Extending Precision Perturbative QCD with Track Functions

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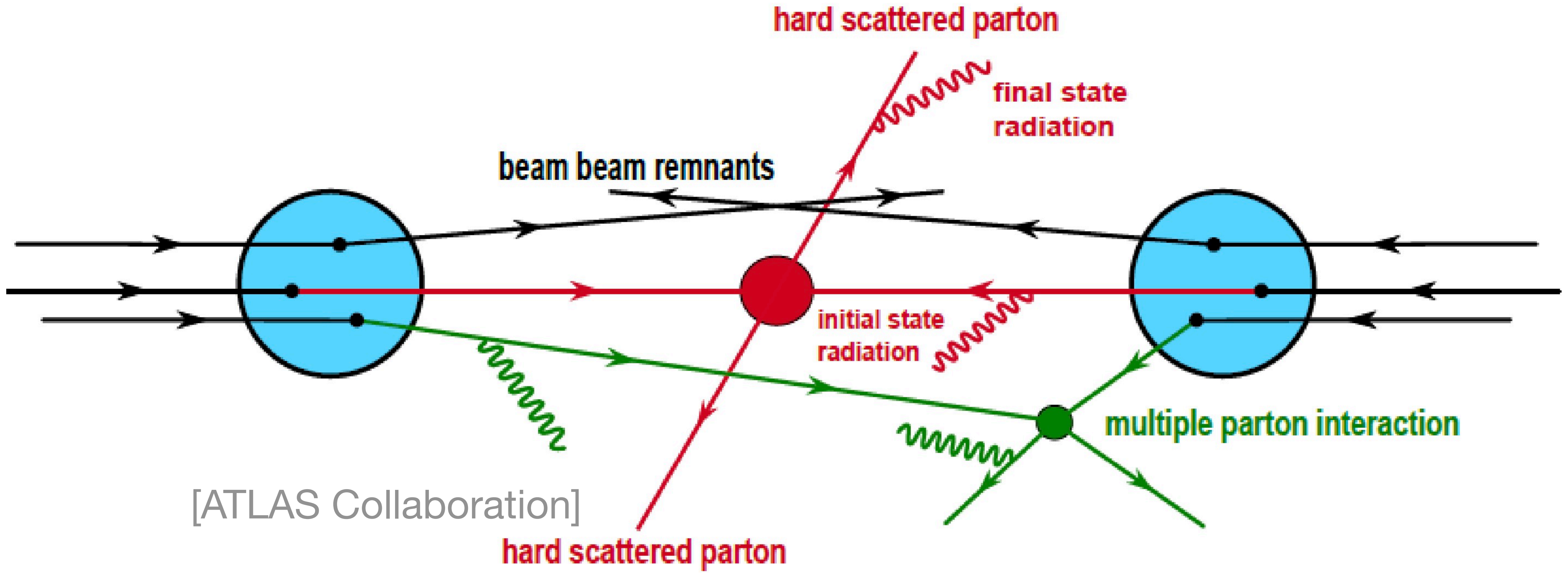
Why track functions?

LHC: ~ 14 TeV



[ATLAS Collaboration]

Future colliders

FCC-hh: ~ 100 TeV

SPPC: ~ 75 TeV

Jet substructure observables

- Jet mass
- Jet angularity, thrust, broadening
- Energy correlation function observable
- N-subjettiness
- Electric charge of a jet

…

→ Use **track-based observables**!

(experimentally cleaner to measure)

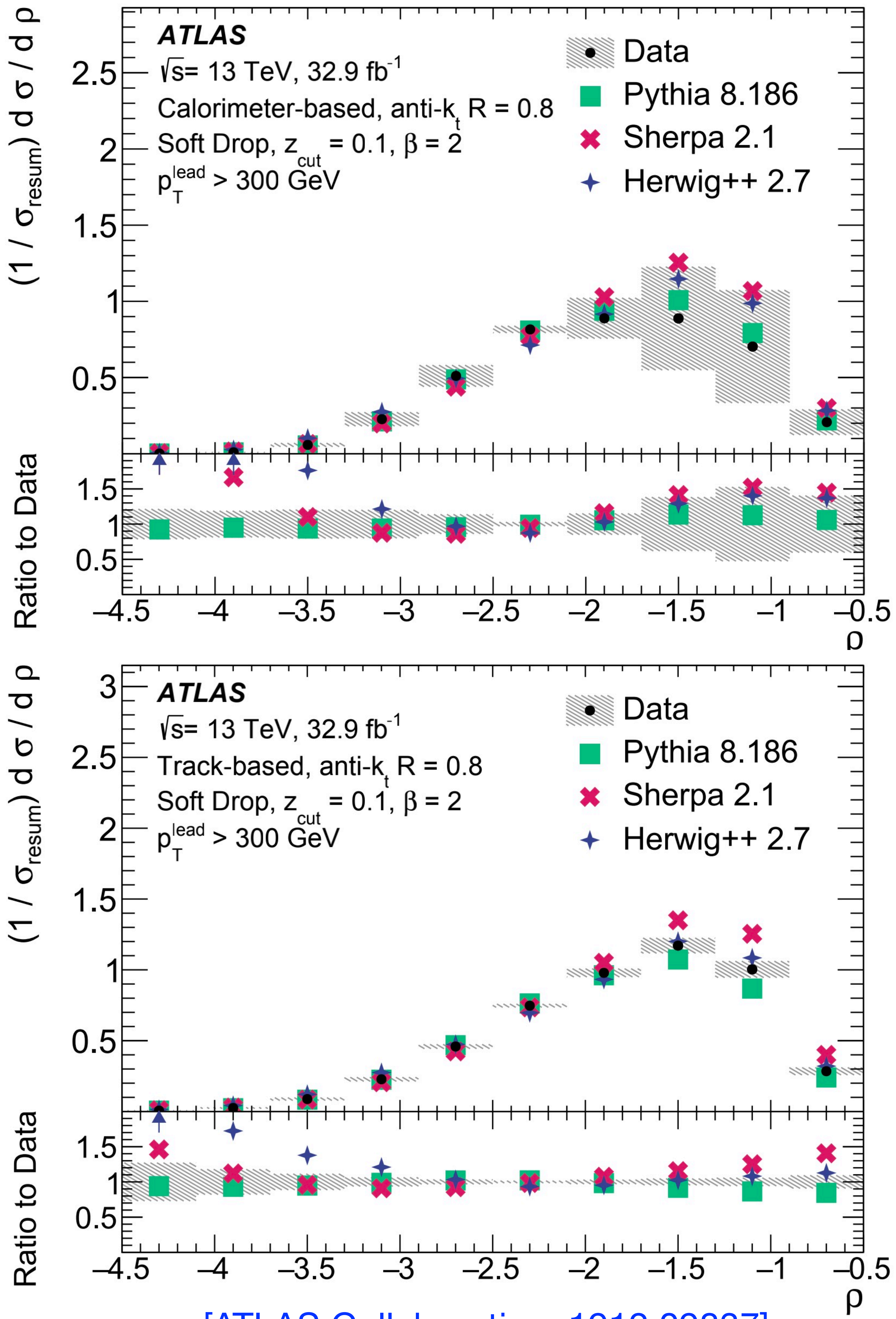
Techniques:
Jet grooming, jet tagging, …

Why track functions?

- Track-based measurements offer:
  - superior angular resolution
  - pileup mitigation.

calorimeter-based
(all-particle)

track-based
(charged-particle)



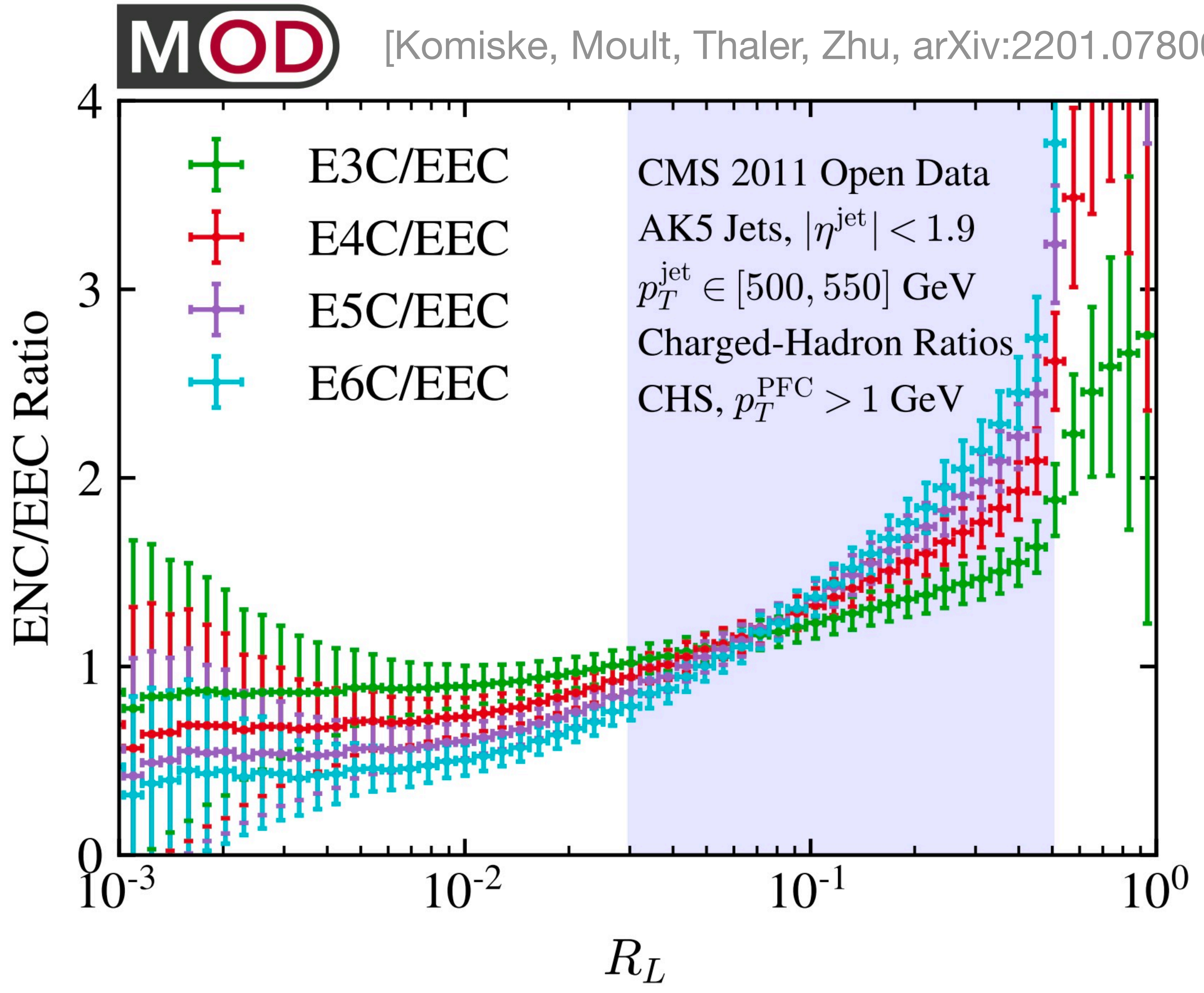
[ATLAS Collaboration, 1912.09837]

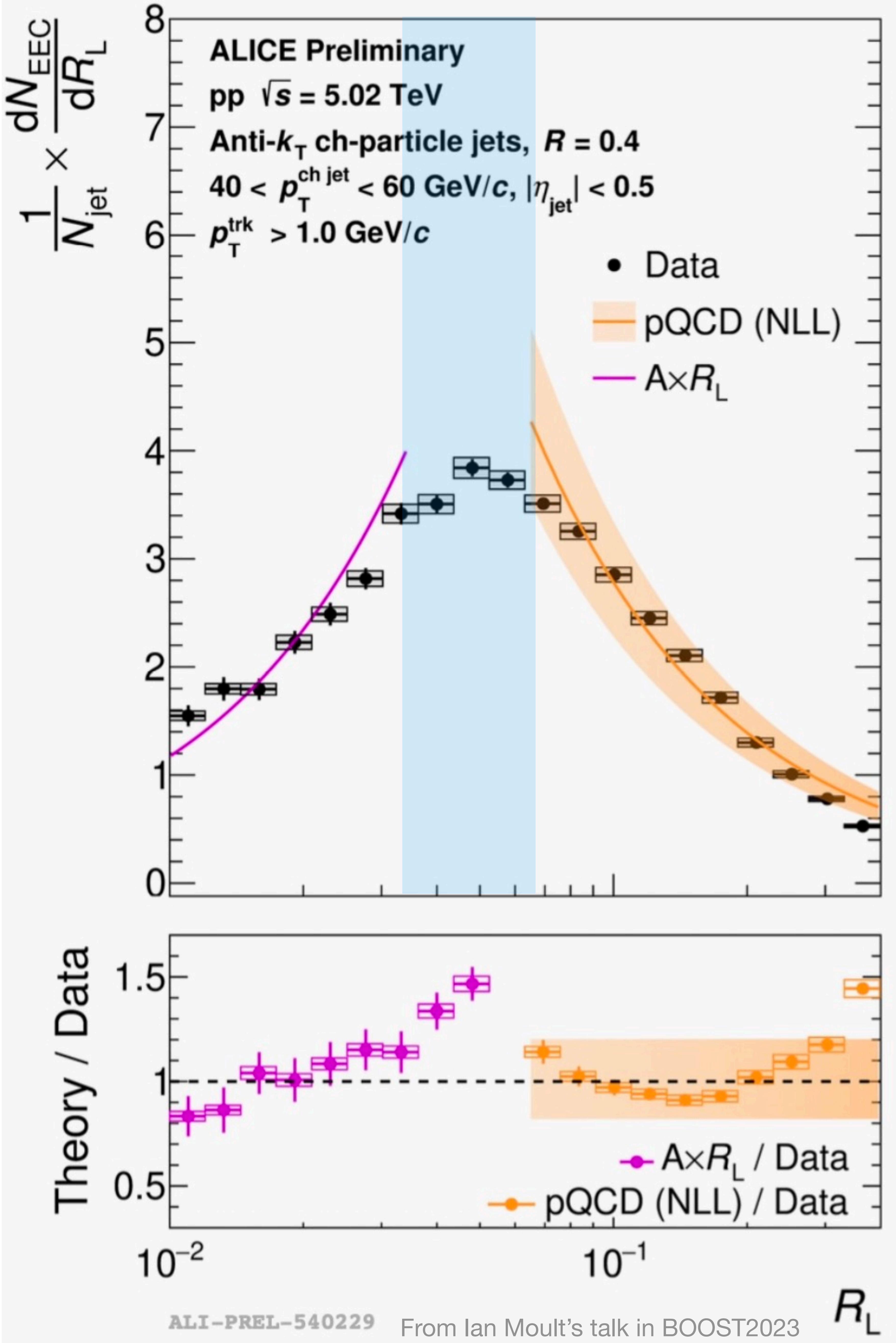
▶ “Imaging” of confinement transition

Why track functions?

