微扰计算方法 → 量子场论

- 基础理论
- 实战策略
- 前沿研究

一、基础理论

1. 路径积分

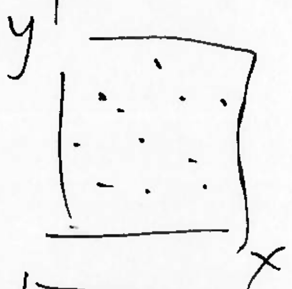
概率幅 ⇒ $\langle\Omega|T\,O(\hat{\varphi})|\Omega\rangle$

$$= \frac{\int D\varphi\, O(\varphi)\, e^{iS[\varphi]}}{\int D\varphi\, e^{iS[\varphi]}}$$ ：算？

$D\varphi \equiv \prod_a D\varphi_a$ ：$\psi, \bar{\psi}, A^\mu, \cdots$

$D\psi \equiv \prod_{x\in\mathbb{R}^4} d\psi(x)$

2. Monte Carlo 计算



有限维（积分变量）

$\int f(x) = \sum f(\vec{x})\,\Delta\vec{x}$

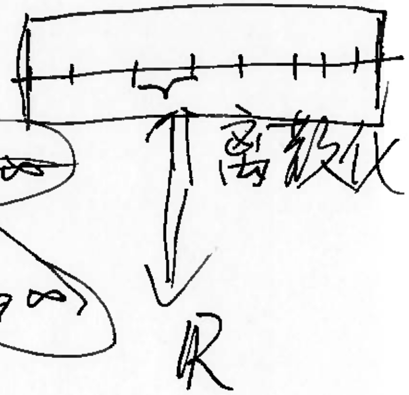
$\lim_{x\to 0} \frac{1}{x}$　$x=0$

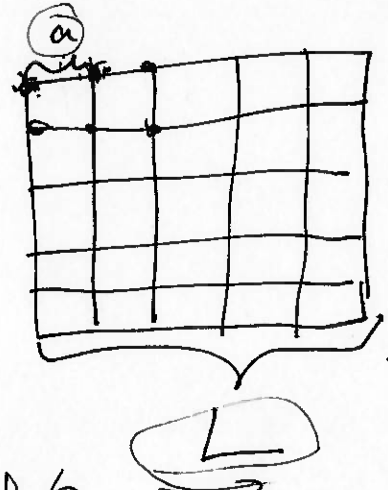
Π

$x_1 \in \{\ldots\}$, x_2　$n\in\mathbb{Z}$

$n\in[-L, L]$, $L\to\infty$

$a\to 0$, $L\to\infty$





⇒ MC积分 ⇒ 非微扰

$\begin{cases} a\to 0 \sim \frac{1}{P_{Large}} \to Q_{ph} \\ L\to\infty \end{cases}$

格点量子场论（Ⅱ）

$L \gg 1\,fm$

$a \ll \frac{1}{Q}$

$Q \sim \frac{1}{\Lambda_{QCD}}$

QCD

$\Lambda_{QCD} \sim \frac{1}{1fm} \sim 300\,MeV$

$\sim 1\,GeV$

$10\,GeV$

$1\,TeV$

$14\,TeV$

$a \to \frac{a}{2} \to 2^4$

当Q很大：格点无效
↓
小Q，低能

欧氏时空：$e^{iHt} \xrightarrow{t \to i\tau} e^{-H\tau}$（$H \to E_n$）

$F_{\mu\nu} F$

3. 微扰论 ②

EW. QCD

EW → $\alpha(\mu) \ll 1$

QCD → $\alpha_s(\mu) \xrightarrow{\text{渐近自由}}$ $\mu \gg \Lambda_{QCD}$ 时 $\alpha_s(\mu) \ll 1$

$\alpha_s \Leftarrow \frac{g_s^2}{4\pi}$，$g_s \ll 1$ 展开

$$\int_{-\infty}^{+\infty} dx\, x^{2n} e^{-ax^2} \sim a^{-n-\frac{1}{2}} \Gamma(n+\tfrac{1}{2})$$

$$\int_{-\infty}^{+\infty} dx\, x^n e^{-ax^2 - \lambda x^4}$$

若 $\lambda \ll 1$：$= \int_{-\infty}^{+\infty} dx\, x^n \left(1 - \lambda x^4 + \frac{1}{2}(\lambda x^4)^2 + \cdots\right) e^{-ax^2}$

$$\int D\phi \; e^{i\int d^4x\left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right]}$$

