parton fragmentation functions.

(部分子碎裂函数) FFs

parton distribution functions.

(部分子分布函数) PDFs

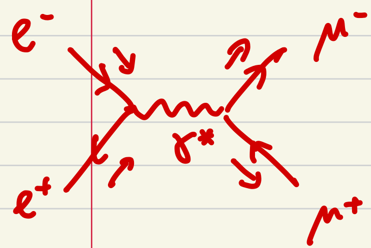
parton $\longleftrightarrow$ hadron

non-perturbative, universal objects.

Naive parton model $\longleftrightarrow$

QCD factorization theorem

1. e^+e^- collision and QCD



, unpolarized, $Q^2 \lesssim M_Z^2$, neglecting muon mass

$s = Q^2$

total cross section

$$\sigma_{\mu\mu} = \frac{1}{2s}\int d\,PS_2\, |M|^2 = \frac{4\pi}{3}\frac{\alpha^2}{Q^2}$$

One can also measure cross sections involving final hadrons.

σ_{tot}

$$= \sigma_{\mu\mu}\left(\sum_i e_{q_i}^2\right)\cdot N_c \quad , \quad i=1,\cdots,6$$

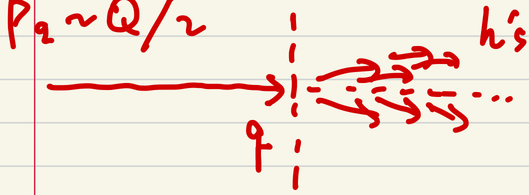
depending on Q^2 vs. $m_{q_i}^2$

Experimentally only observed "hadrons", not quarks

QCD Confinement. quarks $\longleftrightarrow$ hadrons

(hundreds)

At high energy Q^2.



a collection of collimated hadrons.

$\to$ $\boxed{\text{jet}}$

$\sum p_{hi,\parallel} \sim p_q$, $p_{hi\perp} \ll p_q$

$\boxed{\text{Theoretical description of hadron distributions}}$

non-perturbative

many-body

multi-scales

?

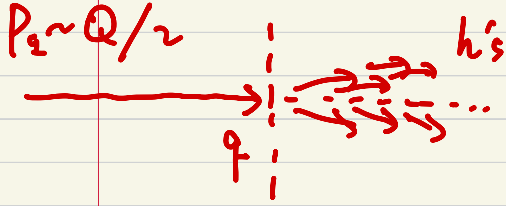
PYTHIA, HERWIG

Lund string, cluster

models

2. Collinear fragmentation function

A simplified observable / description of hadrons inside jet



definition, $z \in [0, 1]$

$$z \equiv \frac{P_{h,i,\parallel}}{P_q} \sim \frac{2E_{h,i}}{Q}$$

FFs, $$D_q^h(z, Q^2) \equiv \frac{\Delta N_h}{\Delta z}$$

momentum conservation

$$\sum_{\text{All } h} \int_0^1 z\, D_q^h(z, Q^2)\, dz = 1$$

Scaling behavior $D_q^h(z, Q^2) \sim D_q^h(z)$, ?

mean number of hadrons ?

$$\langle N_h \rangle = \int_{z_0}^1 D_q^h(z)\, dz \, ,$$ grows logarithmically

Single particle inclusive cross section

$$\frac{d\sigma}{dz}(e^+e^- \to h + X) \equiv \frac{d\sigma_h}{dz}(Q^2)$$

$$= 3\,\sigma_{\mu\mu} \sum_{i=1}^{n_f} e_{q_i}^2 \cdot \left(D_{q_i}^h(z, Q^2) + D_{\bar{q}_i}^h(z, Q^2) \right)$$

normalization

$$\sum_{\text{All } h} \int_0^1 \frac{1}{2} z \frac{d\sigma_h}{dz}(Q^2)\, dz = \sigma_{tot}$$

