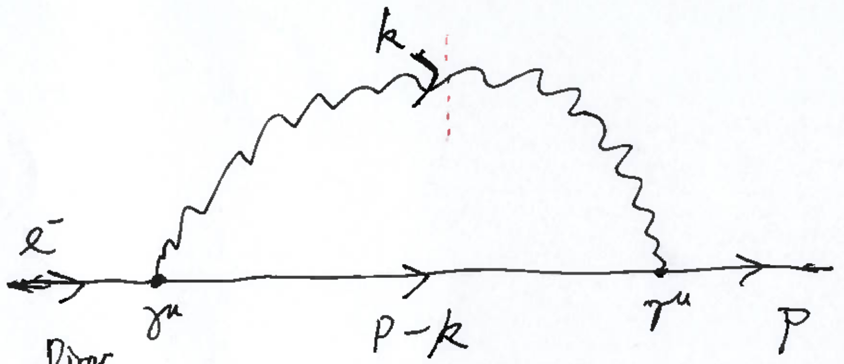
1

Introduction to Renormalization

(Long Chen)

① Oppenheimer's Observation (1930)



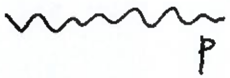
Dirac (1928)

$$\mathcal{L}_{QED} = \frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial\cdot A)^2$$

$$+ \bar{\psi}\,(i \underset{\partial^\mu - ieA^\mu}{D^\mu}\gamma_\mu - m)\,\psi$$

massless γ : 2 : A^μ

QED @ tree → Classifical ~~field theory~~



p

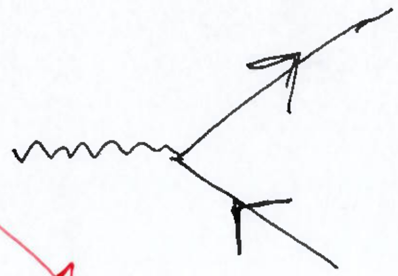
$$\frac{i\left(-g^{\mu\nu} + \cancel{(1-\xi)\frac{p^\mu p^\nu}{p^2+i\varepsilon}}\right)}{p^2+i\varepsilon}$$

Ward identity

p

$$\frac{i}{\not{p} - m + i\varepsilon}$$

$$M^\mu \cdot \varepsilon_\mu(p) \rightarrow \boxed{M^\mu p_\mu = 0}$$



$$ie\gamma^\mu$$

$$e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\varepsilon} \frac{\gamma^\mu((\not{p}-\not{k})+m)\gamma^\mu}{(p-k)^2-m^2+i\varepsilon}$$

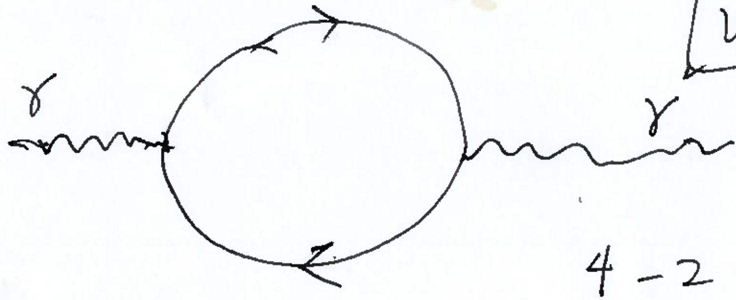
$$\not{k} = k^\mu \gamma_\mu$$

$$\Lambda \int^\infty \boxed{d^4k \frac{1}{k^2} \frac{\not{k}}{k^2}} \sim \Lambda$$

