EFT Perspectives on Feynman Integrals

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Motivation

- Feynman integrals with the multi-scales complicated.
  $\Rightarrow$ Hierarchical scales generally lead to the emergence of EFT.

- Very often computing the full results of Feynman integrals neither mandatory nor necessary.
  - An example from $B \to \gamma \ell \nu_\ell$:

$$F_{V,\mathrm{LP}}(n\cdot p) = \frac{Q_u\, m_B}{n\cdot p}\, \tilde{f}_B(\mu)\, C_\perp(n\cdot p,\mu) \int_0^\infty d\omega\, \frac{\phi_B^+(\omega,\mu)}{\omega}\, J_\perp(n\cdot p,\omega,\mu)\,.$$

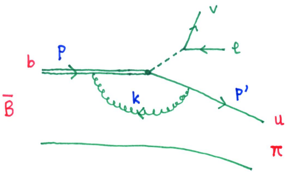
  - Only need to evaluate the hard and jet functions with the strategy of regions.
  - Much easier than evaluating the full QCD amplitude.
  - Even if you obtain the full result, the heavy quark expansion must be then implemented.
    $\hookleftarrow$ QCD/SCET factorization at leading power in Λ/m_b.

- Determine the leading regions of Feynman integrals $\Rightarrow$ Identify the relevant field modes.
  $\hookrightarrow$ Crucial to establish QCD factorization theorems with the traditional and modern formalisms.

- The infrared behaviours of Feynman integrals can be addressed with distinct techniques.
  - Interesting development on the evolution kernels of the light-ray operators.
  - The infrared subtractions generally complicated (in particular for heavy quark physics).

Factorization of Feynman Diagrams

- Example: exclusive weak decay $B \to \pi \ell \bar{\nu}_\ell$ $[b \to u \ell \bar{\nu}_\ell]$.



For illustration take b- and u-quark on-shell and massive gluon

$p^2 = m^2 \quad (p = mv = m\,(1,0,0,0))$

$p'^2 = 0 \quad (p' = En_- = E\,(1,0,0,-1))$

$p \cdot p' = mE$

$\lambda^2 \ll m^2$ provides the small scale, analogous to Λ^2

- Feynman integral:

$$I \equiv m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2-\lambda^2][k^2+2p\cdot k][k^2+2p'\cdot k]}$$

for simplification no numerator (all scalar propagators)

$$\equiv I\left(\frac{\lambda}{m}, E/m\right)$$

small

UV + IR finite integral

Case I: QCD $\to$ HQET

- The EFT description of the process depends on the external momenta.
- For the soft pion (u-quark): $E = \mathcal{O}(\lambda)$.

Even simpler: $E = 0$. Set $m = 1$, $\hat{\lambda} \equiv \lambda/m = \lambda$

$$I(\hat{\lambda}, 0) = \text{logs of square roots} = -\frac{\pi}{\lambda} + \left(-\frac{1}{2}\ln \lambda^2 + 1\right) + \mathcal{O}(\lambda)$$

To factorize the diagram, need to know the relevant integration regions.

Aim: construct expansion in λ ($\simeq \Lambda/m_b$!) from sum of all relevant momentum regions (modes)

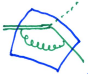
HARD region
loop momentum
$k \sim m$

$$I_h = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2][k^2+2p\cdot k][k^2]} + \text{higher order in } \lambda^2$$

drop λ^2 because $k^2 \sim m^2 \gg \lambda^2$

$p' = 0$

I_h is now IR-divergent $\to$ need auxiliary regularisation; take dim. reg.



off-shell $\sim m^2$
short-distance

Case I: QCD → HQET

$$I_h = -\frac{1}{2\epsilon} - \frac{1}{2}\ln\mu^2 + 1 + \mathcal{O}(\lambda^2)$$

μ scale of dim.reg.

