复习打卡

Particle cosmology

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山东大学 Summer school 讲义 — Fa Peng Huang 中山大学 黄发朋

Refs: 1. The early universe, E. Kolb and M. Turner
2. Modern cosmology, S. Dodelson and F. Schmidt

I. 天问一屈原

遂古之初，谁传道之 (principle)； 上下未形，何由考之 (observation)

0. 宇宙学基本原理：均匀各向同性 homogeneity & isotropy

$\Rightarrow$ FRW metric:

$$ds^2 = dt^2 - a^2(t)\, d\vec{r}^{\,2}$$

1. Einstein field Eq. (general relativity)

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = + 8\pi G\, T_{\mu\nu} \equiv G_{\mu\nu}$$

2. quantum field theory (QFT)

eg. Compton scattering
most important process in cosmology

$\sigma_{\text{compton}} \Rightarrow$ CMB

polarization $\Rightarrow$ primordial GW

$$\gamma + e^- \to \gamma + e^-$$

Qualitative analysis by dimensional analysis

$$\sigma \propto \begin{cases} \dfrac{\alpha^2}{m_e^2} & \text{NR limit} \\[2ex] \dfrac{\alpha^2}{s} & \text{ER limit} \end{cases}$$

3. Boltzmann eq. $\dfrac{df}{dt} = C[f]$

4. 流体力学 hydrodynamics 统计

Radiation:
- 光
  - gamma ray γ
  - 可见光
  - CMB
  - Radio
- 引力波：(new) GW

Matter:
- cosmic ray
- Large scale structure (LSS)
- neutrino
- ---

eg1. CMB $\Rightarrow$ primordial density perturbation (inflation), $T \sim 2.7\,K$, strong constraints on dark matter (DM)

eg2. Radio $\Rightarrow$ ultralight DM, 21 cm $\Rightarrow$ dark energy

eg.3 GW $\Rightarrow$ can detect all the history of universe

photon $\to \alpha \sim \frac{1}{137}$

GW $\to G \sim \frac{1}{M_{pl}^2}$

宇宙即是指南

Compton scattering:

leading-order calculation $\qquad r_c \equiv \frac{\alpha}{m_e}$

$\sigma_c \simeq$ NR: $\frac{8\pi}{3}\left(\frac{\alpha}{m_e}\right)^2 = \frac{8\pi}{3} r_c^2$ $\qquad$ good example to understand cross section

ER: $2\pi \frac{\alpha^2}{s}\ln\frac{s}{m_e^2}$ $\qquad \sigma \sim L^2$ cross section

NLO: Threshold Limit $\qquad x \equiv \frac{s-m_e^2}{m_e^2}$

$$\sigma_c = \frac{\pi\alpha^2}{m_e^2}\left[\frac{8}{3} - \frac{8}{3}x + \dots\right] + \frac{\alpha^3}{m^2}x^2\left[-\frac{16}{9}\ln x + \frac{7}{15} + \dots\right]$$

High energy limit. $\sigma_c = \frac{\pi\alpha^2}{s}\left[2\ln\frac{s}{m_e^2} + 1 + \dots\right]$

$$+\frac{\alpha^3}{s}\left[\frac{1}{3}\ln^3\frac{s}{m^2} - \frac{1}{2}\ln^2\frac{s}{m^2} + \frac{17}{4}\ln\frac{s}{m^2} - \frac{75}{8} - \frac{\pi^2}{2} + 4\zeta_3 + \dots\right]$$

CMB: e^- γ scattering

σ_c determines the mean free path of (MFP) of γ

$l_{MFP} \sim \frac{1}{n_\gamma \sigma_c}$

σ_c (NR) $\rightarrow \frac{8\pi}{3}\frac{\alpha^2}{m_e^2}$

Thomas scattering $\sigma_T = \frac{8\pi}{3} r_c^2$

frequently used in particle cosmology

II 宇宙学第一、二方程 Friedman Eq.