- **Phenomenological applications**
  - ▶ Universal scaling behavior

[Komiske, Moult, Thaler, Zhu, arXiv:2201.07800]





From Ian Moult’s talk in BOOST2023

Why track functions?

- Track-based measurements offer:
  - superior angular resolution
  - pileup mitigation.

- One **problem**: Track-based calculations are not IR safe in perturbation theory.

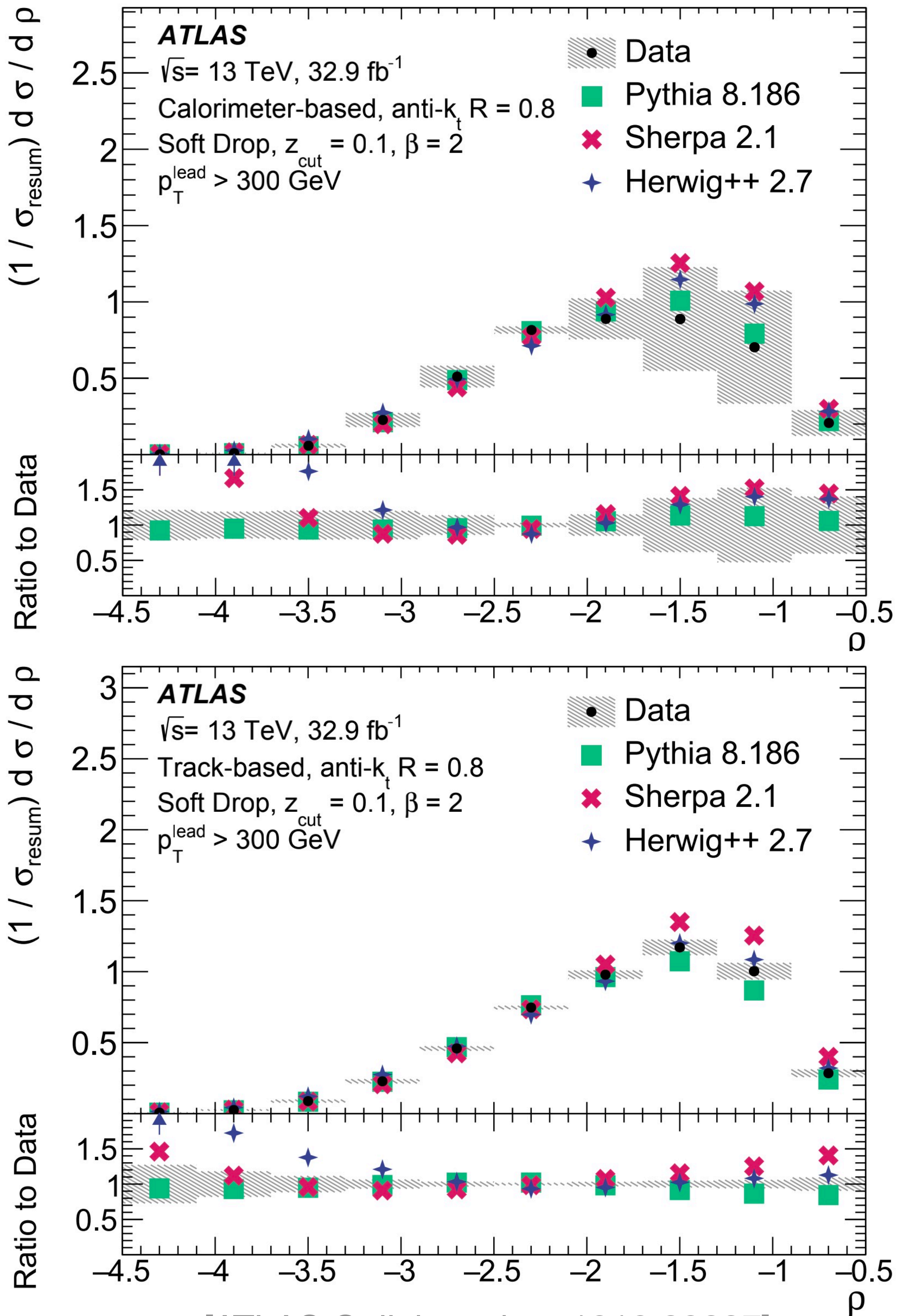
Track Functions

▸ IR divergences are absorbed into these **universal** non-perturbative functions.

(like the case of parton distribution functions and fragmentation functions)

calorimeter-based (all-particle)

track-based (charged-particle)



[ATLAS Collaboration, 1912.09837]

✓ Track functions introduced and studied at $\mathcal{O}(\alpha_s)$.

[H. Chang, M. Procura, J. Thaler, W. Waalewijn, arXiv:1303.6637, arXiv:1306.6630]

- Implementing track functions complicates perturbative calculations, which hinders people to apply that to higher order, while experimentalists urge predictions on tracks.

Eg. NNNLL+NNLO for all-particle thrusts but NLL+NLO for track thrusts.

> observables. For all of these observables, the uncertainties for the track-based observables are significantly smaller than those for the calorimeter-based observables, particularly for higher values of β, where more soft radiation is included within the jet. However, since no track-based calculations exist at the present time, calorimeter-based measurements are still useful for precision QCD studies.

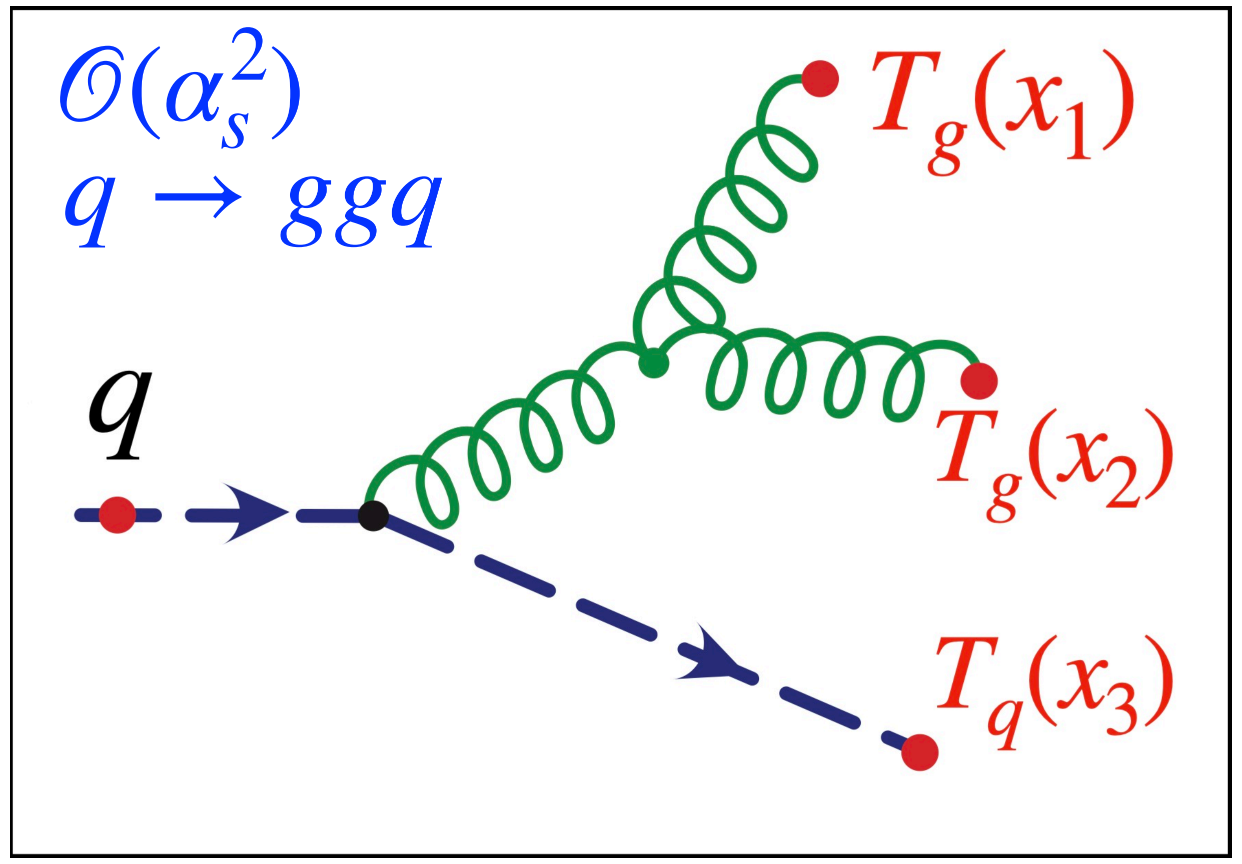
[ATLAS Collaboration, 1912.09837]

> the selection of charged particle jets. Note that track-based observables are IRC-unsafe. In general, nonperturbative track functions can be used to directly compare track-based measurements to analytical calculations [67–69]; however, such an approach has not yet been developed for jet angularities. Two

[ALICE Collaboration, 2107.11303]

- The **complication** due to the RGE: The track function evolution encodes correlations between collinear partons.

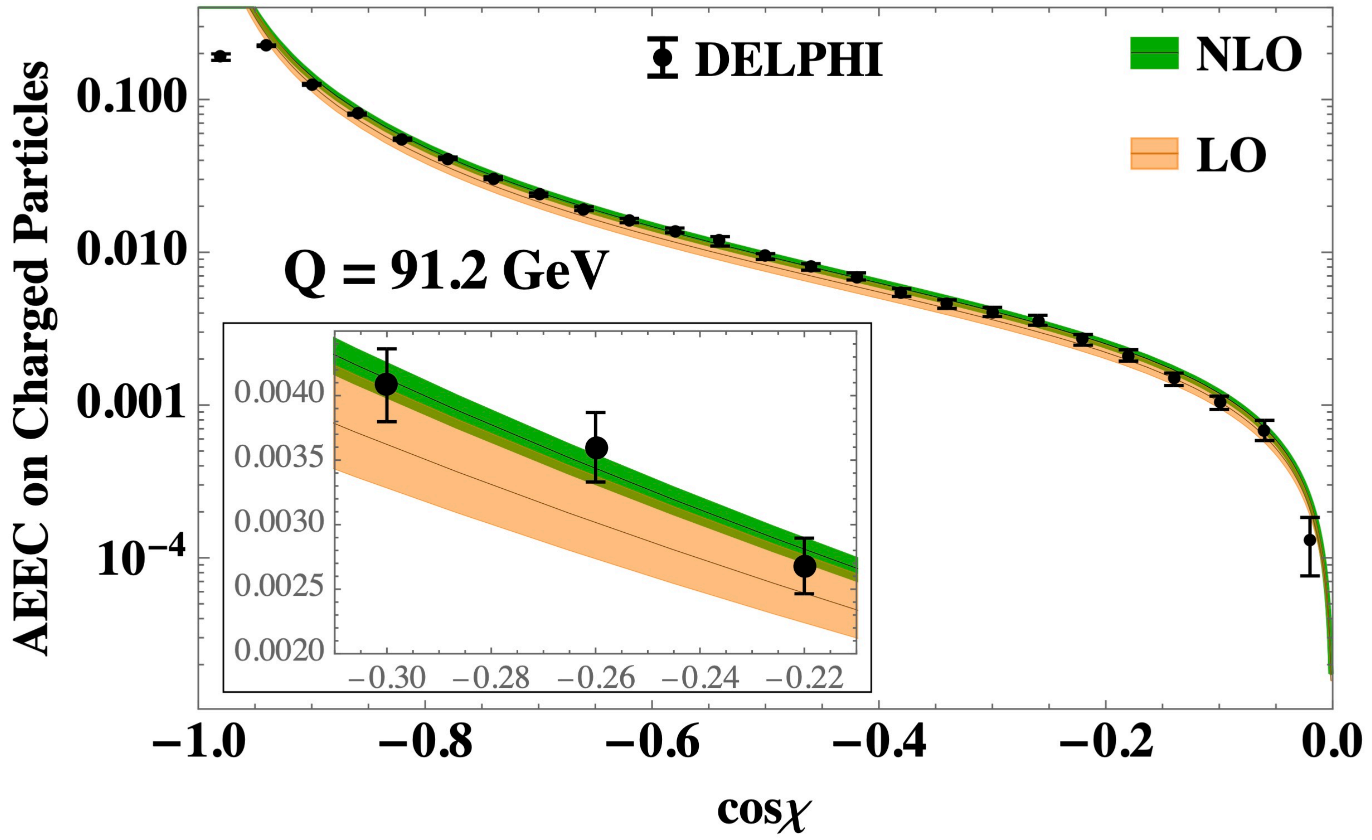
➡ collinear evolution beyond the linear DGLAP paradigm



✓ Our work: Track function formalism beyond leading order.