$$\boxed{\text{Theoretical prediction on } D_q^h(z, Q^2)}$$

non-perturbative, 1-D distribution

simple factorization <u>from QCD</u>

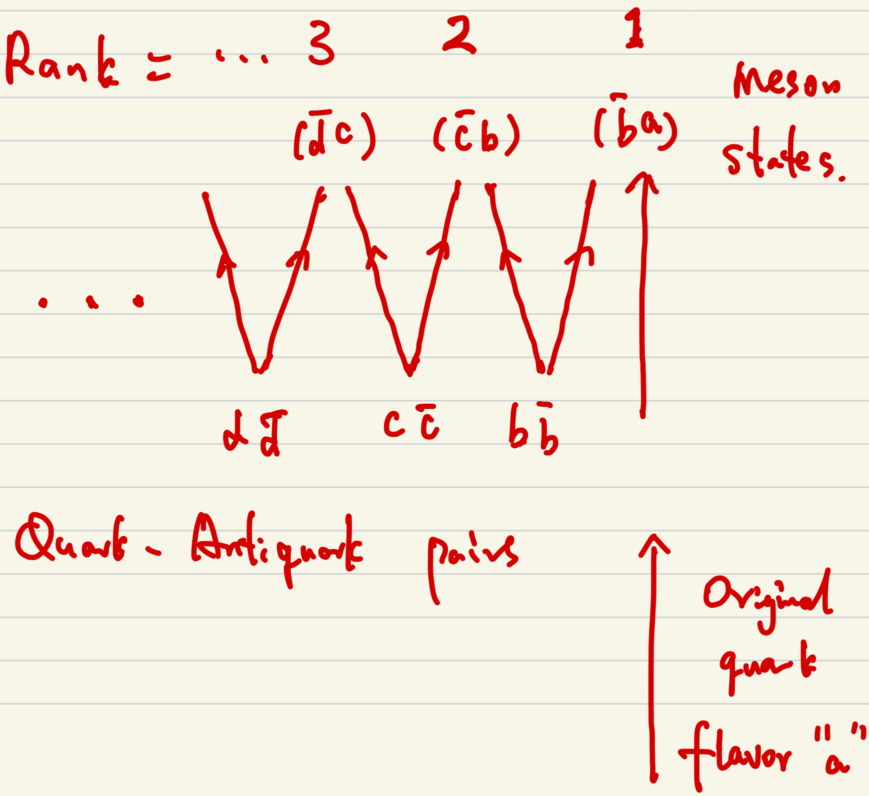
3. A model due to Field & Feynman

taking mesons $(a\bar{b})$ as example.

mostly naive parton model, qualitative feature of D,

exact scaling $D(z, Q^2) = D(z)$.

Hierarchy of Final Mesons



Recursive process controlled by

$f(\eta)\,d\eta$ = probability that the first hierarchy (rank 1) meson leaves fractional momentum $[\eta, \eta + d\eta]$ to the remaining cascade.

by definition $\int_0^1 f(\eta)\,d\eta = 1$.

Now we can derive (independent of hierarchy)

$F(z)\,dz \equiv$ probability of finding a meson with fractional momentum $[z, z+dz]$

Satisfy

$$F(z) = f(1-z) + \underbrace{\int_z^1 \frac{d\eta}{\eta} f(\eta)\, F(z/\eta)}_{\text{convolution}}$$

Solving integral Eq. (using integral transf..)

$$F(z) = f(1-z) + \int_z^1 \frac{d\eta}{\eta} f(\eta)\, F(z/\eta)$$

$$\Rightarrow \boxed{F(z) = f(1-z) + \int_z^1 \frac{d\eta}{\eta} g(\eta)\, f(1-z/\eta)}$$

with $g(\eta)$ defined as.

$$\int_0^1 \eta^r g(\eta)\, d\eta = \frac{C(r)}{1-C(r)}, \quad C(r) = \int_0^1 \eta^r f(\eta)\, d\eta$$

Consistency :

$$\int_0^1 z\, F(z)\, dz = 1 . \qquad z F(z) \underset{z\to 0}{\sim} R$$

example :

$$f(\eta) = (d+1)\, \eta^d \quad \Rightarrow z F(z) = (d+1)(1-z)^d .$$

$d=2$ qualitative description of exp. data.