$$|k^\mu| > \Lambda$$

Vacuum polarization

Dirac 1930s



$$4 - 2 = 2 \qquad \Lambda^2$$

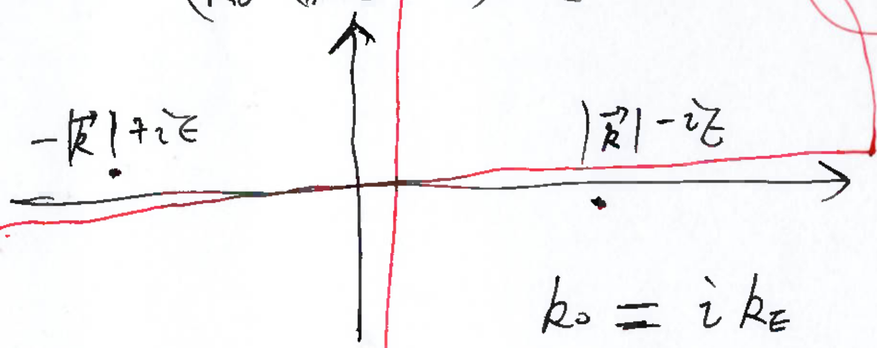
2

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+i\epsilon} \frac{\gamma^\mu((\not p - \not k)+m))\gamma^\mu}{(p-k)^2 - m^2 + i\epsilon}$$

$$\boxed{dk_0\, d^3\vec{k}}$$

$$\int dk_0 \frac{1}{k_0^2 - |\vec{k}|^2 + i\epsilon} \quad \boxed{f(k_0, \vec{k})\, d^3\vec{k}_3}$$

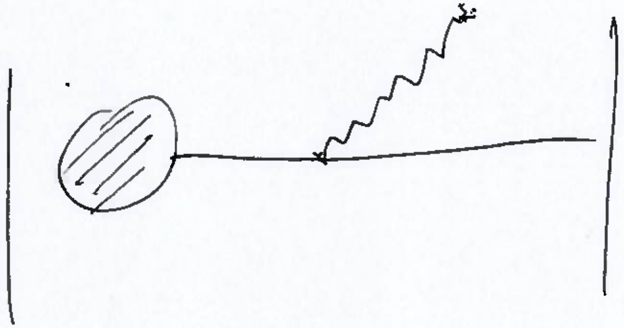
$$\rightarrow \int dk_0 \frac{1}{(k_0 - (|\vec{k}| - i\epsilon))(k_0 + (|\vec{k}| - i\epsilon))} \quad f(k_0,\vec{k})\, d^3\vec{k}$$



$$k_0 = i k_E$$

$$= \int d^3\vec{k} \frac{1}{2|\vec{k}|} \quad f(k_0 = |\vec{k}|, \vec{k})$$

Compared to the phase-space integration of a real radiation contribution, the electron is not on-shell and its energy is not bounded.



2 $d^3\vec{k}$

$$\mathcal{L}_{QED} = -\frac{1}{4} F_B^{\mu\nu} F_{\mu\nu}^B + \bar{\psi}_B (i \not D - m_B) \psi_B$$

$$\not D \to \gamma^\mu - i e_B A_B^\mu.$$

$$O(e_B, m_B) = O(e_B(e_R, m_R), m_B(e_R, m_R))$$

$\ln\Lambda$

$$= \bar{O}(e_R, m_R) \text{ has a limit } \Lambda \to \infty.$$

$e_R(\{O_{i0}\})$ $m_R(\{O_{i0}\})$

$$\boxed{O_{out} = f(\{O_{i0}\})}$$

minimal requirement

Renormalization: idea.

$$f^{\text{observable}}(x, y) \rightarrow \boxed{f^{\text{observable}}(\{O_i\})}$$

unphysical / unobservable

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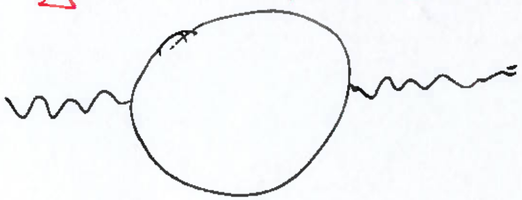
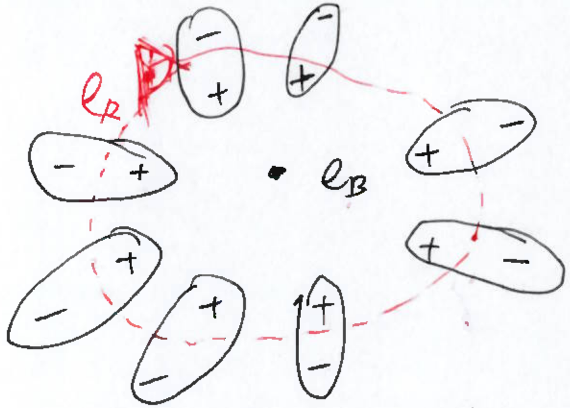
$$e_R = e_R(e_B, m_B)$$



