Electroweak Standard Model （8课时）

一些有益的专题：

✓ 1. Poincaré group, Lorentz group, Little group.

2. Unitarity and its bound on the cross section

3. Gauge symmetry and gauge anomaly

✓ 4. Spontaneously symmetry breaking

✓ 5. Higgs mechanism

✓ 6. The Construction of the Standard Model

7. Hierarchy problem.

Prof. Cao v.s. Mr. Zhang

历史修正主义 v.s. 历史虚无主义

粒子物理标准模型 $\begin{cases} \text{平直时空相对论性量子场论} \\ \text{规范相互作用} \\ \text{对称性自发破缺.} \end{cases}$

第一次课：Poincare 对称性，小群 ⟶ 为什么需要规范"对称性"

Lorentz 群的表示，旋量（补充内容）

相对论性量子场论：

相对论 ———— 狭义相对论的时空对称性.

量子. ———— 态，态空间，力学量算符.

场论 ———— 无穷多自由度.

量子理论：态矢量空间 $H = \{|\psi\rangle\}$

关心的物理观测量：$\forall\ |\psi\rangle, |\varphi\rangle \in H \qquad |\langle\varphi|\psi\rangle|^2 = ?$

对称性 $\Leftrightarrow$ 观测者的自由！

时空对称性：选择惯性运动状态的自由

$U(1)$ 对称性：选择相位零点的自由.

...

数学的语言：设 $\exists$ 映射 $s: H \to H$, s.t. $\forall\ \varphi, \psi \in H$

$$|\langle s\varphi | s\psi\rangle|^2 \equiv |\langle\varphi|\psi\rangle|^2$$

则 s 是系统的一个对称性.

态矢量空间 H 是线性空间，记线性变换 $U: H \to H$

保证 $\forall\ \psi, \varphi \in H$ $\quad \langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle$，则 U 为幺正变换

但是对称性变换 s 既不一定线性，也不保内积（仅模方）

Wigner 神奇的定理：任意对称性 s，一定对应一个 H 上的幺正或反幺正变换 S！

How about a group of symmetry ?

例：Poincaré 变换.

平移：$t \to t + t_0$，$x \to x + x_0$，$y \to y + y_0$，$z \to z + z_0$

Lorentz 变换：$t \to \dfrac{t - xv/c^2}{\sqrt{1 - v^2/c^2}}$，$x \to \dfrac{x - vt}{\sqrt{1 - v^2/c^2}}$，$y \to y$，$z \to z$

定义 $a^\mu = (ct, x, y, z)$ $\quad \mu = 0, 1, 2, 3$

$\Rightarrow$ $a^\mu \to a^\mu + a_0^\mu$ 平移

$a^\mu \to \Lambda^\mu{}_\nu a^\nu$ 转动 + boost.

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \qquad \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \sqrt{\gamma^2-1} & 0 & 0 \\ \sqrt{\gamma^2-1} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\therefore$ $x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu$ ：Poincaré 变换

$\forall\ x, y$，$(x-y)^T \Lambda^T \cdot g \cdot \Lambda (x-y) = (x-y)^T \cdot g \cdot (x-y)$

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

全体 Poincaré 变换构成的集合 $\mathcal{P}$

$$x^\mu \to \Lambda_1{}^\mu{}_\nu x^\nu + a_1^\nu \to \Lambda_2{}^\mu{}_\nu(\Lambda_1{}^\nu{}_\sigma x^\sigma + a_1^\nu) + a_2^\mu$$
$$= \Lambda_2{}^\mu{}_\nu \Lambda_1{}^\nu{}_\sigma x^\sigma + \Lambda_2{}^\mu{}_\nu a_1^\nu + a_2^\mu$$

$\therefore (\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a)$

狭义相对论：Poincaré 变换是系统的对称性.

$\forall\ g \in \mathcal{P}$，$g$ 为 (Λ, a) $\Rightarrow U(g): H \to H$ 是幺正或反幺正变换.

$\forall\ g_1, g_2 \in \mathcal{P}$，$U(g_1)U(g_2)$ 也是对称性.

$$U(g_1)U(g_2) \doteq U(g_1 g_2) \qquad \text{不一定}$$

Projective representation 略.

Poincaré 变换作用在态上 $U(g)|\psi\rangle$

SNAG 定理：平移部分一定可以写作（对于不可约表示）

$$U(\mathbb{1}, a^\mu)|\psi\rangle = \exp(ip_\mu a^\mu)|\psi\rangle$$

这个不可约表示中的任意态都可以写为 $U(\Lambda, b)|\psi\rangle$

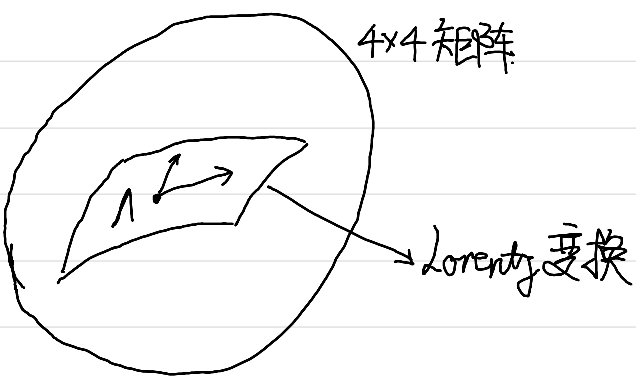
$$\therefore U(\mathbb{1}, a^\mu)\, U(\Lambda, b)|\psi\rangle$$
$$= U(\Lambda, b)U^{-1}(\Lambda, b)U(\mathbb{1}, a)U(\Lambda, b)|\psi\rangle$$
$$= U(\Lambda, b)\, U((\Lambda^{-1}, -\Lambda^{-1}b)(1, a)(\Lambda, b))|\psi\rangle$$
$$= U(\Lambda, b)\cdot U((\Lambda^{-1}, -\Lambda^{-1}b)\ (\Lambda, a+b))|\psi\rangle$$
$$= U(\Lambda, b)\, U(1, \Lambda^{-1}(a+b) - \Lambda^{-1}b)|\psi\rangle$$
$$= U(\Lambda, b)\, U(1, \Lambda^{-1}a)|\psi\rangle = \exp(ip_\mu(\Lambda^{-1}a)^\mu)\, U(\Lambda, b)|\psi\rangle$$
$$= \exp(i(\Lambda^{-1}p)_\mu a^\mu)\, U(\Lambda, b)|\psi\rangle$$

∴ Poincaré 群不可约表示中的态，都对应一个 4-矢量"p^μ"
并且 $p^2=m^2$ 对于一个不可约表示中的所有态是一个常数
p^μ 就是 4 动量.