$\lambda \ll 1$

$$= \int D\phi \left[1 - \int \frac{\lambda}{4!} i\phi^4 d^4x + \frac{1}{2!}\int \cdots\right] e^{i\int d^4x [\phi^2]}$$

$$\int D\psi D\bar{\psi} DA^\mu \; e^{i\int d^4x \left[m\bar{\psi}\psi + \cdots + g\bar{\psi}A\psi + \cdots\right]}$$

$\Rightarrow$ { Feynman 图

- - - [illegible] }

$n \sim \frac{1}{\alpha}$

QED $\alpha \sim \frac{1}{137}$

QCD : $\alpha_s \sim 0.2$, 0.3

渐进展开：

$$\int_{-\infty}^{+\infty} dx \, x^{\cdots} \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda x^4)^n e^{-ax^2}$$

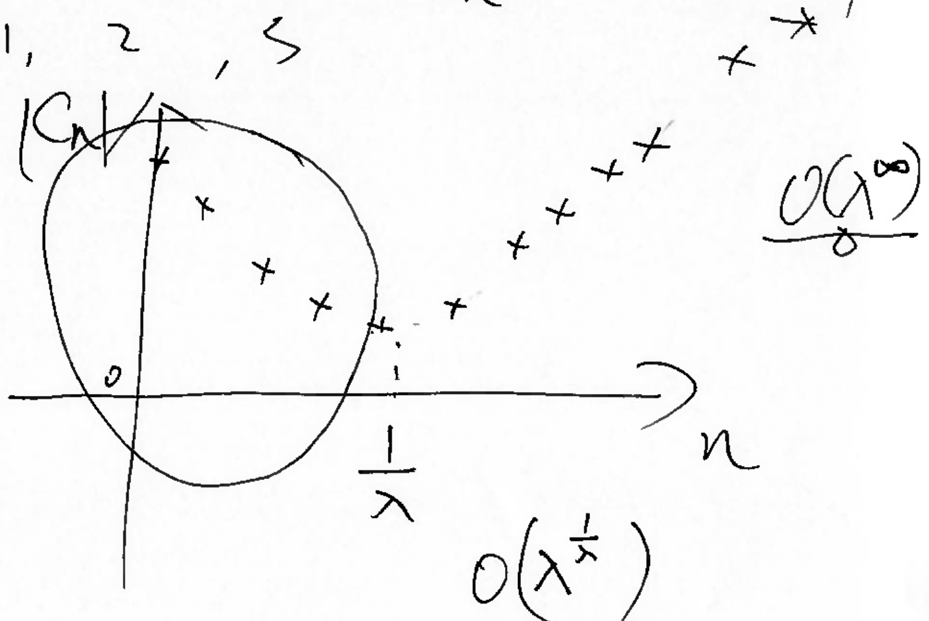
$$\sim \sum_{n=0}^{\infty} \frac{1}{n!}(-\lambda)^n \Gamma(2n)$$

$n! \to \Gamma(n)$

$$\sim \sum_n \underbrace{(-16\lambda)^n n!}_{C_n}$$

$\lambda n < 1$

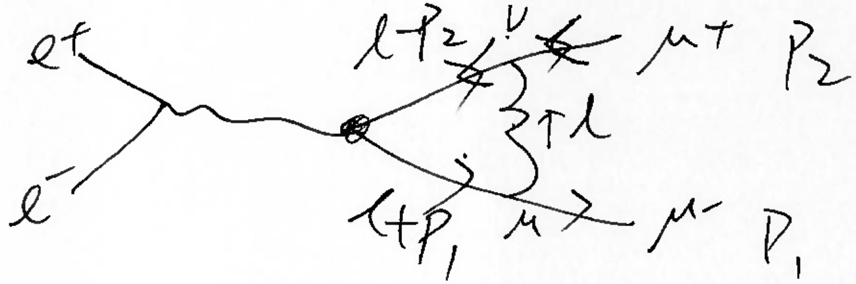
$n = 1, 2, 3$



4. Feynman图 规则

: $\frac{i(\not p+m)}{p^2-m^2+i0^+}$

: $\frac{-g^{\mu\nu}+\frac{p^\mu p^\nu}{p^2+i0^+}(1-\xi)}{p^2+i0^+}$



$$\int\frac{d^4l}{(2\pi)^4}\bar{u}(p_1)\gamma^\mu\frac{\not l+\not p_1}{(l+p_1)^2-m^2}\Gamma\frac{\not l-\not p_2}{(l-p_2)^2-m^2}\gamma^\nu v(p_2)$$

