Soft Theorem to Three Loops in QCD and $\mathcal{N} = 4$ Super Yang-Mills Theory

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Soft Theorem

Soft theorem (QED)

The amplitude for the emission of soft photons takes the factorized form: (Weinberg, QFTt Vol I)

$$\mathcal{M}(q_1, q_2, \cdots; p_1, p_2, \ldots) \simeq \prod_i \left[\sum_j \eta_j Q_j \frac{\varepsilon(q_i) \cdot p_j}{q_i \cdot p_j} \right] \mathcal{M}(p_1, p_2, \ldots). \quad (1)$$

q_i : momenta of soft photons, p_i : momenta of hard-scattering particles

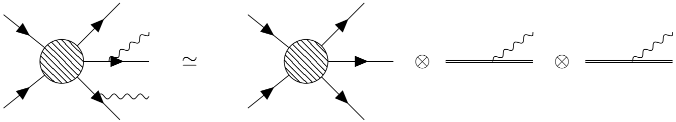


Figure: Soft factorization for the soft-photon emission.

Soft theorem holds to all orders in perturbation theory in massive QED.

Soft Factorization in QCD

$$
\begin{aligned}
&|\mathcal{M}^{\mu_1\mu_2\ldots\mu_m}_{s_1 s_2\ldots s_m, c_1 c_2\ldots c_m}(q_1, q_2, \ldots, q_m; p_1, p_2, \ldots)\rangle \\
&\simeq (g_s\mu^\epsilon)^m \boxed{J}(q_1, q_2, \ldots, q_m)\,|\boxed{\mathcal{M}}(p_1, p_2, \ldots)\rangle\,.
\end{aligned} \tag{2}
$$

soft current | hard-scattering amplitude

The soft current $J(q)$ is process independent and can be calculated perturbatively.

One-loop soft current for a single gluon emission: Catani&Grazzini(2000)

$$
\begin{aligned}
J_a = &-\frac{(g_s\mu^\epsilon)^3}{(4\pi)^2}\frac{1}{\epsilon^2}\frac{\Gamma^3(1-\epsilon)\Gamma^2(1+\epsilon)}{\Gamma(1-2\epsilon)} \\
&\times i f_{abc}\sum_{i\neq j} T_i^b T_j^c \left(\frac{\varepsilon(q)\cdot p_i}{p_i\cdot q} - \frac{\varepsilon(q)\cdot p_j}{p_j\cdot q}\right)\left[\frac{4\pi p_i\cdot p_j e^{-i\lambda_{ij}\pi}}{2p_i\cdot q p_j\cdot q e^{-i\lambda_{iq}\pi}e^{-i\lambda_{jq}\pi}}\right]^\epsilon .
\end{aligned} \tag{3}
$$

Motivations

Soft current with two hard-scattering partons

Phenomenological side: necessary ingredients for both the fixed-order calculation and the resummation of large logarithms

Higgs production in hadron colliders, Drell-Yan, diject production in $e^- e^+$ collision, ...

Theoretical side: reveals some structures of the full amplitude

principle of leading transcendentality, Bern-Dixon-Smirnov (BDS) ansatz Bern et al. (2005), ...

Previous results:

- One loop:
  Bern and Chalmers (1995), Bern et al. (1998), Bern et al. (1999), Catani & Grazzini (2000)
- Two loops:
  Badger & Glover (2004), Li & Zhu (2013), Duhr & Gehrmann (2013) Dixon et al. (2020)

Soft Currents in SCET

soft collinear effective theory (SCET):

A effective theory for the soft can collinear modes.

QCD → integrate out the hard mode → SCET

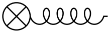
single-gluon soft current in SCET:

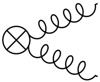
$$J(q) = \langle g(q) | Y_n^\dagger Y_{\bar{n}} | \Omega \rangle \,. \tag{4}$$

Soft Wilson line:

$$Y_n^\dagger(x) \equiv \exp\left(i g_s T^a \int_0^\infty ds\; n \cdot A^a(x + sn) \right) \tag{5}$$

Feynman rules of the Wilson line:





$\cdots$

$$-g_s n^\mu \frac{T^{a_1}}{-n\cdot q_1} \qquad g_s^2 \frac{n^{\mu_1} n^{\mu_2}}{-n\cdot(q_1+q_2)} \left(\frac{T^{a_1} T^{a_2}}{-n\cdot q_1} + \frac{T^{a_2} T^{a_1}}{-n\cdot q_2} \right) \qquad \cdots$$