Poincaré 群的齐次部分：Lorentz 群

$x^\mu \to \Lambda^\mu{}_\nu x^\nu$ $\quad g_{\alpha\beta}x^\alpha y^\beta = g_{\mu\nu}\Lambda^\mu{}_\alpha x^\alpha \Lambda^\nu{}_\beta y^\beta$

∴ $g_{\alpha\beta}=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta g_{\mu\nu}$ 矩阵写法 $\Lambda^T\cdot G\cdot\Lambda = G$



$\Lambda+\delta\Lambda$ 仍然是 Lorentz 矩阵

转动：
$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\cos\theta&\sin\theta\\0&0&-\sin\theta&\cos\theta\end{pmatrix}$$
$\theta\to 0$，单位变换附近.

$$R_x(\theta)\to\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1-\frac{\theta^2}{2}&\theta\\0&0&-\theta&1-\frac{\theta^2}{2}\end{pmatrix} = \mathbb{1}+i\theta L_x$$

$$L_x \equiv \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$$
类似地 $L_y=\begin{pmatrix}0&0&0&0\\0&0&0&i\\0&0&0&0\\0&-i&0&0\end{pmatrix}$

$$L_z=\begin{pmatrix}0&0&0&0\\0&0&-i&0\\0&i&0&0\\0&0&0&0\end{pmatrix}$$

$$\therefore L_x L_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_y L_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$L_x L_y - L_y L_x = i L_z$$

$$\begin{cases} [L_x, L_y] = i L_z \\ [L_y, L_z] = i L_x \\ [L_z, L_x] = i L_y \end{cases}$$

Boost.

$$B_x(\beta) = \begin{pmatrix} \gamma & -\sqrt{\gamma^2-1} & 0 & 0 \\ -\sqrt{\gamma^2-1} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\gamma = \frac{1}{\sqrt{1-\beta^2}}} \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ \frac{-\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\beta \to 0} \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 + i\beta K_x$$

$$K_x \equiv \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$[K_x, K_y] = -i L_z \qquad [K_y, K_z] = -i L_x \qquad [K_z, K_x] = -i L_y$$

$$[K_x, L_x] = \ ?$$

$$L_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x L_x = L_x K_x = 0$$

对易

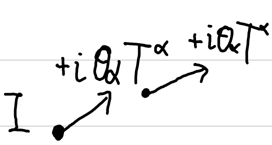
$$K_y L_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad L_x K_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore L_x K_y - K_y L_x = i K_z \qquad [L_x, K_y] = i K_z$

$\therefore$ 所有6个无穷小 Lorentz 变换生成元之间满足

$$\begin{cases} [L_i, L_j] = i\varepsilon_{ijk} L_k \\ [K_i, K_j] = -i\varepsilon_{ijk} L_k \\ [L_i, K_j] = i\varepsilon_{ijk} K_k \end{cases} \qquad i, j, k = 1(x),\ 2(y),\ 3(z)$$

为什么研究无穷小生成元？有限大变换如何得到



$\because \dfrac{dg}{d\theta^\alpha} = iT^\alpha \Big|_{\theta=0}$

$\therefore \dfrac{dg}{d\theta^\alpha} = iT^\alpha \quad \forall\theta$

$$\Rightarrow g(\theta) = \exp(i\theta^\alpha T^\alpha)$$

$\therefore$ G有单位元附近"有限"大的邻域内可以写为 $e^{i\theta^\alpha T^\alpha}$ 的形式！

对于 Lorentz 群 K_i 不是 Hermite 矩阵 !!!

$\therefore$ $e^{i\beta_j K_j}$ 不是幺正矩阵

定义 $J_{\pm i} = \frac{1}{2}(L_i \pm i K_i)$ $\quad L_i = J_{+i} + J_{-i} \quad K_i = \frac{1}{i}(J_{+i} - J_{-i})$

$$[J_{\pm i}, J_{\pm j}] = \frac{1}{4}[L_i, L_j] \pm \frac{1}{4}i[K_i, L_j] \pm \frac{i}{4}[L_i, K_j] - \frac{1}{4}[K_i, K_j]$$
$$= \frac{1}{4} i\varepsilon_{ijk} L_k \pm \frac{i}{4} i\varepsilon_{ijk} K_k \pm \frac{i}{4} i\varepsilon_{ijk} K_k + \frac{1}{4} i\varepsilon_{ijk} L_k$$
$$= i\varepsilon_{ijk} J_{\pm k}$$

$[J_{+i}, J_{-j}] = 0$

$$g(\beta, \theta) = \exp(i\theta_i L_i + i\beta_i K_i)$$
$$= \exp[i\theta_i (J_{+i} + J_{-i}) + \beta_i (J_{+i} - J_{-i})]$$
$$= \exp[i(\theta_i - i\beta_i) J_{+i} + i(\theta_i + i\beta_i) J_{-i}]$$

定义 $z_i = \theta_i + i\beta_i$

$$\Rightarrow g(\beta, \theta) = \exp[i\bar{z}_i J_{+i} + iz_i J_{-i}] \quad \because [J_{+i}, J_{-j}] = 0$$
$$= \exp(i\bar{z}_i J_{+i}) \cdot \exp(iz_i J_{-i})$$
$$= \exp(iz_i J_{-i}) \cdot \exp(i\bar{z}_i J_{+i})$$

$$J_{+x} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad J_{+y} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad J_{+z} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$J_{-x} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \quad J_{-y} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad J_{-z} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$J_+^2 = J_-^2 = \frac{3}{4} = j(j+1)$ $\quad \therefore$ J_+, J_- 是 SU(2) 群 $\frac{1}{2}$ 表示的生成元