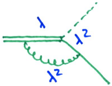
SOFT region
loop momentum
$k \sim \lambda$

$$I_s = m^2 \frac{(4\pi)^2}{i} \underbrace{\int [dk]}_{\lambda^4} \underbrace{\frac{1}{[k^2-\lambda^2][k^2][2p\cdot k]}}_{1/\lambda^5} \Big(\underbrace{1}_{1} - \underbrace{\frac{k^2}{2p\cdot k}}_{\lambda} + \dots \Big)$$

now $k^2 \sim \lambda^2$

I_s is UV-divergent from $\mathcal{O}(1)$ on

$\longrightarrow$ use SAME auxiliary regularization



$$= -\frac{\pi}{\lambda} + \left[\frac{1}{2\epsilon} - \frac{1}{2}\ln\frac{\lambda^2}{\mu^2}\right] + \mathcal{O}(\lambda)$$

internal lines with small virtuality
long-distance

Sum of hard and soft reproduces the expansion of the exact result.

Case I: QCD $\to$ HQET

- First lessons:
  - Expansion in ratio of scales ($\to$ heavy quark expansion) can be obtained by adding contributions from the relevant momentum regions.
  - Direct computation of the expansion is much simpler than the computation of the exact result, because all integrals involve only one scale, not several.
  - Each term scales with a definite power of λ, which is determined before a calculation.

- Diagrammatic factorization:

$$\langle u(p'=0)|\bar{u}\Gamma b|b(p)\rangle = \sum_i C^i_{hard}(m/\mu) \cdot M^i_{soft}(\mu/\lambda)$$

independent on external momenta
$\longrightarrow$ same when $b \to \bar{B}$
$u \to \pi, \rho, \ldots$

depends on Dirac matrix Γ
weak coupling in QCD

depends on soft external momenta (here $p'=0$) and IR scale λ

$\longrightarrow$ strong coupling in QCD

Effective fields and Lagrangians

- Introduce fields for soft modes and construct the Lagrangian that corresponds to the diagrammatic expansion rules ⇒ reproduce the soft matrix element $M^i_{\rm soft}$.

- Effective heavy quark field:

$p = mv + k$

$\sim \lambda$ changes by soft interactions

$h_v(x) \equiv \frac{1+\slashed{v}}{2}\, e^{imv\cdot x}\, \psi(x)$

fixed label v never changes

dual to k $x \sim 1/\lambda$

i.e. typical variations of h_v occur only over large distances

Diagrammatically, in the soft region

$\frac{\slashed{p}+m}{p^2-m^2} \to \frac{1}{v\cdot k} + \dots$ corresponds to $\bar\psi(i\slashed{D}-m)\psi \to \bar h_v\, iv\cdot D\, h_v + \dots$

(HQET)

Interactions of heavy quarks with soft modes are described by heavy quark effective theory.

Heavy Quark Effective Theory

- Effective Lagrangians and currents:

$$
\begin{aligned}
\mathscr{L}_{\text{eff}} &= \left[\bar{h}_v\,(iv\cdot D)\,h_v + \text{NLP}\right] + \sum_{\text{light quarks}} \bar{q}\,\not{D}\,q - \frac{1}{4}\,G^2\,, \\
\bar{u}\Gamma b &= \bar{u}\Gamma h_v + \mathscr{O}(\alpha_s, \lambda)\,.
\end{aligned}
$$

- Scaling rules:

$$h_v(x)\bar{h}_v(y) \sim \int \underbrace{d^4k}_{\lambda^4}\, e^{ik(x-y)}\, \underbrace{\frac{i}{v\cdot k}}_{1/\lambda} \sim \lambda^3 \quad \Rightarrow \quad \text{assign scaling} \quad h_v \sim \lambda^{3/2}$$

Object	Scaling
h_v, q	$\lambda^{3/2}$
A^μ	λ
$iD^\mu = i\partial^\mu + gA^\mu$	λ

$x \sim 1/\lambda \qquad \partial \sim \lambda$

$$S_{\text{eff}} = \int \underbrace{d^4x}_{1/\lambda^4}\, \mathscr{L}_{\text{eff}} = \mathcal{O}(\lambda^0) + \text{corrections}$$

In HQET power counting is simple, because there is only one mode (soft).