4. Flavor of quark / mesons.

creation of $q\bar{q}$ pair follows a probability.

$$1 = \sum_{n=1}^{n_f} \underline{\underline{\beta_n}}, \quad \beta_c, \beta_b, \beta_t \sim 0.$$

When incident quark $\frac{Q}{2} \gg \Lambda_{QCD}$.

approximate $SU(3)/SU(2)$ flavor symmetry.

$$\beta_u = \beta_d = \beta, \quad \beta_s \sim \underset{\triangle}{\frac{1}{2}} \beta_u, \quad \text{(data)}, \quad \beta \sim 0.4$$

Now considering q fragments into $(a\bar{b})$ meson state, probability density

two probability

$$P_q^{a\bar{b}}(z) = \delta_{qa} \beta_b f(1-z) +$$

$$\int_z^1 \frac{d\eta}{\eta} f(\eta) \sum_c \beta_c P_c^{a\bar{b}}(z/\eta),$$

Solving above Eq.,

$$P_q^{a\bar b}(z) = \delta_{qa}\,\beta_b\,\underbrace{f(1-z)}_{} + \beta_a\,\beta_b\,\underbrace{\bar F(z)}_{},$$

with

$$\bar F(z) \equiv F(z) - f(1-z)$$

implications:

① meson state to meson

π^0, $u\bar u \sim \frac{1}{2}$, $d\bar d \sim \frac{1}{2}$, π^+, $u\bar d \sim 1$

π^-, $d\bar u \sim 1$, <u>transition probability</u>

$$D_q^h(z) = A_q^h\, f(1-z) + B^h\, \bar F(z)$$

thus

$$D_u^{\pi^+}(z) = \beta f(1-z) + \beta^2 \bar F(z), \quad D_u^{\pi^-}(z) = \beta^2 \bar F(z)$$

$$D_u^{\pi^0}(z) = \frac{1}{2}\beta f(1-z) + \beta^2 \bar F(z)$$

② average charge distribution inside jet

$$\langle Q_q(z) \rangle = \sum_{a,b} (e_a - e_b)\, P_q^{a\bar{b}}(z)$$

$$= (e_q - e_{\langle q \rangle})\, f(1-z)$$

with

$$e_{\langle q \rangle} = \sum_a \beta_a e_a = \frac{2}{3}\beta - \frac{1}{3}\beta - \frac{1}{6}\beta = \frac{\beta}{6}$$

integrate over z.

$$\langle Q_u \rangle = \frac{2}{3} - \frac{\beta}{6} = 0.6$$

$$\langle Q_d \rangle = \langle Q_s \rangle = -0.4$$

jet / hadrons do "remember" quark charge.

$\boxed{\text{electric charge conservation ?}}$

5. Statement of QCD factorization

Single particle inclusive cross section

$$\frac{d\sigma_h}{dz}(z, Q^2) = \sum_{i=q,\bar{q},g} \frac{d\hat{\sigma}}{dx_i}(x_i, Q^2, \mu^2)$$

$$\otimes \; D_i^h(z, \mu^2)$$

$$\equiv \sum_{i=q,\bar{q},g} \int_z^1 \frac{dx_i}{x_i} \cdot \underbrace{\frac{d\hat{\sigma}}{dx_i}(x_i, \dots)}_{\text{perturbative}} \cdot \underbrace{D_i^h(z/x_i, \mu^2)}_{\text{non-perturbative}}$$

At LO, $\dfrac{d\hat{\sigma}}{dx_i} \sim \delta(1-x_i)$,

$$\frac{d\sigma_h}{dz}(z, Q^2) \sim \sum_{i=q,\bar{q}} D_i^h(z, \mu^2)$$

no Q^2 dependence!! $\Rightarrow$ higer orders in α_s!!

Preparations:

Dimensional regularization, $N = 4 + \epsilon$

Phase space, 2-body, (Center of mass energy $= Q$)

$$\int dPS_2 = \int \frac{d^{N-1} p_1}{(2\pi)^{N-1}(2E_1)} \frac{d^{N-1} p_2}{(2\pi)^{N-1}(2E_2)} \cdot$$

$$(2\pi)^N \delta^N (q - p_1 - p_2),$$

$q \equiv (Q, 0, 0, 0)$, 1, 2 being massless

If $\epsilon = 0$, $\int dPS_2 = \frac{1}{32\pi^2} \int d^2\Omega = \frac{1}{8\pi}$

