Effective Field Theories for Heavy Quark Physics

Yu-Ming Wang

Nankai University

Summer School on Precision Tests of the Standard Model and New Physics
Shandong University, August 4, 2023

There is no royal road to science [Karl Marx]

- From Peter Higgs (Nobel Laureate) [From “My Life as a Boson", 2010]:

What were the reactions there? Before the Princeton seminar I was told by Klaus Hepp, an axiomatic field theorist who had been at the first Scottish summer school, that what I what going to say must be wrong, because Goldstone's theorem had been proved rigorously using C* algebras. I did not understand C* algebras, but I did not think it was possible that they could prove that without hiding something in their axioms. At Harvard the atmosphere was very different, it was more of a dialogue with the audience than a seminar. I was told afterwards when I met Sidney Coleman again they had been looking forward to tearing apart this idiot who thought he could get round the Goldstone theorem, and they were going to have fun. But I had fun too.

There is no royal road to science [Karl Marx]

- Penguin story from Arkady Vainshtein [1999 Sakurai Prize Lecture]:

We had a hard time communicating our idea to the world. Our first publication was a short letter published on July 20, 1975 in the Letters to the Journal of Theoretical and Experimental Physics. Although an English translation of JETP Letters was available in the West we sent a more detailed version to Nuclear Physics shortly after. What happened then was a long fight for publication; we answered numerous referee reports – our paper was considered by quite a number of experts. The main obstacle for referees was to overcome their conviction about the GIM suppression $(m_c^2 - m_u^2)/m_W^2$ and to realize that there is no such suppression at distances larger than $1/m_c$. Probably our presentation was too concise for them to follow.

Eventually the paper was published in the March 1977 issue of Nuclear Physics without any revision, but only after we appealed to David Gross who was then on the editorial board. The process took more than a year and a half! We were so exhausted by this fight that we decided to send our next

There is no royal road to science [Karl Marx]

- Higgs mechanism by Sasha Migdal and Alexander Polyakov:

This article will discuss the subject which occupied most of my scientific life - strong interaction of gauge fields. My first encounter with it happened in 1964 when Sasha Migdal and myself (undergraduates at that time) rediscovered the Higgs mechanism[1]. The idea of this work was given to us by the remarkable condensed matter physicist , Anatoly Larkin. He said that in superconductors there are no massless modes , presumably because of the Coulomb interaction, and advised us to apply this to particle physics with gauge fields. So we did and I still find some non-trivial elements in this old paper. Experts in particle physics thought that our work was a complete nonsense, but because of our age we were excused. However, it delayed the publication of our paper for almost a year. Another year was taken by the English translation of the JETP. As a result our work had no influence on anyone except the authors.

Unfashionable Pursuits [Freeman John Dyson]

- The rare individualist who does not fit into the prevailing pattern:

As an example of a great mathematical physicist whose work is of crucial importance to the development of physics at the present time, I mention the name of Sophus Lie. Lie has been dead for 80 years. His great work was done in the 1870s and 1880s, but it has come to dominate the thinking of particle physicists only in the last 20 years. Lie was the first to understand and state explicitly that the principles of physics have a group-theoretical origin. He constructed almost single-handed a vast and beautiful theory of continuous groups, which he foresaw would one day serve as a foundation of physics. Now, 100 years later, every physicist who classifies particles in terms of broken and unbroken symmetries is, whether he is aware of it or not, talking the language of Sophus Lie. But in his lifetime Lie's ideas remained unfashionable, little understood by mathematicians and not at all by physicists. Felix Klein was one of the few leading mathematicians who understood and supported him.

Unfashionable Pursuits [Freeman John Dyson]

- The interpretation of quantum mechanics [John Stewart Bell]:

John S. Bell (1928–1990, right) and I at CERN in Bell's office 10 years after the neutrino experiment. We were the quasi-official theorists of that experiment. We did not do very well, all things considered, because of inexperience and ignorance. After the experiment, in 1963, we both went to SLAC, where I wrote my computer program Schoonschip and he developed his famous inequalities. We also discussed other things, even wrote a paper together that was never published. He considered his work on the fundaments of quantum mechanics as a hobby, mainly to be done in the evening, at home. He told me that he intended to do away definitely with this nonsense of hidden variables, and so he did. Later he drifted more and more into this subject, and as I consider it as some sort of foolishness not good for anything having to do with the real world, I once asked him: "Why are you doing this? Does it make the slightest difference in the calculations such as I am doing?" To which he answered: "You are right, but are you not interested and curious about the interpretation?" He was right too, up to a point. While his work became very important, as it could be verified by experiment, often in this branch of physics the discussions are on the level of finding out how many angels can dance on the point of a needle. But even so: there are interesting things there.

Unfashionable Pursuits [Freeman John Dyson]

- An excellent example from hadron physics [Bing-Song Zou]:

Fate of the last famous fading pentaquark $\Theta^+(1540)$: $1/2^+$

1997: $Z^+(1530)$ predicted by Diakonov et al., ZPA359, 305

2003: $\Theta^+(1540) \to K^+n$ claimed by LEPS, PRL91, 012002

2003: $\bar{s}$ (ud)(ud) for $\Theta(1540)$ by Jaffe&Wilczek, PRL91, 232003

2003: $\bar{s}$ ud)(ud) for $\Theta(1540)$ by Karliner&Lipkin, PLB575, 249

2004: supported by 10 expts $\to$ $\Theta(1540)$ well-established by PDG

2004: not supported by BESII, PRD70, 012004

2005: not supported by many high stats experiments

2006: removed from PDG

Note: $\Theta^+(1540)$ is not supported by hadronic molecule model & chiral quark model by Huang, Zhang, Yu, Zou, PLB586(2004)69

We live in a heroic, a unique and wonderful age [Feynman]

- Importance of the elementary particle physics [Martinus J. G. Veltman]:

The twentieth century has seen an enormous progress in physics. The fundamental physics of the first half of that century was dominated by the theory of relativity, Einstein's theory of gravitation, and the theory of quantum mechanics. The second half of the century saw the rise of elementary particle physics. In other branches of physics much progress was made also, but in a sense developments such as the discovery and theory of superconductivity are developments in width, not in depth. They do not affect in any way our understanding of the fundamental laws of Nature. No one working in low-temperature physics or statistical mechanics would presume that developments in those areas, no matter how important, would affect our understanding of quantum mechanics.

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- Introduction to QCD sum rules on the light-cone

Part I: General Information and Syllabus

General Information and Syllabus

- Email: wangyuming@nankai.edu.cn

- General textbooks:
  - L. M. Brown, *Renormalization: From Lorentz to Landau (and beyond)*, New York, USA, Springer, 1993.
  - T. Y. Cao, *Conceptual developments of 20th century field theories*, Cambridge University Press, 1997.
  - T. Y. Cao (ed.), *Conceptual foundations of quantum field theory*, Proceedings, Symposium and Workshop, Boston, USA, March 1-3, 1996, Cambridge University Press.
  - S. S. Schweber, *QED and the men who made it: Dyson, Feynman, Schwinger, and Tomonaga*, Princeton, USA: Univ. Pr, 1994.
  - G. 't Hooft, *50 years of Yang-Mills theory*, Hackensack, USA: World Scientific, 2005.

- Advanced textbooks:
  - S. Weinberg, Chapter 18 in *The Quantum Theory of Fields, Vol. 2*, Cambridge University Press, 1995
  - M. Peskin, and D. Schroeder, Chapter 12 in *An Introduction to Quantum Field Theory*, Westview Press, 1995.
  - M. Schwartz, Part V in *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2014.
  - A. Manohar, and M. Wise, *Heavy Quark Physics*, Cambridge University Press, 2007.
  - J. Collins, *Renormalization: An Introduction to Renormalization, the Renormalization Group, and the Operator-Product Expansion*, Cambridge University Press, 1984.
  - J. Collins, *Foundations of perturbative QCD*, Cambridge University Press, 2011.

General Information and Syllabus

- Review articles:
  - J. Polchinski, *Effective field theory and the Fermi surface*, hep-th/9210046.
  - S. Weinberg,*Effective Field Theory, Past and Future*, arXiv:0908.1964 [hep-th].
  - G. P. Lepage, *What is renormalization?*, hep-ph/0506330.
  - H. Georgi, *Effective field theory*, Ann. Rev. Nucl. Part. Sci. 43 (1993) 209.
  - I. Z. Rothstein, *TASI lectures on effective field theories*, hep-ph/0308266.
  - D. B. Kaplan, *Five lectures on effective field theory*, nucl-th/0510023.
  - A. Buras, *Weak Hamiltonian, CP Violation and Rare Decays*, hep-ph/9806471.
  - M. Neubert, *Heavy quark symmetry*, hep-ph/9306320.
  - T. Mannel, *Effective Field Theories in Flavor Physics*, Springer Tracts in Modern Physics 203 (2004) 1.
  - T. Becher, A. Broggio and A. Ferroglia, *Introduction to Soft-Collinear Effective Theory*, arXiv:1410.1892 [hep-ph].
  - G. T. Bodwin, E. Braaten and G. P. Lepage, *Rigorous QCD analysis of inclusive annihilation and production of heavy quarkonium*, hep-ph/9407339.
  - A. V. Manohar, *Introduction to Effective Field Theories*, arXiv:1804.05863 [hep-ph].
  - M. Neubert, *Les Houches Lectures on Renormalization Theory and Effective Field Theories*, arXiv:1901.06573 [hep-ph].
  - A. Pich, *Effective Field Theory with Nambu-Goldstone Modes*, arXiv:1804.05664 [hep-ph].
  - U. van Kolck, *Les Houches Lectures on Effective Field Theories for Nuclear and (some) Atomic Physics*, arXiv:1902.03141 [nucl-th].
  - Iain Stewart, *8.851 Effective Field Theory, Spring 2013*, MIT OpenCourseWare, https://ocw.mit.edu. License: Creative Commons BY-NC-SA.
  - Many more references can be found in Appendix B of arXiv: 1903.03622 [T. Cohen].