To describe the evolution of our Universe, we need firstly have some equations. derived from $\begin{cases} ds^2 = dt^2 - a^2(t)\, d\vec{r}^{\,2} \\ R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu} \end{cases}$

1. We can firstly guess the equations before detailed calculations from dimensional analysis:

$$R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$T_{\mu\nu} = (\rho, -p, -p, -p)$ $\quad [T_{\mu\nu}] = +4 \quad [8\pi G T_{\mu\nu}] = +2$

$[R_{\mu\nu}] = [R] = +2 \quad [a] = 0$

Euler-Lagrange eq. $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$ 2nd eqs.

assume Equation of State $p = w\rho$

$w = \frac{1}{3}$ for photon $\quad w = 0$ for ordinary matter

$w = -1$ for cosmologic constant (C.C.)

Right side $8\pi G T_{\mu\nu} \Rightarrow G\rho$ appear together

(Density is destiny)

left side: dim $= 2 \Rightarrow$ only possible $\begin{cases} \ddot{a} \\ (\dot{a})^2 \end{cases}$

$$\boxed{\text{So guess: } \begin{cases} \ddot{a} \propto G\rho \\ (\dot{a})^2 \propto G\rho \end{cases}}$$

2. Precise calculation in analogy with QFT (non-Abelian gauge field theory)

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\, d\vec{x}^{\,2}$ (1)

$g_{00} = 1 \quad g_{ii} = -a^2(t) \quad \sqrt{-g} = a^3$

connection / gauge field $\quad [A_i^a] = 1 \quad [\Gamma^\lambda_{\mu\nu}] = 1$

gauge theory $\qquad$ connection (GR)

analogy $\begin{cases} \text{dim } 1 & A_i^a & \Gamma^\mu_{\alpha\beta} = \frac{g^{\mu\lambda}}{2}(-\partial_\lambda g_{\alpha\beta} + \partial_\alpha g_{\lambda\beta} + \partial_\beta g_{\alpha\lambda}) \\ \text{dim } 2 & F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu + \dots & R^\mu_{\nu\alpha\beta} = \partial_\alpha \Gamma - \partial_\beta \Gamma + \Gamma\Gamma - \Gamma\Gamma \end{cases}$

$\Gamma^{\mu}_{\mu\nu} \propto \partial \ln\sqrt{g}$ $\Gamma \sim \partial h$

$R \sim \partial\Gamma + \Gamma\Gamma \sim \partial^2 g + \partial g \partial g$

since $g_{00} = 1$ $g_{ii} = -a^2(t)$ $g_{i0} = 0$

most metric components are zero

⇒ most connection Γ vanishes

the only non-zero Γ are $\Gamma^{0}_{ii} \propto \frac{\dot a}{a} g_{ii}$ $\Gamma^{i}_{i0} \sim \frac{\dot a}{a}$

non-zero $R_{\mu\nu}$: $R_{00} = -3\frac{\ddot a}{a}$

$R_{ij} = -g_{ij}\left[\frac{\ddot a}{a} + 2\left(\frac{\dot a}{a}\right)^2\right]$

$\Rightarrow R = -6\left[\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2\right]$

$\Rightarrow G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}$

00-component : $G_{00} = 8\pi G T_{00} = R_{00} - \frac{R}{2} g_{00}$

$\Rightarrow$ $\boxed{H^2 = \frac{8}{3}\pi G \rho \quad \text{宇宙学第一方程}}$ (I)

$p = w\rho$ $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \rho \propto a^{-3(w+1)}$

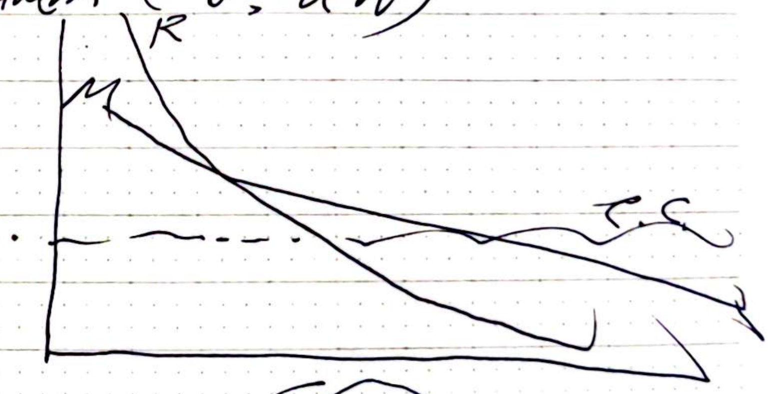
$\rho_R \propto \frac{1}{a^4}$ Radiation (γ, GW)

