The Model and Particle-in-cell Simulation of Three-Dimensional Betatron Oscillation in Plasma Wakefield Acceleration

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1. Introduction

Betatron oscillation is an important phenomenon of electrons in plasma wakefield acceleration. During the process of oscillation, electrons emit synchrotron radiation, which affects the electrons in return. The force reacted on the electrons itself is called the radiation reaction (RR). Theoretically, we have built an three-dimensional betatron oscillation model with RR to describe the long-distance wakefield acceleration. In the model, long-term equations (LTE) are obtained by averaging the motion of the electron in the betatron time scale. In simulation, however, the process has not be simulated yet by QuickPIC, in the particle pusher module of which the longitudinal velocity of an electron is set to the speed of light while the transverse velocity and the radiation reaction are ignored. To solve the problem, we used Boris algorithm to solve for the motion of electrons in the three-dimensional wake, and added radiation reaction to the equations of motion. Finally, two phenomena predicted by the model are observed in simulation results, i.e. the precession movement of the elliptical betatron trajectory in the betatron phase shift dominant regime, and the tapering effect of the elliptical betatron trajectory in the radiation reaction dominant regime.

2. The motion of single electron

Neglecting the interaction between beam particles, the expression of wakefield is:

$$E_z = E_{z0} + \lambda\zeta_1,$$
$$E_r = \kappa^2(1-\lambda)r,$$
$$B_\theta = -\kappa^2\lambda r,$$

Where $\zeta = z - \beta_w t$, E_{z0} is the electric field at $\zeta = \langle\zeta\rangle$, $\langle\zeta\rangle$ is the average of ζ and ζ_1 is the high-frequency term of ζ. λ is the slope of longitudinal electric field at $\zeta = \langle\zeta\rangle$ and κ^2 is the coefficient of restoring force (generally $1/2$).

RR is the dominate force when $k_p r_e\langle\gamma\rangle^{5/2} \gg 1$:

$$F_\mu^{rad} \approx \frac{2}{3} r_e\left(\frac{dP_\nu}{d\tau}\frac{dP^\nu}{d\tau}\right)P_\mu, \quad \text{or } \vec{f}^{rad} = \frac{2}{3} r_e\gamma\left(\dot{\gamma}^2 - \dot{p}_x{}^2 - \dot{p}_y{}^2 - \dot{p}_z{}^2\right)\vec{p}.$$

3. The long-term equations

Area in transverse phase space:
$$\dot{S}_x = -\frac{1}{8}\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}S_xS_y\sin 2\Delta\Phi - \frac{1}{4}r_e\kappa^3\langle\gamma\rangle^{1/2}\left(S_x^2 + \frac{4-\cos 2\Delta\Phi}{3}S_xS_y\right),$$
$$\dot{S}_y = \frac{1}{8}\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}S_xS_y\sin 2\Delta\Phi - \frac{1}{4}r_e\kappa^3\langle\gamma\rangle^{1/2}\left(S_y^2 + \frac{4-\cos 2\Delta\Phi}{3}S_xS_y\right),$$

Transverse phase difference:
$$d\Delta\Phi/dt = \frac{1}{8}\left[\frac{1}{4}\lambda\beta_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}(S_y - S_x)\sin^2\Delta\Phi - \frac{1}{24}r_e\kappa^3\langle\gamma\rangle^{1/2}(S_x+S_y)\sin 2\Delta\Phi,$$

Longitudinal shift: $\langle\dot{\zeta}\rangle = \frac{1}{2}\gamma_w^{-2} - \frac{1}{2}\langle\gamma\rangle^{-2} - \frac{1}{4}\kappa\langle\gamma\rangle^{-3/2}(S_x+S_y),$

Average energy: $\langle\dot{\gamma}\rangle = -E_{z0}(\langle\zeta\rangle)\beta_{z0} - \frac{1}{3}r_e\kappa^3\langle\gamma\rangle^{3/2}(S_x+S_y).$