Vacuum - polarization

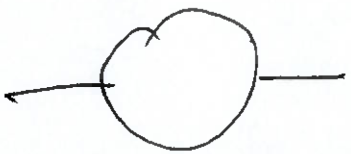


e_B
$\downarrow$
$\ln\Lambda$

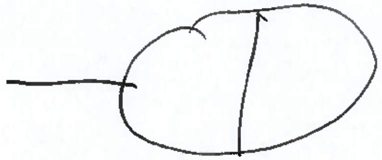
~~Both e_B and the relation $e_R(e_B, m_B)$ contains "divergence", arranged such that e_R is finite.~~

QFT @ tree $\rightarrow$ classical field theory.

~~QFT~~ Quantum effect $\leftarrow$ loops

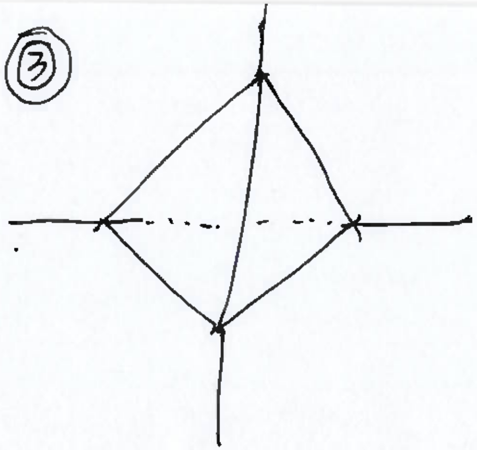


1 loop



2 loop

③

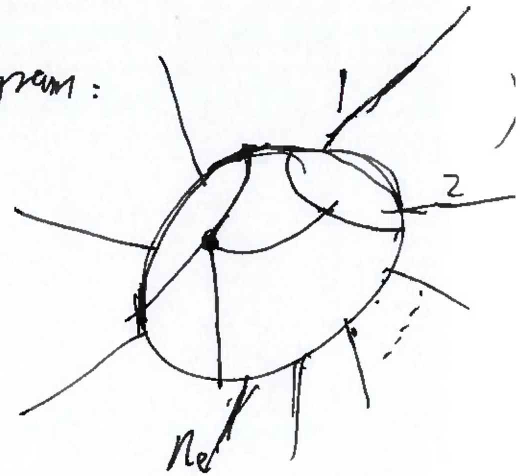


? loop

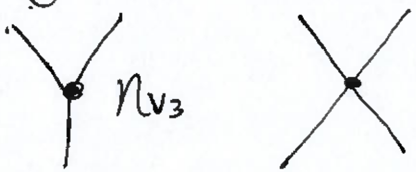
$\boxed{3}$ ✓
4

# of loops : # of loop momenta undetermined by momentum conservation

Consider a Feynman diagram:



n_e leg.
n_i internal lines
n_v vertices

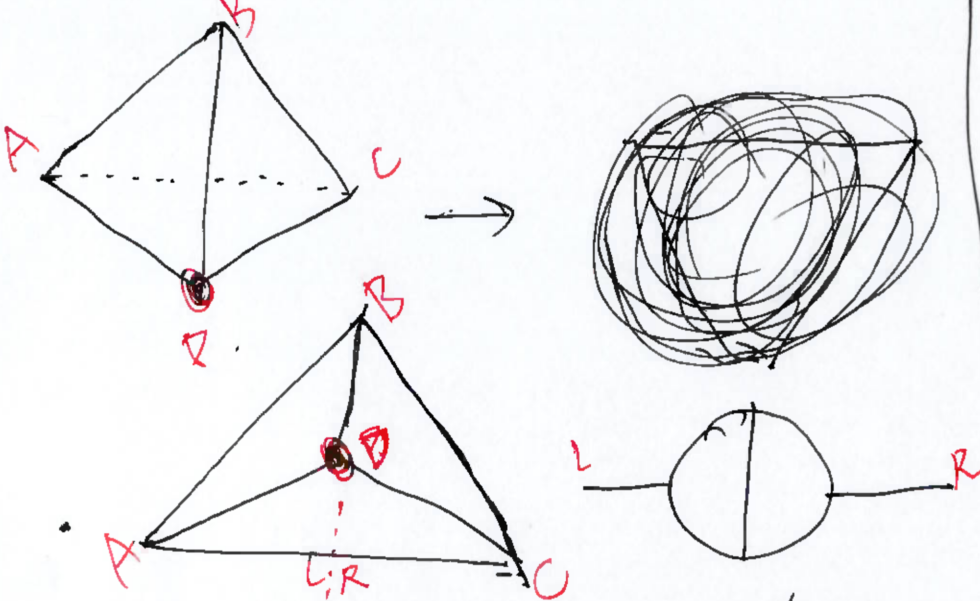


n_{v_3} n_{v_4} $\rightarrow n_v = n_{v_3} + n_{v_4}$.

# of momentum: $n_e + n_i$ p_i

# of mom.... cons... $n_v + n_e - 1$

# of loop-momenta : $\boxed{n_L = n_i - n_v + 1}$



2 loop

↓

3-loop

① $\boxed{i\text{-factor : formula}}$

(1) one i for every propagator : n_i

$$\boxed{e^{iI}} = e^{i\int \mathcal{L}\,dt}$$

$$\mathcal{L} = T - V$$

(2) $-i$ for every vertex.

$$i^{n_i}(-i)^{n_v} = \boxed{i^{n_i - n_v}}$$

$$= i^{n_L - 1}$$

(3) i from the wick-rotation per loop

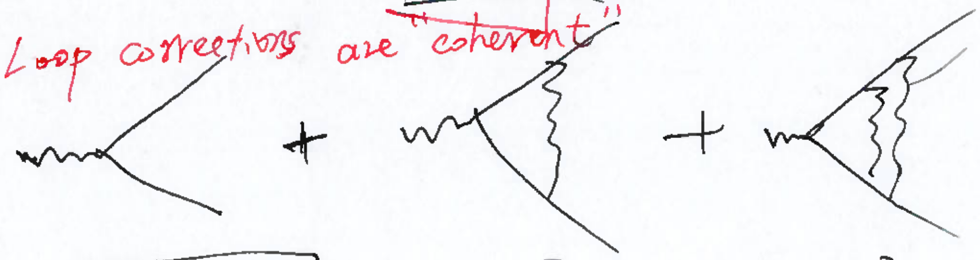
$$d^4k \to i\, d^4 k_E$$

i-factor for a Feynman diagram

$$i^{n_L - 1}\, i^{n_L} = i^{2n_L - 1}$$

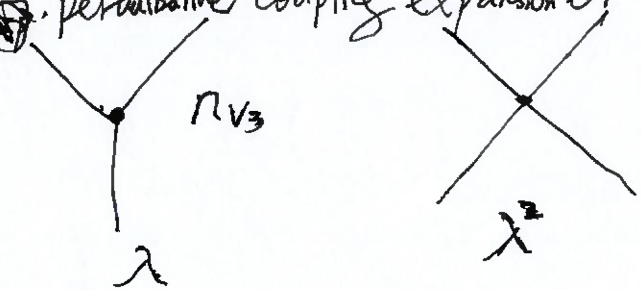
$$= \boxed{(-1)^{n_L}}\, i^{-1}.$$

Loop corrections are "coherent"



$$\left\{ \begin{array}{l} \boxed{A + iB} \to |\quad|^2 = A^2 + B^2 \quad \text{non-coherent} \\ A \pm B \to |A \pm B|^2 = |A^2 + B^2 \pm 2AB| \quad \text{co-herent} \end{array} \right.$$