$\rho_m \propto \frac{1}{a^3}$

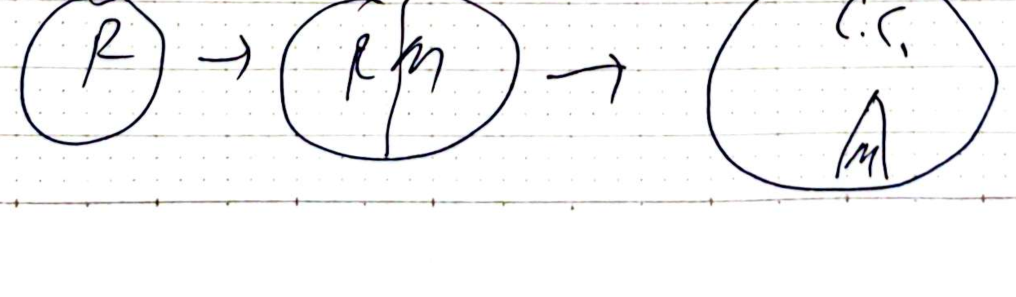
$\rho_K \propto \frac{1}{a^2}$

$\rho_{cc} \propto \frac{1}{a^0}$

⇒



from $H^2 = \frac{8\pi G}{3}\rho$



⇒ ii-component $G_{ii} = 8\pi G T_{ii} = -8\pi G p$
$= 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2$

substitute $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$ ⇒

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \quad \text{自由引力子} = ?\text{理论}} \quad \text{II}$$

对 $\rho + 3p < 0$ 加速膨胀

C.C. $p = -\rho$
dark energy

$$\boxed{\frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho \qquad (\text{II}')}$$

Miracle:
current universe
$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho & (\text{I}) \\ \frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho & \end{cases}$$
C.C. dominate
equation II'

dark energy $\Lambda g_{\mu\nu}$ C.C.

☆ summary:

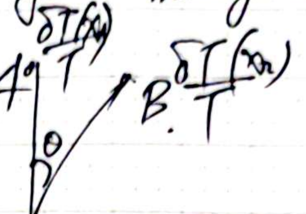
① isotropy homogeneity $ds^2 = dt^2 - a^2(t)\,d\vec{x}^2$
GR: $R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi G T_{\mu\nu}$
$\mathrm{diag}(T_{\mu\nu}) = \{\rho, -p, -p, -p\}$

1.
$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi}{3}G\rho & (\text{I}) \\ \left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G}{3}(\rho + 3p) & (\text{II}) \end{cases} \xrightarrow[p=-\rho]{\text{C.C.}} \frac{8\pi G}{3}\rho \quad (\text{II}')$$

III from inflation to CMB

— quantum fluctuation (density perturbation) beyond homogeneity background

CMB observe anisotropy:



$$C_{AB} = \left\langle \frac{\delta T(x_A)}{T} \frac{\delta T(x_B)}{T} \right\rangle \neq 0.$$

horizon problem: A B are too far away, which seems no communication. While, $C_{AB} \neq 0$

How to solve this: exponential exp. by inflation

other motivation: monopole, flatness.

More importantly: $\boxed{\text{primordial seeds of our universe}}$

our universe: structure & hierarchy

anisotropy at small scale

$\Longleftarrow$ primordial anisotropy

how to generate? quantum fluctuations.

$\Delta E \, \Delta t \geq \hbar$

But Δt is so small

How to freeze out or keep/save these quantum fluctuations? $\exists$ Exponential expansion by inflation.

quantum fluctuation firstly exit the horizon, and then re-enter the horizon to keep the primordial quantum fluctuations.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) > 0 \qquad a \sim e^{Ht}$$

ϕ scalar field $\quad \delta\phi \to$ primordial seeds

By dim. $\quad \delta\phi \sim \frac{H}{2\pi}$ (which can be explicitly calculated by QFT see Yifu Cai's talk)

$\Rightarrow \delta\rho$

density perturbation theory Seen Borteen's paper.

$$\left\langle \frac{\delta\rho(x_1)}{\rho} \frac{\delta\rho(x_2)}{\rho} \right\rangle \propto \int \frac{dk}{k} P_k (\ \ ) \qquad P_k \sim 10^{-5} \text{ CMB}$$

Estimate or Guess P_k ?

