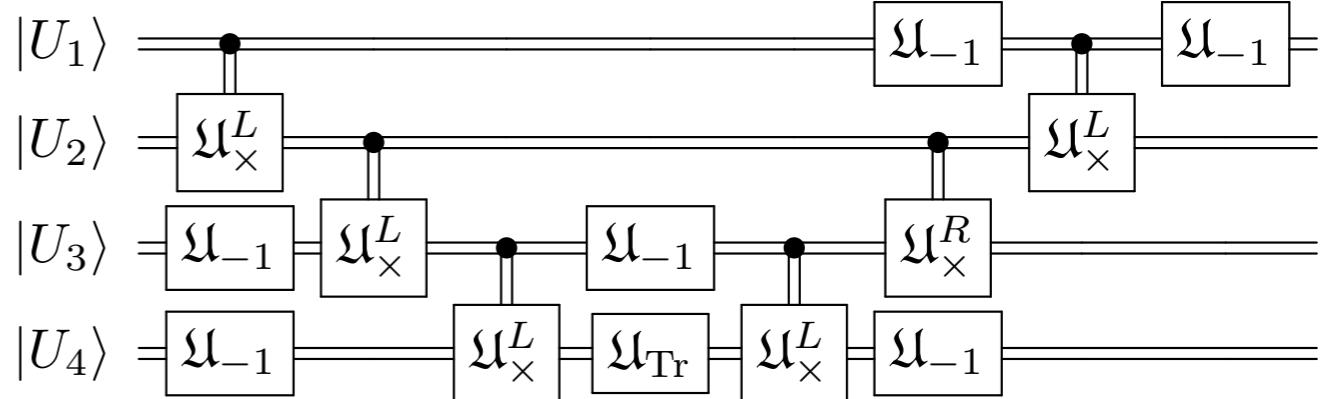
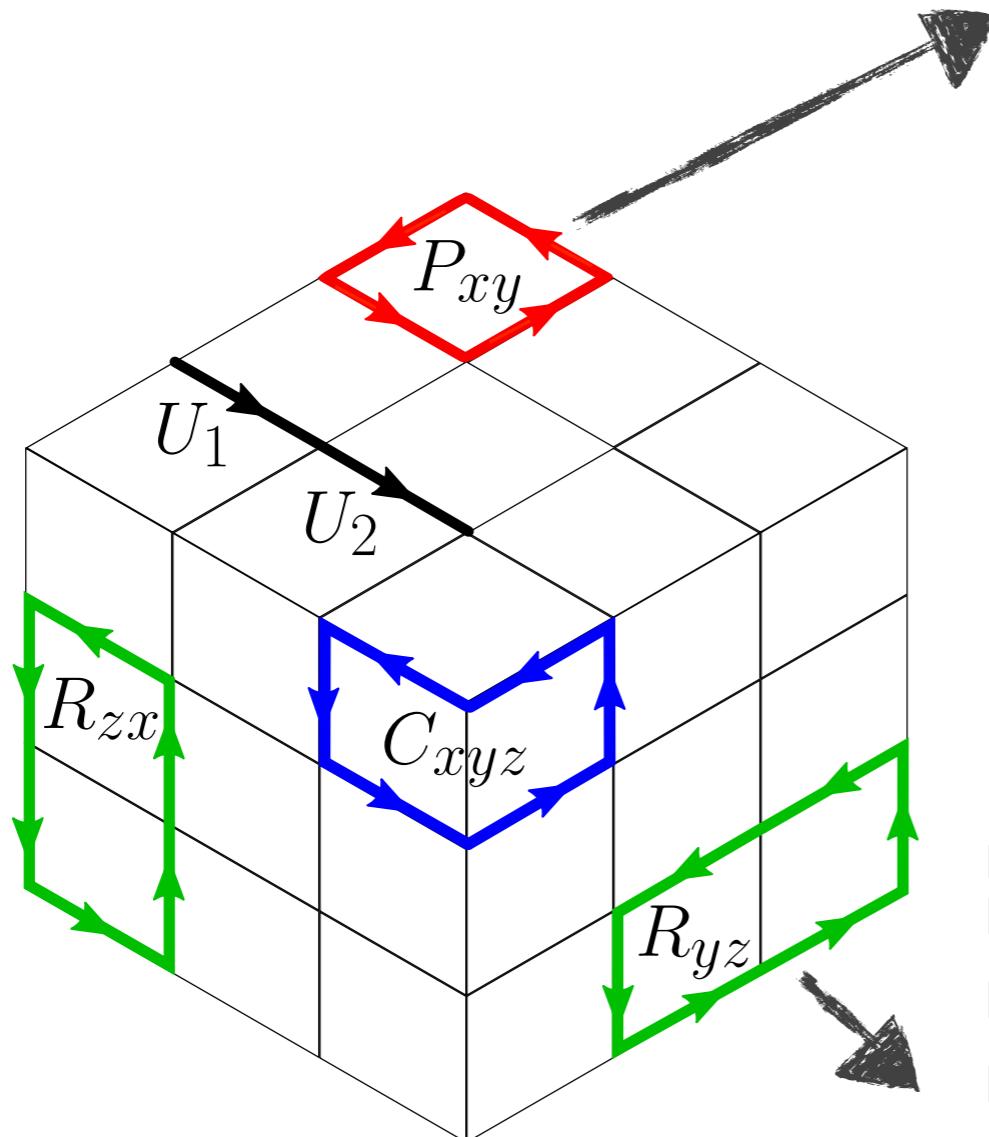
$$|U_1\rangle = \boxed{\mathfrak{U}_F^\dagger} \boxed{\mathfrak{U}_{\text{phase}}} \boxed{\mathfrak{U}_F} =$$

$$P_{xy} \rightarrow$$



$\mathcal{U}_{V_{KS}}$ assuming linear register connectivity

Propagation for Improved Hamiltonian



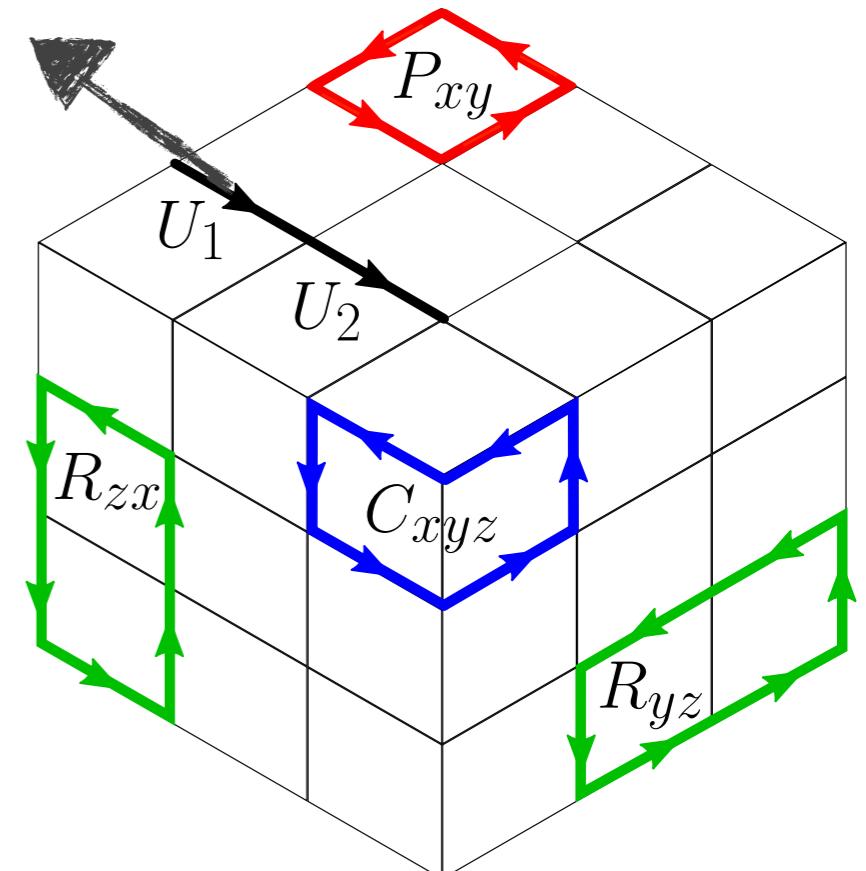
Propagation for Improved Hamiltonian

$$\hat{K}_{2L} = \frac{g_t^2}{a} \sum_{x,i} \text{Tr}[\hat{R}_i(\mathbf{x}) \hat{L}_i(\mathbf{x} + a\mathbf{i})]$$

$$\text{Tr}(\hat{R}_1 \hat{L}_2) = \text{Tr}[\hat{L}_2^2 + \hat{R}_1^2 - (\hat{L}_2 - \hat{R}_1)^2]/2$$