Lorentz 群的表示被 (L_i, K_i) —— 进而被 (J_{+i}, J_{-i}) 的表示所决定，但 J_+, J_- 分别为 $SU(2)$ 的 Lie 代数，于是 Lorentz 群的表示可以被 (j_+, j_-) 代表，其中

$$J_+^2 u = j_+(j_+ + 1) u$$
$$J_-^2 u = j_-(j_- + 1) u$$

j_+, j_- 相互独立. 基础表示为 $(\frac{1}{2}, \frac{1}{2})$

注意: $(\frac{1}{2}, \frac{1}{2}) \neq (\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$

上述分析 ~~为~~ $SO(3,1) = SU(2) \otimes SU(2)$

虽然 J_+, J_- 可以不同，但 $\mathfrak{z}_- \equiv \mathfrak{z}_+$，∴ 相当于 $SU(2)$ 生成元的复化. $SU_{\mathbb{C}}(2)$

$\frac{1}{2}\sigma^x$, $\frac{1}{2}\sigma^y$, $\frac{1}{2}\sigma^z$. $\frac{1}{2}i\sigma^x$, $\frac{1}{2}i\sigma^y$, $\frac{1}{2}i\sigma^z$ 辅以 6 个实参数.

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$i\sigma^x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad i\sigma^y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad i\sigma^z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

无迹 2×2 复矩阵生成的 Lie 代数. $SL(2, \mathbb{C})_R$

$\because \det e^A = e^{\operatorname{tr} A}$ $\quad \therefore \det e^A \equiv 1$

$\therefore (j_+, j_-) = (0, \frac{1}{2})$ 表示. $J_+ u \equiv 0$

$$g(\beta, \theta) = \exp(i \mathfrak{z}_i J_{-i}) = \exp\left(\tfrac{1}{2} i (\theta_i + i\beta_i) \sigma_i\right)$$

$$= \exp\left(\frac{i\theta_x \sigma_x}{2} + \frac{i\theta_y \sigma_y}{2} + \frac{i\theta_z \sigma_z}{2} - \frac{\beta_x \sigma_x}{2} - \frac{\beta_y \sigma_y}{2} - \frac{\beta_z \sigma_z}{2}\right)$$

$$= \exp\begin{pmatrix} -\frac{\beta_z}{2} + \frac{i\theta_z}{2} & -\frac{\beta_x}{2} + \frac{\theta_y}{2} + i\left(\frac{\beta_y}{2} + \frac{\theta_x}{2}\right) \\ -\frac{\beta_x}{2} - \frac{\theta_y}{2} + i\left(-\frac{\beta_y}{2} + \frac{\theta_x}{2}\right) & \frac{\beta_z}{2} - \frac{i\theta_z}{2} \end{pmatrix}$$

$R_x(\theta)$: $\quad \exp\left(i\frac{\theta}{2}\sigma_x\right) = \exp\begin{pmatrix} 0 & \frac{i\theta}{2} \\ \frac{i\theta}{2} & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$

$R_y(\theta)$: $\quad \exp\begin{pmatrix} 0 & \frac{\theta}{2} \\ -\frac{\theta}{2} & 0 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$

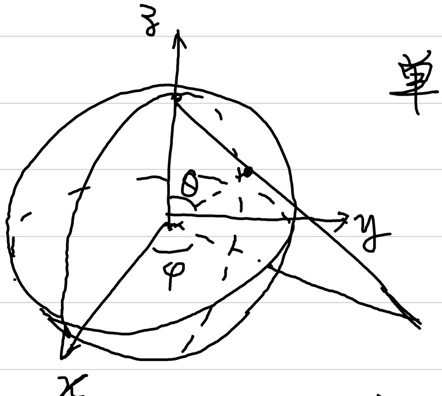
$R_z(\theta)$: $\quad \exp\begin{pmatrix} i\frac{\theta}{2} & 0 \\ 0 & -\frac{i\theta}{2} \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$

$B_x(\beta)$: $\quad \exp\left(-\frac{\beta}{2}\sigma_x\right) = \exp\begin{pmatrix} 0 & -\frac{\beta}{2} \\ -\frac{\beta}{2} & 0 \end{pmatrix} = \begin{pmatrix} \cosh\frac{\beta}{2} & -\sinh\frac{\beta}{2} \\ -\sinh\frac{\beta}{2} & \cosh\frac{\beta}{2} \end{pmatrix}$

$B_y(\beta)$: $\quad \exp\begin{pmatrix} 0 & \frac{i\beta}{2} \\ -\frac{i\beta}{2} & 0 \end{pmatrix} = \begin{pmatrix} \cosh\frac{\beta}{2} & i\sinh\frac{\beta}{2} \\ -i\sinh\frac{\beta}{2} & \cosh\frac{\beta}{2} \end{pmatrix}$

$B_z(\beta)$: $\quad \exp\begin{pmatrix} -\frac{\beta}{2} & 0 \\ 0 & \frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} e^{-\beta/2} & 0 \\ 0 & e^{\beta/2} \end{pmatrix}$

旋量表示的图象



球极投影

单位球面上一点 (x, y, z)

$$z = \sqrt{1-x^2-y^2}$$

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$

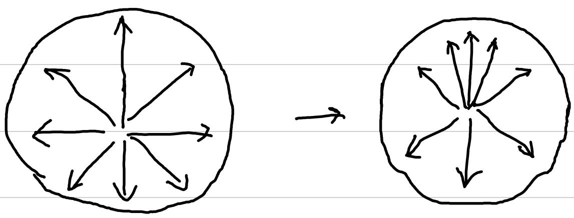
$$\Rightarrow \quad (x, y, z) \to r e^{i\varphi}$$

$$r = \cot\frac{\theta}{2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$\therefore (x, y, z) \to \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} e^{i\varphi} = \frac{e^{i\varphi/2}\cos\frac{\theta}{2}}{e^{-i\varphi/2}\sin\frac{\theta}{2}} = \frac{\xi_1}{\xi_2}$$