λ can only occur as λ/m so $\quad\lambda$- expansion $\simeq$ $1/m$ expansion

$\simeq$ dimensional analysis

Case II: QCD $\to$ SCET

- What happens for the energetic pion (u-quark) with $E = \mathcal{O}(m)$?
- Fail to reproduce the QCD result by adding the hard and soft contributions.

Even simpler $E = E_{max} = \frac{m}{2}$

$$I(\hat{\lambda}, \tfrac{1}{2}) = \text{logs and di-logs} = -\frac{1}{4}\left(\ln^2\lambda^2 + \pi^2\right) + \mathcal{O}(\lambda)$$

Hard contribution

$k \sim m$

$$I_h = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2][k^2 + 2p\cdot k][k^2 + 2p'\cdot k]} + \ldots$$

cannot be dropped now

$$= -\frac{1}{2\varepsilon^2} - \frac{1}{2\varepsilon}\ln\mu^2 - \frac{1}{4}\ln^2\mu - \frac{\pi^2}{24} + \mathcal{O}(\lambda^2)$$

Soft contribution

$k \sim \lambda$

$$I_s = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2 - \lambda^2][2p\cdot k][2p'\cdot k]} + \ldots$$

$$= -\Gamma(\epsilon) \int_0^\infty \frac{dx}{x} (x^2 + \lambda^2)^{-\epsilon} + \ldots$$

ill-defined in dim. reg., so $I \neq I_h + I_s$

A relevant momentum region is missing !

Case II: QCD → SCET

- External kinematics:

External kinematics : additional vector $p' = En_- = E(1,0,0,-1)$ $\quad n_-^2 = 0$

$\longrightarrow$ introduce $n_+ = (1,0,0,1)$ such that $n_+^2 = 0$, $n_+ \cdot n_- = 2$

and write

$$q = n_+ \cdot q \, \frac{n_-}{2} + q_\perp + n_- \cdot q \, \frac{n_+}{2} \qquad \text{(light-cone decomposition)}$$

$\longrightarrow$ $n_+ \cdot p' \sim m$ large

$p'^2 = n_+ \cdot p' n_- \cdot p' + p_\perp'^2 = 0$ small

- Collinear contribution:

COLLINEAR region

loop momentum

$n_+ \cdot k \sim m$

$k^2 \sim \lambda^2$

in general : $k^2 \ll n_+ \cdot k$

$$I_c = m^2 \frac{(4\pi)^2}{i} \int [dk] \frac{1}{[k^2 - \lambda^2][m n_+ \cdot k][k^2 + n_+ \cdot p' n_- \cdot k]} + \ldots$$

- also ill-defined in dim. reg.
  need additional intermediate regularization (analytic - so that scaleless integrals still vanish)
- $I_s + I_c$ is well-defined in dim. reg.

$$I_c + I_s = \frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon} \ln \mu^2 + \frac{1}{4} \ln^2 \mu^2 - \frac{1}{4} \ln^2 \lambda^2 - \frac{5\pi^2}{24} + \mathcal{O}(\lambda)$$

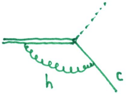
- $I_h + I_c + I_s$ reproduces the expanded exact result. This works to all orders in λ.

Case II: QCD $\to$ SCET

- Key observation:

> When there are energetic, nearly on-shell, nearly massless external lines, one must introduce collinear loop momentum configurations. The infrared structure is different from the case $E \sim \lambda$.

- An important difference is the non-locality.