Prologue I: Factorization in Classical Physics

- Galileo's Leaning Tower of Pisa Experiment:

$$\mathscr{L} = \frac{1}{2} m \dot{h}^2 - m g h .$$

Symmetry of the effective Lagrangian: $h \to h + a$.
Dynamical interpretation: The force acting on the ball, $F = m g$, independent of h.

- Newton's Gravity Theory:

$$V_{\rm full}(h) = -G \frac{M m}{r} = -G \frac{M m}{R+h} .$$

  - Power expansion of the full potential energy:

$$V_{\rm eff}(h) = C_1(R)\, m\, (h/R) + C_2(R)\, m\, (h/R)^2 + \ldots .$$

The general form of the effective potential can be written without knowing $V_{\rm full}$.

  - Matching the full theory and the effective theory:

$$C_1(R) = -C_2(R) = \frac{G M}{R} , \qquad V_{\rm eff}(h) = m g h - \frac{m g}{R} h^2 + \ldots .$$

- Symmetry of the effective Lagrangian broken in the full theory.

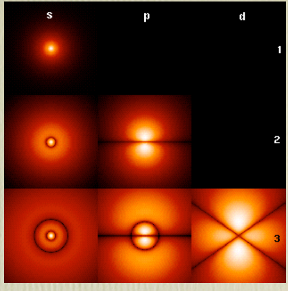
$$g(r) = \frac{G M}{r^2} , \qquad r \frac{\partial}{\partial r} g(r) = \gamma_g\, g(r) .$$

This differential equation is actually a renormalization group equation.

Prologue II: Factorization for the Hydrogen Problem

Example: Hydrogen

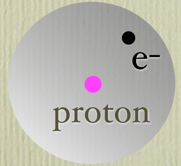
non-relativistic quantum mechanics



parameters:

mass	m_e
charges	$Q_e \, , Q_p$
coupling	$\alpha = \dfrac{1}{137}$

degrees of freedom:



scales:

$$m_p = 938\,\mathrm{MeV} \qquad \longrightarrow \infty$$

$$m_e = 0.511\,\mathrm{MeV}$$

$$p \sim m_e \alpha = 3.7\,\mathrm{keV} \qquad \sim (a_{\mathrm{Bohr}})^{-1}$$

$$E_n = -\frac{m_e \alpha^2}{2n^2} = -\frac{13.6\,\mathrm{eV}}{n^2}$$

+ corrections

Prologue II: Why the Sky is Blue?

Low energy scattering of photons from neutral atoms in their ground state $E_\gamma \ll \Delta E \sim m_e \alpha^2 \ll a_0^{-1} \sim m_e \alpha \ll M_{\text{atom}}$

- Let $v^\mu = (1,0,0,0)$, atoms are static $\mathcal{L} = \phi_v^\dagger \, i\partial^0 \, \phi_v = \phi_v^\dagger \, iv \cdot \partial \, \phi_v$

  ϕ_v is field which destroys an atom, mass dim. $[\phi_v] = 3/2$

- Interactions are constrained by gauge invariance, parity & charge conjugation. Consider

  $$\mathcal{L}^{\text{int}} = \tau_1 \; \phi_v^\dagger \, \phi_v F_{\mu\nu} F^{\mu\nu} + \tau_2 \; \phi_v^\dagger \, \phi_v v^\lambda F_{\lambda\mu} v_\sigma F^{\sigma\mu}$$

  $[F_{\mu\nu}] = 2$ so $[\tau_1] = [\tau_2] = -3$

  even number of $F_{\mu\nu}$'s

  no $\tilde{F}_{\mu\nu}$ here

  $\partial^\mu F_{\mu\nu} = 0, \; v^\mu \partial_\mu \phi_v = 0$

- Very low energy photons do not probe inside the atom, so expect cross section to depend on size of atom: $\tau_1 \sim \tau_2 \sim a_0^3$

  $[\sigma] = -2$

  $\sigma \propto |A|^2 \sim \tau_i^2$

  so $\sigma \propto E_\gamma^4 \, a_0^6$

  blue light is scattered stronger than red light

Prologue III: Factorization in Quantum Physics

- Wigner-Eckart theorem:

$$\langle \tau' j' m' | T^k_q | \tau j m \rangle = \langle j k j' | m q m' \rangle \, \langle \tau' j' || T^k || \tau j \rangle \, .$$

  Separation of geometry and dynamics.

- Generalized Wigner-Eckart theorem in Lie algebra.
  An example from SU(3): u, v and W are all 8s.

$$\langle u | W | v \rangle = \lambda_1 \, \mathrm{Tr}[\bar{u} \, W \, v] + \lambda_2 \, \mathrm{Tr}[\bar{u} \, v \, W] \, .$$

  Notice that $8^3 = 512$ matrix elements expressed in terms of only two parameters.

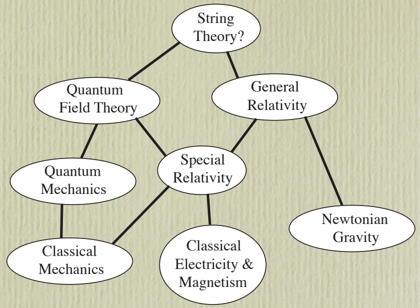
- Factorization for strong interaction physics.
  An example from $B \to \gamma \ell \nu_\ell$:

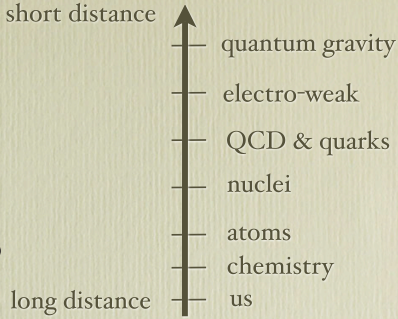
$$F_{V,\mathrm{LP}}(n \cdot p) = \frac{Q_u \, m_B}{n \cdot p} \tilde{f}_B(\mu) \, C_\perp(n \cdot p, \mu) \int_0^\infty d\omega \, \frac{\phi_B^+(\omega, \mu)}{\omega} \, J_\perp(n \cdot p, \omega, \mu) \, .$$

  - Separation of hard, hard-collinear and soft fluctuations.
  - Key input: B-meson light-cone distribution amplitude $\phi_B^+(\omega, \mu)$.

Part II: Why effective field theories?

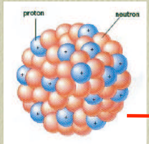
Physics compartmentalized





But, one doesn't need nuclear physics to build a boat

Generality
vs.
Precision



Dynamics at long distance do not depend on the details of what happens at short distance

In the quantum realm, $\lambda \sim \frac{1}{p}$, wavelength and momentum are related, so

Low energy interactions do not depend on the details of high energy interactions

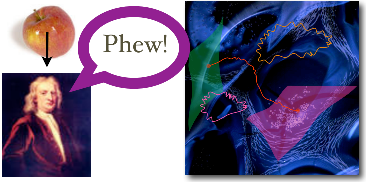
Good:

- we can focus on the relevant interactions & degrees of freedom
- calculations are simpler

Bad:

- we have to work harder to probe the interesting physics at short distances

Newton didn't need quantum gravity for projectile motion



EFT Concepts

1) The ingredients in any EFT: Relevant degrees of freedom, symmetries, scales & power counting

2) Renormalization: Meaning of parameters

3) Decoupling: Effects from Heavy Particles are suppressed

4) Matching: How we can encode dynamics of one theory into another

5) Running: Connecting physics at different momentum scales

Regularization and Renormalization

- Regularization: How do we cutoff UV infinities in loop integrals?
  Practical criteria: Computations are easier if our regulator preserve symmetries and preserves power counting by not mixing up terms of different order in the expansion.

- Renormalization: How we pick a scheme to give definite meaning to parameters of a theory.

- Dimensional regularization [Collins, 1984]:
  - Linearity: For any complex numbers a and b

    $$\int d^d p\, [a f(p) + b\, g(p)] = a \int d^d p f(p) + b \int d^d p\, g(p)\,.$$

  - Scaling: For any number s

    $$\int d^d p f(s p) = s^{-d} \int d^d p f(p)\,.$$

  - Translation invariance: For any vector q

    $$\int d^d p f(p+q) = \int d^d p f(p)\,.$$

Properties of Dimensional Regularization

- Scaleless integral vanishes:

$$\int d^d p \, (p^2)^\alpha = 0 \, .$$

- Exchange of differentiation and integration:

$$\frac{\partial}{\partial q_\mu} \int d^d p f(p,q) = \int d^d p \, \frac{\partial}{\partial q_\mu} f(p,q) \, .$$

- Integration by parts:

$$\int d^d p \, \frac{\partial f(p)}{\partial p_\mu} = 0 \, .$$

- Integration over two (or more) variables:

$$\int d^d p \int d^{d'} k \, f(p,k) = \int d^{d'} k \int d^d p \, f(p,k)$$

holds only if $d = d'$ or $f(p,k)$ is independent of $p \cdot k$.