$R_z(\theta)$ $\quad \varphi \to \varphi + \theta, \quad (\xi_1\ \xi_2) \to (\xi_1\ \xi_2)\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$

沿 z 轴 Boost



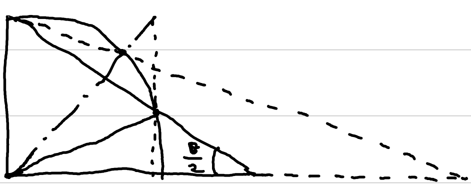
任一速度方向 (x, y, z) 的光线

$$\begin{pmatrix} \gamma & 0 & 0 & -\sqrt{\gamma^2-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sqrt{\gamma^2-1} & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} \gamma - z\sqrt{\gamma^2-1} \\ x \\ y \\ -\sqrt{\gamma^2-1} + z\gamma \end{pmatrix}$$

方向变为 $\left(x,\ y,\ -\sqrt{\gamma^2-1} + z\gamma\right)$

$$= \left(x,\ y,\ \frac{z-\beta}{\sqrt{1-\beta^2}}\right)$$



$$\cot\theta = \frac{z}{\sqrt{x^2+y^2}} \qquad \cot\theta' = \frac{z-\beta}{\sqrt{1-\beta^2}\sqrt{x^2+y^2}}$$

$$\therefore \sin\theta' = \frac{\sqrt{1-\beta^2}\sqrt{x^2+y^2}}{\sqrt{(z-\beta)^2+(1-\beta^2)(x^2+y^2)}}$$

$$\cos\theta' = \frac{z-\beta}{\sqrt{(z-\beta)^2+(1-\beta)(x^2+y^2)}}$$

$$\therefore \sin\theta' = \frac{\sqrt{1-\beta^2}\sin\theta}{\sqrt{\cos^2\theta - 2\beta\cos\theta + \beta^2 + \sin^2 - \beta^2\sin^2\theta}} = \frac{\sqrt{1-\beta^2}\sin\theta}{\sqrt{1-2\beta\cos\theta+\beta^2\cos^2\theta}} = \frac{\sqrt{1-\beta^2}\sin\theta}{1-\beta\cos\theta}$$

$$\cos\theta' = \frac{-\beta+\cos\theta}{1-\beta\cos\theta} \Rightarrow \cot\frac{\theta'}{2} = \frac{1+\cos\theta'}{\sin\theta'} = \frac{1-\beta+\cos\theta-\beta\cos\theta}{\sqrt{1-\beta^2}\sin\theta}$$

$$\therefore \cot\frac{\theta'}{2} = \frac{\sqrt{1-\beta}\,(1+\cos\theta)}{\sqrt{1+\beta}\,\sin\theta} = \sqrt{\frac{1-\beta}{1+\beta}}\cot\frac{\theta}{2}$$

$$\therefore \rightarrow \frac{\sqrt{1-\beta}\cos\frac{\theta}{2}e^{i\varphi/2}}{\sqrt{1+\beta}\sin\frac{\theta}{2}e^{-i\varphi/2}} \qquad \therefore e^{-\tilde\beta} = \frac{\sqrt{1-\beta}}{\sqrt{1+\beta}} \Rightarrow \tilde\beta = \tanh^{-1}\beta$$

生成元参数 β 是 Lorentz boost 的快度!

现在有 Lorentz 群的二维表示. $(j_+\ j_-)=(0,\frac{1}{2})$

$$(\xi_1,\xi_2)\,g_{(0,\frac{1}{2})}(\beta,\theta)=(\xi_1\ \xi_2)\exp\begin{pmatrix}-\frac{\beta_z}{2}+\frac{i\theta_z}{2} & -\frac{\beta_x}{2}+\frac{\theta_y}{2}+\frac{i(\beta_y+\theta_x)}{2}\\ -\frac{\beta_x}{2}-\frac{\theta_y}{2}+\frac{i(-\beta_y+\theta_x)}{2} & \frac{\beta_z}{2}-\frac{i\theta_z}{2}\end{pmatrix}$$

同样地, 有 $(j_+,j_-)=(\frac{1}{2},0)$ 表示

$$(\eta_1,\eta_2)\,g_{(\frac{1}{2},0)}(\beta,\theta)=(\eta_1\ \eta_2)\exp\begin{pmatrix}\frac{\beta_z}{2}+\frac{i\theta_z}{2} & \frac{\beta_x}{2}+\frac{\theta_y}{2}+\frac{i(\theta_x-\beta_y)}{2}\\ \frac{\beta_x}{2}-\frac{\theta_y}{2}+\frac{i(\beta_y+\theta_x)}{2} & -\frac{\beta_z}{2}-\frac{i\theta_z}{2}\end{pmatrix}$$

注意, 如果 η_α 是 $(\frac{1}{2},0)$ 表示的矢量, 则 $(\eta_\alpha)^*$ 的变换行为为:

$$(\eta_1^*\ \eta_2^*)\to(\eta_1^*\ \eta_2^*)\exp\begin{pmatrix}\frac{\beta_z}{2}-\frac{i\theta_z}{2} & \frac{\beta_x}{2}+\frac{\theta_y}{2}+\frac{i(\beta_y-\theta_x)}{2}\\ \frac{\beta_x}{2}-\frac{\theta_y}{2}-\frac{i(\beta_y+\theta_x)}{2} & -\frac{\beta_z}{2}+\frac{i\theta_z}{2}\end{pmatrix}=(\eta_1^*,\eta_2^*)\,g_{(0,\frac{1}{2})}(-\beta,-\theta)^T$$

$$\therefore\ \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}^\dagger\cdot\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=(\eta_1^*,\eta_2^*)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}=\eta_1^*\xi_1+\eta_2^*\xi_2$$ 是 Lorentz 不变量.