In N dimensions

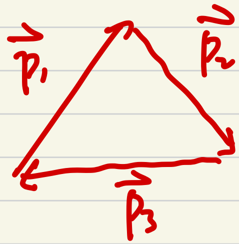
$$\int dPS_2 \ast \left(\frac{Q^2}{4\pi}\right)^{\epsilon/2} \cdot \frac{\Gamma(1+\frac{\epsilon}{2})}{\Gamma(2+\epsilon)}$$

Phase space. 3-body.

$$\int dPS_3 = \int \frac{d^{N-1}p_1}{\cdots}\,\frac{d^{N-1}p_2}{\cdots}\,\frac{d^{N-1}p_3}{\cdots}\,.$$

$$(2\pi)^N\,\delta^N(q-p_1-p_2-p_3)\,,\qquad q=(Q,0,0,0)$$

If $\epsilon = 0$,



$$E_1+E_2+E_3 = Q$$

$$x_1+x_2 > 1$$

$$\int dPS_3 = \frac{Q^2}{16\,(2\pi)^3}\cdot\int dx_1\,dx_2\,,\qquad x_i \equiv \frac{2E_i}{Q}$$

In N dimension,

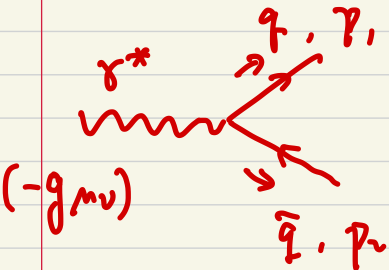
$$\int dPS_3 \;\propto\; \left(\frac{Q^2}{4\pi}\right)^{\epsilon}\cdot\frac{x_1^{\epsilon}\cdot x_2^{\epsilon}}{\Gamma(2+\epsilon)}\cdot\left(\frac{1-z^2}{4}\right)^{\epsilon/2}$$

with $z \equiv 1+\dfrac{2(1-x_1-x_2)}{x_1\,x_2}$, $= \cos\theta_{12}$

MEs for $\gamma^* \to q\bar{q}$, for simplicity

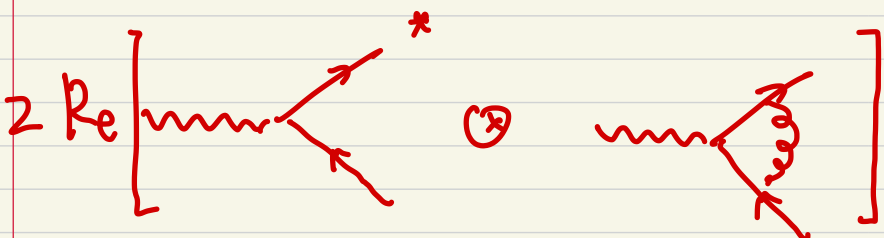
unpolarized, rotation symmetric.

LO



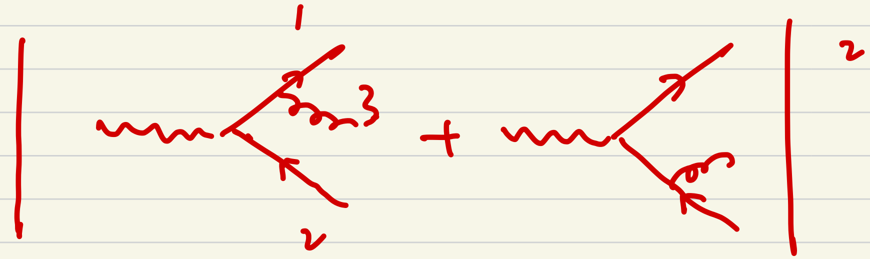
$$|M_B|^2 = 4 N_c \cdot e_q^2 \cdot e^2 \cdot Q^2 \cdot \left(1+\frac{\epsilon}{2}\right) \cdot \frac{1}{\mu^\epsilon}$$

NLO, virtual, $N = 4+\epsilon$



$$2\,\mathrm{Re}\left[M_B^* M_{1\text{-loop}}\right] = |M_B|^2 \cdot \frac{2\alpha_s}{3\pi} \cdot \left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2} \cdot \frac{1}{\Gamma\left(1+\frac{\epsilon}{2}\right)} \left\{-\frac{8}{\epsilon^2} + \frac{6}{\epsilon} + \frac{7\pi^2}{6} - 8\right\}$$

NLO, real, $N = 4+\epsilon$



$$|M_R|^2 = 32\, e_q^2\, e^2 \cdot \frac{4\pi\alpha_s}{\mu^{2\epsilon}} \cdot F(x_1, x_2).$$

with

$$F(x_1,x_2) = \left(1+\frac{\epsilon}{2}\right)^2 \cdot \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

$$+\ \epsilon\left(1+\frac{\epsilon}{2}\right) \cdot \frac{2-2x_1-2x_2+x_1x_2}{(1-x_1)(1-x_2)}$$

Soft & collinear divergences !!