② perturbative coupling expansion ↳ loop expansion



QCD

$$\boxed{\#\text{ of }\lambda} = n_{v_3} + 2n_{v_4}$$

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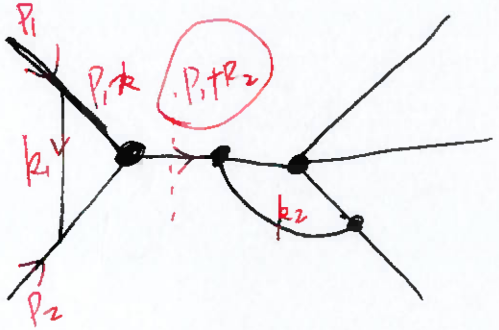


\# of colored nodes:

$$\boxed{3n_{v_3} + 4n_4 = n_e + 2n_i}$$

\# of λ: $n_{v_3} + 2n_{v_4}$

$$= \boxed{2n_L + n_e - 2}$$

(n_e is fixed)

The # of λ for a Feynman amplitude ~~Feynman diagram~~ has a one-to-one correspondance with # of loops.

④ One-particle-irreducible diagram:

$\boxed{\text{1PI}}$

Cutting one propagator can not disconnect the graph

1PI diagram has no tree propagator
bridge

A tree-propagator (bridge). has no loop momentum [redacted] follow ⊛.

A 1PI-subgraph of a Feynman graph:

⑤ Power-counting renomalizability:

① superficial [redacted] degree of UV divergence

② locality of UV divergence (Lagrange counterterm

③ Weinberg convergence theorem.

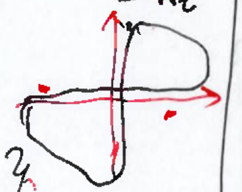
③ BPHZ theorem

④ BPHZ subtraction / renormalization

6 ① Superficial degree of UV divergence.

$$I_G = \int \prod_{i=1}^{n_L} \frac{d^{\boxed{d}} l_i}{i\pi^{d/2}} \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \cdots D_{n_i}^{\alpha_{n_i}}}$$

$d = 4-2\epsilon$



$$D_i = p_i^2 - m_i^2 + i0_+$$

$p_i = (\sum c l_j) + \{k\}$

$\lambda\phi^3$ $\lambda\phi^4$.

The power-scaling transformation for UV analysis.

$\lambda \to \infty$

$$\hat{T} \equiv \begin{cases} l_i^\mu \to \lambda l_i^\mu \\ p_i^\mu \to \lambda^0 p_i^\mu, \quad m_i \to \lambda^0 m_i, \quad e \to \lambda^0 e \end{cases}$$

superficial UV degree.

$$\hat{T}[I_G] \sim \lambda^{\boxed{\quad}} I_G$$

$$\boxed{\Delta_G \equiv d\, n_L - 2\alpha}$$

$$\alpha \equiv \sum_{i=1}^{n_i} \alpha_i$$

$$\int^{\infty}_{\Lambda < |l_i|} \prod_{i=1}^{n_L} \frac{d^d l_i}{i\pi^{d/2}} \frac{1}{\Lambda^{2\alpha_1} \Lambda^{2\alpha_2} \Lambda \cdots \Lambda^{2\alpha_{n_i}}}$$

$\propto \Lambda^{\Delta_G}$.