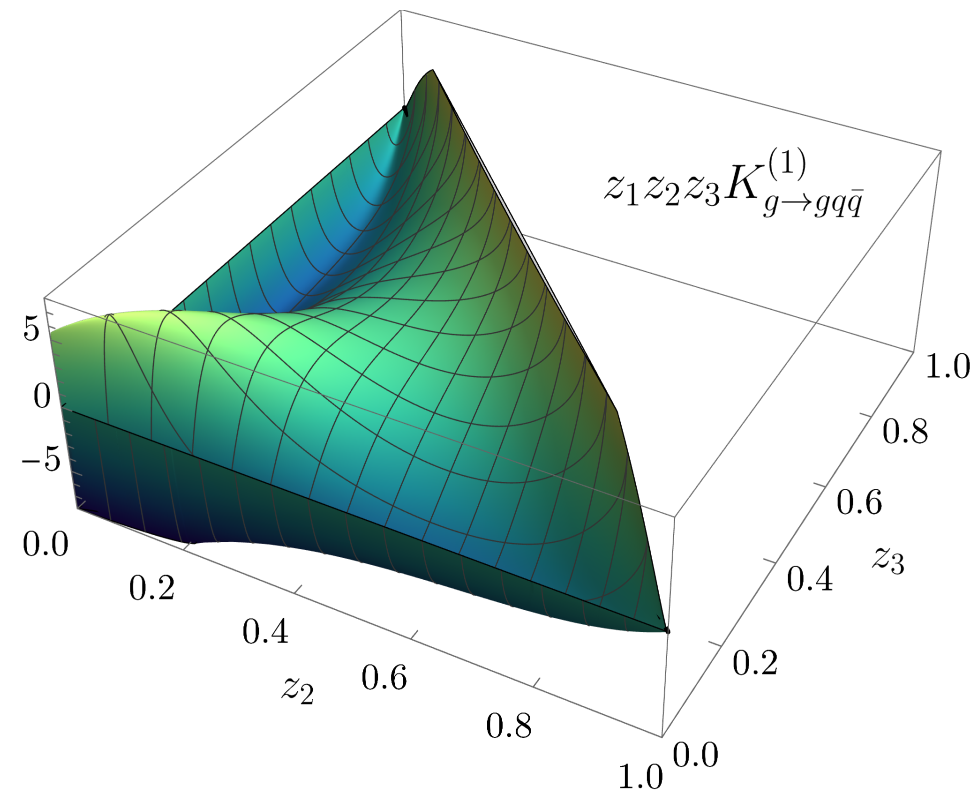
[**Y. Li**, I. Moult, S. S. van Velzen, W. Waalewijn, H. X. Zhu, arXiv:2108.01674;
M. Jaarsma, **Y. Li**, I. Moult, W. Waalewijn, H. X. Zhu, arXiv:2201.05166]

✦ Evolution of track functions in moment space and track EEC at $\mathcal{O}(\alpha_s^2)$.



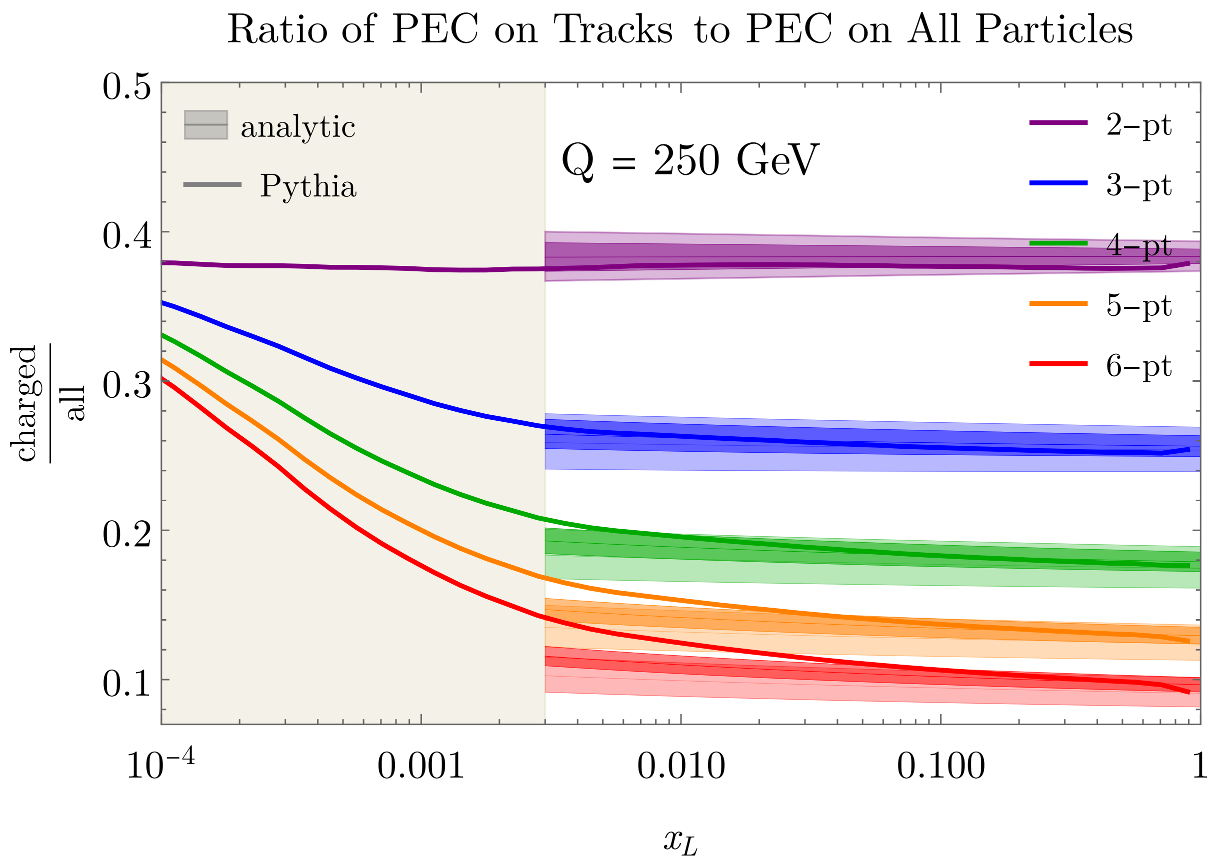
[H. Chen, M. Jaarsma, **Y. Li**, I. Moult, W. Waalewijn, H. X. Zhu, arXiv:2210.10058, arXiv:2210.10061]

✦ Results for the **NLO** non-linear x-space evolution enabling the use of tracks for generic substructure observables!

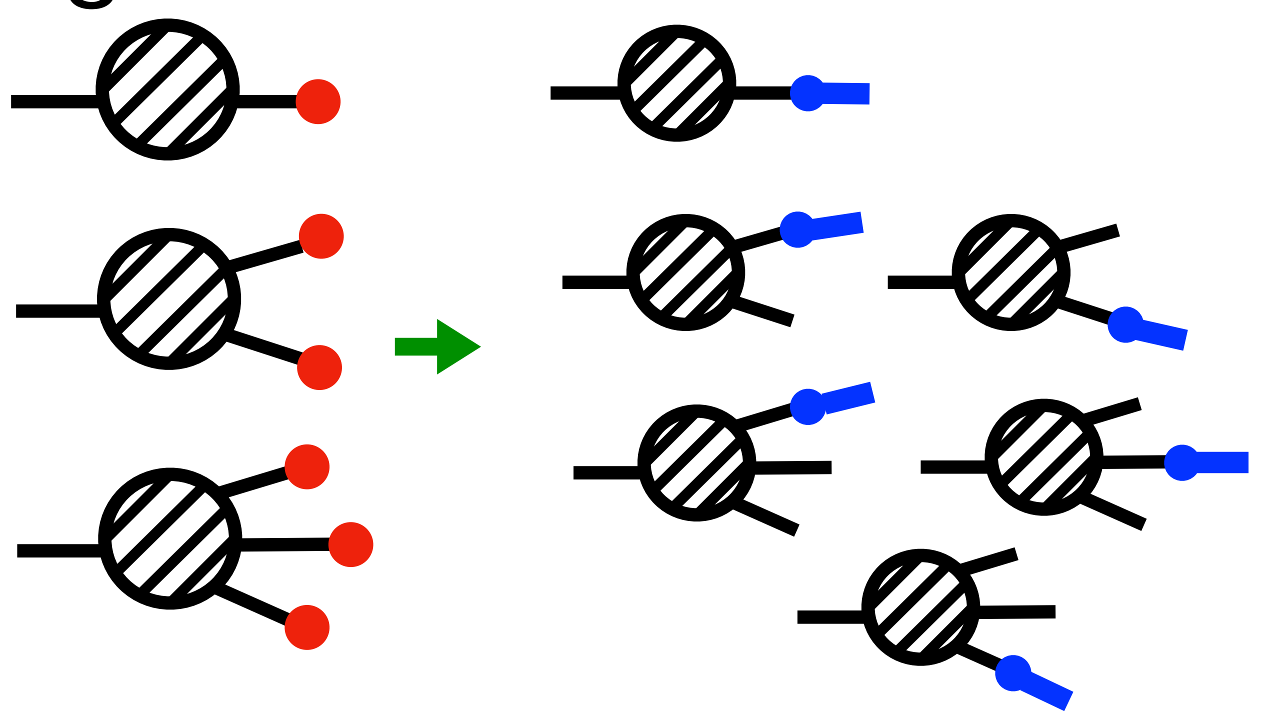


[M. Jaarsma, **Y. Li**, I. Moult, W. Waalewijn, H. X. Zhu, arXiv:2307.15739]

✦ NLL resummation for up to 6-point projected energy correlators on tracks.

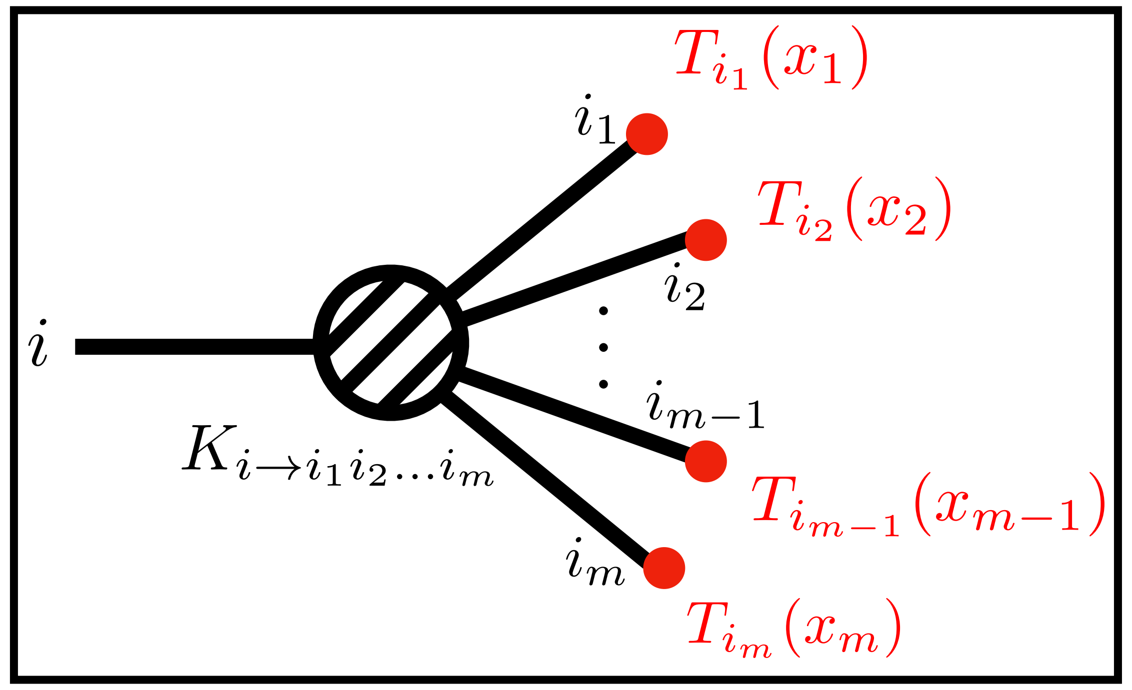


✦ Correspondence between the evolution of track functions and that of single- or multi-hadron fragmentation functions.