Physics $\begin{cases} \text{quantum flu.} \quad \delta\phi \sim \frac{H}{2\pi} \qquad P_k \propto \left(\frac{k}{\delta\phi}\right)^2 \\ \text{exponential expansion: } \underline{\text{c.c.-like}} \text{ energy density} \end{cases}$

$\varepsilon \propto V'(\phi)$ as small as poss

$\Rightarrow$ guess $\quad P_k \sim \dfrac{H^2}{M_{pl}^2} \dfrac{1}{\varepsilon}$

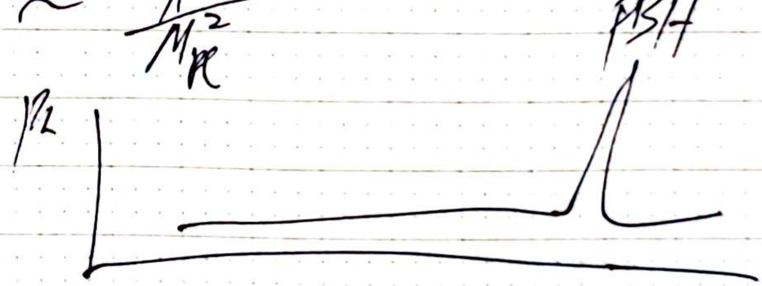
① scalar (density) $\quad P \sim \dfrac{H^2}{M_{pl}^2\, \varepsilon}$

H: quantum fluctuation
ε: c.c.-like energy density

$\varepsilon < 10^{-2}$

② tensor/metric $\quad P_{GW} \sim \dfrac{H^2}{M_{pl}^2}$

eg



$\varepsilon < 10^{-2}$ slow-roll approximation.

eg. Higgs inflation (mostly favored by Planck data and theory)

$$S = \int d^4x \sqrt{-g} \left(R M_{pl}^2 + \xi H^\dagger H R + \frac{\lambda}{4} (H^\dagger H)^2 \right)$$

Jordan frame
more details later

slow-roll inflation $S = \int d^4x \sqrt{-g}\left(\frac{1}{2} g^{\mu\nu} \partial_\mu\phi \partial_\nu\phi - V(\phi)\right)$

E.O.M $\Rightarrow$ $\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$ $\phi = \phi(t)$ $V'(\phi) = \frac{dV}{d\phi}$

$\ddot{\phi} \ll 3H\dot{\phi}$ $\Rightarrow$ $3H\dot{\phi} + V'(\phi) = 0$ $\Rightarrow$ inflation

$a(t) \sim$ exponential solution

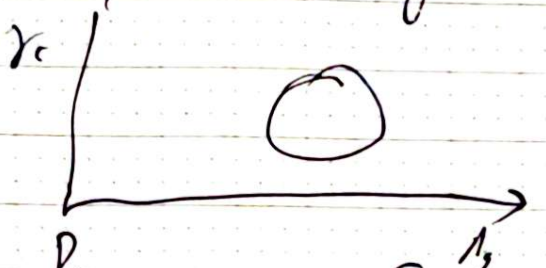
slow-roll parameter $\varepsilon \equiv M_p^2 \left(\frac{V'}{V}\right)^2$ $P_\phi \sim \frac{H^2}{M_p^2}\frac{1}{\varepsilon}$

$N_e \simeq 60$

Higgs inflation: conformal transformation

$V(H) = \frac{\frac{\lambda}{4}(H^\dagger H)^2}{\left[1 + \xi \frac{H^\dagger H}{M_p^2}\right]^2} \sim \frac{\lambda}{4\xi} M_p^4$

nearly flat $\Rightarrow$ good for inflation



$r = \frac{P_{GW}}{P_\phi} \propto \varepsilon \ll 1$

$\sim 10^{-3}$

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IV ~~thermoSS~~. large scale structure formation

1. Jeans instability / criterion.

vs Density destiny.

inflation ⇒ primordial seeds $\xrightarrow{\text{gravity}}$ Jeans $\rightarrow$ structure