Properties of Dimensional Regularization

- Integration over two variables:

$$\int d^d p \int d^{d'} k\, f(p^2+k^2) = \int d^{d+d'} q\, f(q^2)\,.$$

- Symmetry of the Feynman integration:

$$\begin{aligned} \int d^d p\, p_{\mu_1} \dots p_{\mu_t}\, g(p^2) &= 0\,,\ (\text{for odd } t)\,. \\ \int d^d p\, p_{\mu_1} \dots p_{\mu_t}\, g(p^2) &= T_{\mu_1 \dots \mu_t}\, A_t[g]\,,\ (\text{for even } t)\,. \end{aligned}$$

with

$$T_{\mu_1 \dots \mu_t} = \left[g_{\mu_1 \mu_2}\, g_{\mu_3 \mu_4} \dots g_{\mu_{t-1} \mu_t} + \text{all permutations of the } \mu' s \right] / t!\,.$$

and

$$A_t[g] = \frac{\Gamma(d/2)\Gamma(t/2+1/2)}{\Gamma(1/2)\Gamma(d/2+t/2)} \int d^d p\, (p^2)^{t/2}\, g(p^2)\,.$$

Parametrizations of Feynman Integrations

- Schwinger parametrization:

$$\frac{1}{(m^2-k^2)^{\alpha}} = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} dx\, x^{\alpha-1} \exp\left[-x\,(m^2-k^2)\right].$$

- Feynman parametrization:

$$\frac{1}{A^{\alpha}\,B^{\beta}} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, \frac{x^{\alpha-1}\,(1-x)^{\beta-1}}{[A\,x+B\,(1-x)]^{\alpha+\beta}}.$$

- Georgi parametrization:

$$\frac{1}{A^{\alpha}\,B^{\beta}} = \frac{2^{\beta}\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{\infty} d\lambda\, \frac{\lambda^{\beta-1}}{[A+2\,\lambda\,B]^{\alpha+\beta}}.$$

Modern renormalization/EFT paradigm

Old - fashioned renormalization paradigm
($\lesssim$ 1980-1990)

Good theories are renormalizable.

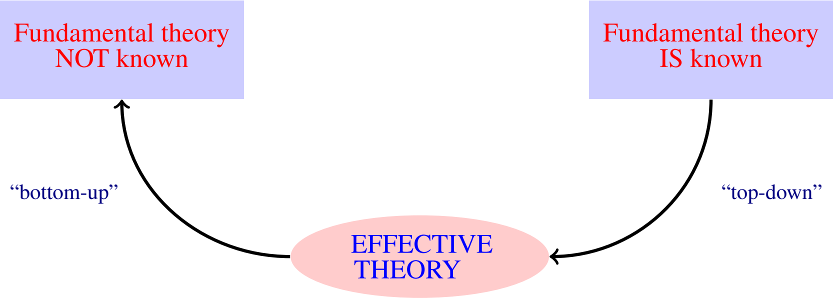
Non-renormalizable theories are BAD (unpredictive ...)

Modern renormalization / EFT paradigm ($\gtrsim$ 1980-1990)

Most theories are probably effective theories and non-renormalizable.

Super-renormalizable interactions are BAD and should be forbidden by symmetries.

Different types of EFTs



- The Standard Model itself
- Gravity (Einstein-Hilbert action)
- Higher-dimensional gauge theories
- ...

Why is this useful?

- Fundamental theory too difficult (QCD $\rightarrow$ HQET, SCET, ...)
- Emergent low-energy symmetries
- Sum large logs

$$(\lambda \ln \frac{M}{E})^n$$

EFT Principles

1) Dynamics at low E does not depend on details of dynamics at high E

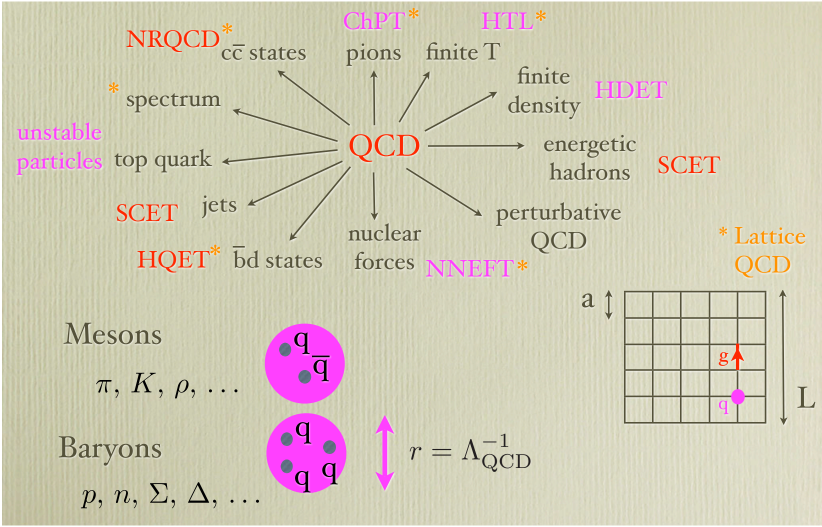
2) Build an EFT using the relevant d.o.f. and known symmetries.

3) EFT has an infinite number of operators, but only a finite number are needed for a given precision as determined by the power counting. With this precision this set closes under renormalization.

4) EFT has same infrared but different ultraviolet than the more fundamental theory.

5) Nature of high energy theory shows up as couplings and symmetries in the low energy EFT.

Effective Field Theories of QCD



HTL: Hard Thermal Loop Effective Theory; HDET: High Density Effective Theory
NNEFT: nucleon-nucleon effective field theory.

Integrating top quarks in QCD

$$\mathcal{L} = -\frac{1}{4} G^A_{\mu\nu} G^{A\mu\nu} + \sum_{f=1}^{5} \bar{\Psi}_f (i\not{D} - m_f)\Psi_f$$

$$+ \boxed{\bar{Q}(i\not{D} - m_Q)Q}$$

Q top field

Assumption: $p_i \cdot p_j \ll m_t^2$ for all external momenta

$\Rightarrow$ no external Q lines

$$A^\mu = A^{\mu(L)} + A^{\mu(H)}$$

$$\Psi_f = \Psi_f^{(L)} + \Psi_f^{(H)}$$

low- and high frequencies

$$Z[J,\eta,\bar\eta] = N \int D[A,\Psi_f,\bar\Psi_f,Q,\bar Q]\, e^{i\int d^4x\, (\mathcal{L} + J^\mu A^{(L)}_\mu + \bar\eta \Psi_f^{(L)} + \bar\Psi_f^{(L)}\eta)}$$

$$= N' \int D[A^{(L)},\Psi_f^{(L)},\bar\Psi_f^{(L)}]\, e^{i\int d^4x\, (\mathcal{L}_{eff} + J^\mu A^{(L)}_\mu + \bar\eta \Psi_f^{(L)} + \bar\Psi_f^{(L)}\eta)}$$

Matching equation for the generating functional:
only light DOFs coupled to external source fields for simplifying the matching calculations.

Green functions

- The n-point Green function:

$$G_n(x_1 \ldots x_n) = \langle 0 | \mathrm{T}\, [\phi(x_1) \ldots \phi(x_n)] | 0 \rangle \,.$$

- Generating functional of the Green function:

$$Z[J] = \int [d\phi] \exp \left\{ i \int d^4 x \, (\mathscr{L} + \phi\, J) \right\} .$$

- Relation between the Green function and the Generating functional:

$$\langle 0 | \mathrm{T}\, [\phi(x_1) \ldots \phi(x_n)] | 0 \rangle = \frac{(-i)^n}{Z(0)} \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0} ,$$

with the definition of the functional derivative (in four dimensions)

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x - y) \,.$$

Matching condition for the effective action

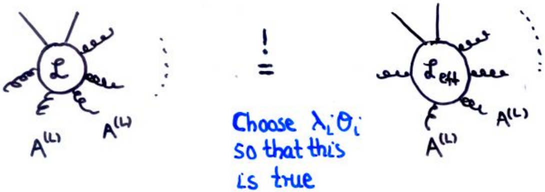
$$e^{iS_{\rm eff}[A^{(L)},\psi_f^{(L)},\bar{\psi}_f^{(L)}]} = \frac{N}{N'}\int D[A^{(H)},\psi_f^{(H)},\bar{\psi}_f^{(H)},Q,\bar{Q}]\; e^{iS[A,\psi_f,\bar{\psi}_f,Q,\bar{Q}]}$$

can be expanded in local operators

"integrating out the heavy fields and high energy modes"

In most cases $S_{\rm eff}$ can be constructed only perturbatively

Diagrammatic representation:



Consider the gluon two-point function

- The first type of diagrams:

EXACTLY reproduced for $\mathcal{L}_{eff} = -\frac{1}{4} G^2 + \sum_f \bar{\psi}_f (i\not{D} - m_f) \psi_f$

Note: renormalize $\mathcal{L}$ and $\mathcal{L}_{eff}$ in $\overline{MS}$ after using dimensional regularization

No explicit high-frequency cut-off

High frequency modes of A, ψ_f appear only in diagrams with Q-lines that contain the scale m_t

- Comment: The great simplification is due to the (soft)-light external fields, otherwise the high frequency modes of A_μ and ψ_f can contribute to these 1-loop diagrams.