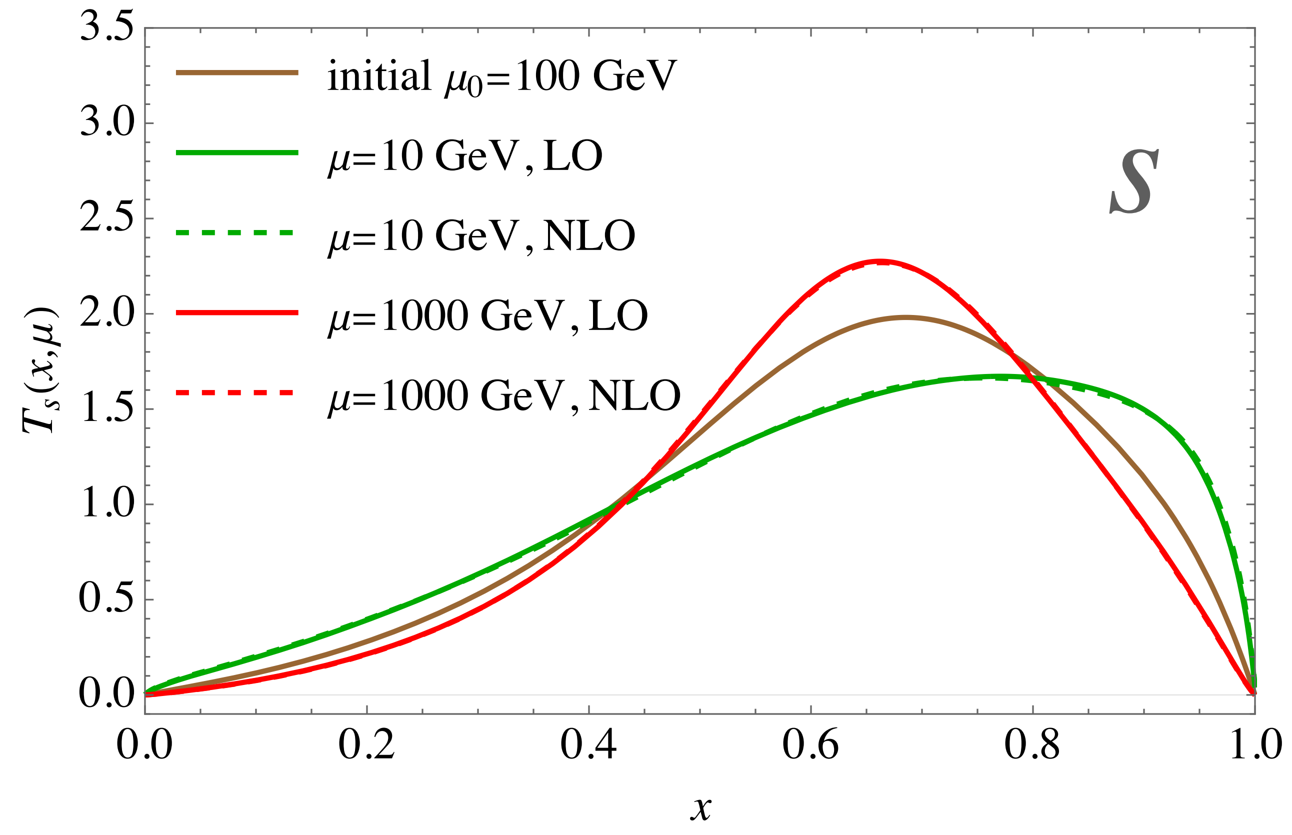


Outline

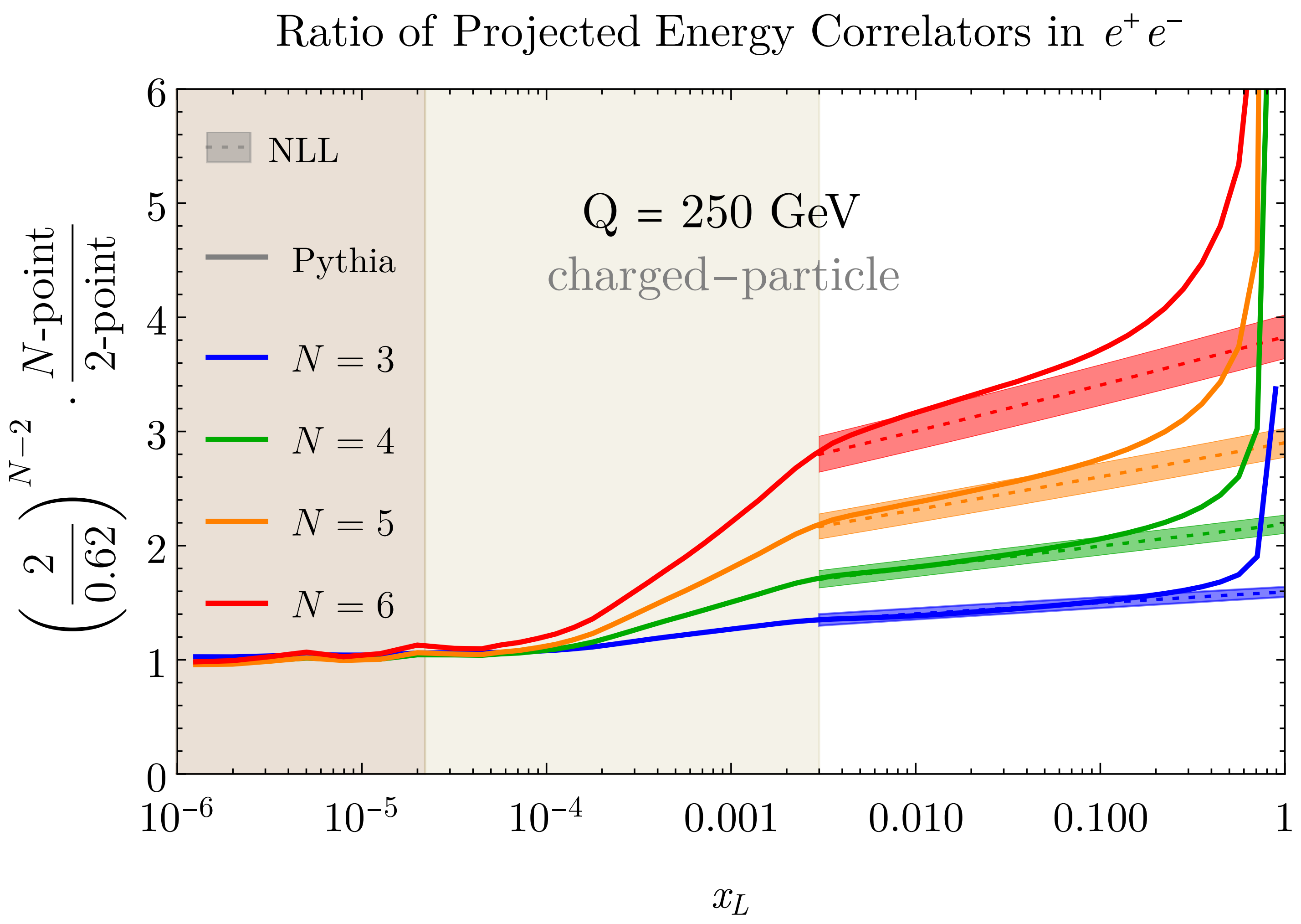
- Introduction to track functions
  - Definition, features and RG evolution



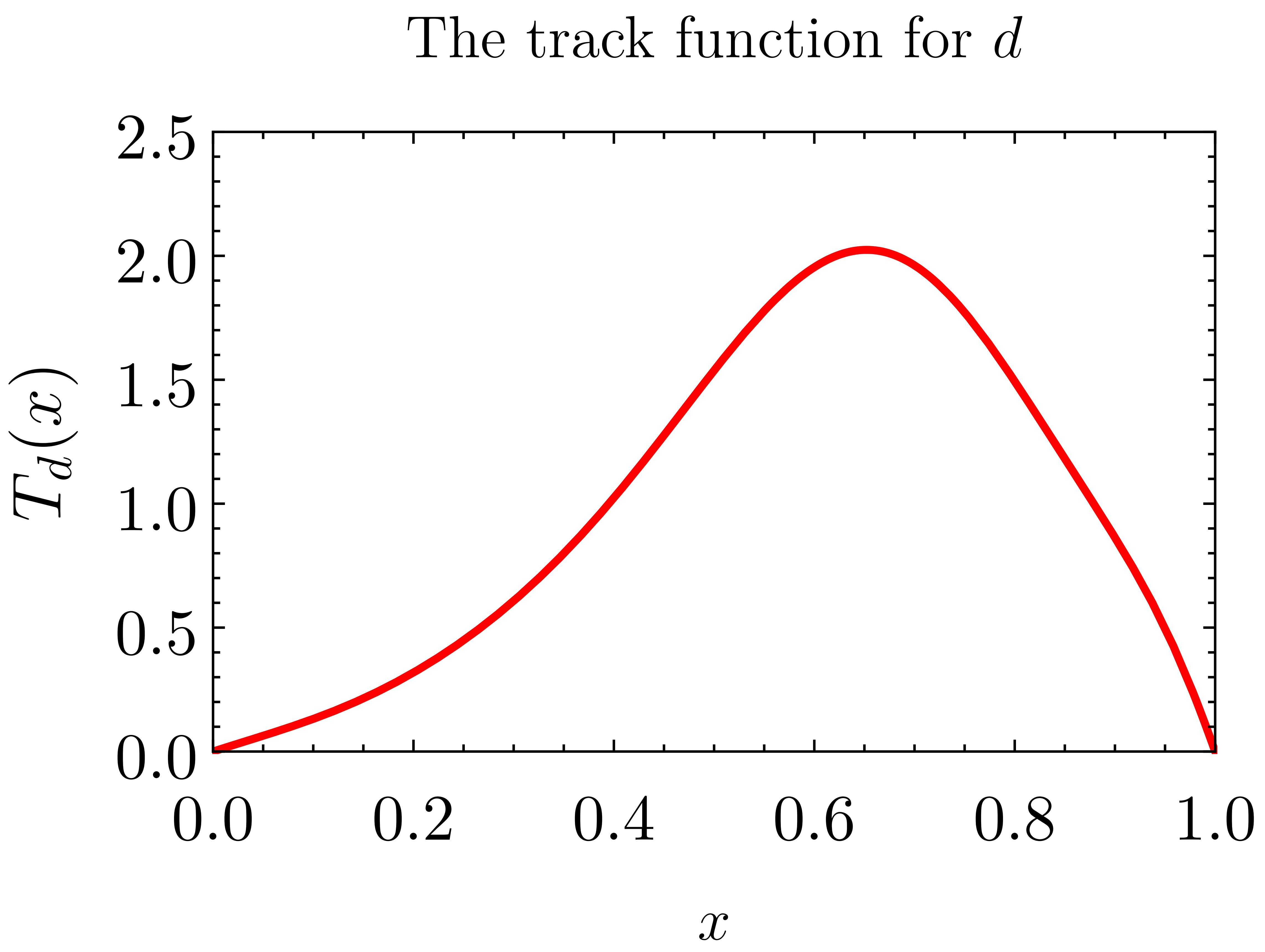
- Calculation & Results for the NLO evolution