旋量指标升降为什么用 $\varepsilon_{\alpha\beta}$?

$$\xi^\alpha\equiv\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}\qquad \xi^\alpha\tilde{\xi}^\beta=\{\xi_1\tilde{\xi}_1,\ \xi_1\tilde{\xi}_2,\ \xi_2\tilde{\xi}_1,\ \xi_2\tilde{\xi}_2\}$$

$$\downarrow$$

$$(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})=(0,0)\oplus(0,1)$$

"单态" $\oplus$ "三重态"

单态 $\to\ \xi_1\tilde{\xi}_2-\xi_2\tilde{\xi}_1$ or $g(\beta,\theta)\cdot\begin{pmatrix}\xi_1 & \tilde{\xi}_1\\ \xi_2 & \tilde{\xi}_2\end{pmatrix}=\begin{pmatrix}\times & \times\\ \times & \times\end{pmatrix}$

$\because\ \det(g(\beta,\theta))=1$ $\quad\therefore\ \det\begin{pmatrix}\xi_1 & \tilde{\xi}_1\\ \xi_2 & \tilde{\xi}_2\end{pmatrix}$ 是 Lorentz 不变量

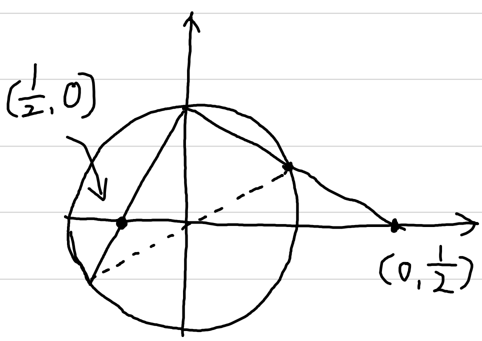
$\det(\xi^\alpha\tilde{\xi}^\beta)=\varepsilon_{\alpha\beta}\xi^\alpha\tilde{\xi}^\beta$

helicity (螺旋度) $\xi\, \vec{n}\cdot\vec{J}_+ = \xi\, h$.

$(0,\frac{1}{2})$ 表示,
$$\left(e^{i\frac{\varphi}{2}}\cos\frac{\theta}{2},\ e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}\right)\cdot\begin{pmatrix}\frac{1}{2}\cos\theta & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & -\frac{1}{2}\cos\theta\end{pmatrix}$$
$$=\frac{1}{2}\left(e^{i\frac{\varphi}{2}}\cos\theta\cos\frac{\theta}{2}+e^{i\frac{\varphi}{2}}\sin\theta\sin\frac{\theta}{2},\ e^{-i\frac{\varphi}{2}}\sin\theta\cos\frac{\theta}{2}-e^{-i\frac{\varphi}{2}}\cos\theta\sin\frac{\theta}{2}\right)$$
$$=\frac{1}{2}\left(e^{i\frac{\varphi}{2}}\cos\frac{\theta}{2},\ e^{-i\frac{\varphi}{2}}\sin\frac{\theta}{2}\right)$$

$\therefore\ h=\frac{1}{2}$ 右手

$(\frac{1}{2},0)$ 表示 $\theta\to\pi-\theta,\ \varphi\to\varphi+\pi$



$$\left(i e^{i\frac{\varphi}{2}}\sin\frac{\theta}{2},\ -i e^{-i\frac{\varphi}{2}}\cos\frac{\theta}{2}\right)$$
$$\to\left(-e^{i\varphi/2}\sin\frac{\theta}{2},\ e^{-i\varphi/2}\cos\frac{\theta}{2}\right)$$

$$\left(-e^{i\varphi/2}\sin\frac{\theta}{2},\ e^{-i\varphi/2}\cos\frac{\theta}{2}\right)\begin{pmatrix}\frac{1}{2}\cos\theta & \frac{1}{2}e^{-i\varphi}\sin\theta \\ \frac{1}{2}e^{i\varphi}\sin\theta & -\frac{1}{2}\cos\theta\end{pmatrix}$$
$$=\frac{1}{2}\left(e^{i\varphi/2}\left(-\sin\frac{\theta}{2}\cos\theta+\sin\theta\cos\frac{\theta}{2}\right),\ e^{-i\varphi/2}\left(-\sin\theta\sin\frac{\theta}{2}-\cos\theta\cos\frac{\theta}{2}\right)\right)$$
$$=\frac{1}{2}\left(e^{i\varphi/2}\sin\frac{\theta}{2},\ -e^{-i\varphi/2}\cos\frac{\theta}{2}\right)$$
$\therefore\ h=-\frac{1}{2}$ 左手

$\exp(2\pi i\,\vec{n}\cdot\vec{J}_+)\to$ 绕自身转 2π 角 $h=\pm\frac{1}{2}\Rightarrow$
差负号

$\Rightarrow(\frac{1}{2},0),(0,\frac{1}{2})$ 是投影表示

思考题：矢量场 A_μ 是 Lorentz 群的 $(\frac{1}{2},\frac{1}{2})$ 表示，
$\partial_\mu A^\mu$ 和 $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ 属于何种表示？

在 Poincaré 变换下.

$$U(\Lambda,a)\,\phi_a(x)\,|\Psi\rangle = U(\Lambda,a)\,\phi_a(x)\,U^{-1}(\Lambda,a)\,U(\Lambda,a)|\Psi\rangle$$
$$= S_a{}^b(\Lambda)\,\phi_b(\Lambda x+a)\,U(\Lambda,a)|\Psi\rangle$$

场算符取值在每一时空点都构成 Lorentz 群的表示.

∴ 人们说"标量场"，"旋量场"，"矢量场"，…

下面我们进入态矢量.