$$\mathcal{U}_{K_{2L}} \equiv e^{i\theta} \text{Tr}(\hat{L}_2 - \hat{R}_1)^2$$

$$[\mathcal{U}_{K_{2L}}, \hat{U}_1 \hat{U}_2] = 0$$



Propagation for Improved Hamiltonian

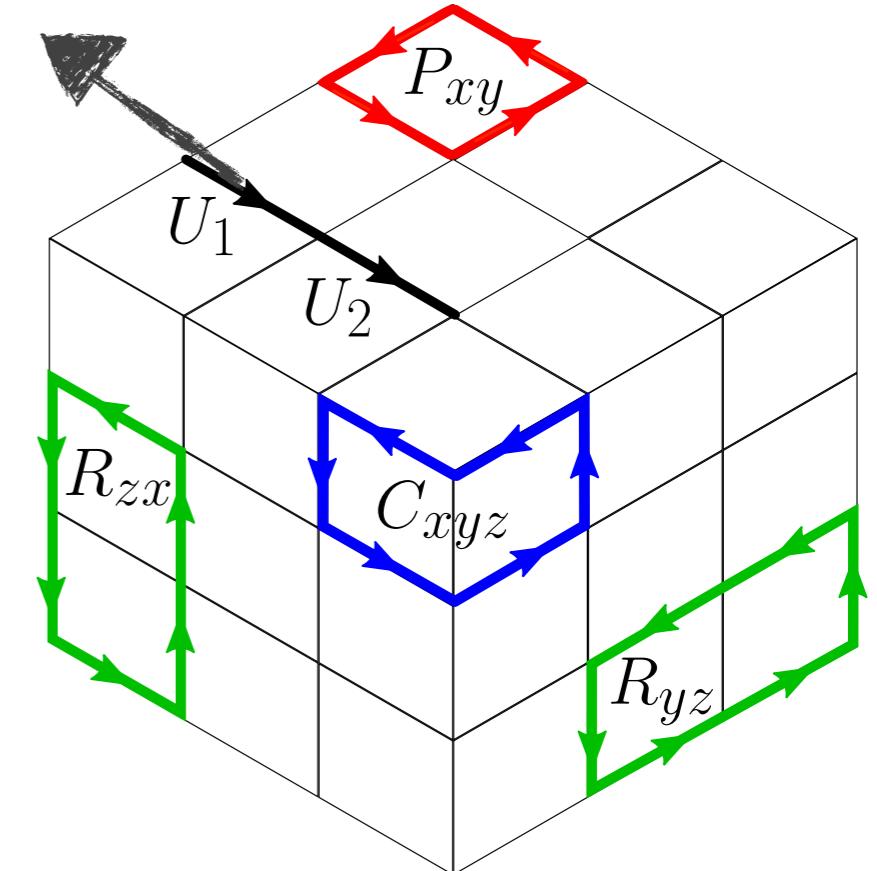
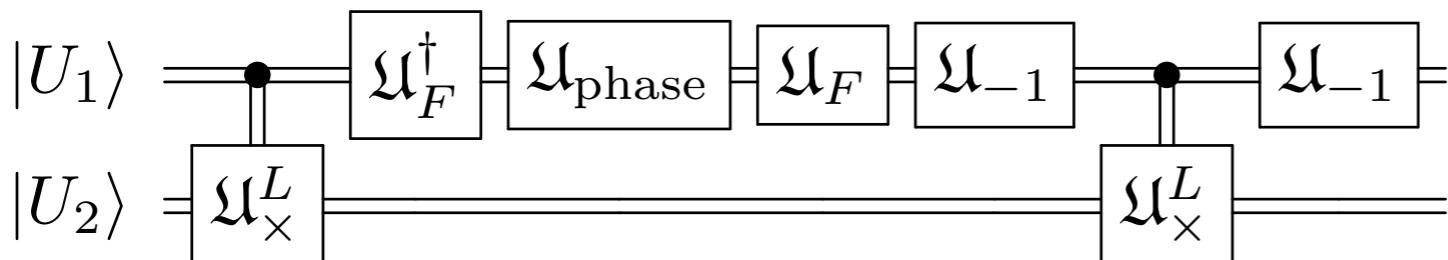
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$$\text{Tr}(\hat{R}_1 \hat{L}_2) = \text{Tr}[\hat{L}_2^2 + \hat{R}_1^2 - (\hat{L}_2 - \hat{R}_1)^2]/2$$

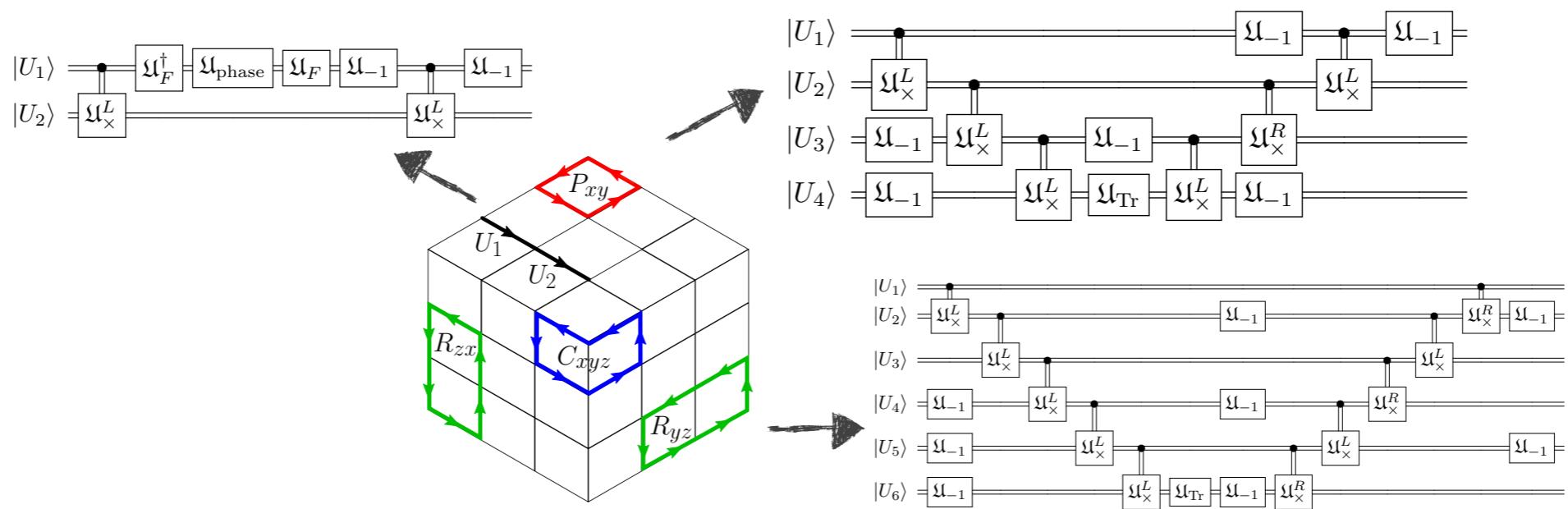
$$\mathcal{U}_{K_{2L}} \equiv e^{i\theta \text{Tr}(\hat{L}_2 - \hat{R}_1)^2}$$

$$[\mathcal{U}_{K_{2L}}, \hat{U}_1 \hat{U}_2] = 0$$

$$\langle U'_1, U'_2 | \mathcal{U}_{K_{2L}} | U_1, U_2 \rangle = \delta_{U'_1 U'_2, U_1 U_2} \langle U'_1 | e^{i\theta \text{Tr} \hat{L}_1^2} | U_1 \rangle$$



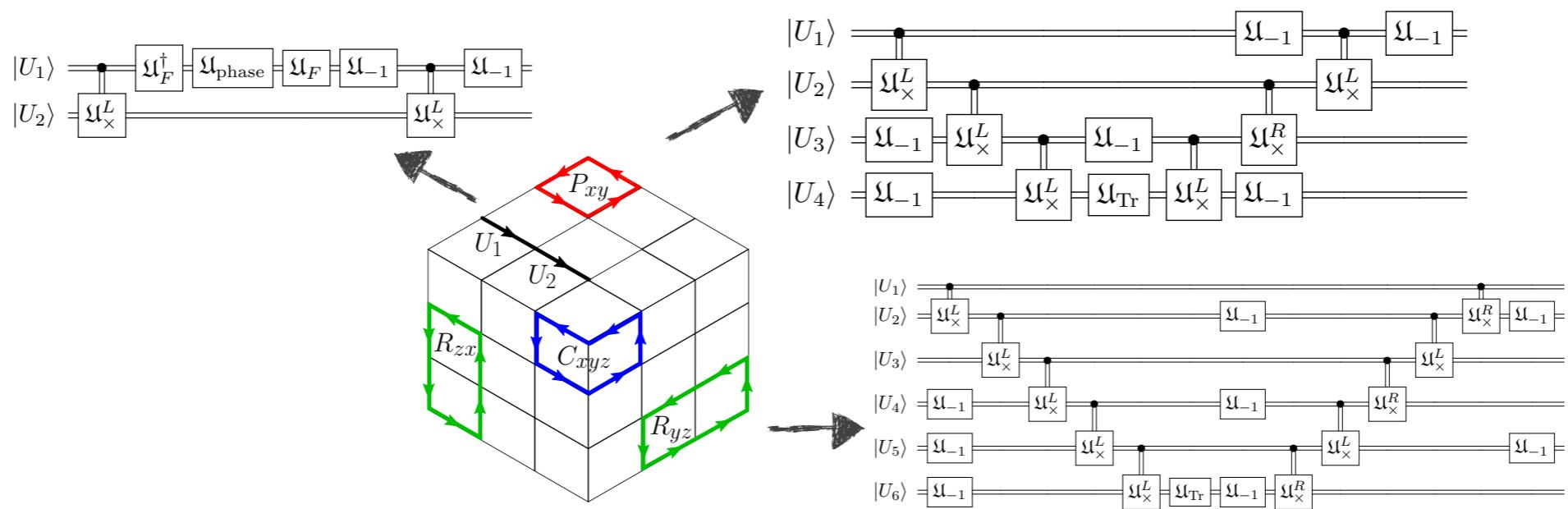
Resource for Improved Hamiltonian



Gate	$N[\hat{K}_{KS} + \hat{V}_{KS}]$	$N[\hat{K}_{2L} + \hat{V}_{\text{rect}}]$
\mathfrak{U}_F	2	2
$\mathfrak{U}_{\text{phase}}$	1	1
\mathfrak{U}_{Tr}	$\frac{d-1}{2}$	$d - 1$
\mathfrak{U}_{-1}	$3(d - 1)$	$2 + 8(d - 1)$
\mathfrak{U}_x	$6(d - 1)$	$4 + 20(d - 1)$

- # of Gates here for a single trotter is increasing only multiplicatively, could be compensated by the decreasing of links.
- Larger trotter steps, instead could be used for improved Hamiltonian.

Resource for Improved Hamiltonian



Gate	$N[\hat{K}_{KS} + \hat{V}_{KS}]$	$N[\hat{K}_{2L} + \hat{V}_{\text{rect}}]$
\mathfrak{U}_F	2	2
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\mathfrak{U}_x	$6(d - 1)$	$4 + 20(d - 1)$

So far, circuits for improved Hamiltonian are designed, reducing the number of qubits required, with comparable or less quantum gates.