Consider the gluon two-point function

- The second type of diagrams:

Expansion in q^2/m_t^2

$$= i\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\delta^{AB}\,\Pi(q^2)$$

$$\Pi(q^2) = +\frac{2T_f\alpha_s}{\pi}\int_0^1 dx\, x(1-x)\,\ln\frac{m_t^2 - x(1-x)q^2}{\mu^2}$$

$T_f = \frac{1}{2}$

$$= \frac{\alpha_s T_f}{3\pi}\ln\frac{m_t^2}{\mu^2} - \frac{\alpha_s T_f}{15\pi}\frac{q^2}{m_t^2} + \mathcal{O}\left(\frac{q^4}{m_t^4}\right)$$

$\mathcal{O}_1 = -\frac{1}{4}G^2$

$\mathcal{O}_2 = G D_\mu D^\mu G$

Consider the gluon two-point function

- The resulting effective Lagrangian:

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}\left(1 - \frac{\alpha_s T_f}{3\pi}\ln\frac{m_f^2}{\mu^2}\right) G^A_{\mu\nu} G^{A\mu\nu} + \sum_f \bar{\Psi}_f (i\not{D} - m_f)\Psi_f + \frac{\alpha_s T_f}{60\pi m_f^2} G^A_{\mu\nu} D^2 G^{A\mu\nu} + \ldots$$

$d_i = 4$ term has been modified

$d_i = 6$ non-renormalizable interaction has been generated

Rescale gluon field to recover canonical kinetic term

$$\hat{A} = \left(1 - \frac{\alpha_s T_f}{6\pi}\ln\frac{m_f^2}{\mu^2}\right) A \quad \Rightarrow \quad -\frac{1}{4}\hat{G}^2$$

but

$$g_s \bar{\Psi}_f \not{A} \Psi_f = \underbrace{\frac{g_s}{1 - \frac{\alpha_s T_f}{6\pi}\ln\frac{m_f^2}{\mu^2}}} \bar{\Psi}_f \hat{\not{A}} \Psi_f$$

$$\equiv \hat{g}_s$$

strong coupling in the effective theory

Five-Flavour QCD as an Effective Field Theory

- use $\mathcal{L}$ $\qquad \mu^2 \dfrac{d\alpha_s}{d\mu^2} = -\beta_0 \dfrac{\alpha_s^2}{4\pi} \qquad \beta_0 = 11 - \dfrac{4}{3}\cdot 6$

- near m_t relate $\qquad \hat{\alpha}_s = \dfrac{\alpha_s}{1 - \frac{\alpha_s T_f}{3\pi} \ln \frac{m_t^2}{\mu^2}} \qquad (*)$

$$\mu^2 \frac{d\hat{\alpha}_s}{d\mu^2} = \mu^2 \frac{d\alpha_s}{d\mu^2} + \frac{\alpha_s^2 T_f}{3\pi}\cdot(-1) = -\beta_0^{(5)} \frac{\hat{\alpha}_s^2}{4\pi} + O(\hat{\alpha}_s^3)$$

$$\beta_0^{(5)} = 11 - \frac{4}{3}\cdot 5$$

- far below m_t MUST use $\mathcal{L}_{eff}$
  Otherwise for $\mu \sim p \ll m_t$ get $\quad \alpha_s \ln \dfrac{m_t^2}{p^2}$
  and perturbation theory breaks down.
  In $\mathcal{L}_{eff}$ these high-energy logs are absorbed
  in $\hat{\alpha}_s(p)$ ↖

  solve renormalization group eq.
  $\mu^2 \dfrac{d\hat{\alpha}}{d\mu^2} = -\beta(\hat{\alpha})$ with
  initial condition $(*)$

Five-Flavour QCD as an Effective Field Theory

And so on ....

 $\longrightarrow \frac{c}{m_t^2} f_{ABC} G^A_{\mu\nu} G^B_{\mu\alpha} G^C_{\nu\alpha}$

.....

Below m_b ↙ two options

- no external bottom lines → repeat above procedure
- external bottom lines must be conserved below m_b
  → Non-relativistic QCD, heavy quark effective theory

- Technical comments for general EFTs:
  - Renormalization-scheme dependence [projection scheme, γ_5 scheme, subtraction scheme (MS, $\overline{\rm MS}$, ...)].
  - Evanescent operators in dimensional regularization [no Fierz transformation in D-dimensions, ...].
  - IR subtraction complicated for many exclusive B-meson decays [a_6 for $B \to h_1\, h_2$, penguin contributions, ...].
  - Loop integrals with three different energy scales at 2 loops [modern amplitude techniques, differential equations, interplay between mathematics and physics, ...].

Effective field theories for heavy quark physics

New Physics: $\mathscr{L}_{NP}$

$\downarrow$

EW scale (m_W): $\mathscr{L}_{SM} + \mathscr{L}_{D>4}$

$\downarrow$

Heavy-quark scale (m_b): $\mathscr{L}_{eff} = -\frac{G_F}{\sqrt{2}} \sum_i C_i \, Q_i + \mathscr{L}_{eff,D>6}$

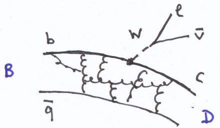
$\downarrow$

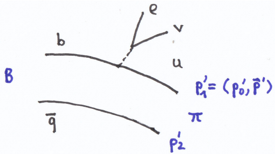
QCD scale ($\Lambda_{\rm QCD}$)

- Aim: $\langle f|Q_i|\bar{B}\rangle = ?$
- QCD factorization [diagrammatic method].
- SCET factorization [operator formalism].
- TMD factorization.
- (Light-cone) QCD sum rules.
- Lattice QCD.

Key Concepts : Factorization, Resummation, Evolution

Why do we need QCD factorization?





$m_b, m_c \to \infty$
no large boost

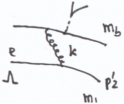
massless fields + heavy quark
fluctuations soft $k^\mu \sim \Lambda$

$$\mathcal{L} = \sum_{Q=b,c} \bar{h}_{Qv_Q} \, i v_Q \cdot D_s \, h_{Qv_Q} + \mathcal{L}_{gluon} + \sum_{q\ \mathrm{light}} \bar{q} \, i \not{D}_s \, q \quad \text{(HQET)}$$

[→ HQ symmetries]

- $p_1' \sim \Lambda$ → soft pion
  heavy-to-light transitions in HQET
- $p_0' \sim m_b$, but $p'^2 = m_\pi^2 \ll m_b^2$
  → energetic pion
  no such field in HQET

$p_1' \sim (m_b, \Lambda, \Lambda, m_b)$ "collinear" (nearly light-like)



$k^2 = (p_2' + e)^2 \sim m_b \Lambda$
$k^0 \sim m_b$
"hard-collinear"

Why do we need QCD factorization?

QCDF deals with B decays that involve energetic light particles (hadrons)

$B \to \pi \ell \nu$
$B \to \gamma \ell \nu$

π, γ energetic

do not involve four-quark operators
→ QCD form factors

$B \to D\pi$
$B \to \pi\pi, \ldots$ [charmless]
$B \to K^* \gamma^{(*)}$
↳ e^+e^-

involve four quark operators
(strong final state interactions)

Modern language

QCDF $\hat{=}$ Soft-collinear effective theory for heavy quark physics
[extends HQET by (hard-) collinear modes]

Wide applications of QCD factorization

HQE/OPE, lattice, (QCD sum rules)

QCD factorization, SCET, flavour symmetries

None – pure quantum interference

$\langle 0|\mathcal{O}|B\rangle$
$\langle B|\mathcal{O}|B\rangle$

$\langle M|\mathcal{O}|B\rangle$

$\langle M_1 M_2|\mathcal{O}|B\rangle$

$\leftarrow$ Easier

Increasingly difficult $\rightsquigarrow$

γ from $B \to DK$	$B \to \tau\nu_\tau$	$B \to D\tau\nu_\tau$	$B \to \rho\gamma$	Direct CP asym
[and related methods]	$B_s \to \mu^+\mu^-$	$\lvert V_{ub}\rvert$	$B \to K^{(*)}\ell\ell$	$B_{(s)} \to \pi K, KK, \ldots$
2β, $2\beta_{B_s}$	$\Delta M_{B_d, B_s}$	$B \to K\nu\bar{\nu}$		$B_s \to \pi\pi$
	$\Delta\Gamma_{B_s}$			$B_s \to \phi\phi, K^{*0}\bar{K}^{*0}$
				All hadronic

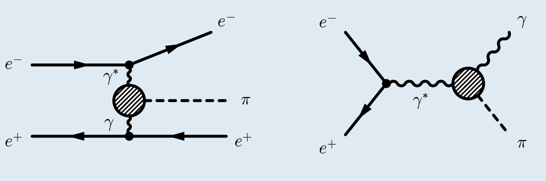
Part III: QCD factorization for the pion form factor

The pion-photon transition form factor

- Theoretically simplest hadronic matrix element:

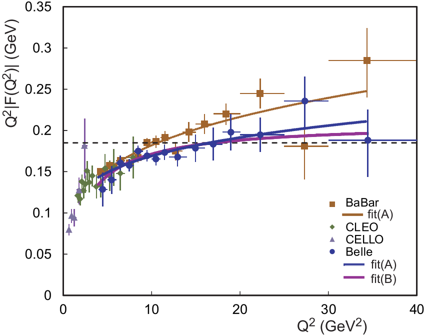
$$\langle \pi(p) | j_\mu^{\rm em} | \gamma(p') \rangle = g_{\rm em}^2 \, \varepsilon_{\mu\nu\alpha\beta} \, q^\alpha \, p^\beta \, \varepsilon^\nu(p') F_{\gamma^*\gamma\to\pi^0}(Q^2) \, .$$

  - Related to the axial anomaly at $Q^2 \equiv -(p-p')^2 = 0$.
  - Golden channel to understand the QCD dynamics.
  - e^+e^- collisions:



The BaBar puzzle

- Status of experimental measurements:

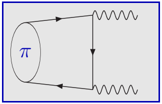


Scaling violation?