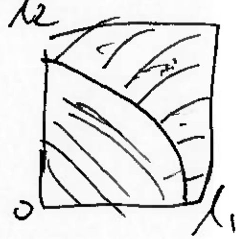
Δ

$$\times\frac{-g^{\mu\nu}+(1-\xi)\frac{l^\mu l^\nu}{l^2}}{l^2}\quad \xi=1$$

$\left\{\begin{array}{l}\boxed{l\to\infty} \Leftrightarrow a\to 0\\ \boxed{l\to 0}\end{array}\right.$

5. 紫外发散 ~~与~~

$l\to\infty$：圈→点



$$\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{l^2[(l+p_1)^2-m^2][(l-p_2)^2-m^2]}$$

$l^\mu\sim(\Lambda,\Lambda,\Lambda,\Lambda)$，$\boxed{\Lambda\to\infty}$ $\gg m$ 除

积分测度：Λ^4

被积函数：$\frac{\Lambda^2}{\Lambda^2\Lambda^2\Lambda^2}$

$\Rightarrow\frac{\Lambda^6}{\Lambda^6}\xrightarrow{\text{对数}}\infty$

$$\int_1^\infty\frac{dx}{x}\longrightarrow\lim_{\Lambda\to\infty}\int_1^\Lambda\frac{dx}{x}=\lim_{\Lambda\to\infty}\ln\Lambda$$

$\sim\frac{\Lambda}{\Lambda}\xrightarrow{\text{对数}}\infty$

对称性Δ：不同消

$$\lim_{\Lambda\to\infty}\int_1^\infty dx=\lim_{\Lambda\to\infty}\Lambda$$

UV：重整化（3步）

$\Gamma_{Ex} = A + B = \boxed{\text{finite}}$

（$A \to \infty$，$B \to \infty$）

定量化无穷大：正规化

保持对称性

无法保持的对称性：量子反常

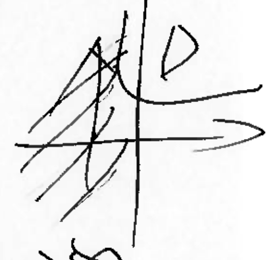
被破坏的对称性：有限重整化，恢复对称性

6. 维数正规化

保持尽可能多的对称性

$$f(D) = \int d^D l \frac{l^2}{l^2[(l+p_1)^2 - m^2][(l-p_2)^2 - m^2]}$$

$$\lim_{D \to 4} f(D)$$



$l^\mu \sim (\Lambda, \Lambda, \cdots)$ $\quad \Lambda \to \infty$

$$\Lambda^D \cdot \frac{\Lambda^2}{\Lambda^6} = \Lambda^{D-4} = \begin{cases} 0, & D < 4 \\ \text{对数}\infty, & D = 4 \\ \text{幂次}\infty, & D > 4 \end{cases}$$

$D = 4 - 2\epsilon$

利用 $D<4$ 半平面：定义 $f(D)$

解析延拓：定义 $\begin{cases} \mathrm{Re}D > 4 \text{ 的值} \\ D \to 4 \text{ 的极限} \end{cases}$

$\to \frac{A}{\epsilon} + a$

$\frac{-A}{\epsilon} + b$

$a + b$

7. 红外发散

$L\to\infty$：长波 ⇒ 小动量，$l^\mu \to 0$

$l^\mu \sim (\lambda, \lambda, \dots)$，$\lambda \ll 1$，$\lambda\to 0$

$p_1^2 = p_2^2 = m^2$

$$g_\lambda(D) = \int d^4 l^D \frac{1}{l^2\left[(l+p_1)^2 - m^2\right]\left[(l-p_2)^2-m^2\right]}$$

$(l+p_1)^2-m^2 = l^2+2l\cdot p_1 \approx 2l\cdot p_1$

$(l-p_2)^2-m^2 = l^2-2l\cdot p_2 \approx -2l\cdot p_2$

$d^4 l \sim \lambda^4 \Rightarrow \lambda^D$

$\frac{1}{\dots} \sim \frac{1}{\lambda^2 \lambda\lambda}$

$\Rightarrow \frac{\lambda^4}{\lambda^4} \xrightarrow{\text{对数}} \infty$

$$\Rightarrow \frac{\lambda^D}{\lambda^4} = \lambda^{D-4} = \begin{cases} 0, & D>4 \\ \text{对数}\,\infty, & D=4 \\ \text{幂}\,\infty, & D<4 \end{cases}$$