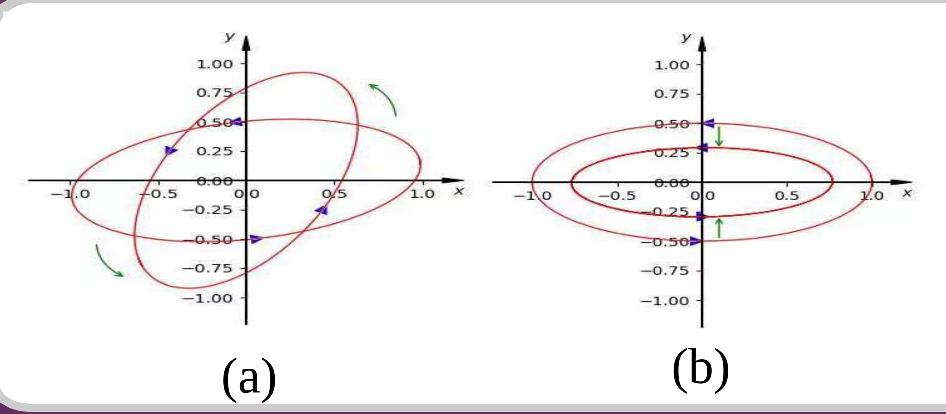

Figure 1. Evolution of the particle trajectory (red ellipse) in the x-y plane. The blue arrows indicate the rotating direction of the particle. (a) In the betatron phase shift dominant regime, (b) In the radiation reaction dominant regime.

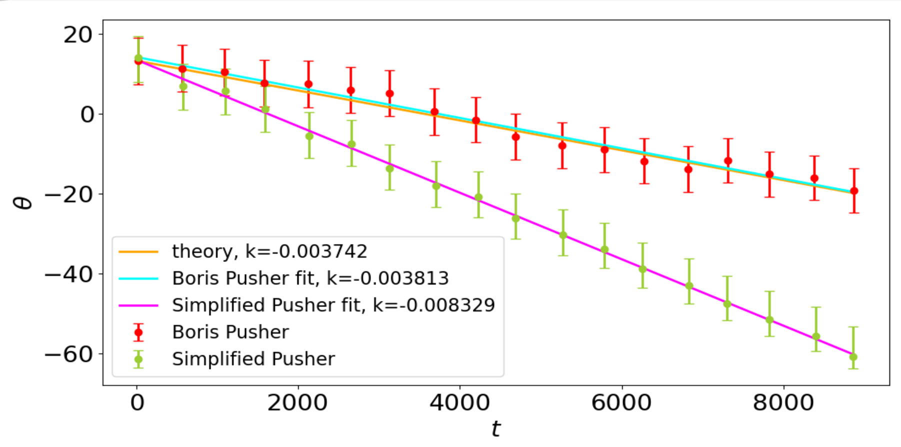

Figure 2. The angle θ of the major axis of the elliptical betatron trajectory changing with time. The difference of results between Boris Pusher and theory model is 1.90% while between Simplified Pusher and theory model is 123%.

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[4] I. Y. Kostyukov et al., Phys. Rev. ST Accel. Beams 15, 111001 (2012).
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[6] W. An et al, J. Comput. Phys. 250 (2013) 165-177.
[7] Viktor K. Decyk et al, CPC 282 (2023).

4. Betatron phase shift dominant regime ($k_p r_e\langle\gamma\rangle^{5/2} \ll 1$)

Angular velocity of the precession movement of the elliptical betatron trajectory :

$$\frac{d\theta}{dt} = -\frac{1}{8}\left[\frac{1}{4}\lambda v_{z0} - \kappa^2(1-2\lambda)\right]\langle\gamma\rangle^{-2}L_z,$$

where θ is the angle of the major axis of the ellipse (from particle trajectory in the betatron time scale) and L_z is the longitudinal angular momentum of the electron.