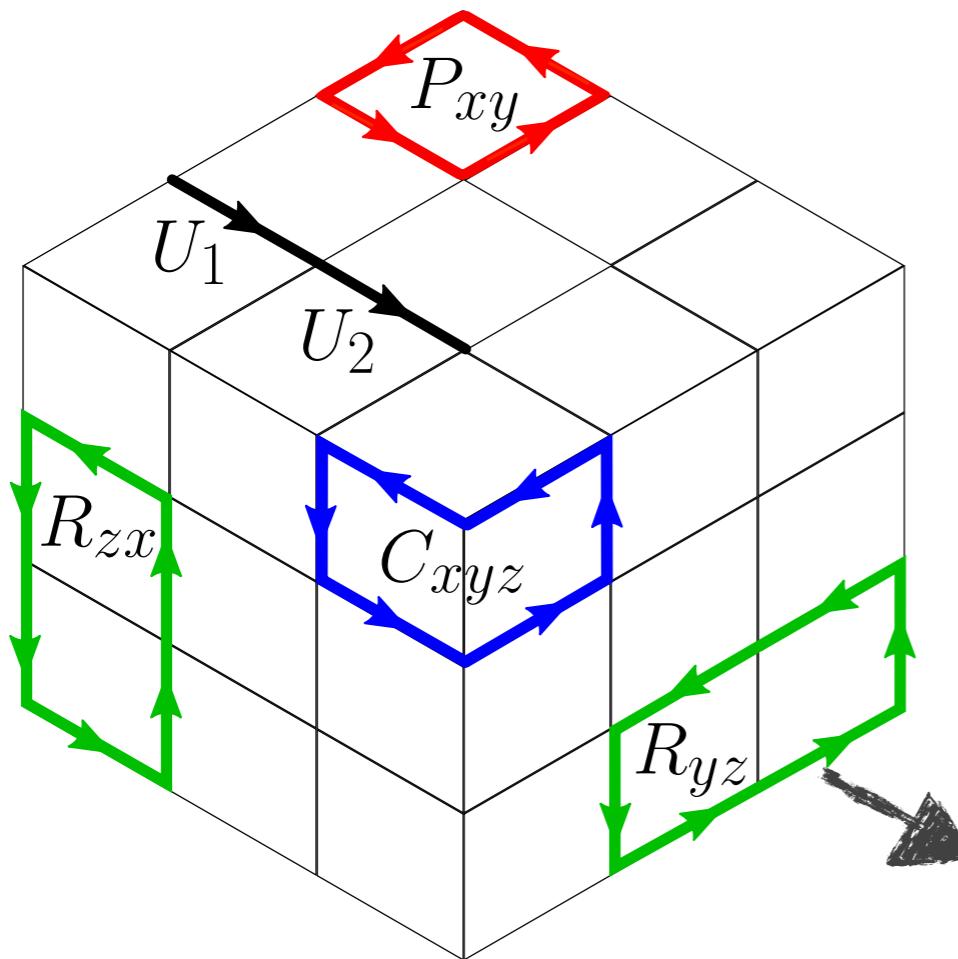
Demonstration

Demonstration for Improved Hamiltonian

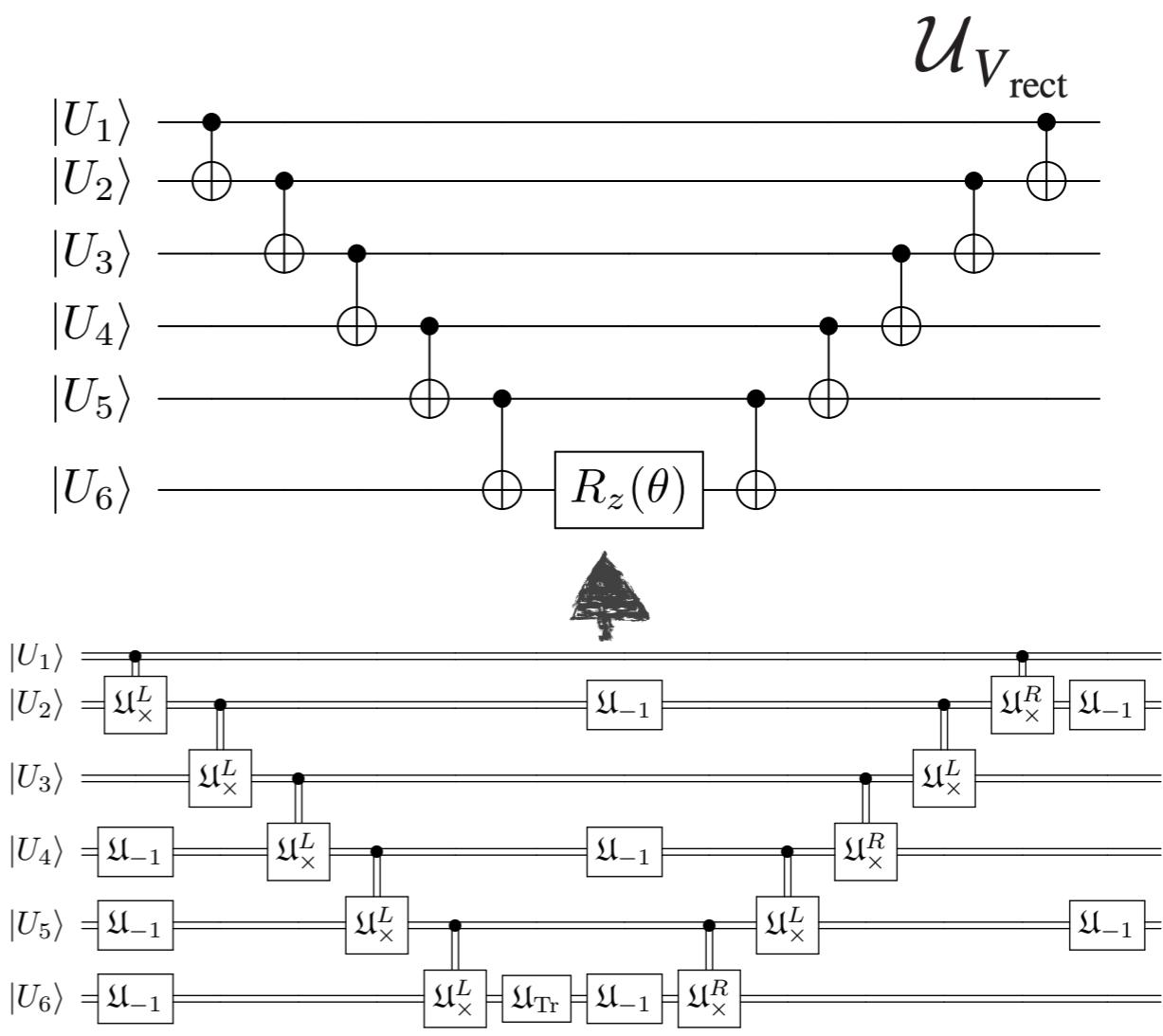
Z₂

$$\begin{array}{c} 1 \rightarrow |0\rangle \\ -1 \rightarrow |1\rangle \end{array}$$

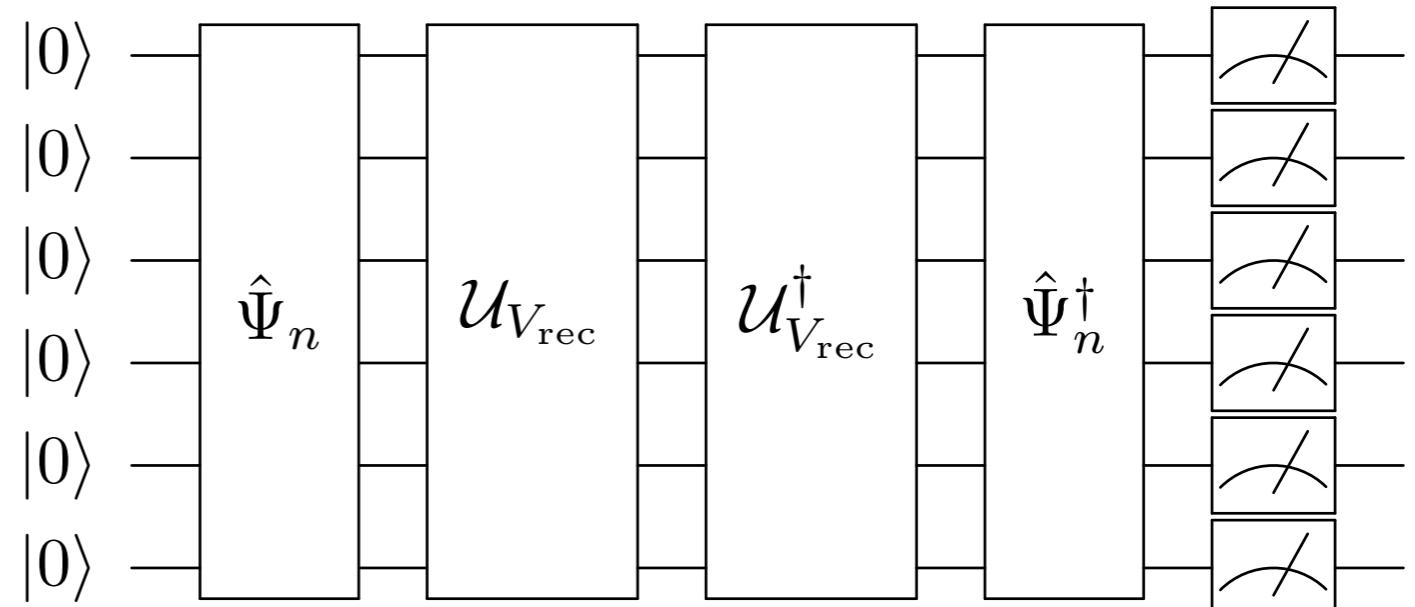
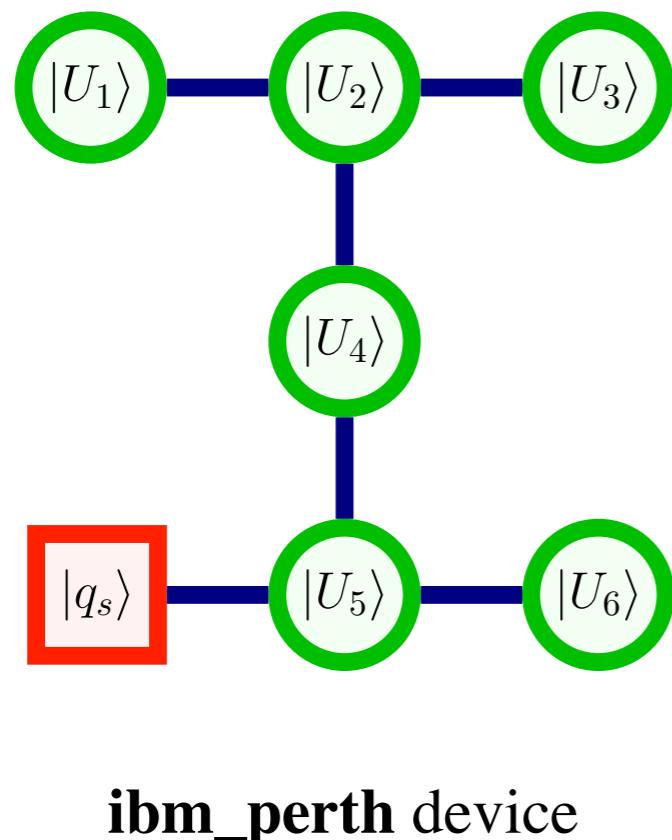
\mathfrak{U}_F	H
$\mathfrak{U}_{\text{phase}}$	$R_z(\theta)$
\mathfrak{U}_{Tr}	$R_z(\theta)$
\mathfrak{U}_{-1}	$\mathbb{1}$
\mathfrak{U}_X	CNOT