$\text{Re}D>4$：定义 $g_\lambda(D) \xrightarrow{\text{延拓}} \frac{1}{\epsilon}$

$J = f_\Lambda + g_\lambda + \dots$　　$D=4-2\epsilon$

f_Λ：$D<4$，ϵ_{UV}；g_λ：$D>4$，ϵ_{IR}　$\Rightarrow D = \text{?}$

共线发散（陈、高、朱）

$\epsilon_{UV}>0$，$\epsilon_{IR}<0$

$\frac{\sigma}{0}$：有限　$\frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{UV}}$，$\frac{1}{\epsilon_{IR}} - \frac{1}{\epsilon_{IR}}$

$\frac{1}{\epsilon} - \frac{1}{\epsilon}$　　$\frac{1}{\epsilon} - \frac{1}{\epsilon}$

一定相消：取 $\epsilon_{UV} = \epsilon_{IR} = \epsilon \to 0$

无标度积分 $= 0$：简化 EFT

$\int d^D l \frac{1}{(l^2)^n} = 0$

$l \to \kappa l \Rightarrow \kappa^{D-2n} \int d^D l \frac{1}{(l^2)^n}$

$n=2$：$\int_{UV} + \int_{IR} = \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \to 0$

⑦

二. 实战策略

$$\sigma = \int dps \overline{\sum} |M|^2$$

1. Feynman 振幅产生

自动化程序：FeynArts, qgraf, …

On-shell 方法 (冯)

2. 振幅简化

· $\overline{\sum}$. 自旋求和：

$$\begin{cases} \sum_{s=\pm\frac{1}{2}} u_s(p)\bar{u}_s(p) = \not{p}+m \\ \sum_{s=\pm\frac{1}{2}} v_s(p)\bar{v}_s(p) = \not{p}-m \end{cases}$$

自旋=1： $\varepsilon_\mu(p,h)\cdot p^\mu = 0$

$h=\pm 1$ 引入参考矢量：n^μ

$\varepsilon_\mu \cdot n^\mu = 0$

$$\sum_{h=\pm} \varepsilon_\mu(p,h)\,\varepsilon_\nu^*(p,h) = -g^{\mu\nu} + \frac{p^\mu n^\nu + p^\nu n^\mu}{p\cdot n} - n^2\frac{p^\mu p^\nu}{p^2}$$

$$p^\mu M_{\mu\ldots} \overset{QCD}{\underset{QED}{=}} 0 \;\rightarrow\; -g^{\mu\nu}$$

· Color 代数 f^{abc} $Tr(T^d T^e T^a \ldots)$

Dirac 代数 $Tr(\not{p}\gamma^\mu\gamma^5 \ldots)$

· 张量积分分解

$$F^{\mu\nu} = \int d^D l \, \frac{l^\mu l^\nu}{l^2 (l+p)^2} = g^{\mu\nu} A + p^\mu p^\nu B$$

→ 张量积分　　A = ?　　B = ?

$$\begin{cases} \int d^D l \, \frac{l^2}{l^2 (l+p)^2} = D A + p^2 B \\ \int d^D l \, \frac{(p\cdot l)^2}{l^2 (l+p)^2} = p^2 A + (p^2)^2 B \end{cases} \Rightarrow \begin{cases} A = \cdots \\ B = \cdots \end{cases}$$

→ 标量积分

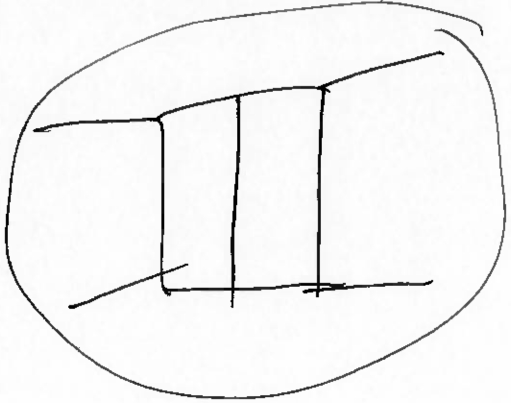
- 自动化程序：FeynCalc、FormCalc、CalcLoop, …

gitlab.com/multiloop-pku: AMFlow, Blade, CalcLoop

3. Feynman圈积分计算

路径积分 ⇒ 圈动量积分

微扰论中最本质困难（前沿部分）



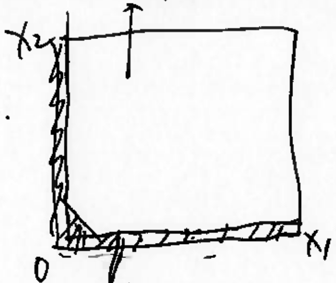
4. 相空间积分计算

$$dPS \equiv (2\pi)^D \delta^{(D)}(P_{in} - P_{out}) \prod_{i=1}^{n_{out}} \frac{d^D p_i}{(2\pi)^D} (2\pi)\delta_+(p_i^2 - m_i^2)$$