SNAG Thm $\Rightarrow$ $U(1,a)|\Psi\rangle = \int e^{ip_\mu a^\mu}\, d\mu_p(\Psi)\,|\Psi\rangle$
$$= \int d^4p\; e^{ip_\mu a^\mu}\,|p\rangle\langle p|\Psi\rangle$$

形式上，对于动量"本征态" $|p\rangle$

$$U(1,a)|p\rangle = e^{ip_\mu a^\mu}|p\rangle$$

考虑某种粒子的单粒子动量本征态 $|p,\sigma\rangle$

$$U(1,a)|p,\sigma\rangle = e^{ip_\mu a^\mu}|p,\sigma\rangle$$

$\because$ $U(1,a)\,U(\Lambda,b)|p,\sigma\rangle$

$= U(\Lambda,b)\,U(\Lambda,b)^{-1}\,U(1,a)\,U(\Lambda,b)|p,\sigma\rangle$

$= U(\Lambda,b)\,U(\Lambda^{-1},-\Lambda^{-1}b)\,U(1,a)\,U(\Lambda,b)|p,\sigma\rangle$

$= U(\Lambda,b)\,U(\Lambda^{-1},-\Lambda^{-1}b)\,U(\Lambda,a+b)|p,\sigma\rangle$

$= U(\Lambda,b)\,U(1,\Lambda^{-1}a)|p,\sigma\rangle$

$= \exp(ip_\mu(\Lambda^{-1}a)^\mu)\,U(\Lambda,b)|p,\sigma\rangle$

$= \exp(i(\Lambda p)_\mu a^\mu)\,U(\Lambda,b)|p,\sigma\rangle$

$\Rightarrow$ $U(\Lambda,b)|p,\sigma\rangle$ 是动量为 Λp 的动量本征态

$$U(\Lambda,b)|p,\sigma\rangle = |\Lambda p,\sigma'\rangle$$

Question： σ 与 σ' 是什么关系？ $U(\Lambda, b)|p,\sigma\rangle = |\Lambda p, \sigma'\rangle$

选择一个标准动量"k_μ"，则粒子的各种动量都可以通过 Lorentz 变换由 k_μ 到达，

$$p_\mu = L(p)_\mu{}^\nu k_\nu$$

注意！ $L(p)_\mu{}^\nu$ 是不唯一的，k_μ 和 $L(p)_\mu{}^\nu$ 的选择都有任意性，

(1) $m>0$ 的粒子，

方便起见，选 $k_\mu = (m, 0, 0, 0)$

$$U(\Lambda, 0)|p,\sigma\rangle = |\Lambda p, \sigma'\rangle$$

$$\Rightarrow U(\Lambda,0)U(L(p),0)U(L(p)^{-1},0)|p,\sigma\rangle$$

$$= U(L(\Lambda p),0)U(L(\Lambda p)^{-1},0)|\Lambda p,\sigma'\rangle$$

$$\therefore U(L(\Lambda p)^{-1},0)U(\Lambda,0)U(L(p),0)\ U(L(p)^{-1},0)|p,\sigma\rangle$$
$$= U(L(\Lambda p)^{-1},0)|\Lambda p,\sigma'\rangle$$

以 $|k,\sigma\rangle$ 定标 p 处的 σ 态

$$|p,\sigma\rangle \equiv U(L(p),0)|k,\sigma\rangle$$

$$\therefore U(L(\Lambda p)^{-1},0)U(\Lambda,0)U(L(p),0)\ |k,\sigma\rangle = |k,\sigma'\rangle$$

$\therefore$ $|k,\sigma\rangle$ 构成所有 $U(L(\Lambda p)^{-1}\Lambda L(p), 0)$ 的一个表示，

$L(\Lambda p)^{-1}\Lambda L(p)$ 是保证 k_μ 不变的全体 Lorentz 变换

$\Rightarrow$ 小群 (Little Group)

对于 $k_\mu=(m,0,0,0)$. 显然. 小群为 $SO(3)$ (或 $SU(2)$)
即粒子静止系的全体空间转动,

$$U(L(\Lambda p)^{-1}\Lambda L(p),0)\,|k,\sigma\rangle=|k,\sigma'\rangle=D_{\sigma'\sigma}\,|k,\sigma\rangle$$

$D_{\sigma'\sigma}$ 为 $SU(2)$ 的转动矩阵. 其 J, 为粒子的自旋

(2) $m=0$.

需考察 $k^\mu=(1,0,0,1)$ 的不变 Lorentz 变换

方法一. $(k^\mu)_{\alpha\dot\beta}=\begin{pmatrix}2&0\\0&0\end{pmatrix}$ $\quad A\in SL(2,\mathbb{C})$

$$A(k^\mu)_{\alpha\dot\beta}A^\dagger=(k^\mu)_{\alpha\dot\beta}$$

$$\because\ \begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\begin{pmatrix}2&0\\0&0\end{pmatrix}\begin{pmatrix}a_{11}^*&a_{21}^*\\a_{12}^*&a_{22}^*\end{pmatrix}=\begin{pmatrix}2a_{11}&0\\2a_{21}&0\end{pmatrix}\begin{pmatrix}a_{11}^*&a_{21}^*\\a_{12}^*&a_{22}^*\end{pmatrix}$$

$$=\begin{pmatrix}2|a_{11}|^2&2a_{11}a_{21}^*\\2a_{21}a_{11}^*&2|a_{21}|^2\end{pmatrix}=\begin{pmatrix}2&0\\0&0\end{pmatrix}$$

$$\Rightarrow a_{21}=0,\quad |a_{11}|=1,\quad \because \det A=1\quad \therefore a_{22}=a_{11}^*$$

$$\therefore\ A=\begin{pmatrix}e^{i\theta}&e^{-i\theta}z\\0&e^{-i\theta}\end{pmatrix}$$ $z\in\mathbb{C}$ 为任意复数

$$A(\theta_1,z_1)\cdot A(\theta_2,z_2)=\begin{pmatrix}e^{i\theta_1}&e^{-i\theta_1}z_1\\0&e^{-i\theta_1}\end{pmatrix}\begin{pmatrix}e^{i\theta_2}&e^{-i\theta_2}z_2\\0&e^{-i\theta_2}\end{pmatrix}$$