$n_+ \cdot p_c \sim m$

collinear momentum

$\Rightarrow$ hard subgraphs cannot be expanded in $n_+ \cdot p_c$ of external collinear momenta

$\Rightarrow$ non-polynomial in $n_+ \cdot p_c$ $\cong$ non-local in position space

expect $\frac{1}{i n_+ \cdot \partial}$

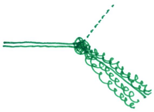
This is indeed true for the collinear Lagrangian in SCET.

$$\mathscr{L}_c = \bar{\xi} \left(i n_- \cdot D + i \not{D}_\perp \frac{1}{i n_+ \cdot D} i \not{D}_\perp \right) \frac{\not{n}_+}{2} \xi \,.$$

Case II: QCD $\to$ SCET

- Next lessons:
  - For a given process (external kinematics) find the relevant momentum regions to construct the expansion.
  - Introduce effective fields and vertices which reproduce this $\Rightarrow$ effective Lagrangian.

- Relevant field modes in SCET$_\mathrm{I}$.



assume:
jet with
$E \sim m_b$
invariant mass
$p^2 \sim m_b^2 \lambda$

Relevant modes (set $m_b = 1$)

hard-collinear	$n_+ \cdot p \sim 1$	$n_- \cdot p \sim \lambda$	$p_\perp \sim \lambda^{1/2}$
soft	$p \sim \lambda$		
later add collinear	$n_+ \cdot p \sim 1$	$n_- \cdot p \sim \lambda^2$	$p_\perp \sim \lambda$

Note: $p_{hc}^2 \sim \lambda \gg p_s^2 \sim p_c^2 \sim \lambda^2$

Can add a soft line to a jet $(p_{\rm hc} + p_{\rm s})^2 \sim \lambda$, but not a line with $p \sim \lambda^{1/2}$.

Soft-collinear effective theory: power counting

- Light quarks:

collinear quark $\quad \psi_c = \xi + \eta \qquad \xi \equiv \frac{\not{n}_-\not{n}_+}{4}\psi_c$ large $\qquad \eta \equiv \frac{\not{n}_+\not{n}_-}{4}\psi$ small

since $\not{p}' u_c(p') = 0$

$\longrightarrow$ η is integrated out (see below)

$$\underbrace{\xi(x)\bar{\xi}(y)}\ \sim \int \underbrace{d^4p}_{\lambda^2}\, e^{-ip(x-y)}\ \underbrace{\frac{i\not{n}_+ p}{p^2}}_{\frac{1}{\lambda}} \sim \lambda \qquad \Rightarrow \qquad \text{Field } \xi \text{ with } \not{n}_-\xi = 0 \text{ and } \xi \sim \lambda^{1/2}$$

soft quark

$$\underbrace{q(x)\bar{q}(y)}\ \sim \int \underbrace{d^4k}_{\lambda^4}\, e^{-ik(x-y)}\ \underbrace{\frac{i\not{k}}{k^2}}_{1/\lambda} \sim \lambda^3 \qquad q \text{ with } q \sim \lambda^{3/2}$$

- Heavy quarks:

can never be collinear since always $n_+ \cdot p \sim 1$ and $n_- \cdot p \sim 1$

soft heavy quark (as in HQET)

$$Q = e^{-im v\cdot x}\,(h_v + \underbrace{H_v}_{\text{integrate out}}) \qquad \Rightarrow \qquad h_v \sim \lambda^{3/2} \quad (\text{as before})$$

$\not{v} h_v = h_v$

Soft-collinear effective theory: power counting

- Gluons:

collinear gluon $\quad A_c^\mu(x) A_c^\nu(y) \sim \int d^4p \, e^{-ip(x-y)} \, \frac{i}{p^2}\left(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$