[M. Carena, H. Lamm, YYL, W. Liu, PRL. 129, 051601]



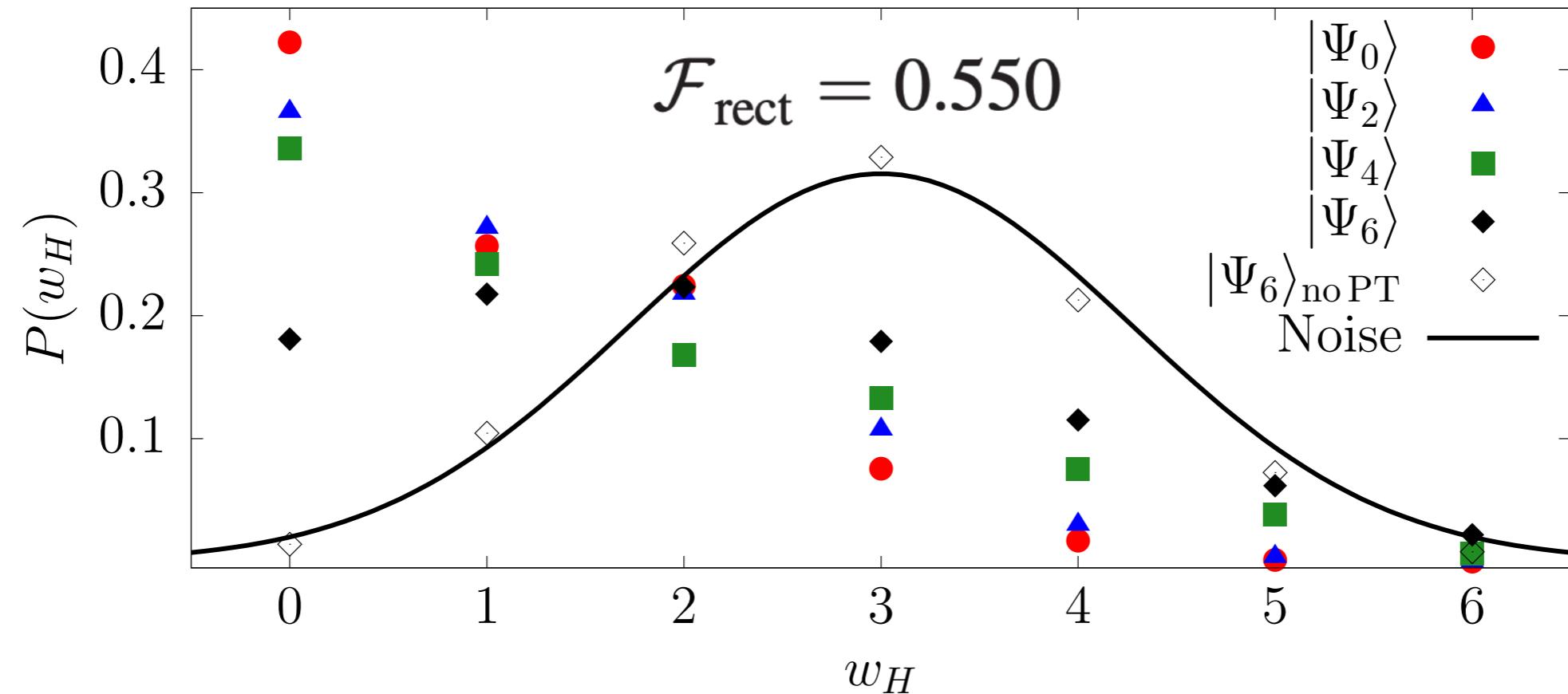
Demonstration for Improved Hamiltonian



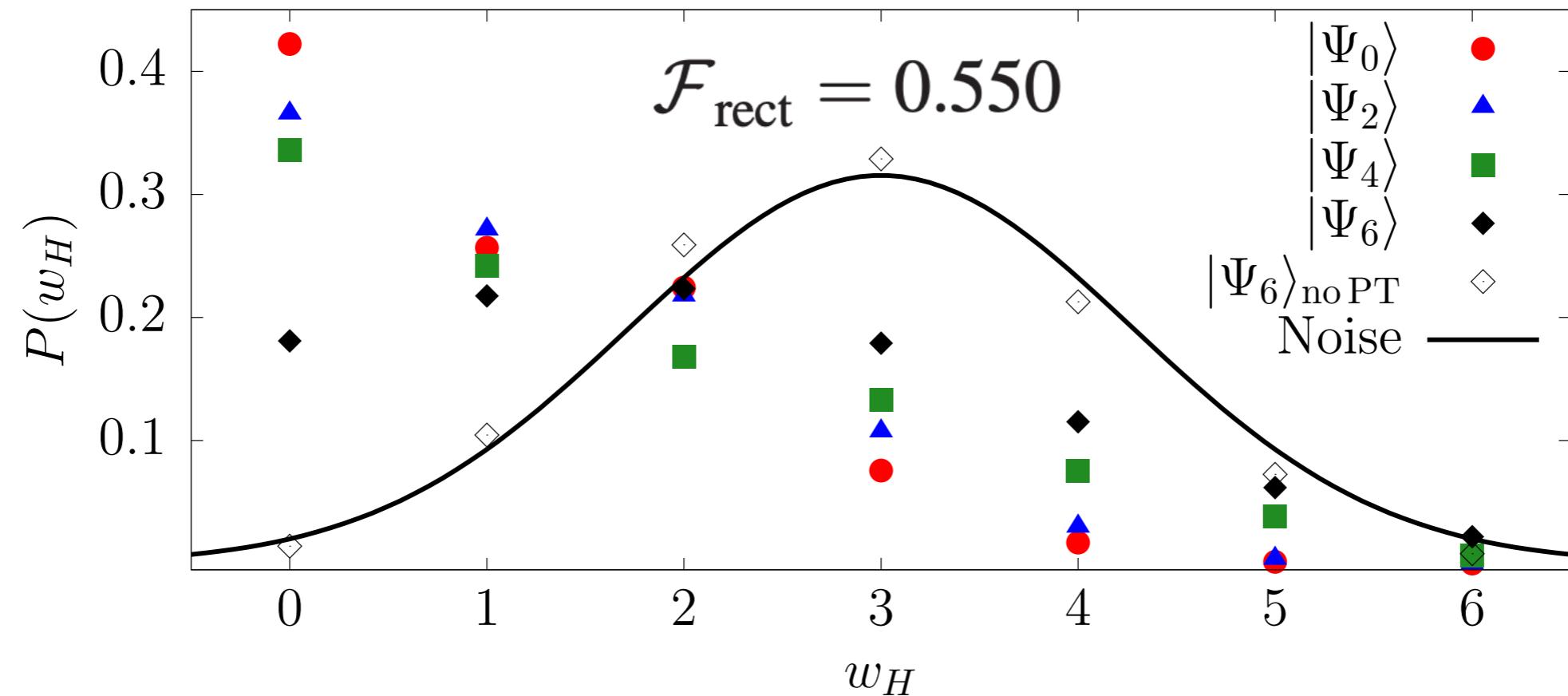
$$\hat{\Psi}_n = \prod_{m \leq n} H_m^{\otimes}$$
$$\left[\prod_i (\sigma_i^{b_i})^{\otimes} \right] \text{CNOT} \otimes \mathbb{1}_4 \left[\prod_i (\sigma_i^{a_i})^{\otimes} \right] = \text{CNOT} \otimes \mathbb{1}_4$$