Shape of pion wave function?

The onset of QCD factorization?

- Asymptotic limit:



$$\longrightarrow \quad \lim_{Q^2 \to \infty} Q^2 F_{\pi\gamma\gamma^*}(Q^2) = 2 f_\pi$$

Brodsky, Lepage

QCD factorization works perfectly (against by [Manohar, 1990] and resolved by [Grossman, König, Neubert, 2015]).

Some popular explanations:

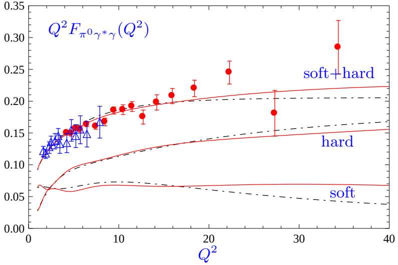
- Non-vanishing pion wave function at the end points [Radyushkin, 2009; Polyakov, 2009].

$$F(Q^2) \quad = \quad \frac{\sqrt{2}}{3}\int_0^1 \frac{\varphi_\pi(x)}{xQ^2}\left[1-\underbrace{\exp\left(-\frac{xQ^2}{2\bar{x}\sigma}\right)}\right].$$

$\Uparrow$

from k_T dependence of pion wave function

- Large soft corrections at moderate Q^2 [Agaev, Braun, Offen, Porkert, 2011].

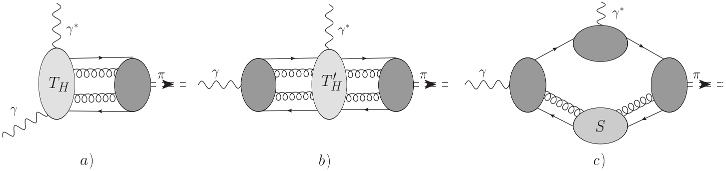


The "hard" and "soft" contributions to the $\pi^0\gamma^*\gamma$ form factor for model I (solid curves) and model III (dash-dotted curves). The experimental data are from BaBar (full circles) and CLEO (open triangles).

- Threshold resummation generates power-like $[x(1-x)]^{c(Q^2)}$ distribution [Li and Mishima 2009]. $c(Q^2)$ is around 1 for low Q^2, but small for high Q^2.

The general picture

- Schematic structure of the distinct mechanisms [Agaev, Braun, Offen, Porkert, 2011]:



A: hard subgraph that includes both photon vertices $\qquad \dfrac{1}{Q^2} + \dfrac{1}{Q^4} + \ldots$

B: real photon emission at large distances $\qquad \dfrac{1}{Q^4} + \ldots$

C: Feynman mechanism: soft quark spectator $\qquad \dfrac{1}{Q^4} + \ldots$

- Operator definitions of different terms needed for an unambiguous classification.

Region A: Leading Twist Contribution

- QCD factorization formula:

$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2} \int_0^1 dx\, T^\Delta(x, Q^2, \mu)\, \phi_\pi^\Delta(x, \mu)\,.$$

  - Renormalization-scheme dependence due to the IR subtraction.
    $\Rightarrow$ Verify scheme independence of $F_{\gamma^*\gamma\to\pi^0}$ at NLO (this lecture).
  - NLO hard function in the NDR scheme [Braaten, 1983 + many others]:

$$T^{(1)} = \frac{\alpha_s C_F}{4\pi} \left\{ \frac{1}{\bar{x}} \left[-(2\ln\bar{x} + 3)\ln\frac{\mu^2}{Q^2} + \ln^2\bar{x} + (-1)\,\frac{\bar{x}\ln\bar{x}}{x} - 9 \right] + (x \leftrightarrow \bar{x}) \right\}.$$

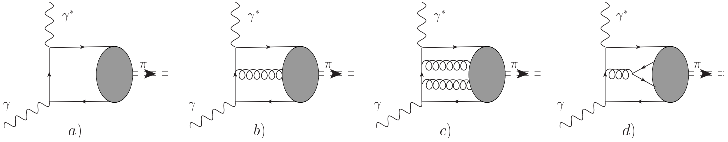
  - Twist-2 pion DA (collinear matrix element):

$$\langle \pi(p) | \bar{\xi}(y)\, W_c(y,0)\, \gamma_\mu\, \gamma_5\, \xi(0) | 0 \rangle = -i f_\pi\, p_\mu \int_0^1 du\, e^{iup\cdot y}\, \phi_\pi(u,\mu) + \mathcal{O}(y^2)\,.$$

The ERBL evolution implies Gegenbauer expansion.

Region A: Twist-4 Contribution

- Subleading power correction at $\mathcal{O}(1/Q^4)$:



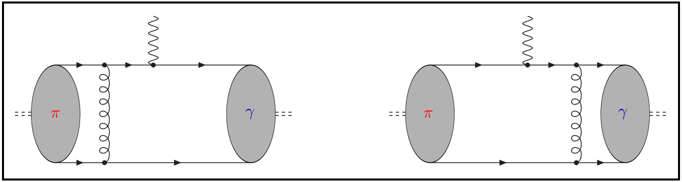
- QCD factorization at tree level [Khodjamirian, 1999]:

$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2} \left[\int_0^1 dx\, \frac{\phi_\pi(x)}{x} - \frac{80}{9}\, \frac{\delta_\pi^2}{Q^2} \right], \qquad \delta_\pi^2 \simeq 0.2\,{\rm GeV}^2\,.$$

  - Asymptotic twist-4 contribution at LO in QCD.
  - Two-particle and three-particle twist-4 corrections related by the EOM.
  - Four-particle twist-4 correction not included and assumed to be tiny.
  - Asymptotic twist-4 correction at $\mathcal{O}(\alpha_s)$ in progress.

Region B: Photon emission at large distances

- Hadronic photon correction:



- QCD factorization at tree level:

$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2} \, \frac{16\,\pi\,\alpha_s\,\chi(\mu)\,\langle \bar{q} q \rangle^2}{9 f_\pi^2\, Q^2} \int_0^1 dx \, \frac{\phi^p_{3;\pi}(x)}{x} \int_0^1 dy \, \frac{\phi_\gamma(y)}{\bar{y}^2} \, .$$

Breakdown of QCD factorization due to rapidity singularities.

- QCD calculation of the hadronic photon effect beyond the VMD approximation.

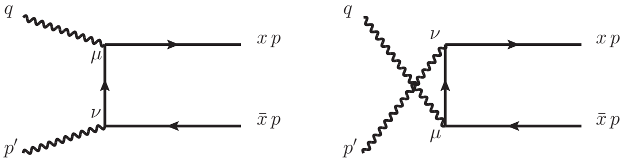
  ⇒ The NLL LCSRs for the long-distance photon effect.

  ⇒ QCD factorization for the correlation function with a pseudoscalar current.

  ⇒ γ_5 prescription in dimensional regularization.

Part I: Leading twist contribution

- Tree diagrams [YMW and Shen, 2017]:



Kinematics:

$$p'_\mu = \underbrace{\frac{n \cdot p'}{2}}_{\mathcal{O}(\sqrt{Q^2})} \bar{n}_\mu \,, \qquad p_\mu = \underbrace{\frac{\bar{n} \cdot p}{2}}_{\mathcal{O}(\sqrt{Q^2})} n_\mu + \underbrace{\frac{n \cdot p}{2}}_{\mathcal{O}(\Lambda^2/\sqrt{Q^2})} \bar{n}_\mu \,.$$

- The four-point partonic matrix element at LO:

$$\begin{aligned} \Pi_\mu &= \langle q(xp)\, \bar{q}(\bar{x}p) | j_\mu^{\rm em} | \gamma(p') \rangle \\ &= -\frac{i\, g_{\rm em}^2\, (Q_u^2 - Q_d^2)}{2\sqrt{2}\, n \cdot p}\, \varepsilon^\nu(p') \left\{ \frac{\left[\bar{u}(xp)\, \gamma_{\mu,\perp}\, \not{\bar{n}}\, \gamma_{\nu,\perp}\, v(\bar{x}p)\right]}{\bar{x}} - \frac{\left[\bar{u}(xp)\, \gamma_{\nu,\perp}\, \not{\bar{n}}\, \gamma_{\mu,\perp}\, v(\bar{x}p)\right]}{x} \right\} \\ &\quad + \mathcal{O}(\alpha_s^2)\,. \end{aligned}$$

QCD matrix element free of the γ_5 ambiguity, however not the IR subtraction.