The soft function can be calculated perturbatively in SCET:

$$J(q) = g_s \sum_{i=0}^{\infty} \left[\left(4\pi e^{-\gamma_E}\right)^{\epsilon} \frac{\alpha_s}{4\pi} \right]^i \varepsilon(q) \cdot J^{(i)}(q). \tag{6}$$

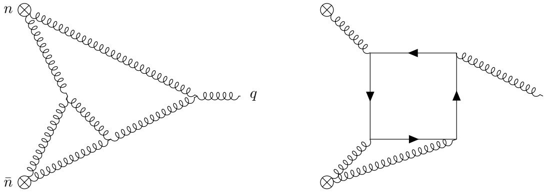


Figure: Representative diagrams for the three-loop soft current.

Methods to Calculate Feynman Integrals

- Mellin-Barnes method
- Sector decomposition
- Dimensional-recurrence-and-analyticity method Tarasov (1996), Laporta (2000), Lee (2010)
- Differential-equation method Kotikov (1991), Remiddi (1997), Henn (2013)

Introduce auxiliary scales for single-scale integrals

Drinfeld-associator method Henn et al. (2014)

Auxiliary mass flow Liu, Ma, Wang (2018), Liu&Ma (2022), Liu&Ma (2023)

Parametric Representation

Schwinger alpha parametrization

$$\frac{1}{D_i^{\lambda_i+1}} = \frac{e^{-\frac{\lambda_i+1}{2}i\pi}}{\Gamma(\lambda_i+1)} \int_0^\infty dx_i\, e^{ix_iD_i} x_i^{\lambda_i}, \qquad \mathrm{Im}\{D_i\} > 0.$$

parametrization

$$\int \mathrm{d}^d l_1 \mathrm{d}^d l_2 \cdots \mathrm{d}^d l_L \frac{1}{D_1^{\lambda_1+1} D_2^{\lambda_2+1} \cdots D_n^{\lambda_n+1}}$$

$$\to \cdots \to \frac{\Gamma(-\lambda_0)}{\prod_{i=1}^{n+1} \Gamma(\lambda_i+1)} \int d\Pi^{(n+1)} \mathcal{F}^{\lambda_0} \prod_{i=1}^{n+1} x_i^{\lambda_i} \equiv \int d\Pi^{(n+1)} \mathcal{I}^{(-n-1)}.$$

Homogeneous polynomial: $\mathcal{F} = F + U x_{n+1}$

integration measure: $\mathrm{d}\Pi^{(n+1)} \equiv \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \delta(1 - E^{(1)})$

$E^{(n)}$: a positive definite homogeneous function of x_i of degree n.

parametric IBP Chen (2020a,2021,2020b)

$$0 = \int d\Pi^{(n+1)} \frac{\partial}{\partial x_i} \mathcal{I}^{(-n)} + \delta_{\lambda_i 0} \int d\Pi^{(n)} \, \mathcal{I}^{(-n)} \Big|_{x_i=0} . \tag{7}$$

differential equations

$$\frac{\partial}{\partial y} I_m = \int d\Pi^{(n+1)} \frac{\partial}{\partial y} \mathcal{I}^{(-n-1)} = \sum_n M_{mn} I_n . \tag{8}$$

introduce an auxiliary scale

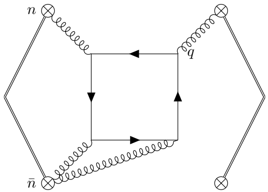
$$I(\lambda_0, \lambda_1, \dots, \lambda_n) = \int \mathrm{d}y \mathrm{d}\Pi^{(n+1)} \, \delta(y - E^{(0)}(x)) \mathcal{I}^{(-n-1)} . \tag{9}$$

$E^{(0)}(x) = \frac{x_i}{x_j}$ is equivalent the method of combining two propagators with a Feynman parameter Hidding&Usovitsch (2022).

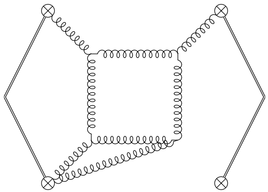
The boundary integrals are again parametric integrals. Thus this method allows us to calculate Feynman integrals recursively.