$\Rightarrow \quad n_+ \cdot A_c \sim 1 \,, \quad A_{\perp c} \sim \lambda^{1/2} \,, \quad n_- \cdot A_c \sim \lambda$ $\quad$ (same as collinear momentum)

soft gluon $\quad A_s \sim \lambda$

- Derivatives on fields:

soft fields vary over distances $x \sim 1/\lambda \quad \Rightarrow \quad \partial \phi_s \sim \lambda \phi_s$

collinear fields vary differently in different directions

$$n_+ \cdot x \sim 1/n_- \cdot p_c \sim 1/\lambda \qquad x_\perp \sim 1/p_{c\perp} \sim 1/\lambda^{1/2}$$

$$n_- \cdot x \sim 1/n_+ \cdot p_c \sim 1$$

$$\Rightarrow \quad \begin{aligned} n_+ \cdot \partial \phi_c &\sim \phi_c \\ \partial_\perp \phi_c &\sim \lambda^{1/2} \phi_c \\ n_- \cdot \partial \phi_c &\sim \lambda \phi_c \end{aligned}$$

(same as momentum)

$\Longrightarrow$ Power counting for operators (field products)

Constructing the soft-collinear Lagrangian

- Here we aim at deriving the $\mathrm{SCET_I}$ Lagrangian and first neglect the soft quark field.
- Step I: Integrate out small components of the collinear quark field.

$$\mathcal{L}_c = \bar{\psi}_c i \not{D} \psi_c \underset{\psi_c = \xi + \eta}{=} \bar{\xi} \frac{\not{n}_+}{2} in_- \cdot D \xi + \bar{\eta} \frac{\not{n}_-}{2} in_+ \cdot D \eta + \bar{\xi} i \not{D}_\perp \eta + \bar{\eta} i \not{D}_\perp \xi$$

Gaussian path integral over η can be done exactly and sets $\eta = -\frac{\not{n}_+}{2} \frac{1}{in_+ \cdot D + i\varepsilon} i\not{D}_\perp \xi$

definition!

Functional determinant $\det[in_+ \cdot D] = \det[in_+ \cdot \partial]$ is a field-independent (irrelevant constant)

$$\Rightarrow \quad \mathcal{L}_c = \bar{\xi} \left(in_- \cdot D + i\not{D}_\perp \frac{1}{in_+ \cdot D} i\not{D}_\perp \right) \frac{\not{n}_+}{2} \xi$$

non-local

This is exact!

- No degrees of freedom have been integrated out! $\mathscr{L}_c$ is simply the QCD Lagrangian in a frame where all particles are boosted to large momentum.
- Non-locality can be made explicit in terms of Wilson lines.

$$\frac{1}{in_+ \cdot D} = W \frac{1}{in_+ \cdot \partial} W^\dagger .$$

Constructing the soft-collinear Lagrangian

- Step II: Expansion of the exact Lagrangian in $\lambda^{1/2}$.

$$in_-\cdot D = \underset{\lambda}{in_-\cdot\partial} + \underset{\lambda}{gn_-\cdot A_c} + \underset{\lambda}{gn_-\cdot A_s}$$

$$iD_\perp = \underset{\lambda^{1/2}}{i\partial_\perp} + \underset{\lambda^{1/2}}{gA_{\perp c}} + \underset{\lambda}{gA_{\perp s}} \equiv iD_{\perp c} + gA_{\perp s}$$

$$\frac{1}{in_+\cdot D} = \underset{\lambda^0}{\frac{1}{in_+\cdot D_c}} - \underset{\lambda}{\frac{1}{in_+\cdot D_c}\, gn_+\cdot A_s \, \frac{1}{in_+\cdot D_c}} + \ldots$$

Multipole expansion

$$\int d^4x \; \bar{\xi}(x)\, n_-\cdot A_s(x)\, \frac{\not{n}_+}{2}\, \xi(x)$$

dominated by rapid variations of collinear fields
$n_-\cdot x \sim 1$, $x_\perp \sim 1/\lambda^{1/2}$
$n_+\cdot x \sim 1/\lambda$

slowly varying $x \sim 1/\lambda$ in all components
$\Rightarrow$ Taylor expansion in $n_-\cdot x, x_\perp$ around
$$x_-^\mu \equiv n_+\cdot x\, \frac{n_-^\mu}{2}$$