Twist-2 factorization at tree level

- Operator matching automatically:

$$\Pi_\mu^{(0)} = -\frac{i\, g_{\rm em}^2\, (Q_u^2 - Q_d^2)}{2\sqrt{2}\, n\cdot p}\, \varepsilon^\nu(p') \left[\frac{1}{\bar{x}'} * \langle O_{A,\mu\nu}(x,x')\rangle^{(0)} - \frac{1}{x'} * \langle O_{B,\mu\nu}(x,x')\rangle^{(0)} \right].$$

- Collinear operators in the momentum space:

$$\begin{aligned} O_{j,\mu\nu}(x') &= \frac{\bar{n}\cdot p}{2\pi} \int d\tau\, e^{i x' \tau \bar{n}\cdot p}\, \bar{\xi}(\tau\bar{n})\, W_c(\tau\bar{n},0)\, \Gamma_{j,\mu\nu}\, \xi(0)\,. \\ \Gamma_{A,\mu\nu} &= \gamma_{\mu,\perp}\, \not{\bar{n}}\, \gamma_{\nu,\perp}\,, \qquad \Gamma_{B,\mu\nu} = \gamma_{\nu,\perp}\, \not{\bar{n}}\, \gamma_{\mu,\perp}\,. \end{aligned}$$

- Matrix elements of the collinear operators:

$$\langle O_{j,\mu\nu}(x,x')\rangle \equiv \langle q(xp)\,\bar{q}(\bar{x}p)|O_{j,\mu\nu}(x')|0\rangle = \bar{\xi}(xp)\,\Gamma_{j,\mu\nu}\,\xi(\bar{x}p)\, \delta(x-x') + \mathcal{O}(\alpha_s)\,.$$

- Operator matching with the collinear operator defining the standard pion DA:

$$\begin{aligned} O_{A,\mu\nu} &= -\left(O_{1,\mu\nu} + O_{2,\mu\nu} + O_{E,\mu\nu}\right), \qquad O_{B,\mu\nu} = -\left(O_{1,\mu\nu} - O_{2,\mu\nu} - O_{E,\mu\nu}\right). \\ \underbrace{\Gamma_{1,\mu\nu}}_{\text{wrong projector}} &= g_{\mu\nu}^{\perp}\, \not{\bar{n}}\,, \qquad \Gamma_{2,\mu\nu} = i\,\varepsilon_{\mu\nu}^{\perp}\, \not{\bar{n}}\, \gamma_5\,, \qquad \underbrace{\Gamma_{E,\mu\nu}}_{\text{evanescent operator}} = \not{\bar{n}} \left(\frac{[\gamma_{\mu,\perp}, \gamma_{\nu,\perp}]}{2} - i\,\varepsilon_{\mu\nu}^{\perp}\, \not{\bar{n}}\, \gamma_5 \right). \end{aligned}$$

Twist-2 factorization at tree level

- Operator matching with the evanescent operator:

$$\Pi_\mu = \left[\frac{i\, g_{\rm em}^2 \, (Q_u^2 - Q_d^2)}{2\sqrt{2}\, n \cdot p} \, \varepsilon^\nu(p') \right] \sum_i T_i(x') * \langle O_{i,\mu\nu}(x,x') \rangle \,.$$

Expansion $\Downarrow$ at tree level

$$T_1^{(0)}(x') = \frac{1}{x'} - \frac{1}{\bar{x}'} \,, \qquad T_2^{(0)}(x') = T_E^{(0)}(x') = \frac{1}{x'} + \frac{1}{\bar{x}'} \,.$$

- Hard-collinear factorization at tree level:

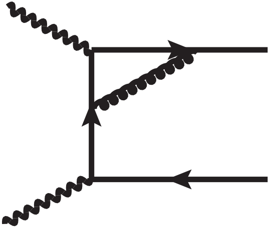
$$F^{\rm LP}_{\gamma^* \gamma \to \pi^0}(Q^2) = \frac{\sqrt{2}\,(Q_u^2 - Q_d^2) f_\pi}{Q^2} \int_0^1 dx \, T_2^{(0)}(x) \, \phi_\pi(x,\mu) + \mathcal{O}(\alpha_s) \,.$$

Evanescent operator does not mix into the physical operator at LO.

$\Rightarrow$ Trivial IR subtraction here.

Twist-2 factorization at NLO: A sample calculation

- Virtual-photon vertex correction:



- Partonic amplitude:

$$\Pi_\mu^{(1a)} = \left[\frac{g_{\rm em}^2 \, (Q_u^2 - Q_d^2)}{\sqrt{2}\,(p' - \bar{x} p)^2} \, \varepsilon^\nu(p') \right] g_s^2 \, C_F \int \frac{d^D l}{(2\pi)^D} \frac{1}{[(xp+l)^2 + i0][(p' - \bar{x}p + l)^2 + i0][l^2 + i0]} \bar{u}(xp)\, \gamma_\rho \, (x\not{p} + \not{l}) \, \gamma_\mu^\perp \, (\not{p}' - \bar{x}\not{p} + \not{l}) \, \gamma^\rho \, (\not{p}' - \bar{x}\not{p}) \, \gamma_\nu^\perp \, v(\bar{x}p) \, .$$

- Performing the loop-momentum integration with the method of regions [Beneke, Smirnov, 1997].
  - Leading contributions only come from the hard and anti-collinear (i.e., $l \parallel p$) regions.
  - The anti-collinear contribution corresponds to the scaleless integral, and hence vanishes in dimensional regularization.

- The hard contribution to the scalar 1-loop:

$$\begin{aligned} I &= \int \frac{d^D l}{(2\pi)^D} \frac{1}{[l^2 + x\,\bar{n}\cdot p\, n\cdot l + i0][n\cdot(l+p')\,\bar{n}\cdot(l - \bar{x}\,p) + l_\perp^2 + i0][l^2 + i0]} \\ &= -\frac{i}{16\pi^2}\frac{1}{x\,Q^2}\left[\ln\bar{x}\left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{Q^2}\right) - \frac{1}{2}\ln^2\bar{x}\right]. \end{aligned}$$

Only hard scale $Q^2 = n\cdot p' \; \bar{n}\cdot p$ can appear in the hard function!

Strategy of regions

- For an excellent review, Smirnov, *Applied asymptotic expansions in momenta and masses*, Springer Tracts Mod. Phys. 177 (2002) 1.

- The general procedure:
  - Identify all regions of the integration which lead to singularities in the limit under consideration.
  - Expand the integrand in each region and integrate over the full phase space.
  - Summing the contribution from the different regions gives the expansion of the original integral.

- Problems in the strategy of regions [Smirnov, 1999]:
  - Need to identify all regions of the integration which lead to singularities.

    $\Downarrow$

    There are examples where additional regions appear at higher loop order.
  - Make sure the expanded integrals are regularized.

    $\Downarrow$

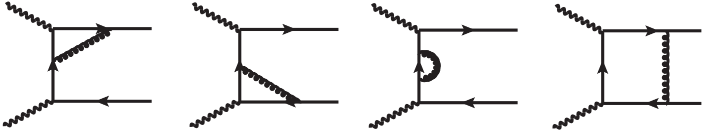
    Sometimes dimensional regularization is not enough.
  - More discussions can be found in [Jantzen, 2008; Semenova, Smirnov, Smirnov, 2019].

    $\Downarrow$

    A rigorous proof only exists for the space-like momentum configuration.

Twist-2 factorization at NLO

- The four-point partonic matrix element at NLO:



- Extracting the hard contribution with the method of regions:

$$\Pi_\mu^{(1)} = \frac{i\, g_{\rm em}^2\,(Q_u^2 - Q_d^2)}{2\sqrt{2}\, n\cdot p}\, \varepsilon^\nu(p')\, \langle O_{2,\mu\nu}(x,x')\rangle^{(0)} * A_{2,\rm hard}^{(1)}(x') + \ldots .$$

$$A_{2,\rm hard}^{(1)}(x') = \frac{\alpha_s\, C_F}{4\pi} \left\{ \frac{1}{\bar{x}'} \left[-\left(2\ln\bar{x}' + 3\right) \left(\frac{1}{\varepsilon} + \ln\frac{\mu^2}{Q^2} \right) + \ln^2\bar{x}' + 7\,\frac{\bar{x}'\,\ln\bar{x}'}{x'} - 9 \right] + \left(x' \leftrightarrow \bar{x}'\right) \right\}.$$

Only the hard and collinear regions relevant at leading power in $1/Q^2$.
Independent of the γ_5 prescription!

- However the IR subtraction is not trivial any longer due to the operator mixing.