$\frac{\delta\rho}{\rho} \sim 10^{-5}$ (anisotropy) v.s. pressure criterion LSS,

gravity | v.s. pressure

$\lambda > \lambda_J$	$\lambda_J \sim \sqrt{\frac{1}{G\rho}}$	$\lambda < \lambda_J$
gravity wins	dim analysis	oscillation
structure formation	$G\rho \rightarrow \sqrt{\frac{1}{G\rho}}$	BAO, baryon acoustic oscillation

1.1. 大一水平的估计：问一团气体满足凝聚成星体的条件？

- criterion: free-fall time < sound pro. time

$$t_{ff} < \frac{r_0}{c_s}$$

$$t_{ff} = \int dt = \int \frac{dr}{\frac{dr}{dt}}$$

$$\frac{m}{2}\left(\frac{dr}{dt}\right)^2 = GMm\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

$$\frac{dr}{dt} = \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

$$t_{ff} = \int dr \Big/ \sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)} \qquad r = r_0 \sin^2\theta \quad \theta \in [0, \tfrac{\pi}{2}]$$

$$= \int_0^{\frac{\pi}{2}} \frac{r_0\, 2\sin\theta\, d\sin\theta}{\sqrt{\frac{2GM}{r_0}\frac{1 - \sin^2\theta}{\sin^2\theta}}} = \int_0^{\frac{\pi}{2}} \sqrt{\frac{r_0^3}{2GM}}\, \frac{2\sin^2\theta\, d\sin\theta}{\cos\theta}$$

$$t_{ff} < r_0/c_s \Rightarrow \lambda > \lambda_J = c_s\sqrt{\frac{3\pi}{32G\rho}}$$

Jean length

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Perturbation theory in Newtonian gravity

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 & \text{continuum eq.} \\ \frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla\psi - \frac{\nabla p}{\rho} & \text{Euler eq.} \\ \nabla^2\psi = 4\pi G\rho & \text{Poisson eq.} \end{cases}$$

Perturb. $x \to x_0 + x_1$ x_1 小量

counting order. x_1^2 略掉

$\rho = \rho_0 + \rho_1$ goal: 得到 ρ_1 满足的 eq.!!

$p = p_0 + p_1$

$\psi = 0 + \psi_1$

$v = 0 + v_1$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \Rightarrow \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla\cdot\vec{v}_1 = 0 \quad (1)$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\nabla)\vec{v} = -\rho\nabla\psi - \nabla p.$$

$$\Rightarrow \rho_0 \frac{\partial \vec{v}_1}{\partial t} + 0 = -\rho_0\nabla\psi_1 - \nabla p_1 \quad (2)$$

$$c_s^2 = \frac{p_1}{\rho_1}$$

$$\rho_0\frac{\partial \vec{v}_1}{\partial t} = -\rho_0\nabla\psi_1 - c_s^2\nabla\rho_1 \quad (2)$$

$\nabla\cdot(2)$ $$\rho_0\frac{\partial}{\partial t}\nabla\cdot\vec{v}_1 = -\rho_0\nabla^2\psi_1 - c_s^2\nabla^2\rho_1 \quad (2)$$

$\frac{\partial}{\partial t}(1)$ $$\frac{\partial^2\rho_1}{\partial t^2} + \rho_0\frac{\partial}{\partial t}\nabla\cdot\vec{v}_1 = 0 \quad (1)$$

$(1)\to(2)$ $$\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\nabla^2\rho_1 - \rho_0\nabla^2\psi_1 = 0 \quad (4)$$

$\nabla^2\psi_1 = 4\pi G(\rho_0+\rho_1)$ cancel $\nabla^2\psi_0 = 4\pi G\rho_0$

$\nabla^2\psi_1 = 4\pi G\rho_1$ (3)

$$\Rightarrow \boxed{\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\nabla^2\rho_1 - \rho_0 4\pi G\rho_1 = 0}$$

Scanned with CamScanner

density perturbation eq.

$$\ddot{\rho}_1 - c_s^2 \nabla^2 \rho_1 - 4\pi G \rho_0 \rho_1 = 0$$

Fourier transformation $\rho_1(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \tilde{\rho}_1(k,t) e^{-ik\cdot x}$

$$\Rightarrow \ddot{\tilde{\rho}}_1(\vec{k},t) + (k^2 c_s^2 - 4\pi G \rho_0)\tilde{\rho}_1(k,t) = 0$$