① $m^2 \Lambda^{\Delta_G - 2}$

Mass-dimension of I_G:

$MD[I_G]$:

power-counting transformation:

$$\begin{cases} l_i^\mu \to \lambda l_i^\mu & e \to \lambda^0 e. \\ p_i^\mu \to \lambda p_i^\mu, & m_i \to \lambda m_i \end{cases}$$

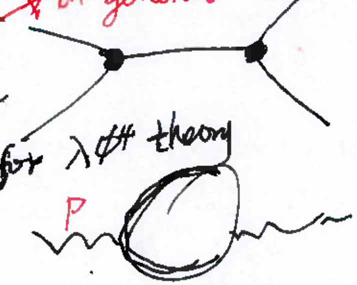
$$\boxed{MD[I_G] = d\, n_L - 2\alpha}$$

① overall dimensional consumption

② symmetry consequence

$$\Delta_G = MD[I_G] + \underset{0 \text{ for } \lambda\phi^4 \text{ theory}}{\delta}$$

in general not zero



$$\begin{cases} \Delta_G \geq 0 : \text{UV} \checkmark \\ \Delta_G < 0 \text{ . no UV} \end{cases}$$

$\propto (p \cdot p\, g^{\mu\nu} - p^\mu p^\nu)$

$* \square$

② Weinberg convergence theorem. (1960s)

("1949" Dyson. "1951" Salam):

If the $\Delta(\gamma_{1PI})$ of any 1PI-subgraph γ_{1PI} of G (including $\gamma_{1PI} = G$) is < 0, then I_G is $\boxed{\text{convergent}}$ (in the Euclidean region $(k_i + k_j)^2 < 0$ with $m_i \neq 0$) → every propagator is massive

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$$I = \int d^4x \, \mathcal{L}(x) \rightarrow \int \delta d^4 y$$

③ Locality of UV divergence:

The UV-divergent part of $I_G(\{k_i\}, m_i)$ has a $\boxed{\text{polynomial dependence}}$ on external momenta $\{k_i\}$ (若 I_G has no γ_{1PI} with $\Delta_{\gamma_{1PI}} \geq 0$).

$$\frac{\partial}{\partial k_i^{\mu}} I_G = \frac{\partial}{\partial k_i} \int dL \, f(\{D_i\})$$

$$= \int dL \sum_{i=1}^{n_i} \frac{\partial f}{\partial D_i} \frac{\partial D_i}{\partial k_i} \propto \boxed{l_j}$$

$$d x^a = a \, d x^{a-1}$$

$$\boxed{\Delta_{\frac{\partial}{\partial k} I_G} = \Delta_{I_G} - 1}$$

$$\mathcal{L}_{QED}(\psi_B, A_B, e_B, m_B) = \mathcal{L}_{QED}(\psi_R, A_R, e_R, m_R) + \boxed{\delta \mathcal{L}}$$

④ BPHZ – subtraction

$$I_G \longrightarrow I_G^{fin}$$

UV.

$$\boxed{I_G(x)} = I_G(x=0) + x \left.\frac{\partial I_G}{\partial x}\right|_{x=0} + \frac{x^2}{2!} \frac{\partial I_G}{\partial x^2} + \cdots$$

$$\boxed{I_G(x) - \sum_{n=0}^{N} \frac{x^n}{n!} \left.\frac{\partial I_G}{\partial x^n}\right|_{x=0}}$$

a systematic zero-momentum subtraction

$\| \; I_G^{fin}(x)$ is free of UV divergence

⑤ BPHZ – theorem:

The counter-term-Lagrangian $\delta\mathcal{L}$ corresponds the 1PI amplitude with $\Delta_G \geq 0$.

$$\Delta_G = MD[G] - \delta$$

$$\boxed{\langle 0 | \hat{T} \, \varphi_1(x_1) \, \varphi_2(x_2) \cdots \varphi_n(x_n) | 0 \rangle}$$

e^{iI} : I is mass-dimensionless

$$I = \int d^4x \, \mathcal{L}(x)$$

$MD[p] = 1$. $\quad e^{ipx}$

$MD[x] = -1$.