Twist-2 factorization at NLO

- Expanding the matching equation at NLO:

$$\left[\frac{i\,g_{\rm em}^2\,(Q_u^2-Q_d^2)}{2\sqrt{2}\,n\cdot p}\,\varepsilon^{\nu}(p')\right]\sum_i A_i^{(1)}(x')*\langle O_{i,\mu\nu}(x,x')\rangle^{(0)}$$
$$=\left[\frac{i\,g_{\rm em}^2\,(Q_u^2-Q_d^2)}{2\sqrt{2}\,n\cdot p}\,\varepsilon^{\nu}(p')\right]\sum_i\left[T_i^{(1)}(x')*\langle O_{i,\mu\nu}(x,x')\rangle^{(0)}+T_i^{(0)}(x')*\langle O_{i,\mu\nu}(x,x')\rangle^{(1)}\right].$$

- One-loop renormalized matrix elements of collinear operators:

$$\langle O_{i,\mu\nu}\rangle^{(1)}=\sum_j\left[M_{ij,\rm bare}^{(1)R}+Z_{ij}^{(1)}\right]*\langle O_{j,\mu\nu}\rangle^{(0)}.$$

Vanishing bare matrix element $M_{ij,\rm bare}^{(1)R}$ in dimensional regularization $\Rightarrow$

$$T_2^{(1)}=A_2^{(1)}-\sum_i T_i^{(0)}*Z_{i2}^{(1)}=\underbrace{A_2^{(1)}-T_2^{(0)}*Z_{22}^{(1)}}_{A_{2,\rm hard}^{(1)}}-T_E^{(0)}*\underbrace{Z_{E2}^{(1)}}_{\text{operator mixing}}.$$

- The IR finite matrix element of the evanescent operator vanishes [Dugan and Grinstein, 1991].

$$Z_{E2}^{(1)}=-M_{E2}^{(1)\rm off}.$$

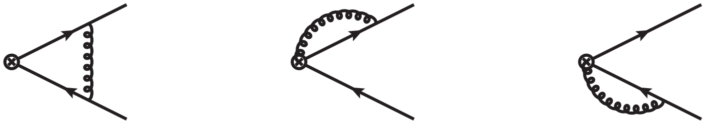
IR singularities regularized by any parameter other than the dimensions of spacetime.

Twist-2 factorization at NLO

- Master formula for the hard function:

$$T_2^{(1)} = A_2^{(1)} - T_2^{(0)} * Z_{22}^{(1)} + T_E^{(0)} * M_{E2}^{(1)\text{off}} = A_{2,\text{hard}}^{(1)} + T_E^{(0)} * M_{E2}^{(1)\text{off}} .$$

- The IR subtraction:



γ_5-scheme dependent subtraction term:

$$T_E^{(0)} * M_{E2}^{(1)\text{off}} \Big|_{\text{NDR}} = \frac{\alpha_s C_F}{2\pi} (-4) \left(\frac{\ln \bar{x}'}{x'} + \frac{\ln x'}{\bar{x}'} \right) .$$

$$T_E^{(0)} * M_{E2}^{(1)\text{off}} \Big|_{\text{HV}} = 0 .$$

$M_{E2}^{(1)\text{off}}$ proportional to the spin-dependent term of the Brodsky-Lepage evolution kernel.

A simple example :

$$\gamma_\alpha \not{\bar{n}} \gamma_5 \gamma^\alpha = \not{\bar{n}} \gamma_5 \begin{cases} D-2 , & \text{[NDR scheme]} \\ 6-D . & \text{[HV scheme]} \end{cases}$$

Twist-2 factorization at NLO

- The NLO hard matching coefficient:

$$T_2^{(1)}(x',\mu) = \frac{\alpha_s C_F}{4\pi} \left\{ \frac{1}{\bar{x}'} \left[-\left(2\ln\bar{x}'+3\right) \ln\frac{\mu^2}{Q^2} + \ln^2\bar{x}' + \delta\,\frac{\bar{x}'\ln\bar{x}'}{x'} - 9 \right] + \left(x' \leftrightarrow \bar{x}'\right) \right\}.$$

$\delta = -1$ in NDR scheme and $\delta = +7$ in HV scheme.

- The NLO factorization formula:

$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{\sqrt{2}(Q_u^2 - Q_d^2) f_\pi}{Q^2} \int_0^1 dx \left[T_2^{(0)}(x) + T_2^{(1),\Delta}(x,\mu) \right] \phi_\pi^\Delta(x,\mu) + \mathcal{O}(\alpha_s^2).$$

- Scheme dependence of the pion DA [Melic, Nizic and Passek, 2002]:

$$\phi_\pi^{\rm HV}(x,\mu) = \int_0^1 dy\, Z_{\rm HV}^{-1}(x,y,\mu)\, \phi_\pi^{\rm NDR}(y,\mu),$$

$$Z_{\rm HV}^{-1}(x,y,\mu) = \delta(x-y) + \frac{\alpha_s C_F}{2\pi}\, 4 \left[\frac{x}{y}\,\theta(y-x) + \frac{\bar{x}}{\bar{y}}\,\theta(x-y) \right] + \mathcal{O}(\alpha_s^2).$$

$$\Downarrow$$

$$\int_0^1 dx\, T_2^{(0)}(x) \left[\phi_\pi^{\rm HV}(x,\mu) - \phi_\pi^{\rm NDR}(x,\mu) \right] = \frac{\alpha_s C_F}{2\pi} (-4) \int_0^1 dy \left(\frac{\ln\bar{y}}{y} + \frac{\ln y}{\bar{y}} \right) \phi_\pi^{\rm NDR}(x,\mu) + \mathcal{O}(\alpha_s^2).$$

$\Rightarrow$ Scheme independence of $F_{\gamma^*\gamma\to\pi^0}$ at NLO.

Twist-2 factorization at NLL

- RG evolution of the pion LCDA:

$$\mu^2 \frac{d}{d\mu^2} \phi_\pi(x,\mu) = \int_0^1 dy\, V(x,y)\, \phi_\pi(y,\mu)\,, \qquad V(x,y) = \sum_{n=0} \left(\frac{\alpha_s}{4\pi}\right)^{n+1} [V_n(x,y)]_+ \,.$$

$$V_0(x,y) = 2\, C_F \left[\frac{1-x}{1-y} \left(1 + \frac{1}{x-y}\right) \theta(x-y) + \frac{x}{y} \left(1 + \frac{1}{y-x}\right) \theta(y-x) \right].$$

Multiplicative renormalization at LO:

$$\phi_\pi(x,\mu) = 6 x \bar{x} \sum_{n=0}^{\infty} \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{\gamma_{V,n}^{(0)}/(2\beta_0)} a_n(\mu_0)\, C_n^{3/2}(2x-1)\,, \qquad a_0(\mu) = 1\,.$$

- NLL resummation needs two-loop evolution kernel [Mikhailov and Radyushkin, 1985; etc]:

$$V_1(x,y) = 2 N_f\, C_F\, V_N(x,y) + 2\, C_F\, C_A\, V_G(x,y) + C_F^2\, V_F(x,y)\,.$$

Triangular evolution matrix [Müller, 1994/1995 + many others]:

$$a_n(\mu) = E_{V,n}^{\rm NLO}(\mu,\mu_0)\, a_n(\mu_0) + \frac{\alpha_s(\mu)}{4\pi} \sum_{k=0}^{n-2} E_{V,n}^{\rm LO}(\mu,\mu_0)\, d_{V,n}^{k}(\mu,\mu_0)\, a_k(\mu_0)\,.$$

Construction from the forward anomalous dimensions and the special conformal anomaly matrix.

Twist-2 factorization at NLL

- NLL resummation improved factorization formula:

$$F^{\rm LP}_{\gamma^*\gamma\to\pi^0}(Q^2) = \frac{3\sqrt{2}\,(Q_u^2-Q_d^2)}{Q^2}\, f_\pi \sum_{n=0}^{\infty} a_n(\mu)\, C_n(Q^2,\mu) + \mathcal{O}(\alpha_s^2)\,.$$

  - NNLO hard kernel in the large β_0 limit [Melic, Müller and Passek-Kumericki, 2003].
  - Full NNLO hard kernel in the $\overline{\rm MS}$ scheme [Gao, Huber, Ji, YMW, 2021].

- Generating function of the Gegenbauer polynomials:

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0} C_n^{(\alpha)}(x)\, t^n\,.$$

The NLO hard coefficients:

$$C_n(Q^2,\mu) = 1 + \frac{\alpha_s(\mu)\,C_F}{4\pi}\left\{\left[4H_{n+1} - \frac{3n(n+3)+8}{(n+1)(n+2)}\right]\ln\frac{\mu^2}{Q^2} + 4H_{n+1}^2 - 4\,\frac{H_{n+1}+1}{(n+1)(n+2)} + 2\left[\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}\right] + 3\left[\frac{1}{(n+1)}-\frac{1}{(n+2)}\right] - 9\right\}.$$

Part II: Hadronic photon effect

- Breakdown of the direct QCD factorization for the long-distance photon effect
  $\Rightarrow$ Construct the sum rules for the hadronic photon correction.

- Correlation function [YMW and Shen, 2017]:

$$G_\mu(p',q) = \int d^4z\, e^{-iq\cdot z} \langle 0|\mathrm{T}\left\{ j^{\rm em}_{\mu,\perp}(z), j_\pi(0) \right\} |\gamma(p')\rangle = -g^2_{\rm em}\, \varepsilon_{\mu\nu\alpha\beta}\, q^\alpha\, p'^\beta\, \varepsilon_\nu(p')\, G(p^2,Q^2)\,.$$

$$j_\pi = \frac{1}{\sqrt{2}}\left(\bar u\, \gamma_5\, u - \bar d\, \gamma_5\, d\right)\,, \qquad p^2 = (p'+q)^2\,.$$

Explicit dependence on γ_5 $\Rightarrow$ scheme dependence of the QCD amplitude.

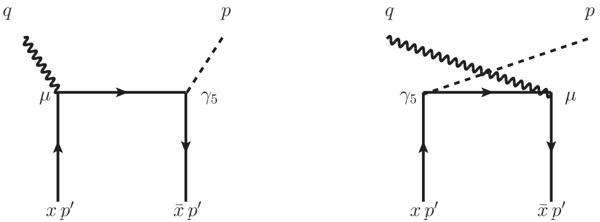
- Standard strategies:
  - QCD factorization for the correlator $G_\mu(p',q)$ with the power-counting scheme:

$$|n\cdot p| \sim \bar n\cdot p \sim n\cdot p' \sim \mathcal{O}(\sqrt{Q^2})\,.$$

  - Write down the hadronic dispersion relation based upon the analyticity and unitarity.
  - Constructing the sum rules by matching the QCD and hadronic representations.
  - Need the semi-global parton-hadron duality ansatz.