6. total cross section at $\mathcal{O}(\alpha_s)$

LO cross section. (width)

$$\sigma_0 = \frac{1}{2Q} \cdot \int d\,PS_2 \, |M_B|^2 = N_c \cdot 2\, e_q^2 \cdot Q \cdot$$

$$\underline{\frac{\Gamma(2+\frac{\epsilon}{2})}{\Gamma(2+\epsilon)} \cdot \left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2}}$$

Virtual, $\mathcal{O}(\alpha_s)$

$$\sigma_V = \sigma_0 \cdot \frac{2\alpha_s}{3\pi}\left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2} \cdot \frac{1}{\Gamma(1+\frac{\epsilon}{2})}\left\{\frac{-8}{\epsilon^2} + \frac{6}{\epsilon} + \pi^2 - 8\right\}$$

real, $\mathcal{O}(\alpha_s)$

$$\sigma_R = \frac{1}{2Q} \cdot \int d\,PS_3 \, |M_R|^2$$

define

$$\frac{d\sigma_R}{dx_1\, dx_2} \equiv \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2} \frac{F(x_1, x_2)}{\Gamma(2+\frac{\epsilon}{2})} \cdot$$

$$x_1^{\epsilon}\, x_2^{\epsilon} \cdot \left(\frac{1-z^2}{4}\right)^{\epsilon/2}$$

and

$$\sigma_R = \int_0^1 dx_1 \int_{1-x_1}^1 dx_2 \cdot \frac{d\sigma_R}{dx_1\, dx_2}$$

introducing $x_2 = 1 - v x_1$, $v \in [0, 1]$,

$$\Rightarrow \int_0^1 dx_1 \int_0^1 dv \ \cdots .$$

$$\int_0^1 dy\, y^{\alpha-1} (1-y)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

arriving at

$$\sigma_R = \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \cdot \left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2} \cdot \frac{1}{\Gamma(1+\frac{\epsilon}{2})} \cdot \left\{ \frac{8}{\epsilon^2} - \frac{6}{\epsilon} - \pi^2 + \frac{57}{6} \right\}$$

NLO cross section

$$\sigma_{tot} = \sigma_0 + (\sigma_V + \sigma_R)$$

$$= \sigma_0 + \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \cdot \left(\frac{57}{6} - 8 + \mathcal{O}(\epsilon) \right)$$

$$= \sigma_0 \cdot \left(1 + \frac{\alpha_s}{\pi} \right)$$

Δ

7. Single particle inclusive cro... at $\mathcal{O}(\alpha_s)$

LO cross section, (1 flavor of quark)

$$\frac{d\sigma_h}{dz}(z, Q^2) = \sigma_0 \cdot \left(D_q^h(z, \mu^2) + D_{\bar{q}}^h(z, \mu^2) \right)$$

virtual, $(\mathcal{O}(\alpha_s))$

$$\frac{d\sigma_{h,v}}{dz}(z, Q^2) = \sigma_v \cdot \left(D_q^h(z, \mu^2) + D_{\bar{q}}^h(z, \mu^2) \right)$$

real, $(\mathcal{O}(\alpha_s))$

$$\frac{d\sigma_{h,R}}{dz}(z, Q^2) = \int_z^1 \left\{ \frac{1}{x_1} \frac{d\hat{\sigma}_R}{dx_1} \cdot D_q^h\left(\frac{z}{x_1}, \mu^2\right) + \text{''} \right\}$$

need to know $\frac{d\hat{\sigma}_R}{dx_i}$ in N dimensions.