$k^2 c_s^2 < 4\pi G \rho_0$ Jeans instability $\Rightarrow$ structure form

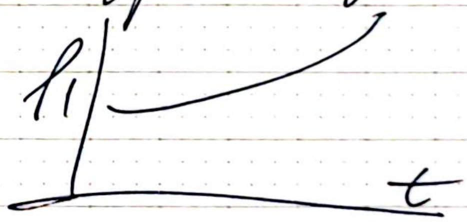
$$k_J = \sqrt{\frac{4\pi G \rho_0}{c_s^2}} \qquad \lambda_J = \frac{2\pi}{k_J}$$

$$\lambda_J \sim c_s \sqrt{\frac{\pi}{G\rho_0}}$$ Jeans length

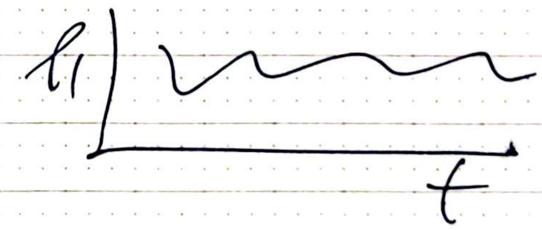
$$M_J = \frac{4\pi}{3}\rho_0 \left(\frac{\lambda_J}{2}\right)^3$$ Jeans mass.

$\lambda > \lambda_J$ ρ_1 指数增长 exponential growth

$\Rightarrow$ structure formation



$\lambda < \lambda_J$



Including cosmic expansion: $a(t)$ enters in

$\Box$ or $\nabla^2 = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij}\partial_j)$

$$\lambda_J \sim \frac{c_s}{a}\sqrt{\frac{\pi}{G\rho_0}}$$

More general: cosmological perturbation theory.

V - thermal universe

$H^2 = \frac{8\pi G}{3}\rho$: Question $\rho \propto T^n$ $n = ?$

1. From statistical method

$$\rho_i = \frac{g_i}{2\pi^2}\int \frac{E^3}{e^{E/T} \mp 1} dE = \begin{cases} g_i \frac{\pi^2}{30} T^4 & \text{Bose} \\ \frac{7}{8} g_i \frac{\pi^2}{30} T^4 & \text{Fermi} \end{cases}$$

2. Method from QFT 量子场论的方法

Free energy $e^{-\beta F} \equiv Tr(e^{-\beta H}) \equiv Z$ $\beta \equiv \frac{1}{T}$

(1) For single harmonic oscillator $\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}) = \varepsilon(\hat{n} + \frac{1}{2})$

$$Z = \sum_{n=0}^{\infty} \langle n | e^{-\beta\hat{H}} | n \rangle = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega} = \frac{e^{-\beta\varepsilon/2}}{1 - e^{-\beta\varepsilon}} \qquad \varepsilon \equiv \hbar\omega$$

$$F = -T\ln Z = \frac{\varepsilon}{2} + T\ln(1 - e^{-\beta\varepsilon})$$

(2) scalar field theory: collection of oscillators.

$$\varepsilon \to \varepsilon_k = \sqrt{k^2 + m^2}$$

$$Z_\phi = \prod_k Z_k \Rightarrow \ln Z_\phi = \sum_k \ln Z_k$$

$$F_\phi = -T\ln Z_\phi = -\sum_k T\ln Z = \sum_k F_k$$

$$F_k = \frac{\varepsilon_k}{2} + T\ln(1 - e^{-\beta\varepsilon_k})$$

$$F_\phi = \sum_k \left[\frac{\varepsilon_k}{2} + T\ln(1 - e^{-\beta\varepsilon_k})\right]$$

Define free energy density $f_\phi = \frac{F_\phi}{V}$

$V \to \infty$ 时 $f_\phi = \lim_{V\to\infty} \frac{1}{V}\sum_k \left[\frac{\varepsilon_k}{2} + T\ln(1 - e^{-\beta\varepsilon_k})\right]$

$$= \int_k \left[\frac{\varepsilon_k}{2} + T\ln(1 - e^{-\beta\varepsilon_k})\right]$$

$$f_\phi = V_0 + T\int \frac{d^3k}{(2\pi)^3}\left[\ln(1 - e^{-\beta\varepsilon_k})\right]$$

$$V_0 = \int \frac{d^3k}{(2\pi)^3}\frac{\varepsilon_k}{2} \quad \frac{\text{UV div.}}{\text{contour integral}} \to \text{Coleman Weinberg potential}$$