$$=\begin{pmatrix}e^{i(\theta_1+\theta_2)}&e^{i(\theta_1-\theta_2)}z_2+e^{-i(\theta_1+\theta_2)}z_1\\0&e^{-i(\theta_1+\theta_2)}\end{pmatrix}$$

$$=\begin{pmatrix}e^{i(\theta_1+\theta_2)}&e^{-i(\theta_1+\theta_2)}(z_1+e^{2i\theta_1}z_2)\\0&e^{-i(\theta_1+\theta_2)}\end{pmatrix}$$

$$=A(\theta_1+\theta_2,\ z_1+e^{2i\theta_1}z_2)$$

定义. $E(\alpha, x, y) = A(\frac{\alpha}{2}, x+iy)$

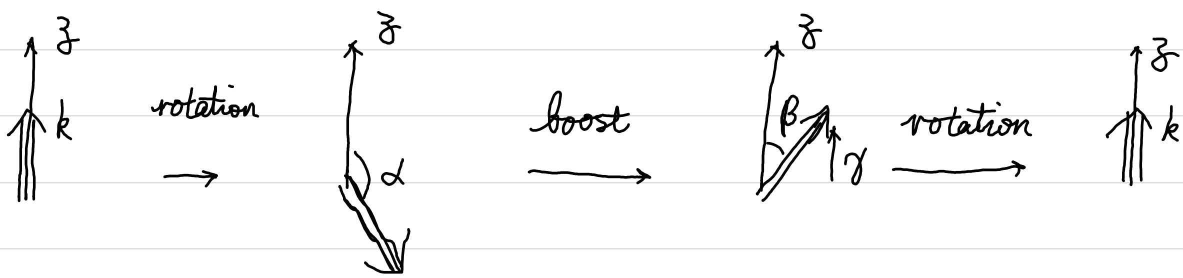
$\Rightarrow E(\alpha_1, x_1, y_1)E(\alpha_2, x_2, y_2) = E(\alpha_1+\alpha_2, X, Y)$

$$X+iY = x_1+iy_1+e^{i\alpha_1}(x_2+iy_2)$$
$$= x_1+iy_1+x_2\cos\alpha_1 - y_2\sin\alpha_1 + i(x_2\sin\alpha_1+y_2\cos\alpha_1)$$

$$\therefore E(\alpha_1, x_1, y_1)E(\alpha_2, x_2, y_2) = E(\alpha_1+\alpha_2, x_1+x_2\cos\alpha_1-y_2\sin\alpha_1, y_1+x_2\sin\alpha_1+y_2\cos\alpha_1)$$

∴ 所有的2维转动和平移变换, E_2, or $ISO(2)$
2维欧几里得群.

方法二:



$$(1,0,0,1)\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&C_\alpha&S_\alpha\\0&0&-S_\alpha&C_\alpha\end{pmatrix}\begin{pmatrix}\gamma&0&0&-\sqrt{\gamma^2-1}\\0&1&0&0\\0&0&1&0\\-\sqrt{\gamma^2-1}&0&0&\gamma\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&C_\beta&-S_\beta\\0&0&S_\beta&C_\beta\end{pmatrix}$$

$$= (\gamma-\sqrt{\gamma^2-1}\cos\alpha,\ 0,\ (\gamma\cos\alpha-\sqrt{\gamma^2-1})\sin\beta-\sin\alpha\cos\beta,\ \cos\beta(\gamma\cos\alpha-\sqrt{\gamma^2-1})+\sin\alpha\sin\beta)$$

$$\Rightarrow \cos\alpha = \sqrt{\frac{\gamma-1}{\gamma+1}} \qquad \sin\alpha = \sqrt{\frac{2}{\gamma+1}}$$

$$\therefore \gamma\cos\alpha-\sqrt{\gamma^2-1} = \frac{\gamma}{\gamma+1}\sqrt{\gamma^2-1}-\sqrt{\gamma^2-1} = -\sqrt{\frac{\gamma-1}{\gamma+1}} = -\cos\alpha.$$

$$\therefore -\cos\alpha\sin\beta - \sin\alpha\cos\beta = 0 \Rightarrow \alpha+\beta = \pi$$

即 $(1,0,0,1)\begin{pmatrix} \gamma & 0 & \sqrt{2\gamma-2} & \gamma-1 \\ 0 & 1 & 0 & 0 \\ \sqrt{2\gamma-2} & 0 & 1 & \sqrt{2\gamma-2} \\ 1-\gamma & 0 & -\sqrt{2\gamma-2} & 2-\gamma \end{pmatrix}$

$= (1,0,0,1)$

可以验证.
$$\begin{pmatrix} \gamma & 0 & \sqrt{2\gamma-2} & 1-\gamma \\ 0 & 1 & 0 & 0 \\ \sqrt{2\gamma-2} & 0 & 1 & -\sqrt{2\gamma-2} \\ \gamma-1 & 0 & \sqrt{2\gamma-2} & 2-\gamma \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$\times\begin{pmatrix} \gamma & 0 & \sqrt{2\gamma-2} & \gamma-1 \\ 0 & 1 & 0 & 0 \\ \sqrt{2\gamma-2} & 0 & 1 & \sqrt{2\gamma-2} \\ 1-\gamma & 0 & -\sqrt{2\gamma-2} & 2-\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

这个变换，以 $\gamma=\cosh\beta$ 中的 β 为变量(快度)

生成元为 $S_x = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$　　辅以绕z的转动 R_z.