Result in QCD

$$J_\mu^a(q) = -\frac{g_s}{2}\left(\frac{n_1^\mu}{n_1\cdot q} - \frac{n_2^\mu}{n_2\cdot q}\right)\Bigg[\left(\mathbf{T}_1^a - \mathbf{T}_2^a\right) A_{12} + 2if^{abc}\left(\mathbf{T}_1^b\mathbf{T}_2^c - \mathbf{T}_2^b\mathbf{T}_1^c\right) B_{12} - \left(\mathbf{T}_1^b\mathbf{T}_1^c\mathbf{T}_2^d - \mathbf{T}_2^b\mathbf{T}_2^c\mathbf{T}_1^d\right)\left(C_{12}\, d_A^{abcd} + D_{12}\, d_F^{abcd} N_f\right)\Bigg] + \mathcal{O}(\alpha_s^4)\,, \tag{10}$$



$d_F^{abcd} d_F^{abcd}$



$d_F^{abcd} d_A^{abcd}$

$$S_{12}^{(l)}(q) = \frac{1}{4N_R C_R} \mathsf{Tr}\left\{ \left[\varepsilon^\mu J_\mu^{a(l)}\right] \left[\varepsilon^\nu J_\nu^{a(0)}\right]^* (q) \right\}, \tag{11}$$

$$S_\epsilon = \left(4\pi S_{12}^{(0)} \mu^2 e^{-\gamma_E} \frac{e^{-i\lambda_{12}\pi}}{e^{-i\lambda_{1q}\pi} e^{-i\lambda_{2q}\pi}} \right)^\epsilon ,$$

$$S_{12}^{(l)}(q) = S_{12}^{(0)}(q) S_\epsilon^l \, r_{12}^{(l)} ,$$

$$B_{12}^{(l)} = S_\epsilon^l \, b_{12}^{(l)} , C_{12}^{(l)} = S_\epsilon^l \, c_{12}^{(l)} , D_{12}^{(l)} = S_\epsilon^l \, d_{12}^{(l)} .$$

$$
\begin{aligned}
b_{12}^{(3)} =& C_A^2\Bigg\{-\frac{1}{6\epsilon^6}+\frac{11}{12\epsilon^5}+\frac{1}{\epsilon^4}\left(\frac{119}{324}-\frac{3\zeta_2}{4}\right)+\frac{1}{\epsilon^3}\left(\frac{649\zeta_2}{216}+\frac{2\zeta_3}{3}-\frac{1517}{486}\right) \\
&+\frac{1}{\epsilon^2}\left(\frac{2501\zeta_2}{648}-\frac{2101\zeta_3}{108}-\frac{1487\zeta_4}{288}-\frac{7271}{486}\right) \\
&+\frac{1}{\epsilon}\left(\frac{11\zeta_3\zeta_2}{18}+\frac{437\zeta_2}{972}+\frac{2575\zeta_3}{36}-\frac{22583\zeta_4}{576}+\frac{98\zeta_5}{5}-\frac{446705}{8748}\right)+\dots\Bigg\} \\
&+C_A N_f\Bigg\{-\frac{1}{6\epsilon^5}+\frac{43}{162\epsilon^4}+\frac{1}{\epsilon^3}\left(\frac{895}{486}-\frac{59\zeta_2}{108}\right) \\
&+\frac{1}{\epsilon^2}\left(-\frac{31\zeta_2}{324}+\frac{239\zeta_3}{54}+\frac{2603}{486}\right) \\
&+\frac{1}{\epsilon}\left(\frac{3265\zeta_2}{972}-\frac{4945\zeta_3}{162}+\frac{2437\zeta_4}{288}+\frac{24169}{2187}\right)+\dots\Bigg\} \\
&+C_F N_f\Bigg\{\frac{1}{9\epsilon^3}+\frac{1}{\epsilon^2}\left(\frac{55}{54}-\frac{8\zeta_3}{9}\right)+\frac{1}{\epsilon}\left(\frac{\zeta_2}{6}-\frac{76\zeta_3}{27}-\frac{4\zeta_4}{3}+\frac{1819}{324}\right)+\dots\Bigg\} \\
&+N_f^2\Bigg\{-\frac{4}{81\epsilon^4}+-\frac{40}{243\epsilon^3}+\frac{1}{\epsilon^2}\left(-\frac{2\zeta_2}{27}-\frac{8}{27}\right) \\
&+\frac{1}{\epsilon}\left(-\frac{20\zeta_2}{81}+\frac{260\zeta_3}{81}-\frac{704}{2187}\right)+\dots\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
c_{12}^{(3)} =& \frac{-32\zeta_2\zeta_3 - 16\zeta_5}{\epsilon} - 192\zeta_3^2 + \frac{64\zeta_3}{3} - 64\zeta_2 + \frac{1760\zeta_5}{3} - 940\zeta_6 + \dots\,, \\
d_{12}^{(3)} =& 128\zeta_2 - \frac{128\zeta_3}{3} - \frac{640\zeta_5}{3} + \dots\,.
\end{aligned}
$$