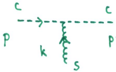
Hence, in products of collinear and soft fields, expand the soft fields:

$$\phi_s(x) = \underset{\lambda^0}{\phi_s(x_-)} + \underset{\lambda^{1/2}}{[x_\perp\cdot\partial\phi_s](x_-)}$$
$$+ \underset{\lambda}{\frac{n_-\cdot x}{2}[n_+\cdot\partial\phi_s](x_-)} + \underset{\lambda}{\frac{1}{2}x_\perp^\mu x_\perp^\nu[\partial_\mu\partial_\nu\phi_s](x_-)}$$
$$+ \;\ldots\ldots$$

Constructing the soft-collinear Lagrangian

- Remarks on the multipole expansion.

Interpretation :



$p' = p + n_- \cdot k \, \frac{n_+}{2}$

$k_\perp, n_+ \cdot k$ are expanded , because $k_\perp \ll p_\perp$, $n_+ \cdot k \ll n_+ \cdot p$ (diagrammatic interpretation)

cf. atomic (non-relativistic) physics : $e^{i\vec{k}\cdot\vec{x}} \approx 1 + i\vec{k}\cdot\vec{x} + \ldots$

phase of light wave

$\frac{\text{size of atom}}{\text{wavelength of light}} \ll 1$

" dipole approximation "

Here x_- plays the role of time t , which is not expanded

- The expanded collinear Lagrangian:

$$\mathcal{L}_c = \mathcal{L}_\xi^{(0)} + \mathcal{L}_\xi^{(1)} + \ldots = \bar{\xi} \left(i n_- \cdot D + i \not{D}_{\perp c} \frac{1}{i n_+ \cdot D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi$$

$$+ \bar{\xi} \left(x_\perp \cdot \partial \, g n_- \cdot A_s + i \not{D}_{\perp c} \frac{1}{i n_+ D_c} g \not{A}_{\perp s} + g \not{A}_{\perp s} \frac{1}{i n_+ D_c} i \not{D}_{\perp c} \right) \frac{\not{n}_+}{2} \xi + \ldots$$

- all soft fields are taken at x_-
- translation invariance not manifest (no momentum conservation at vertices)
- $\mathcal{L}_\xi^{(0)}$ contains only the $n_- \cdot A_s$ component (at x_-) – key property for factorization proofs
- gauge - invariance not manifest

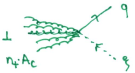
Constructing the soft-collinear Lagrangian

- Step III: Include soft quarks.

$$\bar\Psi \slashed{A} \Psi \ni \bar q \slashed{A}_c \xi + \dots \approx \bar q \slashed{A}_{\perp c} \xi + \dots$$

$$\slashed{A}_c = \underset{\substack{\lambda^0 \\ \text{but } \slashed{n}_- \xi = 0}}{n_+ \cdot A_c \frac{\slashed{n}_-}{2}} + \underset{\lambda^{1/2}}{\slashed{A}_{\perp c}} + \underset{\lambda}{n_- \cdot A_c \frac{\slashed{n}_+}{2}}$$

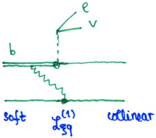
but this is not all :



$$\mathcal{L}^{(1)}_{\xi q} = \underbrace{\underset{\lambda^{3/2}}{\bar q}\; W_c^\dagger\; \underset{\lambda^{1/2}}{i\slashed{D}_{\perp c}}\; \underset{\lambda^{1/2}}{\xi}}_{\lambda^{5/2}} + h.c.$$

gauge-invariant

- soft quark interactions are power-suppressed , $\mathcal{O}(\lambda^{1/2})$
  higher-order terms are known
- $\mathcal{L}^{(1)}_{\xi q}$ is very important for exclusive B decays
  Soft spectator quark must be converted into collinear.



- More discussions can be found in the review article [arXiv: 1410.1892] and references therein.