- Incorporating tracks
  - Up to 6-point projected energy correlators



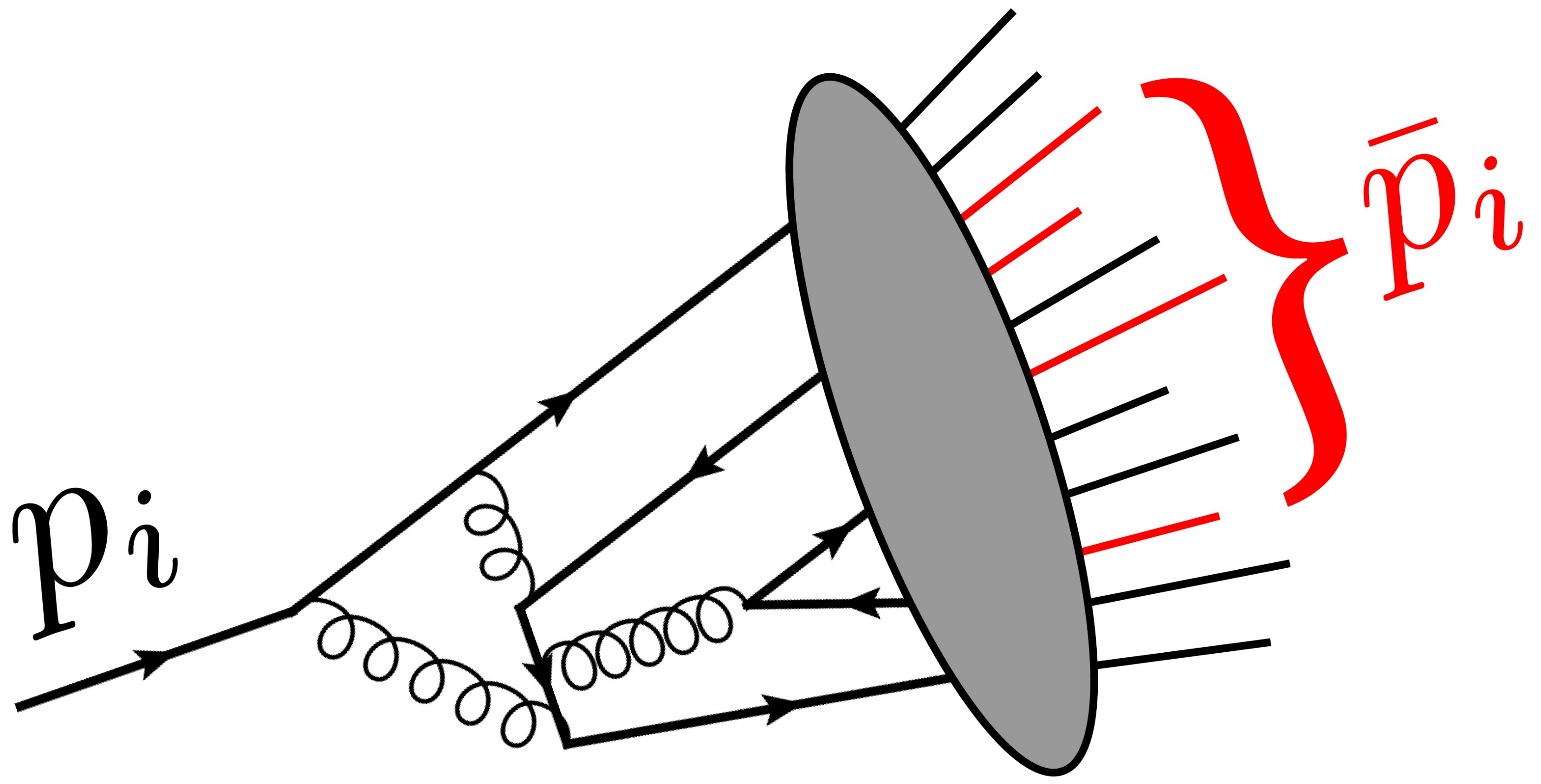
Introduction to Track Functions



Track Functions $T_i(x, \mu)$ [H. Chang, M. Procura, J. Thaler, W. Waalewijn, arXiv:1303.6637, arXiv:1306.6630]

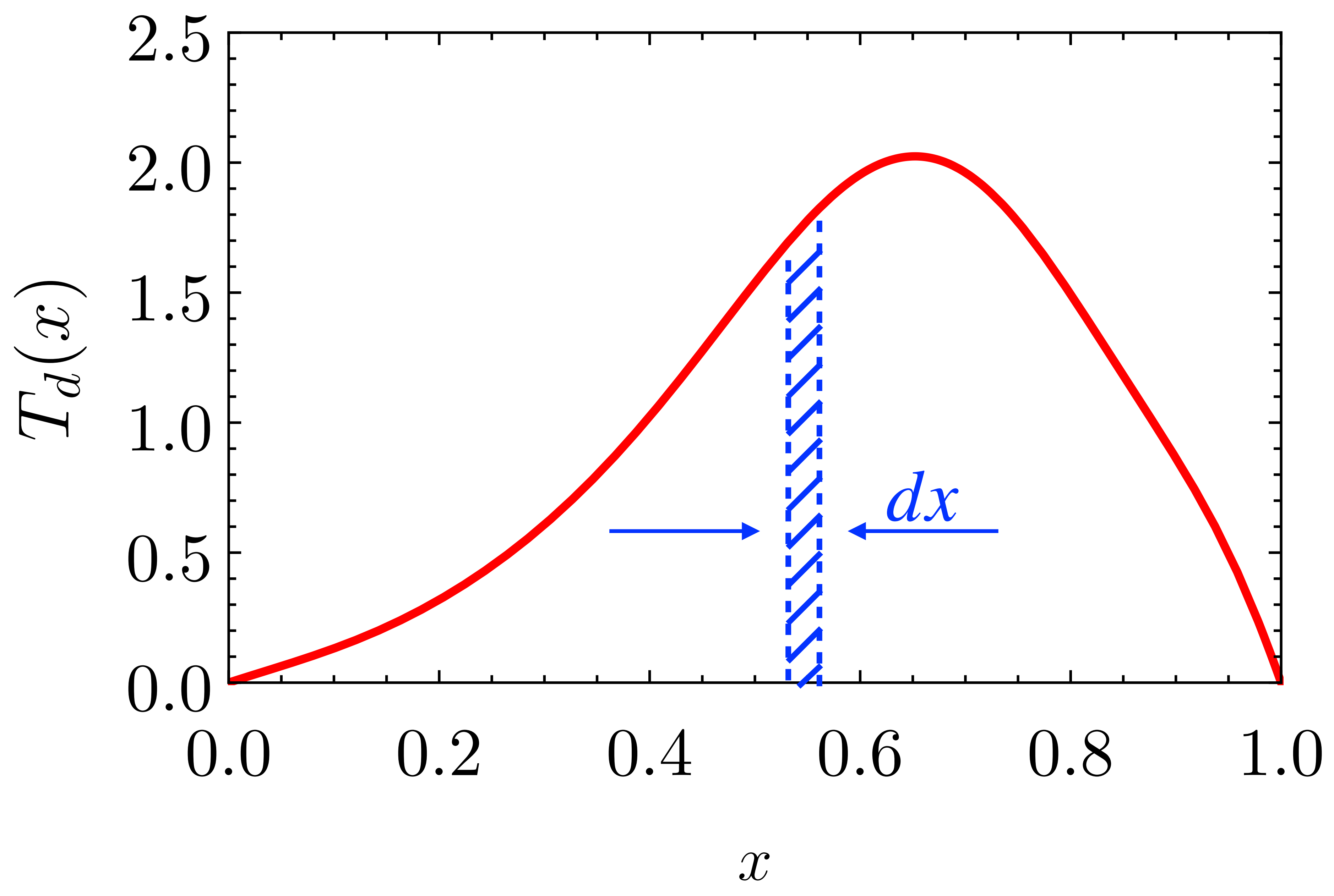
Definition

- $T_i(x, \mu)$ describes the total momentum fraction x of *all* charged particles (tracks) in a jet initiated by a hard parton i.



$$\bar{p}_i^{\mu} = x p_i^{\mu} + O(\Lambda_{\mathrm{QCD}}) \, , (0 \leq x \leq 1) \ .$$

- Sum rule: $\int_0^1 dx \; T_i(x, \mu) = 1$.



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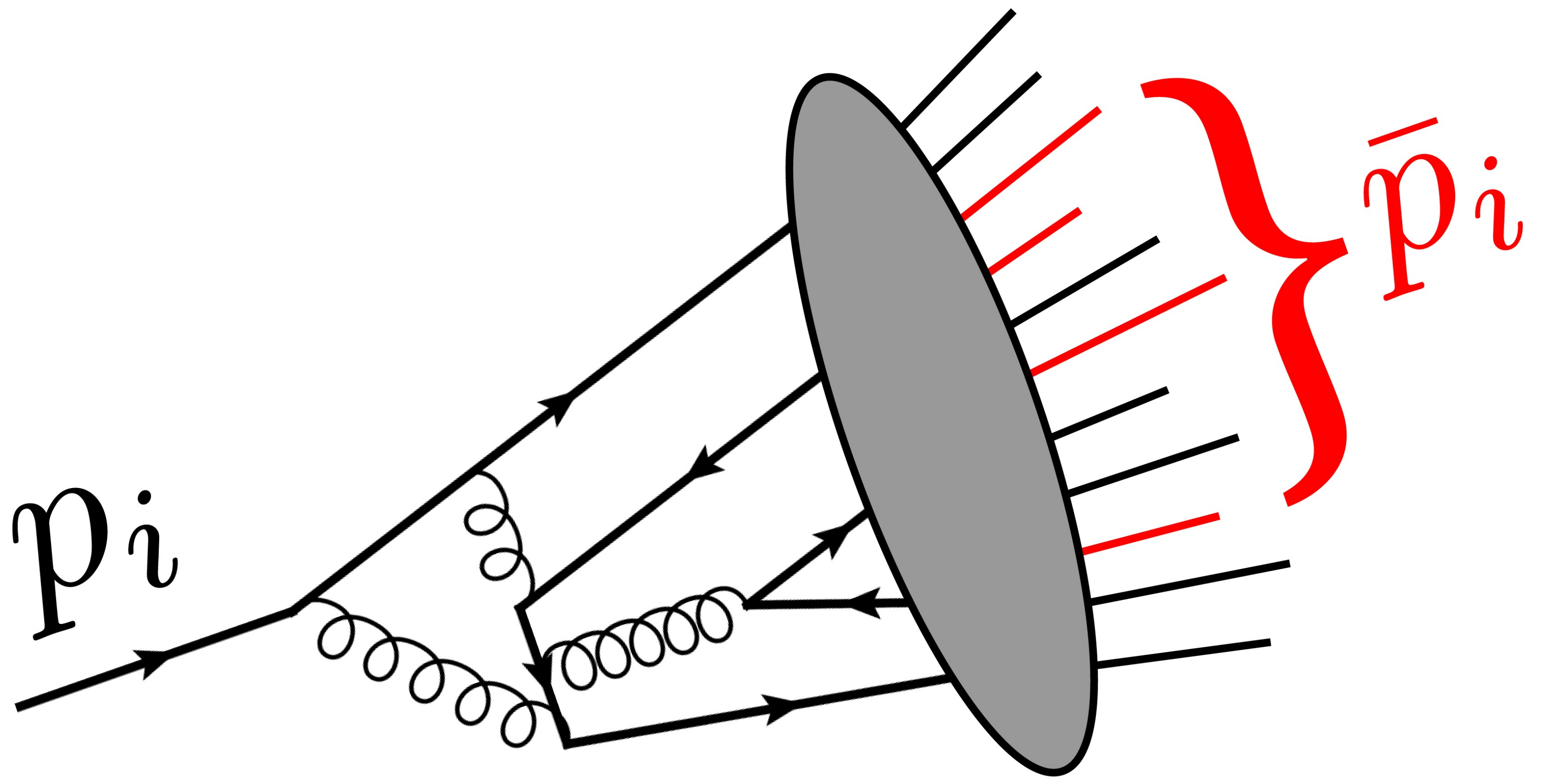
Definition



- $T_i(x, \mu)$ describes the total momentum fraction x of *all* charged particles (tracks) in a jet initiated by a hard parton i.

$$\bar{p}_i^\mu = x p_i^\mu + O(\Lambda_{\text{QCD}}) \, , (0 \le x \le 1) \ .$$

- This formalism applies to other subsets of hadrons (positively-charged, strange, etc).

Track Functions [H. Chang, M. Procura, J. Thaler, W. Waalewijn, 1303.6637, 1306.6630]

Features

- A generalization of fragmentation functions (FFs).
  - Independent of hard process.
  - Fundamentally non-perturbative, with a perturbatively calculable scale (μ) dependence.
  - Incorporating correlations between final-state hadrons, like multi-hadron FFs. ⟷

- The **single-hadron** fragmentation function:
  - The probability of a parton to produce a single-hadron state considered.
  - The momentum sum rule:

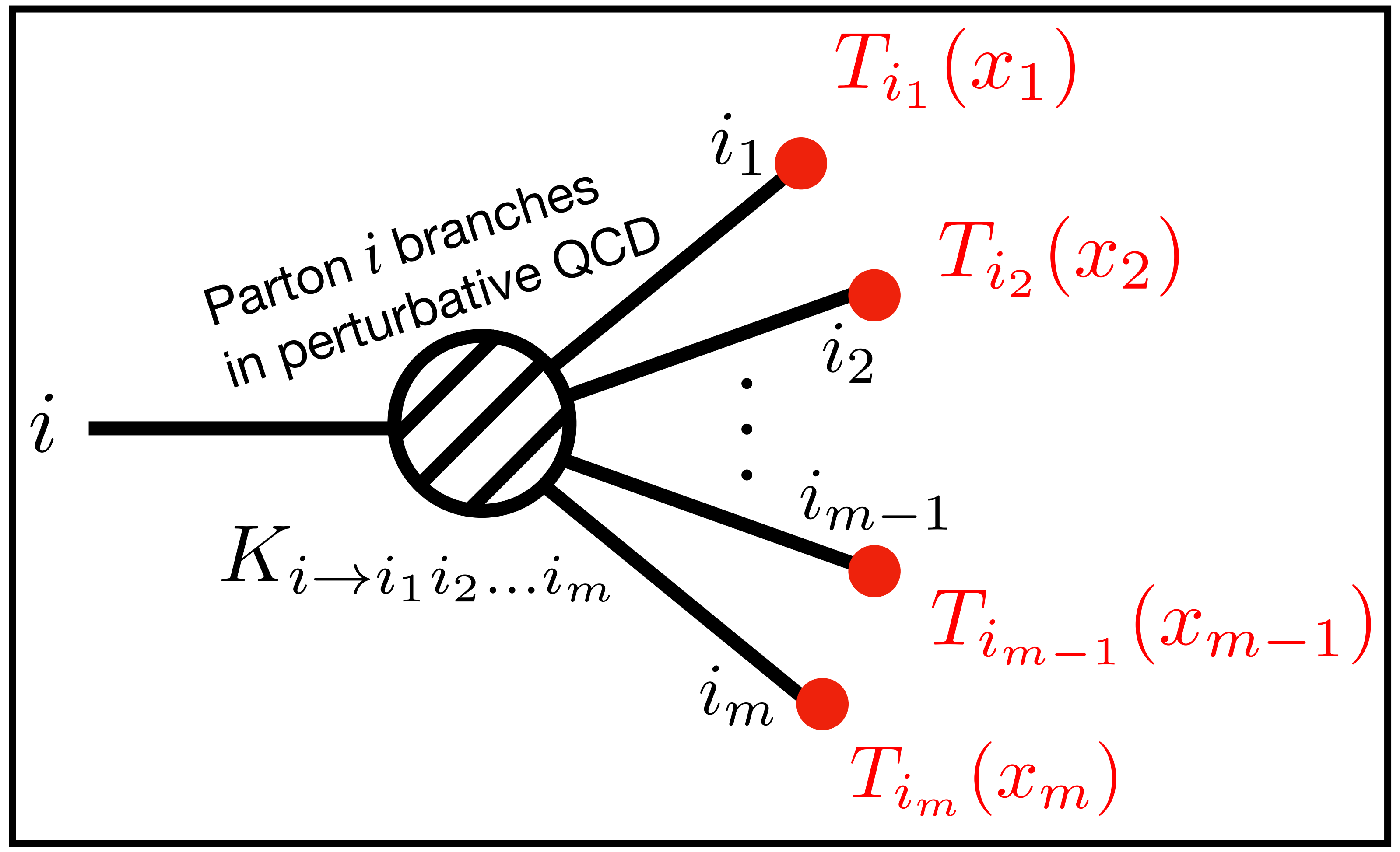
$$\sum_h \int_0^1 dz\ z\ D_{i\to h}(z,\mu) = 1\ .$$

Track Function Evolution

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x) = \sum_M \sum_{\{i_f\}} \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}z_m\right]\delta\left(1-\sum_{m=1}^{M} z_m\right) K_{i\to\{i_f\}}(\{z_f\})$$

$$\times \left[\prod_{m=1}^{M}\int_0^1 \mathrm{d}x_m\, T_{i_m}(x_m)\right]\delta\left(x-\sum_{m=1}^{M} z_m x_m\right)$$

$(i, i_f = g, u, \bar{u}, d, \cdots)$

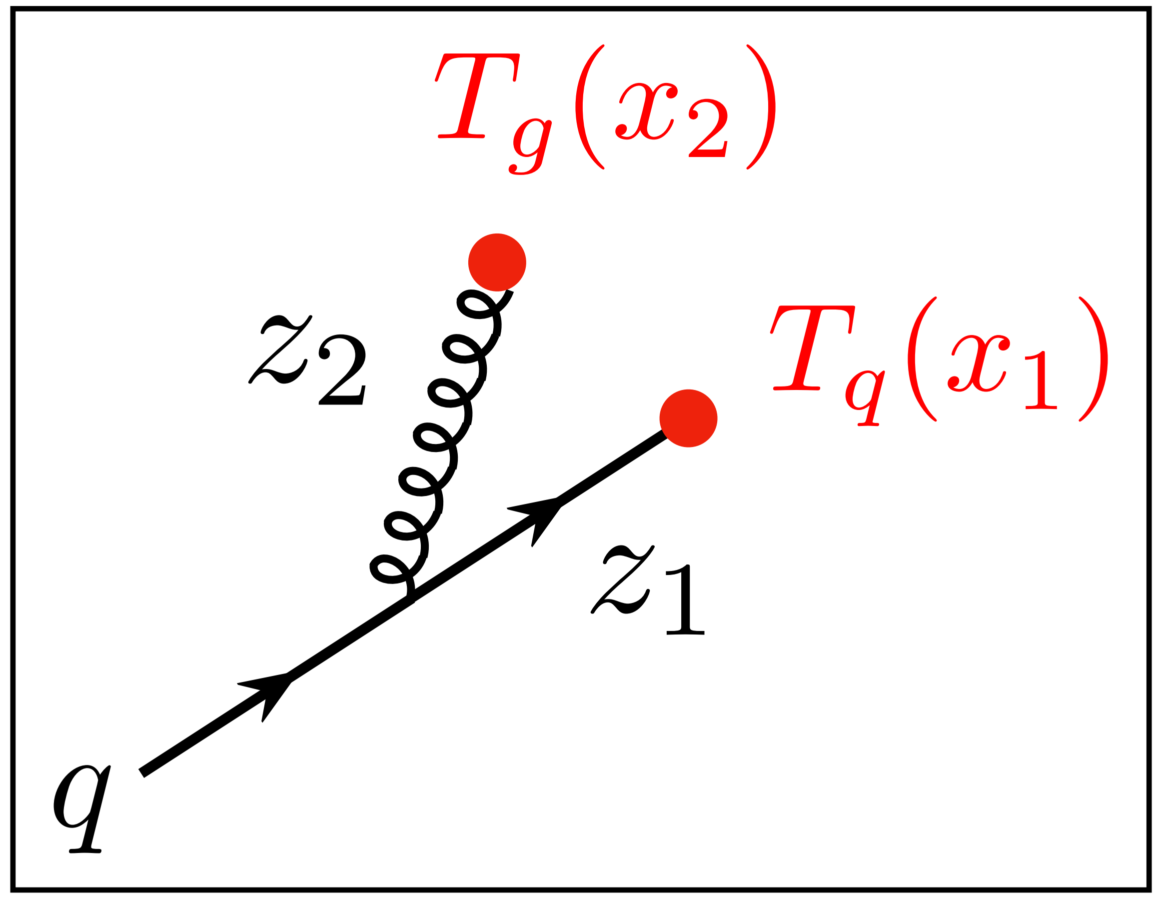


- **Nonlinear**, involving contributions from all branches of splittings.
- E.g., LO evolution:

- For single-hadron FFs: Only one branch observed → **Linearity**

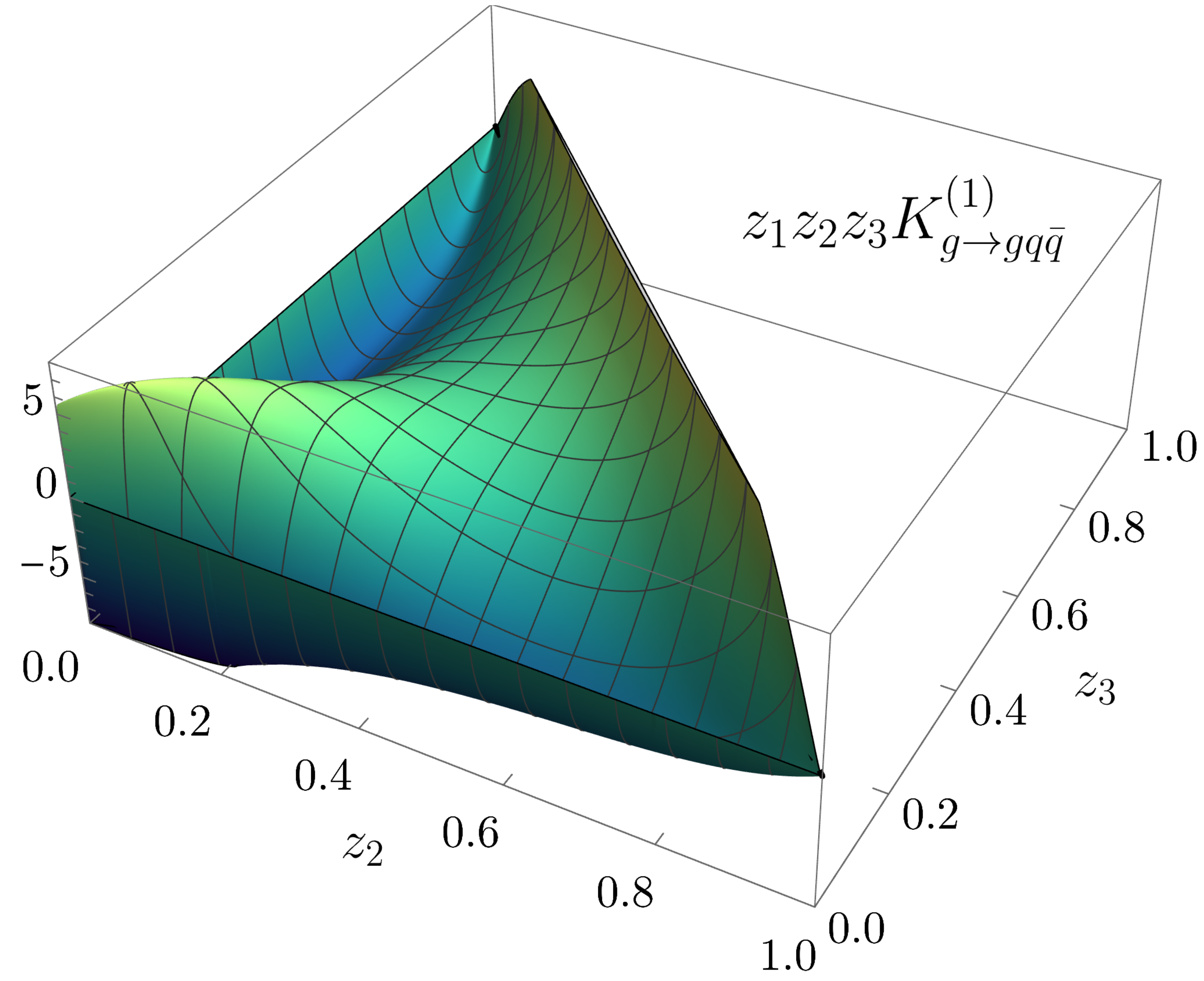
$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}T_i(x,\mu) = a_s(\mu)\sum_{\{jk\}}\int \mathrm{d}z_1\mathrm{d}z_2\, K^{(0)}_{i\to jk}(z_1,z_2)\delta(1-z_1-z_2)$$

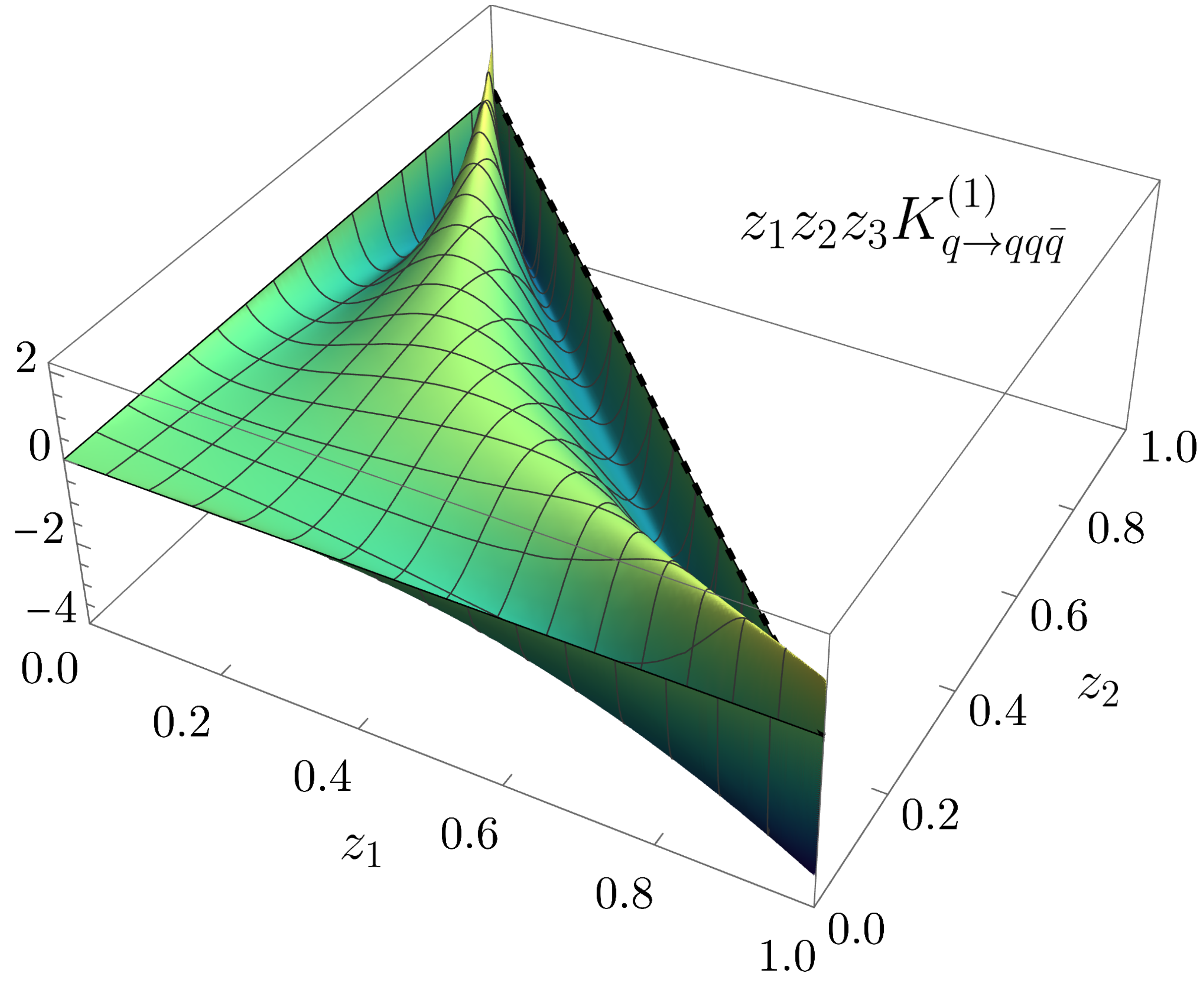
$$\times \int \mathrm{d}x_1\mathrm{d}x_2 T_j(x_1,\mu)T_k(x_2,\mu)\delta[x-z_1x_1-z_2x_2]\ .$$



Involving contributions from both the branches of the splitting.