IR Singularities

IR singularities of massless gauge theories Catani (1998), Becher&Neubert (2009a,2009b)

$$|\mathcal{M}_n(\{p\},\mu)\rangle = \lim_{\epsilon\to 0} \boldsymbol{Z}^{-1}(\epsilon,\{p\},\mu)|\mathcal{M}_n(\epsilon,\{p\})\rangle, \tag{12}$$

$$\boldsymbol{Z}(\epsilon,\{p\},\mu) = \mathbf{P}\exp\left[\int_\mu^\infty \frac{d\mu'}{\mu'}\boldsymbol{\Gamma}(\{p\},\mu')\right] \tag{13}$$

$$\boldsymbol{\Gamma}(\{p\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i\cdot\boldsymbol{T}_j}{2}\gamma_{\text{cusp}}(\alpha_s)\ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) + \boldsymbol{\Delta_3} + \mathcal{O}(\boldsymbol{\Delta}_4) \tag{14}$$

soft factorization:

$$\lim_{p_3^0\to 0} |\mathcal{M}_3(p_1,p_2,p_3,\mu)\rangle = J(p_3,\mu)|\mathcal{M}_2(p_1,p_2,\mu)\rangle\,, \tag{15}$$

IR singularities of the soft current:

$$Z_s^{-1}(\epsilon,p_3,\mu) = \lim_{p_3\to 0} Z_3^{-1}(\epsilon,p_1,p_2,p_3,\mu)Z_2(\epsilon,p_1,p_2,\mu) \tag{16}$$

Result in $\mathcal{N} = 4$ SYM

$$S^{(3)}_{12,\,\mathcal{N}=4}(q) = S^0_{12}(q) S^3_\epsilon \Bigg[C_A^3 \Bigg\{ -\frac{1}{6\epsilon^6} - \frac{3\zeta_2}{4\epsilon^4} + \frac{2\zeta_3}{3\epsilon^3} - \frac{1487\zeta_4}{288\epsilon^2} + \frac{1}{\epsilon}\left(\frac{284\zeta_5}{15} - \frac{13\zeta_2\zeta_3}{18}\right) + \frac{5\zeta_3^2}{36} + \frac{174959\zeta_6}{6912} + \ldots \Bigg\} + \frac{3}{2} C_A \Bigg\{ \frac{-32\zeta_2\zeta_3 - 16\zeta_5}{\epsilon} - 192\zeta_3^2 - 940\zeta_6 + \ldots \Bigg\} \Bigg] ,$$

Bern-Dixon-Smirnov (BDS) ansatz Bern et al. (2005)

$$S^{(l)}_{12,\,\mathcal{N}=4}(q) = 2^l S^0_{12}(q) S^l_\epsilon C^l_A r^{(l)}_S(\epsilon) + \text{sub-leading color contribution} ,$$

$$r^{(3)}_S(\epsilon) = -\frac{1}{3}\left(r^{(1)}_S(\epsilon)\right)^3 + r^{(1)}_S(\epsilon) r^{(2)}_S(\epsilon) + f^{(3)}(\epsilon) r^{(1)}_S(3\epsilon) + \mathcal{O}(\epsilon) , \qquad (17)$$

$$f^{(3)}(\epsilon) = \frac{11\zeta_4}{2} + (5\zeta_2\zeta_3 + 6\zeta_5)\epsilon + a\epsilon^2 + \mathcal{O}(\epsilon^3) ,$$

$$a = 31\zeta_3^2 + \frac{1909\zeta_6}{48} \simeq 85.25374611 ,$$ Li & Zhu (2013)

Summary

We calculated the single-gluon soft current with two hard-scattering partons to three loops.

We developed a systematic method to calculate Feynman integrals recursively based on the parametric representation.

We confirmed the prediction on the three-loop soft current in SYM based on the BDS ansatz.

Our results provide an indispensable ingredient for the N^4LO QCD corrections.

Thanks for your patience!

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