Transverse phase space area is conserved: $\frac{d(S_x+S_y)}{dt} = 0.$

5. Radiation reaction dominant regime ($k_p r_e\langle\gamma\rangle^{5/2} \gg 1$)

Area in transverse phase space decreases and the difference between Simplified_RR Pusher and Boris_RR Pusher can be ignored:

$$\frac{d(S_x+S_y)}{dt} = -\frac{1}{4}r_e\kappa^3\langle\gamma\rangle^{\frac{1}{2}}\left[S_x^2 + S_y^2 + \frac{2(4-\cos 2\Delta\Phi)}{3}S_xS_y\right].$$

6. Five types of particle pushers

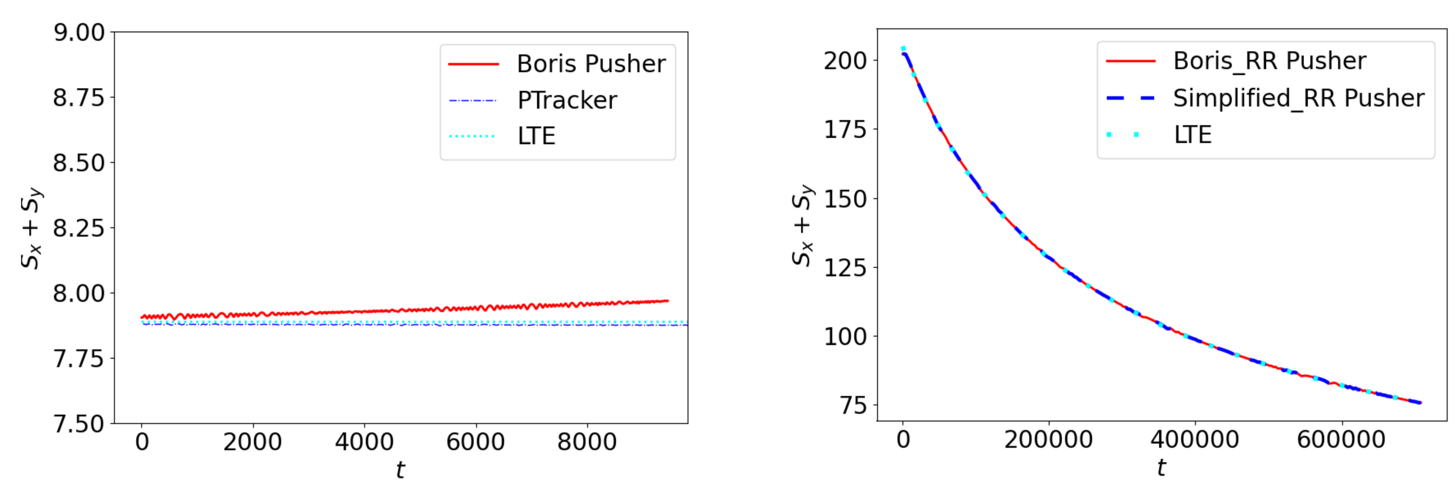
Frozen Pusher: keeping uniform motion in the longitudinal direction and stationary motion in the transverse direction;

Simplified Pusher ($v_z = c$): $f_x = -E_x + B_y$, $f_y = -E_y - B_x$, $f_z = -E_z$;

Simplified_RR Pusher: considering RR based on Simplified Pusher;

Boris Pusher: using Boris algorithm, $\vec{f} = -\vec{E} - \vec{v}\times\vec{B}$;

Boris_RR Pusher: considering RR based on Boris Pusher.



(a) (b)

Figure 3. Area in transverse phase space ($S_x + S_y$) changing with time: (a) In the betatron phase shift dominant regime, comparison among Boris Pusher, Ptracker [2] and long-term evolution equation (LTE) [3]; (b) In the radiation reaction dominant regime, comparison among Boris_RR Pusher, Simplified_RR Pusher, and Long-Term Evolutionary Equation (LTE) [3].

Conclusion:

1. A three-dimensional betatron oscillation model is established; Long-term equations give information on amplitude, phase and average energy changing with time;
2. Simulation based on three-dimensional betatron oscillation model can be done by five particle pushers;
3. When $k_p r_e\langle\gamma\rangle^{5/2} \ll 1$, the precession movement of the elliptical betatron trajectory is observed, and the angular velocity of precession is consistent with theoretical expectations;
4. When $k_p r_e\langle\gamma\rangle^{5/2} \gg 1$, RR reduces the area in transverse phase space and makes the ellipse become thinner.

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