类似有 $S_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$

$\Rightarrow$ $[S_x, S_y]=0$, $[S_x, R_z]= iS_y$ $[S_y, R_z]=-iS_x$

S_x, S_y, R_z 构成 $ISO(2)$ 的平移生成元

所有动量为 k^μ 的光子态，$|k,\sigma\rangle$ 一定构成 $ISO(2)$ 的表示，并且是幺正表示，

∴ 或者 该表示空间为无穷维线性空间

$\longrightarrow$ 光子的极化"矢量"构成无穷维线性空间

或者，该表示的空间平移部分是平凡的.！

真实世界中，人们只看到光子的两种极化模式（左旋，右旋）

$$\therefore \quad U(\exp(ixS_x + iyS_y))|k,\sigma\rangle = |k,\sigma\rangle$$

$$U(\exp(i\theta R_3))|k,\sigma\rangle = e^{ih\theta}|k,\sigma\rangle \qquad SO(2), \text{ or } U(1)\text{表示}$$

h 为光子的 helicity

∵ 左旋光子（右旋光子）本身构成一个表示，可以独立于右旋光子（左旋光子）存在，

当结合了微观因果性（局域对易性，弱局域对易性）后，场论满足 PCT 对称性，左旋和右旋态必须同时出现。

∴ 一般的 $U(\Lambda,a)$

$$\begin{aligned}
U(\Lambda,a)|p,\sigma\rangle &= U(L(\Lambda p),a)\,U(L(\Lambda p)^{-1}\Lambda L(p),0)|k,\sigma\rangle \\
&= D(\Lambda p,p)_{\sigma\sigma'}\,U(L(\Lambda p),a)|k,\sigma'\rangle \\
&= e^{i(\Lambda p)_\mu a^\mu} D(\Lambda p,p)_{\sigma\sigma'}\,U(L(\Lambda p),0)|k,\sigma'\rangle \\
&= e^{i(\Lambda p)_\mu a^\mu} D(\Lambda p,p)_{\sigma\sigma'}|\Lambda p,\sigma'\rangle
\end{aligned}$$

例1：标量粒子． $D_{\sigma'\sigma}$ 为 SU(2) 或 U(1) 的平凡表示．

$$U(\Lambda, a)|p\rangle = e^{i(\Lambda p)_\mu a^\mu}|\Lambda p\rangle$$

例2：$m>0$ 的自旋 $\frac{1}{2}$ 粒子．

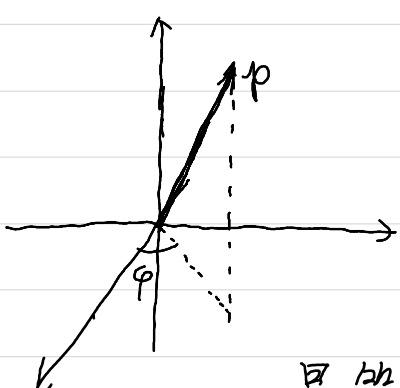
$D_{\sigma'\sigma}$ 为 SU(2) 的 $j=\frac{1}{2}$ 表示．

选 $k^\mu = (m, 0, 0, 0)$ 任意动量为

$$p^\mu = (\gamma m, \sqrt{\gamma^2-1}\, m\sin\theta\cos\varphi, \sqrt{\gamma^2-1}\, m\sin\theta\sin\varphi, \sqrt{\gamma^2-1}\, m\cos\varphi)$$

$$p^\mu = L(p)^\mu{}_\nu k^\nu$$

$$L(p) = B_z(\gamma)R_y(\theta)R_z(\varphi) = \begin{pmatrix} \gamma & 0 & 0 & \sqrt{\gamma^2-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{\gamma^2-1} & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & -\sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



$$= \begin{pmatrix} \gamma & \sqrt{\gamma^2-1}\sin\theta\cos\varphi & \sqrt{\gamma^2-1}\sin\theta\sin\varphi & \sqrt{\gamma^2-1}\cos\theta \\ 0 & \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ \sqrt{\gamma^2-1} & \gamma\sin\theta\cos\varphi & \gamma\sin\theta\sin\varphi & \gamma\cos\theta \end{pmatrix}$$

最简单的例子．如果 Λ 为绕 p 方向的转动 α

则 $L(\Lambda p)^{-1}\Lambda L(p) = L(\Lambda p)^{-1} R_z(\alpha) L(p)$

$$= R_p(\alpha)$$

$|k,\sigma\rangle$ 为 SU(2) 的 $j=\frac{1}{2}$ 表示．

$|k,\uparrow\rangle$ ，$|k,\downarrow\rangle$

$$|p,\uparrow\rangle \equiv U(L(p),0)|k,\uparrow\rangle$$

$$|p,\downarrow\rangle \equiv U(L(p),0)|k,\downarrow\rangle$$

($(p^0)^{-1/2}$ 的来源: 我们采用的归一化与Peskin不同.

Peskin ${}_n\langle \vec{p} | \vec{q} \rangle_n = \delta^3(\vec{p}-\vec{q})$, δ函数作为 distribution 应在积分下理解. $\therefore \int d^3q \, {}_n\langle \vec{p} | \vec{q} \rangle_n = 1$

然而 $\int d^3q$ 不是 Poincaré 不变的积分 故而 ${}_n\langle \vec{p} | \vec{q} \rangle_n$ 不是 Poincaré 不变的, ${}_n\langle \vec{p} | \vec{q} \rangle_n = \delta^3(\vec{p}-\vec{q})$ 不是合适的归一化.

$\because \int \frac{d^3q}{2q^0}$ 是 Lorentz 不变积分. $\therefore \langle \vec{p} | \vec{q} \rangle = 2p^0 \, \delta^3(\vec{p}-\vec{q})$

$$|p,\sigma\rangle = \sqrt{2p^0}\, |p,\sigma\rangle_{Peskin}$$

a^+, a 与 Peskin定义相同. 故 $|p,\sigma\rangle = \sqrt{2p^0}\, a^+_\sigma(p) |0\rangle$)

$$\begin{aligned} U(\Lambda,a)|p,\sigma\rangle &= U(\Lambda,a)\sqrt{2p^0}\, a^+_\sigma(p)\,|0\rangle \\ &= U(\Lambda,a)\sqrt{2p^0}\, a^+_\sigma(p)\, U(\Lambda,a)^+ |0\rangle \end{aligned}$$