w_H :number of states measured in the 1 state

Demonstration for Improved Hamiltonian



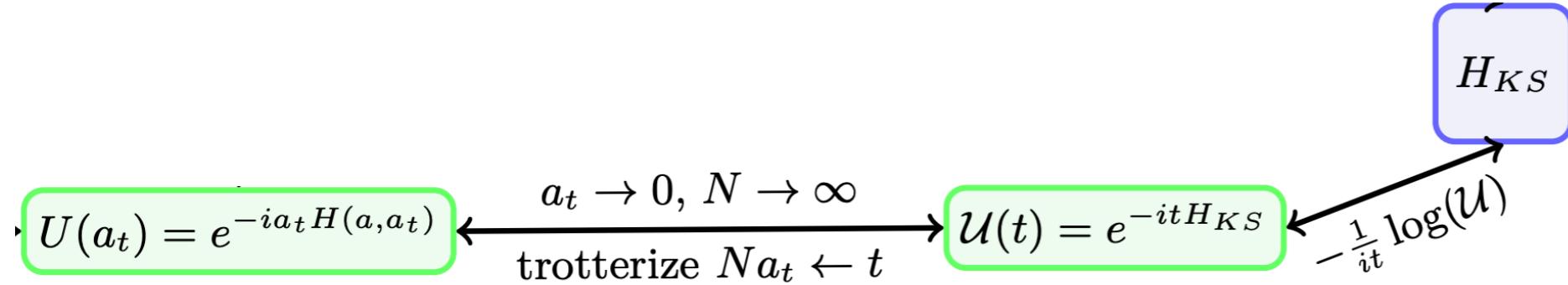
Demonstration for Improved Hamiltonian



$$\mathcal{F}_\delta \approx 0.25$$

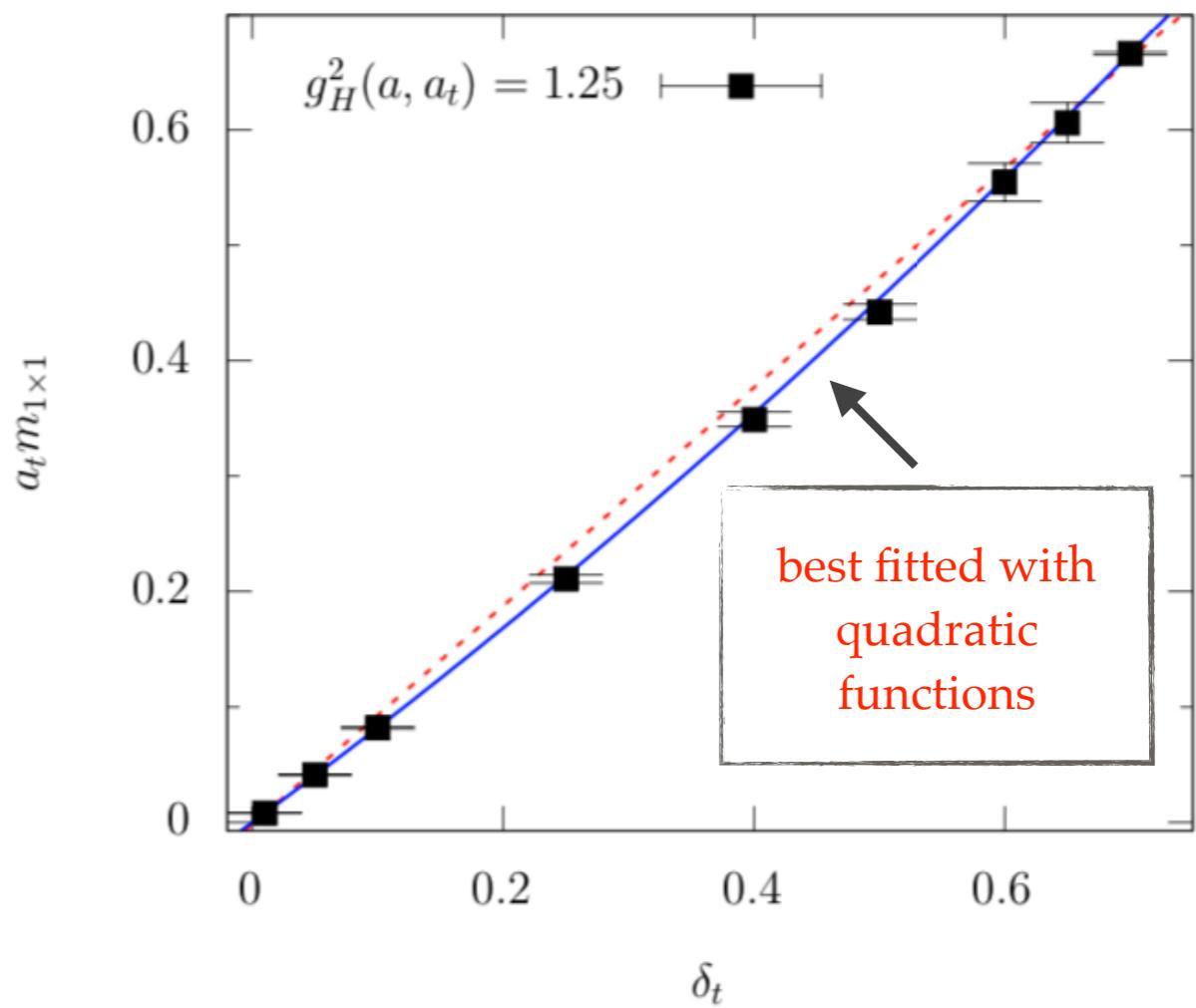
demonstration of improved Hamiltonian is allowed in the near future

Propagation– Discretization in Time



$$\begin{aligned}\mathcal{U}(t) &= e^{-iH_{KS}t} \\ &\approx [e^{ix_1 H_V} e^{ix_2 H_K} e^{ix_3 H_V} e^{ix_4 H_K} \dots]^N + O(a_t^p)\end{aligned}$$

Anisotropic lattices $\xi = \frac{a}{a_t}$, introducing finite temporal lattice spacing renormalized via scale setting

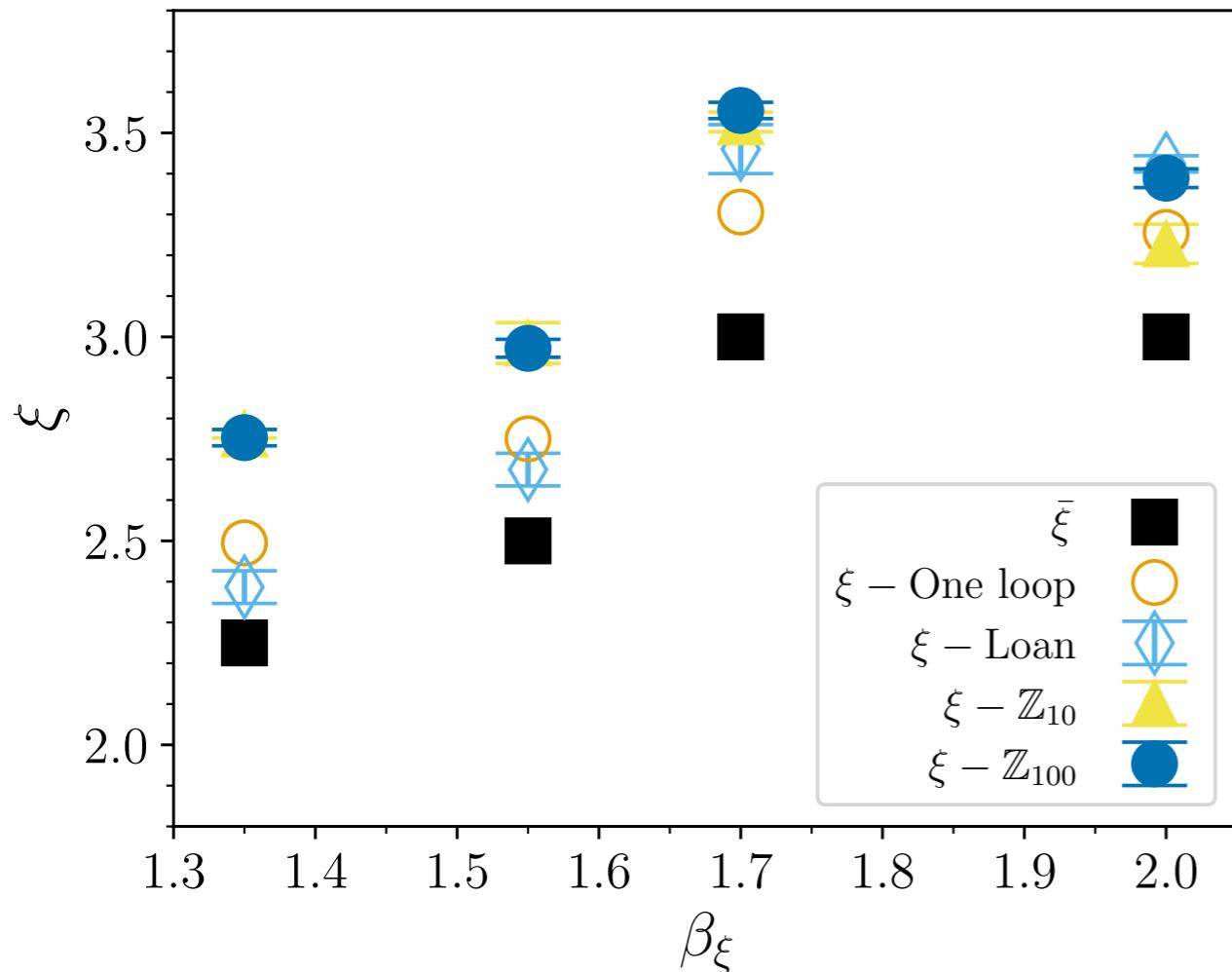


[M. Carena, H. Lamm, YYL, W. Liu, PRD. 104, 094519]

Propagation– Discretization in Time

Anisotropic Parameter Renormalization

- numerical results is pretty tedious—saving measurement on the Euclidean side
- Preferred for analytical continuation
- Determine the fixed anisotropic trajectory
- Continuous group agrees quite well with their discrete subgroups



β_ξ	N_s	N_t	$\bar{\xi}$	$\xi_{\text{1-loop}}$		ξ $SU(2)$ [111]
				BII	...	
$D = 3$						
2.00	36	72	2.00	2.097	2.099(1)	...
2.00	12 ^a	60 ^a	4.00	4.278	...	4.35(19)
2.65	16 ^a	64 ^a	4.00	4.207	...	4.22(11)
3.00	36	72	1.33	1.351	1.369(19)	...
4.00	24 ^a	96 ^a	4.00	4.136	...	4.08(9)
$D = 4$						
3.0	36	72	1.33	1.351	1.36(1)	...

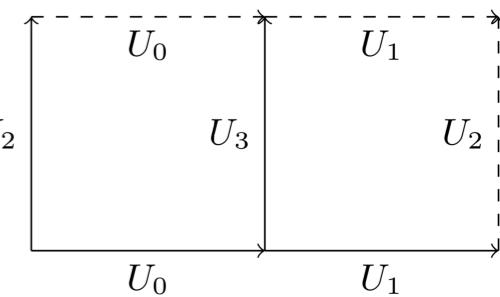
[M. Carena, E. J. Gustafson, H. Lamm, YYL, W. Liu, PRD 106 11, 114504]