Indeed

$$\frac{d\hat{\sigma}_R}{dx_1} = \int_{1-x_1}^{1} dx_2 \frac{d\sigma_R}{dx_1 dx_2}$$

$$= \sigma_0 \cdot \frac{2\alpha_s}{3\pi} \cdot \left(\frac{Q^2}{4\pi\mu^2}\right)^{\epsilon/2} \cdot \frac{1}{\Gamma(1+\frac{\epsilon}{2})} \cdot$$

$$\left\{ \frac{1+x_1^2}{1-x_1}\left(\frac{2}{\epsilon} + \ln\left(x_1^2(1-x_1)\right)\right) - \frac{3}{2}\frac{1}{1-x_1} - \frac{3}{2}x_1 + \frac{5}{2} \right\}$$

now taking only quark contribution as example,

$$\int_z^1 dx_1 \frac{d\hat{\sigma}_R}{dx_1} \cdot \left(\frac{D_q^h(z/x_1, \mu^2)}{x_1} - D_q^h(z, \mu^2) + D_q^h(z, \mu^2) \right)$$

thus

$$\frac{d\sigma_h}{dz}(z, Q^2) = \sigma_R \cdot D_q^h(z, \mu^2)$$

$$+ \int_z^1 dx_1 \frac{d\hat{\sigma}_R}{dx_1} \left(\frac{D_q^h(z/x_1, \mu^2)}{x_1} - D_q^h(z, \mu^2) \right)$$

$$- \left(\int_0^z dx_1 \frac{d\hat{\sigma}_R}{dx_1} \right) \cdot D_q^h(z, \mu^2) + \cdots$$

$$= \sigma_R \cdot D_q^h(z, \mu^2) + \left[\frac{d\hat{\sigma}_R}{dx_1} \right]_+ \otimes D_q^h(z, \mu^2)$$

"plus function", $[F(x)]_+$

$$\equiv \lim_{\beta \to 0} \left\{ F(x)\, \theta(1-x-\beta) - \delta(1-x-\beta) \int_0^{1-\beta} F(y)\, dy \right\}$$

satisfy $[F(x)]_+ = F(x)$, $x < 1$, $\int_0^1 [F(x)]_+ dx = 0$

$$\int_0^1 [F(x)]_+ g(x)\, dx = \int_0^1 F(x) \cdot (g(x) - g(1))\, dx$$

adding together,

$$\frac{d\sigma_h}{dz}(z,Q^2) = \sigma_0 \cdot \Big\{ \left(1+\frac{\alpha_s}{\pi}\right)\cdot\left(D_q^h(z,\mu^2)+D_{\bar{q}}^h(z,\mu^2)\right)$$

$$+ \frac{2\alpha_s}{3\pi}\cdot\frac{1}{\Gamma(1+\frac{\epsilon}{2})}\cdot\left(\left[\frac{1+x_1^2}{1-x_1}\right]_+\cdot\left(\frac{2}{\epsilon}+\ln\frac{Q^2}{\mu^2}\right)\right.$$

$$\left.+\cdots\right)\otimes D_q^h(z,\mu^2)+\cdots\Big\}$$

mass factorization, have fragmentation f.

$$D_q^h(z) = D_q^h(z,\mu^2)+\frac{\alpha_s}{2\pi}\cdot\frac{1}{\Gamma(1+\frac{\epsilon}{2})}\cdot\frac{-2}{\epsilon}\cdot$$

$$C_F\cdot\left[\frac{1+x^2}{1-x}\right]_+\otimes D_q^h(z,\mu^2)+\cdots$$

Physical prediction.

$$\frac{d\sigma_h}{dz}(z, Q^2) = \sigma_0 \cdot \left\{ \left(1+\frac{\alpha_s}{\pi}\right) \cdot \left(D_q^h(z, \mu^2) + D_{\bar{q}}^h(z, \mu^2)\right) \right.$$

$$\left. + \frac{2\alpha_s}{3\pi} \cdot \left(\left[\frac{1+x_1^2}{1-x_1}\right]_+ \otimes D_q^h(z, \mu^2) \cdot \ln \underline{\underline{\frac{Q^2}{\mu^2}}} + \ldots \right) \right\}$$

Scaling violation appears at $\mathcal{O}(\alpha_s)$!!

Reference :

Applications of Perturbative QCD

R. D. Field