$\Downarrow$ $\frac{d^{D-1}\vec{p}_i}{2E_i (2\pi)^{D-1}}$; $\delta_+ \to p_i^0 > 0$

- Monte Carlo

红外、共线发散 / 处理. 有限, MC 积分 $\sim \frac{1}{\sqrt{N}}$



- 相空间切割: two-cutoff...
- 发散减除: Dipole. Antenna. STRIPER; ...

$\Downarrow$ 利用因子化性质 $\Rightarrow$ 变为更简单问题

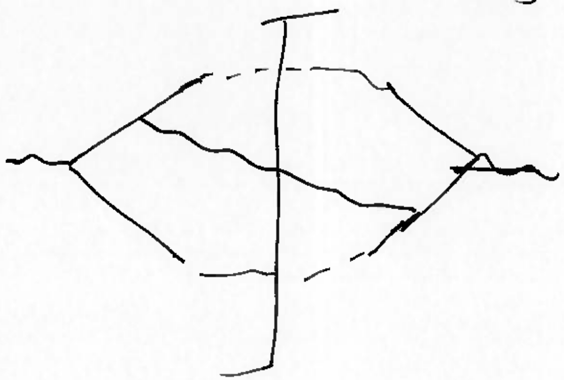
e.g.: $\int_0^1 dx \frac{f(x)}{x^{1+\epsilon}} \to \int_0^\delta + \int_\delta^1$

$\hookrightarrow \underbrace{\int_0^1 dx \frac{f(x)-f(0)}{x^1}}_{MC} + \underbrace{\int_\delta^1 dx \frac{f(0)}{x^{1+\epsilon}}}_{\text{解析计算}}$ $\ln^2\delta$ $+ O(\delta)$

- Reverse Unitarity

$$2\pi\delta(p^2 - m^2) = \lim_{\epsilon \to 0^+} \left(\frac{i}{p^2 - m^2 + i\epsilon} + \frac{-i}{p^2 - m^2 - i\epsilon} \right)$$

相空间积分 $\Rightarrow$ 圈积分



三、前沿研究：圈积分计算

$$J(\vec{\nu}) = \int \underbrace{\left(\prod_{i=1}^{L} \frac{d^D l_i}{i\pi^{D/2}}\right)}_{\equiv [dl]} \frac{D_{K+1}^{-\nu_{K+1}} \cdots D_N^{-\nu_N}}{D_1^{\nu_1} \cdots D_K^{\nu_K}}$$

$$D_i = \sum_{j,k=1}^{L} a_{ijk}\, l_j \cdot l_k + \sum_{j,k=1}^{L,E} b_{ijk}\, l_j \cdot p_k + c_i$$

p_i：为外动量，$i = 1, \cdots, E$

$$\begin{cases} \nu_1 \cdots \nu_K \in \mathbb{N} \\ -\nu_{K+1}, \cdots, -\nu_N \in \mathbb{Z} \end{cases}$$

$$N = \frac{1}{2}L(L+1) + LE$$

（$\frac{1}{2}L(L+1) \Rightarrow l_i \cdot l_j$，$LE \Rightarrow l_i \cdot p_j$）

$$\begin{cases} \text{数值：UV, IR} \\ \text{解析：函数表示} \end{cases}$$

1、Feynman 参数化表示

$$J(\vec{\nu}) = \int [dl] \prod_{i=1}^{n} \frac{1}{D_i^{\nu_i}} \qquad n \le N$$

$$= \frac{\Gamma(\nu)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} \int_0^1 \left(\prod_{i=1}^{n} dx_i\, x_i^{\nu_i - 1}\right) \delta\left(1 - \sum_{i=1}^{n} x_i\right) \int [dl] \frac{1}{\left(\sum_i x_i D_i\right)^{\nu}}$$

$$\nu \equiv \sum_{i=1}^{n} \nu_i$$

$$= \frac{\Gamma\left(\nu - \frac{LD}{2}\right)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)} (-1)^{\nu} \int_0^1 \left(\prod_{i=1}^{n} dx_i\, x_i^{\nu_i - 1}\right) \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{U^{\nu - \frac{(L+1)D}{2}}}{F^{\nu - \frac{LD}{2}}}$$