Hadronic photon effect at tree level

- Tree diagrams:



p for the four-momentum of the pion current and q for the transfer momentum.

- The four-point QCD amplitude at LO:

$$\begin{aligned}\widetilde{\Pi}_\mu &= \int d^4z\, e^{-iq\cdot z}\, \langle 0|\mathrm{T}\left\{ j^{\mathrm{em}}_{\mu,\perp}(z), j_\pi(0)\right\} |q(xp')\,\bar{q}(\bar{x}p')\rangle \\ &= -\frac{i\, g_{\mathrm{em}}}{2\sqrt{2}}\, \frac{\bar{n}\cdot p}{Q^2} \left[\frac{1}{x\,r+\bar{x}} + \frac{1}{\bar{x}\,r+x}\right] \sum_{q=u,d} \eta_q\, Q_q\, \bar{q}(\bar{x}p')\,\gamma_5\, \not{n}\, \gamma_{\mu,\perp}\, q(xp')\,,\end{aligned}$$

where $r = -p^2/Q^2$, $\eta_u = 1$ and $\eta_d = -1$.

Hadronic photon effect at tree level

- Operator matching automatically:

$$\widetilde{\Pi}_{\mu}^{(0)} = -\frac{i\, g_{\rm em}}{2\sqrt{2}} \frac{\bar{n}\cdot p}{Q^2} \sum_{q=u,d} \eta_q \, Q_q \left[\frac{1}{x' \, r + \bar{x}'} + \frac{1}{\bar{x}' \, r + x'} \right] * \langle \widetilde{O}_{A,\mu}(x,x') \rangle^{(0)} .$$

- (Anti)-collinear operators in the momentum space:

$$\begin{aligned} \widetilde{O}_{j,\mu}(x') &= \frac{\bar{n}\cdot p'}{2\pi} \int d\tau \, e^{i x' \tau \bar{n}\cdot p'} \, \bar{\chi}(0) \, W_{\bar{c}}(0, \tau n) \, \widetilde{\Gamma}_{j,\mu} \, \chi(\tau n) , \\ \widetilde{\Gamma}_{A,\mu} &= \gamma_5 \, \not{n} \, \gamma_{\mu,\perp} . \end{aligned}$$

- Matrix element of the (anti)-collinear operators:

$$\langle \widetilde{O}_{j,\mu}(x,x') \rangle \equiv \langle 0 | \widetilde{O}_{j,\mu}(x') | q(xp') \, \bar{q}(\bar{x}p') \rangle = \bar{\chi}(\bar{x}p') \, \widetilde{\Gamma}_{j,\mu} \, \chi(xp') \, \delta(x - x') + \mathscr{O}(\alpha_s) .$$

- Operator matching with the effective operator defining the photon DA:

$$\begin{aligned} \widetilde{O}_{A,\mu} &= \widetilde{O}_{1,\mu} + \widetilde{O}_{E,\mu} , \\ \widetilde{\Gamma}_{1,\mu} &= \frac{n^{\alpha}}{2} \, \varepsilon^{\perp}_{\mu\nu\alpha\beta} \, \sigma^{\nu\beta} , \qquad \underbrace{\widetilde{\Gamma}_{E,\mu}}_{\text{evanescent operator}} = \gamma_5 \, \not{n} \, \gamma_{\mu,\perp} - \frac{n^{\alpha}}{2} \, \varepsilon^{\perp}_{\mu\nu\alpha\beta} \, \sigma^{\nu\beta} . \end{aligned}$$

Hadronic photon effect at tree level

- Operator matching with the evanescent operator:

$$\widetilde{\Pi}_\mu = -\frac{i\, g_{\rm em}}{2\sqrt{2}}\,\frac{\bar{n}\cdot p}{Q^2}\sum_{q=u,d}\eta_q\, Q_q \sum_i \widetilde{T}_i(x') * \langle \widetilde{O}_{i,\mu}(x,x')\rangle\,.$$

Expansion $\Downarrow$ at tree level

$$\widetilde{T}_1^{(0)}(x') = \widetilde{T}_E^{(0)}(x') = \frac{1}{x'\, r + \bar{x}'} + \frac{1}{\bar{x}'\, r + x'}\,.$$

- Leading twist photon DA [Ball, Braun and Kivel, 2002]:

$$\langle 0|\bar{\chi}(0)\, W_{\bar{c}}(0,y)\, \sigma_{\alpha\beta}\, \chi(y)|\gamma(p')\rangle$$
$$= i\, g_{\rm em}\, Q_q\, \chi(\mu)\, \langle \bar{q} q\rangle(\mu)\, \left[p'_\beta\, \varepsilon_\alpha(p') - p'_\alpha\, \varepsilon_\beta(p')\right] \int_0^1 du\, e^{-iu p'\cdot y}\, \phi_\gamma(u,\mu)\,.$$

The magnetic susceptibility of the quark condensate $\chi(\mu)$ as the key nonperturbative parameter.

- Hard-collinear factorization at LO:

$$G(p^2,Q^2) = -\frac{Q_u^2 - Q_d^2}{\sqrt{2}\, Q^2}\, \chi(\mu)\, \langle \bar{q} q\rangle(\mu) \int_0^1 dx\, \widetilde{T}_1^{(0)}(x)\, \phi_\gamma(x,\mu) + \mathcal{O}(\alpha_s)\,.$$

Hadronic photon effect at tree level

- The hadronic dispersion relation of $G(p^2, Q^2)$:

$$G(p^2, Q^2) = \frac{f_\pi \, \mu_\pi(\mu)}{m_\pi^2 - p^2 - i0} \, F^{\rm NLP}_{\gamma^* \gamma \to \pi^0}(Q^2) + \int_{s_0}^{\infty} ds \, \frac{\rho^h(s, Q^2)}{s - p^2 - i0} \, .$$

- The tree-level LCSR:

$$F^{\rm NLP}_{\gamma^* \gamma \to \pi^0}(Q^2) = - \frac{\sqrt{2}\, (Q_u^2 - Q_d^2)}{f_\pi \, \mu_\pi(\mu)} \, \chi(\mu) \, \langle \bar{q} q \rangle(\mu) \int_{u_0}^{1} \frac{du}{u} \exp\left[- \frac{\bar{u}\, Q^2 + u m_\pi^2}{u M^2} \right] \phi_\gamma(u, \mu) + \mathcal{O}(\alpha_s) \, .$$

  - The power counting scheme for the sum rule parameters:

$$s_0 \sim M^2 \sim \mathcal{O}(\Lambda^2) \, , \qquad \bar{u}_0 \sim \mathcal{O}(\Lambda^2 / Q^2) \, , \qquad u_0 \equiv Q^2 / (s_0 + Q^2) \, .$$

  - The scaling behaviour of the hadronic photon correction:

$$\frac{F^{\rm NLP}_{\gamma^* \gamma \to \pi^0}(Q^2)}{F^{\rm LP}_{\gamma^* \gamma \to \pi^0}(Q^2)} \sim \mathcal{O}\left(\frac{\Lambda^2}{Q^2} \right) \, .$$

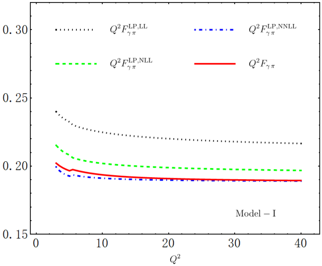
- NLO QCD correction to the hadronic photon effect [YMW and Shen, 2017].

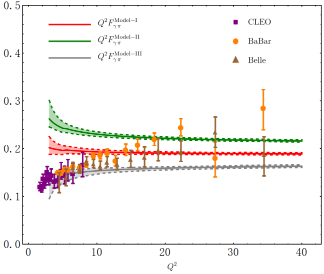
NNLO prediction of the pion-photon form factor

- Models of the pion DA [Gao, Huber, Ji, YMW, 2021]:

$$\begin{aligned}
\text{Model I}: &\quad \phi_\pi(x,\mu_0) = \frac{\Gamma(2+2\,\alpha_\pi)}{\Gamma^2(1+\alpha_\pi)}\,(x\bar{x})^{\alpha_\pi}\,, \qquad \alpha_\pi(\mu_0) = 0.422^{+0.076}_{-0.067}\,,\\
&\quad [\text{Holographic} \oplus \text{Lattice QCD 2020}]\,;\\
\text{Model II}: &\quad \{a_2, a_4\}(\mu_0) = \{0.269(47), 0.185(62)\}\,,\\
&\quad \{a_6, a_8\}(\mu_0) = \{0.141(96), 0.049(116)\}\,, \qquad [\text{CKR}, 2020]\,;\\
\text{Model III}: &\quad \{a_2, a_4\}(\mu_0) = \{0.203^{+0.069}_{-0.057}\,, -0.143^{+0.094}_{-0.087}\}\,, \qquad [\text{BMS}, 2020]\,.
\end{aligned}$$

- NNLO QCD predictions [Gao, Huber, Ji, YMW, 2021]:





Very sizeable $\mathcal{O}(\alpha_s^2)$ correction and the golden process to distinguish the various models!