To the Continuum

$$a \rightarrow 0, a_t \rightarrow 0$$

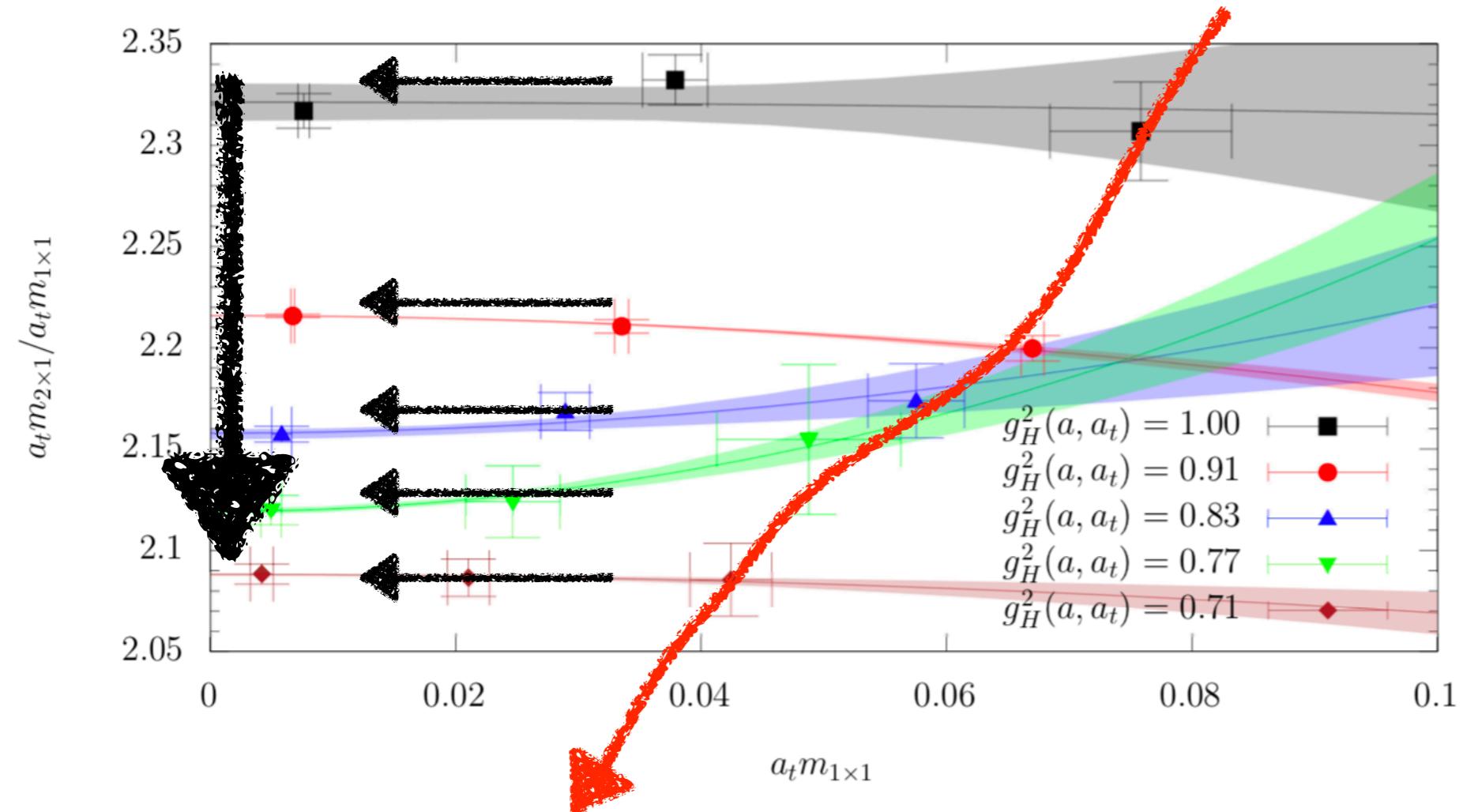
To the Continuum

D4
group



Eco-trajectory: extrapolation to the continuum at fixed $\xi = a/a_t$

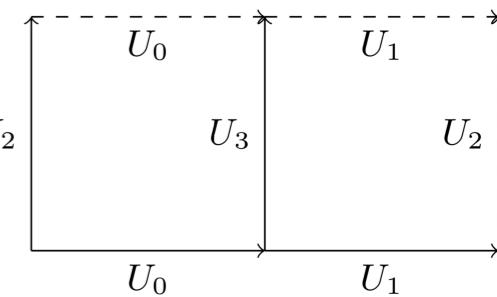
weak coupling input: $\xi^{-1} \sim \delta_t$, if $\delta_t \ll 1$



[M. Carena, H. Lamm, YYL, W. Liu, PRD. 104, 094519]

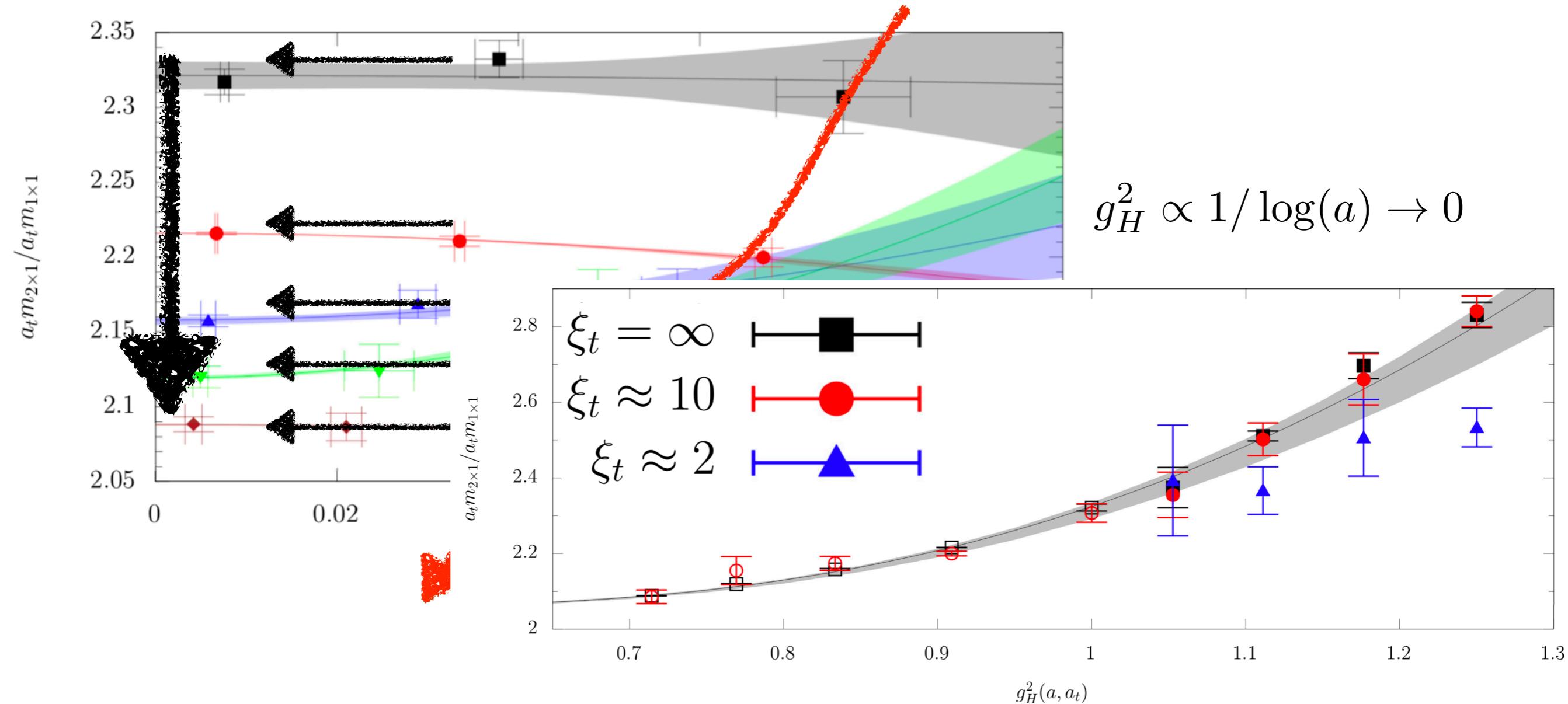
To the Continuum

D4
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Eco-trajectory: extrapolation to the continuum at fixed $\xi = a/a_t$

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[M. Carena, H. Lamm, YYL, W. Liu, PRD. 104, 094519]

Conclusions

- ❖ Techniques on real-time simulation of lattice field theory:
*improved Hamiltonian: matrix elements for the improved terms,
circuits designed, fidelity for current device.
scale setting, fixed anisotropic trajectory.*
- ❖ More to explore
*demonstration of different Hamiltonians, renormalization in
error mitigation, discrete group validity regime, ...*

For High Energy Physics

- ❖ All of these techniques could be used for simulations of high energy processes in the future!!

Thank you

BACK UP

Discretization of space

$$\hat{K}_{2L} = \frac{g_t^2}{a} \sum_{x,i} \text{Tr}[\hat{R}_i(\mathbf{x}) \hat{L}_i(\mathbf{x} + a\mathbf{i})]$$

?

[M. Carena, H. Lamm, YYL, W. Liu, PRL. 129, 051601]

Discretization of space

$$\mathrm{Tr}(\hat{R}_1 \hat{L}_2) = \mathrm{Tr}[\hat{L}_2^2 + \hat{R}_1^2 - (\hat{L}_2 - \hat{R}_1)^2]/2$$

$$\mathcal{U}_{K_{2L}} \equiv e^{i\theta \mathrm{Tr}(\hat{L}_2 - \hat{R}_1)^2}$$

$$[\mathcal{U}_{K_{2L}}, \hat{U}_1 \hat{U}_2] = 0$$

$$\langle U'_1, U'_2 | \mathcal{U}_{K_{2L}} | U_1, U_2 \rangle = \delta_{U'_1 U'_2, U_1 U_2} \langle U'_1 | e^{i\theta \mathrm{Tr} \hat{L}_1^2} | U_1 \rangle$$

[M. Carena, H. Lamm, YYL, W. Liu, PRL. 129, 051601]

$$\langle U'_1, U'_2 | \mathcal{U}_{K_{2L}} | U_1, U_2 \rangle = \delta_{U'_1 U'_2, U_1 U_2} \mathcal{A}(U_1, U_2, U'_1).$$

$$\mathcal{A}(U_1, U_2, U'_1) = \int dU'_2 \langle U'_1, U'_2 | \mathcal{U}_{K_{2L}} | U_1, U_2 \rangle.$$

$$\mathcal{A}(U_1, U_2, U'_1) = \int dV \langle U'_1, U_2 | e^{i\alpha_c \hat{L}_2^c} \mathcal{U}_{K_{2L}} | U_1, U_2 \rangle.$$