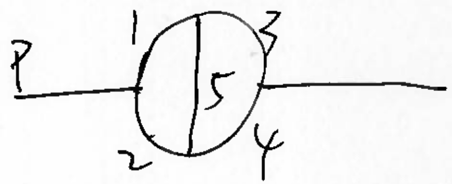
U, F: Symanzik 多项式

U ↓ 1-tree cut

$F = F_0 + U\sum x_i m_i^2$

F_0 → 2-tree cut

E.g.:



$p^2 = s$

$x_{abc\ldots} \equiv x_a + x_b + \ldots$

$U = x_{12}x_{34} + x_{1234}x_5$

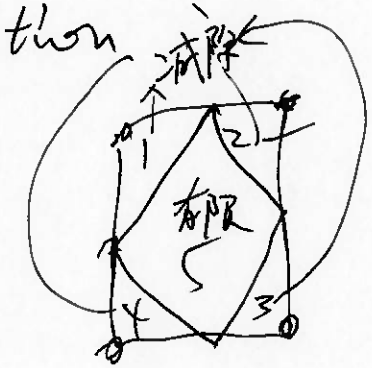
$F = -s\left(x_1x_2x_{345} + x_{125}x_3x_4 + x_1x_4x_5 + x_2x_3x_5\right)$

解析、数值

2. Monte Carlo

需先减除发散

Sector Decomposition



(py) SecDec

FIESTA

#Sector ↑ 10^6

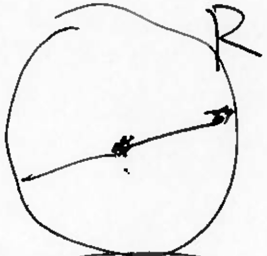
$\frac{1}{\sqrt{N}}$

3. IBP 约化

l^μ, p^μ

$$\int d^D l \frac{\partial}{\partial l^\mu}\left[\frac{1}{(l^2)^a((l+p)^2)^b}\, q^\mu\right] = 0$$

$$\oint_S d^{D-1} l \;\cdot\; R^{D-1}\cdot R^{n}$$



$n < -D+1$ → 0

$$\lim_{R\to\infty}\int dR\, d\Omega\, R^{D-1} = 0$$

$$= \int d^D l \frac{2 l^\mu \cdot p_\mu}{(l^2)^{a+1}((l+p)^2)^b} + \cdots$$

$$\Rightarrow \circ f_{a+1,b} + \circ f_{a+1,b-1} + \circ f_{a,b+1} + \cdots = 0$$

← IBP 关系

$$f_{a,b} = \int d^D l \frac{1}{(l^2-m^2)^a((l+p)^2)^b}$$

\# 线性关系：很多 → 有限线性空间

$$f_{a,b} = C^1_{ab} f_{11} + C^2_{ab} f_{10}$$ （二维）

基：主积分

一般积分 → 线性方程组（难）

Kira, Blade

4. 计算主积分

微分方程：

$$f_{a,b}^{(m^2,p^2)} \equiv \int d^D l \frac{1}{(l^2-m^2)^a [(l+p)^2]^b}$$

IBP约化

$$\begin{cases} \frac{\partial}{\partial m^2} f_{1,1} = f_{2,1} \stackrel{\downarrow}{=} A_{11} f_{11} + A_{12} f_{10} \\ \frac{\partial}{\partial m^2} f_{10} = f_{2,0} = A_{21} 0 + A_{22} f_{10} \end{cases}$$

$$\frac{\partial}{\partial p^2} \vec{f} = B_{p^2} \vec{f}$$

90s

边界条件 $f(m^2=a, p^2=b)$ ✓

· 2013 Henn：

$$\frac{\partial \vec{f}}{\partial s_i} = \epsilon A \vec{f}$$

A 不依赖 ϵ

$D=4-2\epsilon$

一类积分 GPL

· AMFlow

$f(m^2=a, p^2=b)$ ✓

$$\int d^D l \, f(l) \xlongequal{l \to l+\delta} \int d^D l \, f(l) + \delta(\,)$$

$$\frac{\partial f(\,)}{\partial a} \, a^\mu$$