Calculation & Results for the NLO Evolution





Track Jet Functions

In DR: $T_i^{(0)} = T_i^{\text{bare}}$

LO track jet function: $J_i^{(0)} = \delta(s) T_i^{(0)}$

- We use the jet function to extract the track function evolution.
- The definition for the jet function on tracks is

$$J_{\mathrm{tr},i}^{\mathrm{bare}}(s,x) = \sum_N \sum_{\{i_f\}} \int \mathrm{d}\Phi_N^c \delta(s-s')\sigma^c_{i\to\{i_f\}}(\{i_f\},\{s_{ff'}\},s') \int \left[\prod_{m=1}^{N} \mathrm{d}x_m T_{i_m}^{(0)}(x_m)\right] \delta\left(x - \sum_{m=1}^{N} x_m z_m\right)$$

- After integration over angular variables,

$$J_{\mathrm{tr},i}^{\mathrm{bare}}(s,x) \supset \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \int_0^1 \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \delta(1-z_1-z_2-z_3) P_{i\to i_1 i_2 i_3}(z_1,z_2,z_3)$$

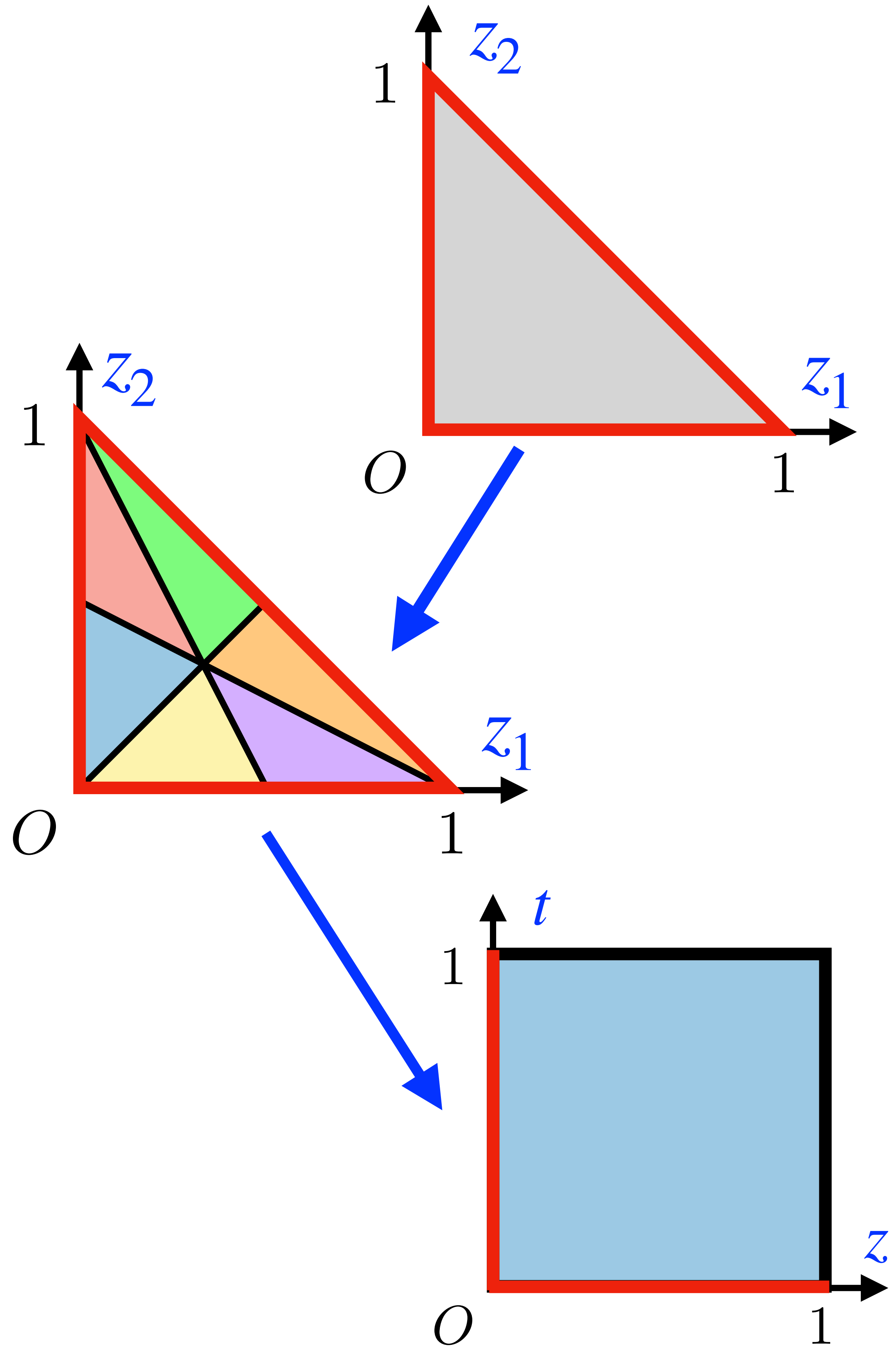
have not been expanded in ϵ

$$\times T_{i_1}^{(0)}(x_1) T_{i_2}^{(0)}(x_2) T_{i_3}^{(0)}(x_3) \delta(x - z_1 x_1 - z_2 x_2 - z_3 x_3)$$

- For $z_{i_1} < z_{i_2} < z_{i_3}$ $(i_1, i_2, i_3 = 1,2,3)$, do the coordinate transformation

[Sector decomposition (Heinrich, arXiv:0803.4177)]

$$t = \frac{z_{i_1}}{z_{i_2}}, z = \frac{z_{i_2}}{z_{i_3}} \quad \text{, i.e.,} z_{i_1} \to \frac{zt}{1+z+zt}, z_{i_2} \to \frac{z}{1+z+zt}, z_{i_3} \to \frac{1}{1+z+zt}$$



Results in $\mathcal{N} = 4$ SYM

a: t' Hooft coupling constant

$$\frac{d}{d\ln\mu^2}T(x) = a^2 \Bigg\{ K^{(1)}_{1\to1}\, T(x) + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z\; K^{(1)}_{1\to2}(z)\; T(x_1)T(x_2)\; \delta\left(x - x_1\frac{1}{1+z} - x_2\frac{z}{1+z}\right)$$

$$+ \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\; K^{(1)}_{1\to3}(z,t)\; T(x_1)T(x_2)T(x_3)$$

$$\times\, \delta\left(x - x_1\frac{1}{1+z+zt} - x_2\frac{z}{1+z+zt} - x_3\frac{zt}{1+z+zt}\right)\Bigg\}$$

where $K^{(1)}_{1\to1} = -25\zeta_3$ $\quad K^{(1)}_{1\to2}(z) = \frac{8}{3}\pi^2\left[\frac{1}{z}\right]_+ + \frac{32\ln^2(z+1)}{z} - \frac{16\ln(z)\ln(z+1)}{z}$

$$K^{(1)}_{1\to3}(z,t) = 8\Bigg\{\frac{4\ln(1+z)}{z}\left[\frac{1}{t}\right]_+ + \left[\frac{1}{z}\right]_+\left(4\left[\frac{\ln t}{t}\right]_+ - \frac{\ln t}{1+t} - \frac{7\ln(1+t)}{t}\right)$$

$$+\frac{2\left[\ln(1+tz) - \ln(1+z+tz)\right]}{(1+t)(1+z)(1+tz)} + \frac{10\left[\ln(1+z+tz) - \ln(1+z)\right]}{tz} + \frac{\ln(1+tz)}{(1+t)z(1+z)}$$

$$-\frac{7\ln(1+tz)}{tz} + \frac{\ln(1+t) - \ln t}{(1+t)(1+tz)} + \frac{\ln(1+z) + \ln(1+t)}{(1+t)(1+z)} - \frac{\ln(1+z)}{(1+t)z} - \frac{z\ln(1+z)}{(1+z)(1+tz)}\Bigg\}$$

Results in QCD

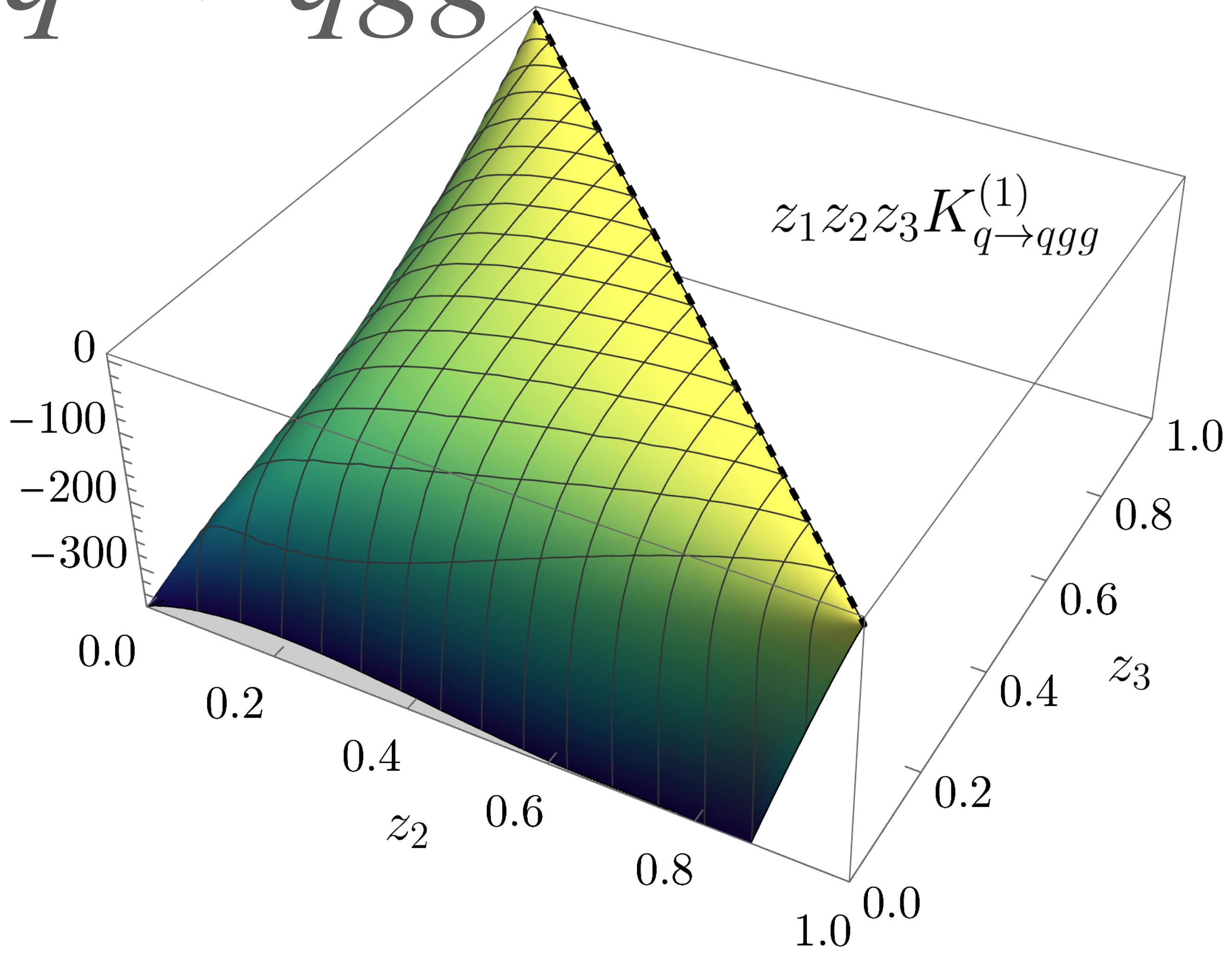
E.g. Gluon case:

For brevity, $a_s^2 = [\alpha_s(\mu)/(4\pi)]^2$ is suppressed.