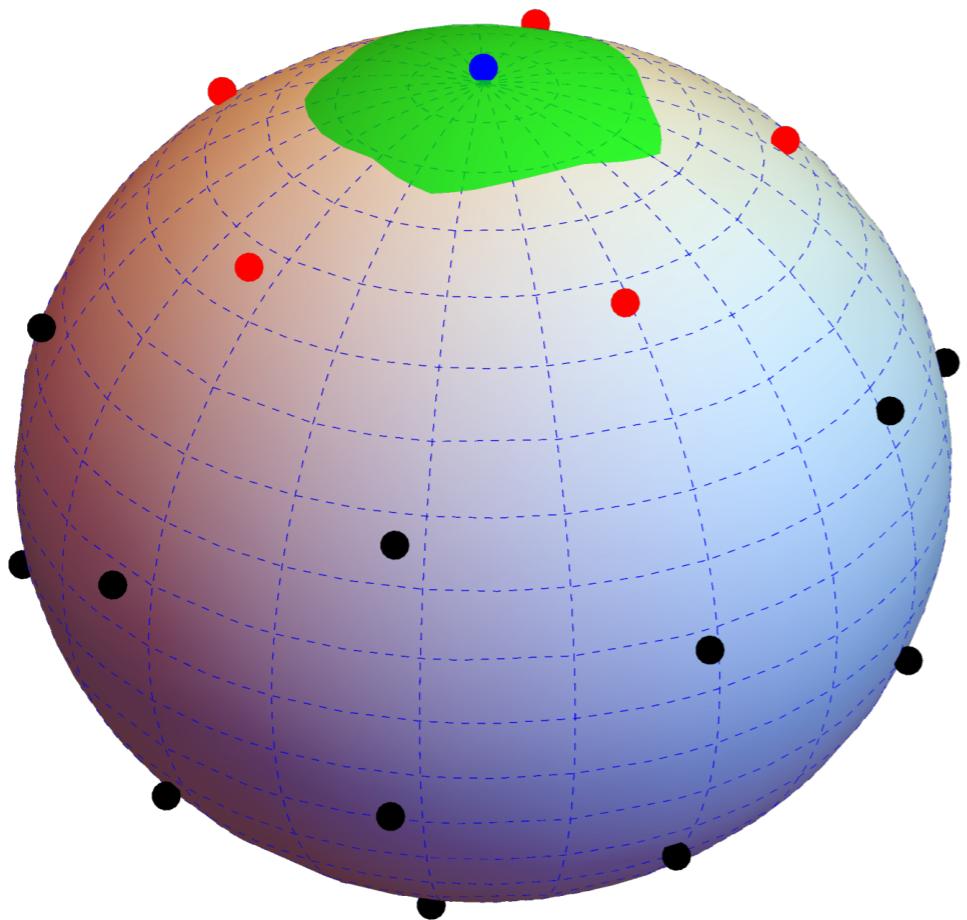
$$\int dV e^{i\alpha_a \hat{L}_2^a} = |G| |J_2 = 0\rangle \langle J_2 = 0| = |G| \hat{P}_{J_2=0}.$$

$$\begin{aligned} \mathcal{A}(U_1, U_2, U'_1) &= \langle U'_1, J_2 = 0 | \mathcal{U}_{K_{2L}} | U_1, J_2 = 0 \rangle \\ &= \langle U'_1 | e^{i\theta \text{Tr } \hat{R}_1^2} | U_1 \rangle, \end{aligned}$$

Digitization and Propagation

digitize gluon:

qubit regularization, quantum link models,
discrete subgroups, etc



Digitization

G -register : $U_i(x) \rightarrow |g\rangle$

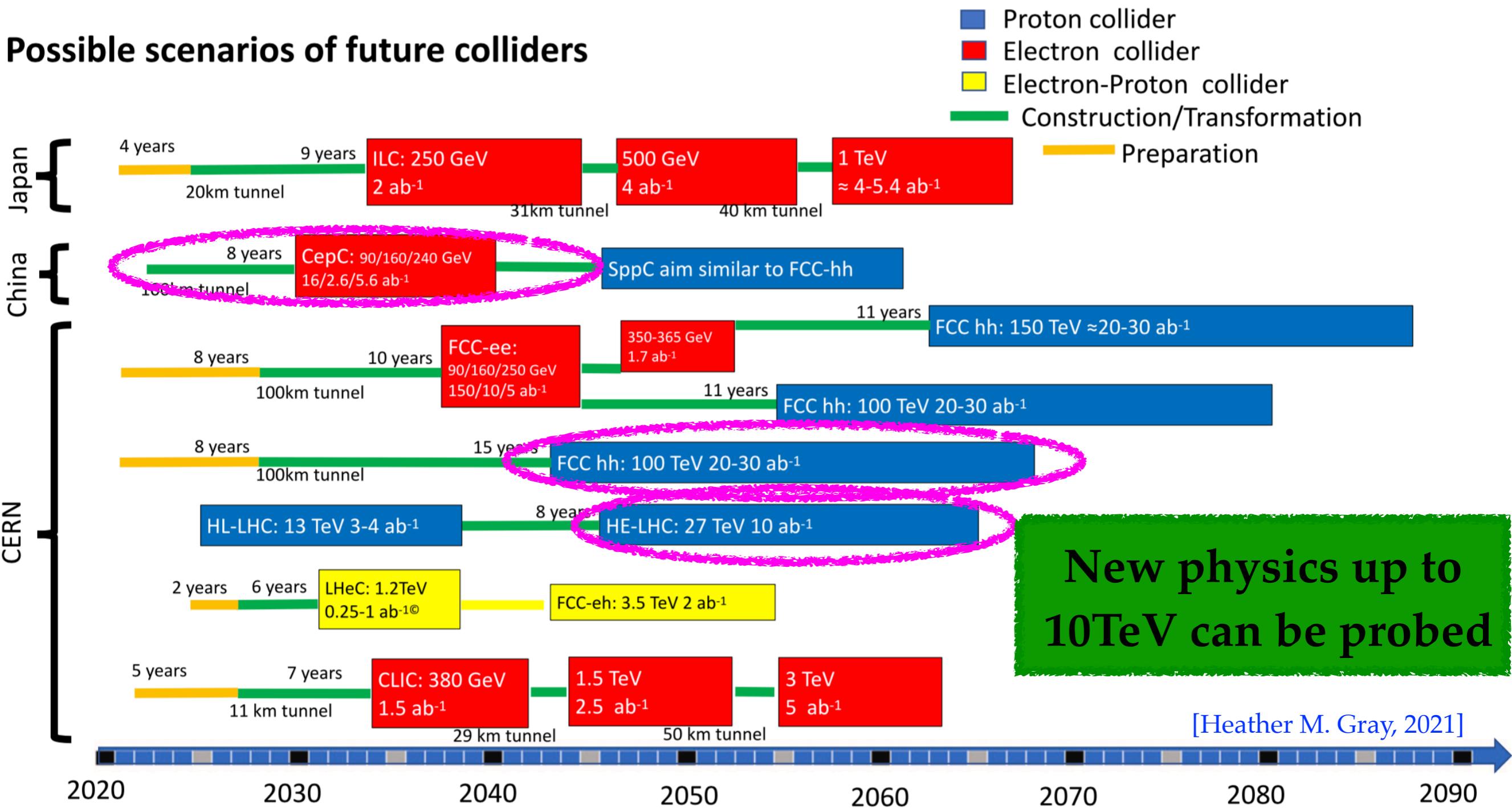
D4 group: 4 rotations, 4
reflections

group element g: $|abc\rangle$

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right]^a \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right]^{2b+c}$$

-Now-: precision measurement

Possible scenarios of future colliders

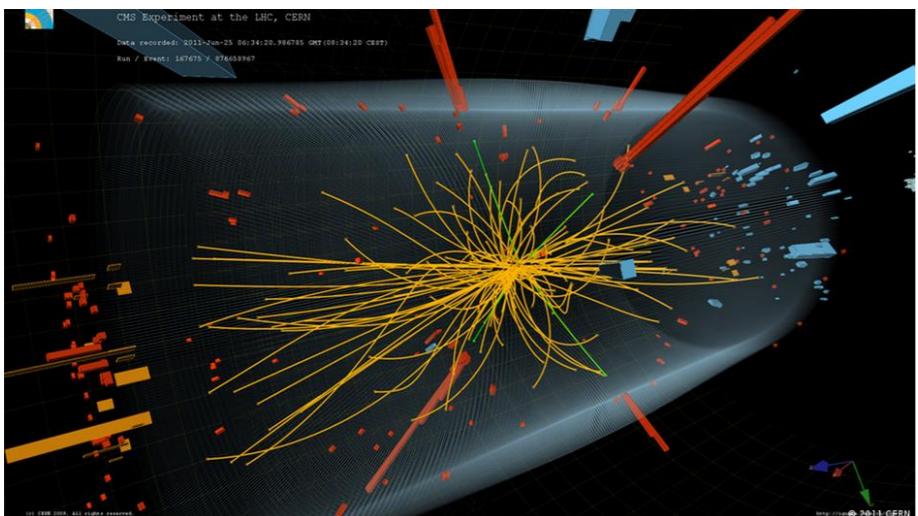


[Heather M. Gray, 2021]

-Now:- precision measurement

New physics up to TeV can be probed

HL-Large Hadron Colliders (LHC)



HL-LHC CMS & ATLAS (3 ab ⁻¹) in %									
$\mu_{gg h}^{\gamma\gamma}$	4.20	$\mu_{VBF}^{\gamma\gamma}$	12.8	$\mu_{W h}^{\gamma\gamma}$	13.9	$\mu_{Z h}^{\gamma\gamma}$	23.3	$\mu_{t h}^{\gamma\gamma}$	9.40
	4.52		8.93		14.1		16.5		8.92
$\mu_{gg h}^{ZZ}$	4.00	μ_{VBF}^{ZZ}	13.4	$\mu_{W h}^{ZZ}$	47.8	$\mu_{Z h}^{ZZ}$	78.6	$\mu_{t h}^{ZZ}$	24.6
	4.64		11.8		43.8		83.3		19.7
$\mu_{gg h}^{WW}$	3.70	μ_{VBF}^{WW}	7.30	$\mu_{W h}^{WW}$	13.8	$\mu_{Z h}^{WW}$	18.4	$\mu_{t h}^{WW}$	9.70
	6.16		6.68		—		—		114
$\mu_{gg h}^{\tau\tau}$	5.50	$\mu_{VBF}^{\tau\tau}$	4.40					$\mu_{t h}^{\tau\tau}$	14.9
	8.79		8.06						73.3
$\mu_{gg h}^{\mu\mu}$	13.8	$\mu_{VBF}^{\mu\mu}$	54.0						
	18.5		36.1						
$\mu_{gg h}^{Z\gamma}$	—	$\mu_{VBF}^{Z\gamma}$	—						
	33.3		68.2						
$\mu_{gg h}^{bb}$	24.7			$\mu_{W h}^{bb}$	9.40	$\mu_{Z h}^{bb}$	6.5	$\mu_{t h}^{bb}$	11.6
	—				10.1		5.85		14.8