$MD\left[\frac{\partial}{\partial p}\right] = -1 \qquad MD\left[\frac{\partial}{\partial x}\right] = +1$

$MD[\mathcal{L}(x)] = 4 \to d = 4 - 2\epsilon$

$$\mathcal{L}_{QED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i(\partial^\mu - ieA^\mu)\gamma_\mu)\psi + m\bar{\psi}\psi$$

$2 + 2MD[A] = d \Rightarrow \boxed{MD[A] = \frac{d-2}{2}}$

$MD[\psi] = \frac{d-1}{2}$; $\frac{3}{2}$

$\boxed{\bar{\psi} e A \psi} \to d - \frac{d-2}{2} - (d-1) = \frac{2d - d + 2 - 2d + 2}{2} = \frac{4-d}{2}$

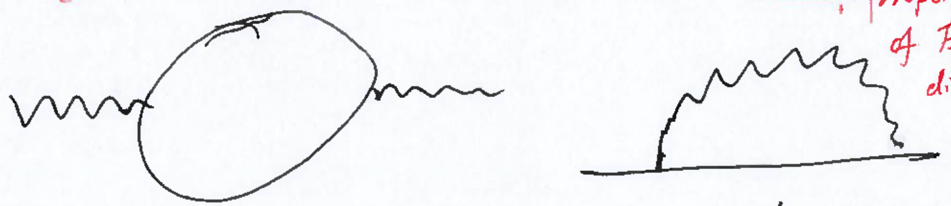
$MD[e] = \epsilon$

$MD[d^d x] = -d$

$\boxed{QED}$. The superficial UV degree of 1PI-Green functions in the momentum space ☆

$$\boxed{\Delta_G = 4 - n_A - 3 n_{\psi\bar{\psi}}}$$

following from the topological properties of Feynman diagrams



$\Delta_G \geq 0$ has the follow solutions:

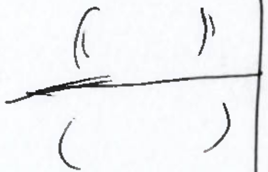
(One can also derive this using the functional method)

(1) $n_{\psi\bar{\psi}} = 0$; $n_A \begin{cases} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{cases}$

(2) $n_{\bar{\psi}\psi} = 1$:

$n_A = \begin{cases} 0 \\ 1 \end{cases}$

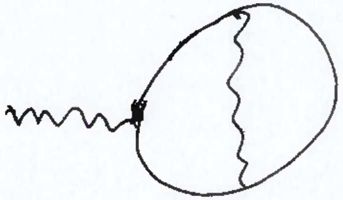
① $\boxed{n_{\psi\bar\psi}=0}$, $n_A=0$.



Itzykson - Zuber (1980)

② $n_{\psi\bar\psi}=0$, $n_A=1$.

$\langle 0|\hat{T}A^{\mu}(x)|0\rangle$



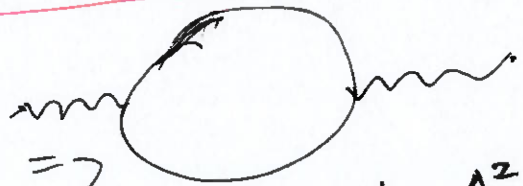
(1) Lorentz invariance of the vacuum state

(2) special case of the Furry theorem

$n_{\psi\bar\psi}=0$, $n_A=2$

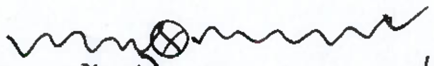
1

③ $\Delta_G = 4-2-0=2$



naively Λ^2

But only logarithmic $(g^{\mu\nu}p\cdot p-p^{\mu}p^{\nu})\,C$ owing to transversality