$\varepsilon_k = \sqrt{k^2 + m^2}$

$$f_\phi \equiv f_\phi^T + V_0 \qquad f_\phi^T \xrightarrow[\text{expansion}]{\text{high-temperature}} -\frac{\pi^2}{90}T^4 + \frac{m^2 T^2}{24} + \cdots$$

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$$p \Rightarrow -f_\phi^T = \frac{\pi^2}{90}T^4$$

for radiation $p = \frac{\rho}{3}$ $\Rightarrow \rho = 3p = \frac{\pi^2 T^4}{30}$

Thus $\rho_{Bose} = \frac{\pi^2 T^4}{30}$ for one D.O.F

for Fermion, we use the same method but Fermion has only need to sum to eigenstate

So $f_F = -V_{0,F} - T\int\frac{d^3k}{(2\pi)^3}\ln(1+e^{-\beta E_k})$

$\equiv -V_{0,F} + f_F^T$

$$f_F^T \xrightarrow[\text{expansion}]{\text{high temperature}} -\frac{7}{8}\frac{\pi^2}{90}T^4 + \frac{m^2}{48}T^2 + \dots$$

$$\Rightarrow p = \frac{7}{8}\frac{\pi^2}{90}T^4 \Rightarrow \rho = \frac{7}{8}\frac{\pi^2}{30}T^4$$

without using finite temperature effective field theory, we can obtain ~~the~~ same result by using quantum statistics and quantum field theory.

Example on thermal universe. BBN

$H^2 = \frac{8\pi}{3}G\rho$ strongly constrain new physics

$T < 1\,\text{MeV}$ BBN $\quad m_n - m_p = 1.3\,\text{MeV} = \Delta m$

miracle when neutron freeze out?

$\Delta m \sim T_{NF}$

BBN $T < 1M \rightarrow O(10\,\text{keV})$ $\quad n + p \rightarrow$ light nuclei

mostly ^4He, small but measurable mixture of D, ^3He, ^7Li

At relevant temperature. neutrons are less abundant than protons. "extra" protons remain free and later form hydrogen atoms.

Estimate T_{NF}? $\quad \frac{n_n}{n_p} = e^{-\frac{m_n - m_p}{T_{NF}}} < 1$

$n + e \to p + \bar{\nu}_e$

$\Gamma \leq H$

$\Gamma \sim G_F^2 T^5 \qquad H \sim \sqrt{G_N T^4}$

$G_F^2 T_{NF}^5 \sim \sqrt{G_N T_{NF}^4} \Rightarrow T_{NF} \sim \frac{1}{(M_{pl}^2 G_F^2)^{1/3}}$

$T_{NF} \sim 1.4$ MeV $\sim \Delta m = m_n - m_p \sim 1.3$ MeV

Miracle! Because if $\Delta m >> T_n$: n_n would be exponentially suppressed

$n_n \propto e^{-\Delta m / T_n}$

if $\Delta m << T_n$: $n_n \sim n_p \Rightarrow$ lack hydrogen.

Precise calculation gives $T_{NF} \sim 0.75$ MeV $\frac{n_n}{n_p} \sim 0.18$

$t \sim 1.1$ s

Summary

cosmology = GR + FRW + QFT/QED + quantum statistics + hydrodynamics.

Key point: how to estimate important quantities by quantum field theory

eg1. inflation density pert $\delta\phi \sim \frac{H}{2\pi}$

eg2 Compton scattering, $\begin{cases} NR & \frac{8\pi}{3}\frac{\alpha^2}{m_e^2} \\ ER & \frac{2\pi\alpha^2}{s} \end{cases}$ $\begin{cases} \text{CMB} \\ \text{primordial} \end{cases}$

eg3. QFT $\Rightarrow \rho_B = \frac{\pi T^4}{30} \qquad \rho_F = \frac{7}{8} \times \frac{\pi T^4}{30}$

eg Without calculation, just by estimation

$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \xrightarrow{P = -\rho} \frac{8\pi G \rho}{3}$

$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \rho}{3}$