[Jorge de Blas, et al, arXiv:1907.04311]

$$\sim \mathcal{O}(10\%)$$

$$\begin{aligned} [E_i^a(\mathbf{x}), A_j^b(\mathbf{y})] &= i\delta_{ij}\delta_{ab}\delta(\mathbf{x} - \mathbf{y}) \\ [A_i^a(\mathbf{x}), A_j^b(\mathbf{y})] &= [E_i^a(\mathbf{x}), E_j^b(\mathbf{y})] = 0. \end{aligned}$$

$$\begin{aligned} \mathcal{A}_l &= \frac{1}{a} \int_{-a/2}^{a/2} dt A_i(\mathbf{x} + t\hat{\mathbf{i}}), \\ &= \frac{1}{a} \int_{-a/2}^{a/2} dt \left[A_i(\mathbf{x}) + t\partial_i A_i(\mathbf{x}) + \frac{1}{2}t^2\partial_i^2 A_i(\mathbf{x}) + \mathcal{O}(a^3) \right] \\ &= A_i(\mathbf{x}) + \frac{a^2}{24}\partial_i^2 A_i(\mathbf{x}) + \frac{a^4}{1920}\partial_i^2 A_i(\mathbf{x}) \dots \end{aligned}$$

$$\mathcal{E}_i^{(1)\alpha}(\mathbf{x}) = -\frac{a^{d-1}}{e} \left[E_i^\alpha(\mathbf{x}) - \frac{a^2}{24}\partial_i^2 E_i^\alpha(\mathbf{x}) \right]$$

$$\begin{aligned} [\hat{L}_i^a(\mathbf{x}), \hat{U}_i(\mathbf{x})] &= \lambda_a \hat{U}_i(\mathbf{x}), \\ [\hat{L}_i^a(\mathbf{x}), \hat{L}_i^b(\mathbf{x})] &= -if_{abc}\hat{L}_i^c(\mathbf{x}), \end{aligned}$$

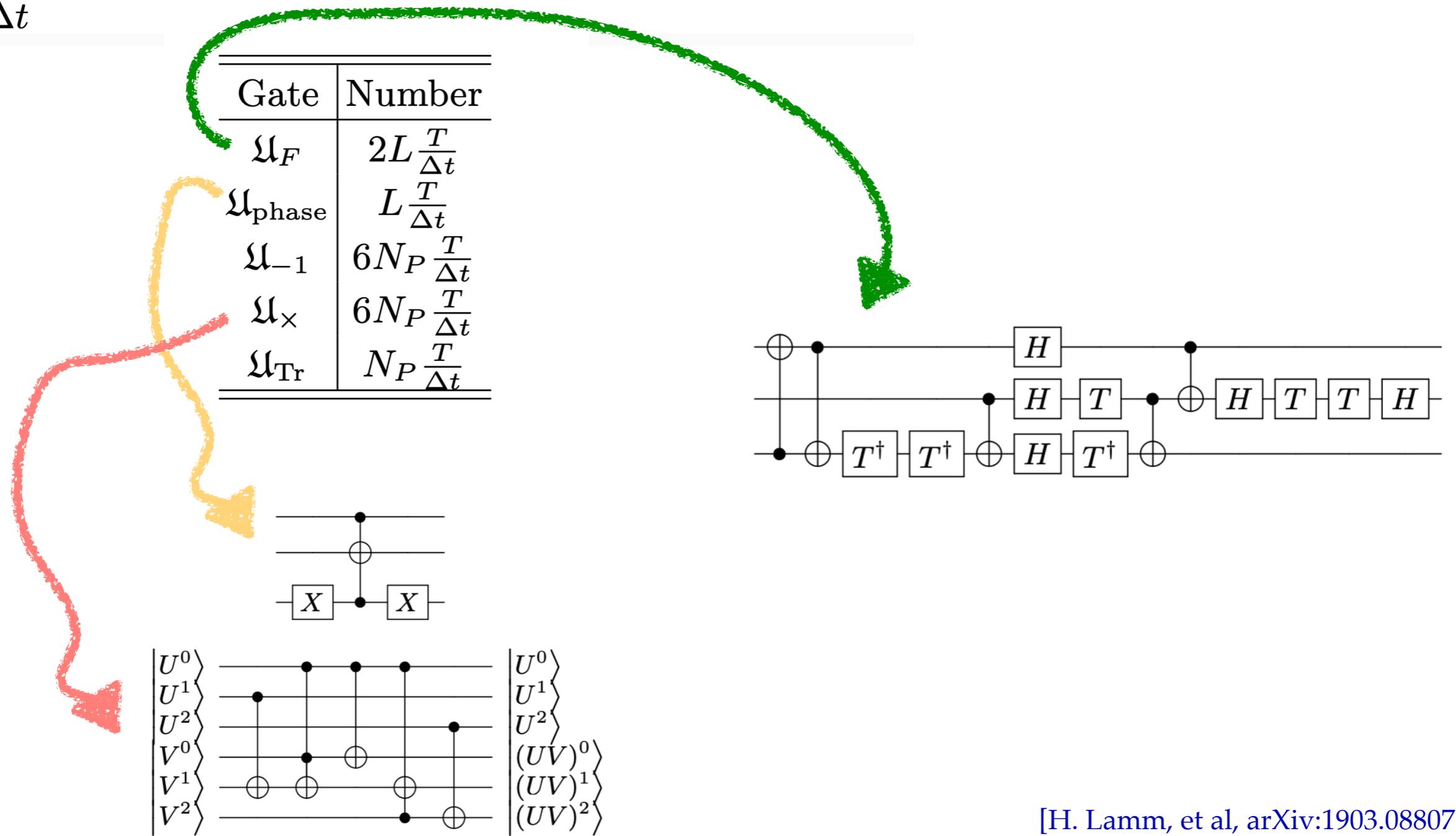
Discretization of space

$$\begin{aligned} [E_i^a(\mathbf{x}), A_j^b(\mathbf{y})] &= i\delta_{ij}\delta_{ab}\delta(\mathbf{x} - \mathbf{y}) \\ [A_i^a(\mathbf{x}), A_j^b(\mathbf{y})] &= [E_i^a(\mathbf{x}), E_j^b(\mathbf{y})] = 0. \end{aligned}$$

$$\begin{aligned} [\hat{L}_i^a(\mathbf{x}), \hat{U}_i(\mathbf{x})] &= \lambda_a \hat{U}_i(\mathbf{x}), \\ [\hat{L}_i^a(\mathbf{x}), \hat{L}_i^b(\mathbf{x})] &= -if_{abc} \hat{L}_i^c(\mathbf{x}) \end{aligned}$$

Gate counting for D4

TABLE I. Gate requirements for the propagation of a lattice with N_P plaquettes and L links, for a time T with time-steps of size Δt



Discretization of space

[J. Carlsson, et al, hep-lat/0105018]

$$[L_i^a(\mathbf{x}), U_i(\mathbf{x})] = T_a U_i(\mathbf{x})$$

$$[L_i^a(\mathbf{x}), L_i^b(\mathbf{x})] = -i f_{abc} L_i^c(\mathbf{x})$$

$$[R_i^a(\mathbf{x}), U_i(\mathbf{x})] = U_i(\mathbf{x}) T_a$$

$$[R_i^a(\mathbf{x}), R_i^b(\mathbf{x})] = i f_{abc} R_i^c(\mathbf{x})$$

$$[L_i^a(\mathbf{x}), R_i^b(\mathbf{x})] = 0$$

Anisotropic Parameter Renormalization

Moreover, one-loop calculation of the renormalization effects on the Euclidean side

$$S_E \xleftrightarrow{U_t \leftrightarrow l_{ij}^2} T(a_0) = e^{-a_0 H(a, a_0)}.$$

SU(N) For $\xi \rightarrow \infty$

[C. J. Hamer, Phys. Rev. D 53, 7316, ...]

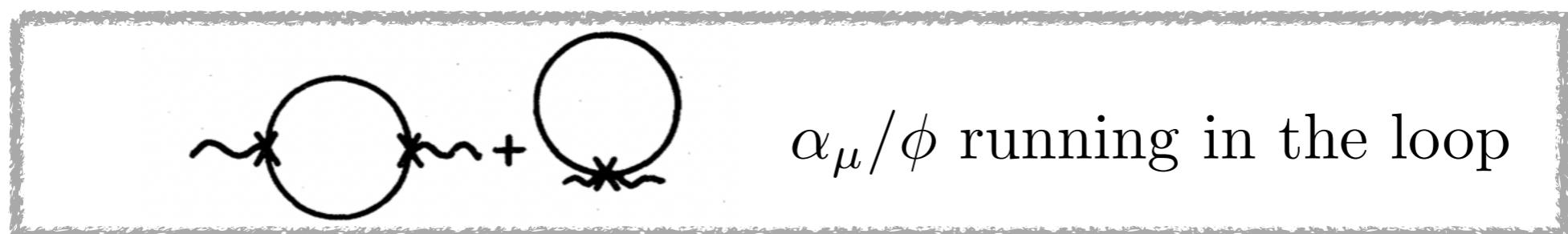
For finite ξ in 3 + 1

[F. Karsch, Nuclear Physics B205 (1982) 285-300, ...]

Background field method at one loop order [R. Dashen and D. Gross, Phys. Rev. D 23, 2340]

$$U_{x,x+\mu} = e^{ieg_E\alpha_\mu(x)} U_{x,x+\mu}^{(0)}, \quad U_{x,x+\mu}^{(0)} = e^{iea_\mu A_\mu(x)}$$

$$S_{gf} = a^{D-1} a_\tau \sum_x \text{Tr} \left(\sum_\mu \overline{D_\mu^{(0)}} \alpha_\mu(x) \right)^2 \quad S_{gh} = 2a^{D-1} a_\tau \sum_{x,\mu} \text{Tr}[(D_\mu^{(0)} \phi(x))^\dagger (D_\mu^{(0)} \phi(x))]$$



Independent of regularization

$$\Delta S_{\text{eff}} = S_{\text{eff}}^{(\xi=1)} - S_{\text{eff}}^{(\xi \neq 1)} = 0$$