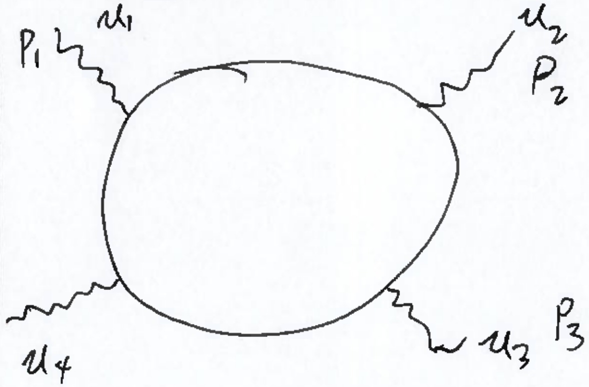
④ $n_{\psi\bar\psi}=0$, $n_A=3$

$\Delta_G=1$

Furry theorem

$\langle 0|\hat{T}A^{\mu_1}(x_1)A^{\mu_2}(x_2)A^{\mu_3}(x_3)|0\rangle$

⑤ $n_{\psi\bar\psi}=0$, $n_A=4$



$\Delta_G=0$.

$$A^{\mu_1\mu_2\mu_3\mu_4} = C_1\left(\boxed{g^{\mu_1\mu_2}g^{\mu_3\mu_4}+\cdots}\right) + C_2\left(\cdot\{p^{\mu}\}\right)$$

$\{p_1, p_2, p_3\}$

$g^{\mu\nu}$

$\Delta_{C_1}=0$

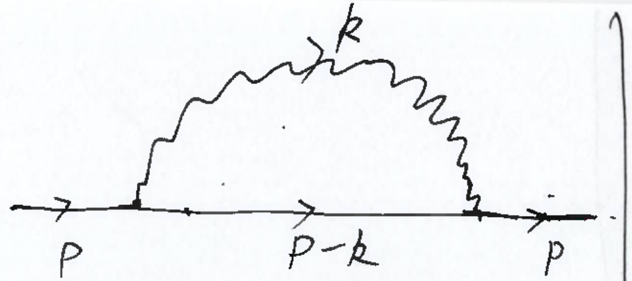
$\Delta_{C_2}=0-1<0$

Ward Identity

$$\boxed{p_1^{\mu_1}A_{\mu_1\mu_2\mu_3\mu_4}=0}$$

It is UV finite.

⑥ $\boxed{n_{\psi\bar\psi} = 1 \quad n_A = 0}$ ②



$\Delta_G = 1$. naively linearly divergent

$$\int d^4k \left[\frac{1}{k^2+i\varepsilon} \gamma^\mu \frac{1}{(\not p - \not k) - m + i\varepsilon} \gamma^\mu \right]$$

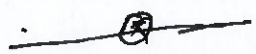
$$= \underset{\downarrow\, 0}{c_1}\, \not p + \underset{\downarrow\, 0}{c_2}\, m + \underset{\downarrow\, 1}{c_0}\, 1$$

$\boxed{c_0 = 0}$ → chirality conservation in the massless limit.

only logarithmic

$\psi_i = \psi_L$

ψ_0



if $\boxed{\text{massless}}$, $\psi_0 = \psi_L$

$$\mathcal{L}_{QED} = \bar\psi\, i D^\mu \gamma_\mu \psi - m \bar\psi \psi$$

$$= \bar\psi_L i D^\mu \gamma_\mu \psi_L + \bar\psi_R i D^\mu \gamma_\mu \psi_R$$

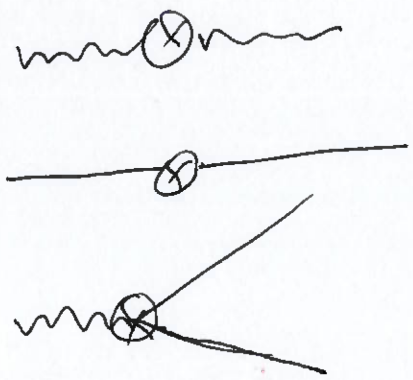
$$- m(\bar\psi_L \psi_R + \bar\psi_R \psi_L)$$

⑦

$\Delta_G = 0$.

$\boxed{n_{\psi\bar\psi} = 1 \, , \, n_A = 1}$ ③

QED: only $\boxed{3}$ counter-term vertices.



The corresponding ~~QED~~ local operators belong to the class of local composite operators appearing in the original $\mathcal{L}_{QED}$, hence the QED Lagrangian is multiplicatively renormalizable.