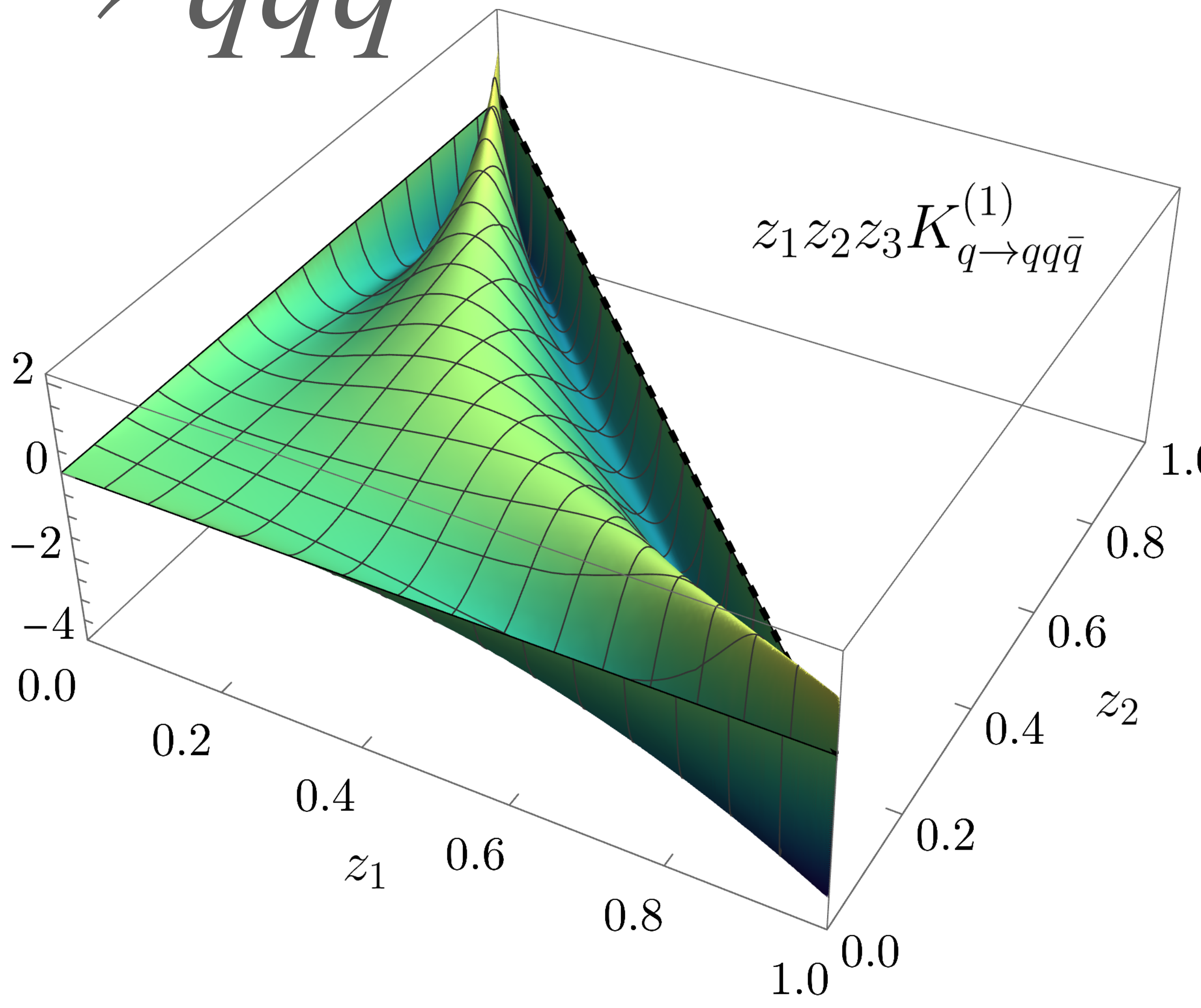
$$
\begin{aligned}
\frac{d}{d\ln\mu^2} T_g(x) = {} & T_g(x)\; K_g^{(1)} \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}z \delta\left(x - x_1 \frac{1}{1+z} - x_2 \frac{z}{1+z}\right) \Big[T_g(x_1)T_g(x_2)\; K_{gg,1}^{(1)}(z) \\
& + \sum_q \left(T_q(x_1)T_{\bar{q}}(x_2) + T_q(x_2)T_{\bar{q}}(x_1)\right)\; K_{q\bar{q},1}^{(1)}(z)\Big] \\
& + \int_0^1 \mathrm{d}x_1 \int_0^1 \mathrm{d}x_2 \int_0^1 \mathrm{d}x_3 \int_0^1 \mathrm{d}z \int_0^1 \mathrm{d}t\; \delta\left(x - x_1 \frac{1}{1+z+zt} - x_2 \frac{z}{1+z+zt} - x_3 \frac{zt}{1+z+zt}\right) \\
& \times \Big\{ 6\; T_g(x_1)T_g(x_2)T_g(x_3)\; K_{ggg,1}^{(1)}(z,t) \\
& \quad + \sum_q \Big[T_g(x_3)\left(T_q(x_2)T_{\bar{q}}(x_1) + T_q(x_1)T_{\bar{q}}(x_2)\right)\; K_{gq\bar{q},1}^{(1)}(z,t) \\
& \quad + T_g(x_2)\left(T_q(x_3)T_{\bar{q}}(x_1) + T_q(x_1)T_{\bar{q}}(x_3)\right)\; K_{gq\bar{q},2}^{(1)}(z,t) \\
& \quad + T_g(x_1)\left(T_q(x_3)T_{\bar{q}}(x_2) + T_q(x_2)T_{\bar{q}}(x_3)\right)\; K_{gq\bar{q},3}^{(1)}(z,t)\Big]\Big\}.
\end{aligned}
$$

Results in QCD, Pictorially
the 1→3 Kernels

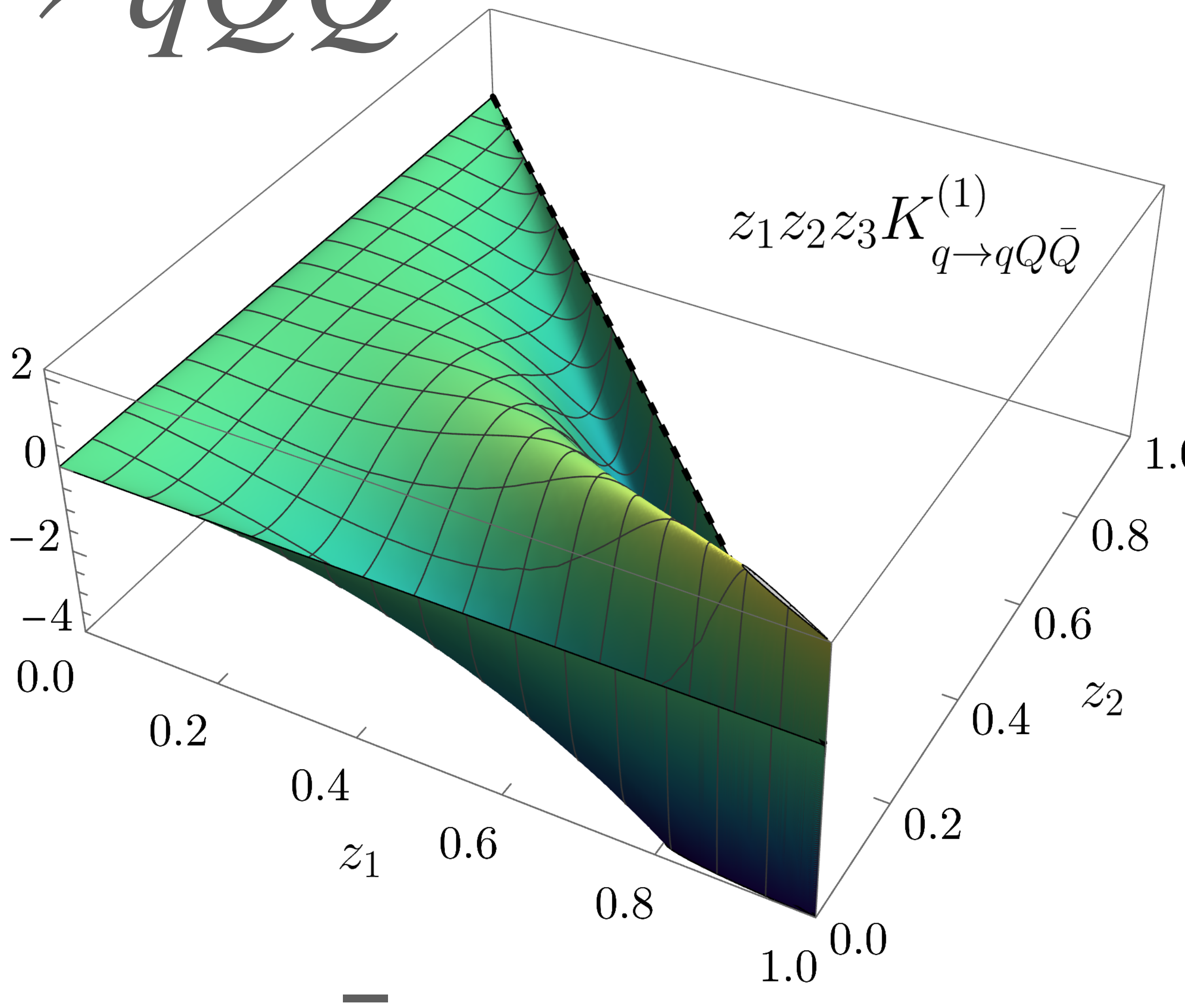
$q \to qgg$



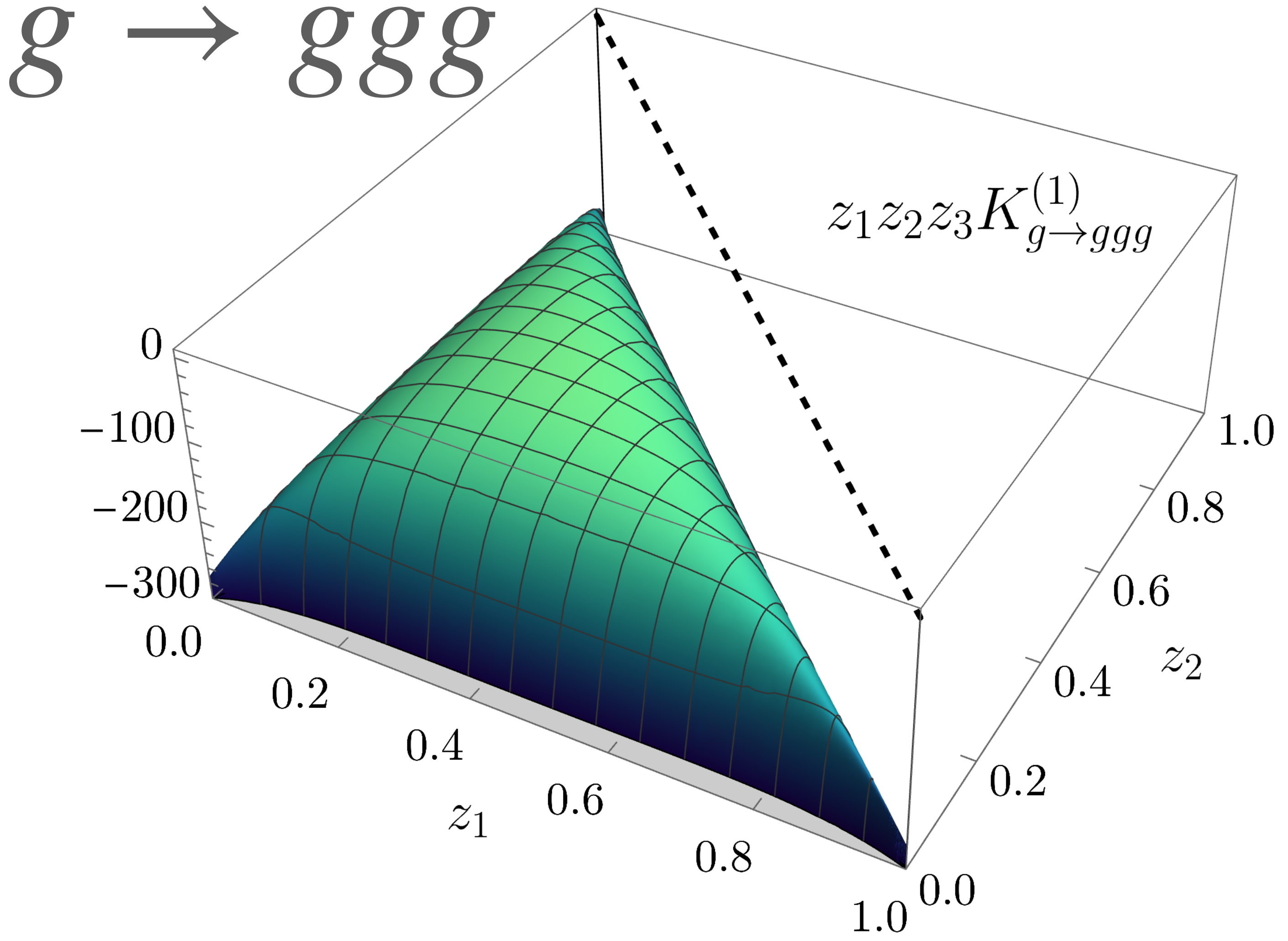
$q \to qq\bar{q}$



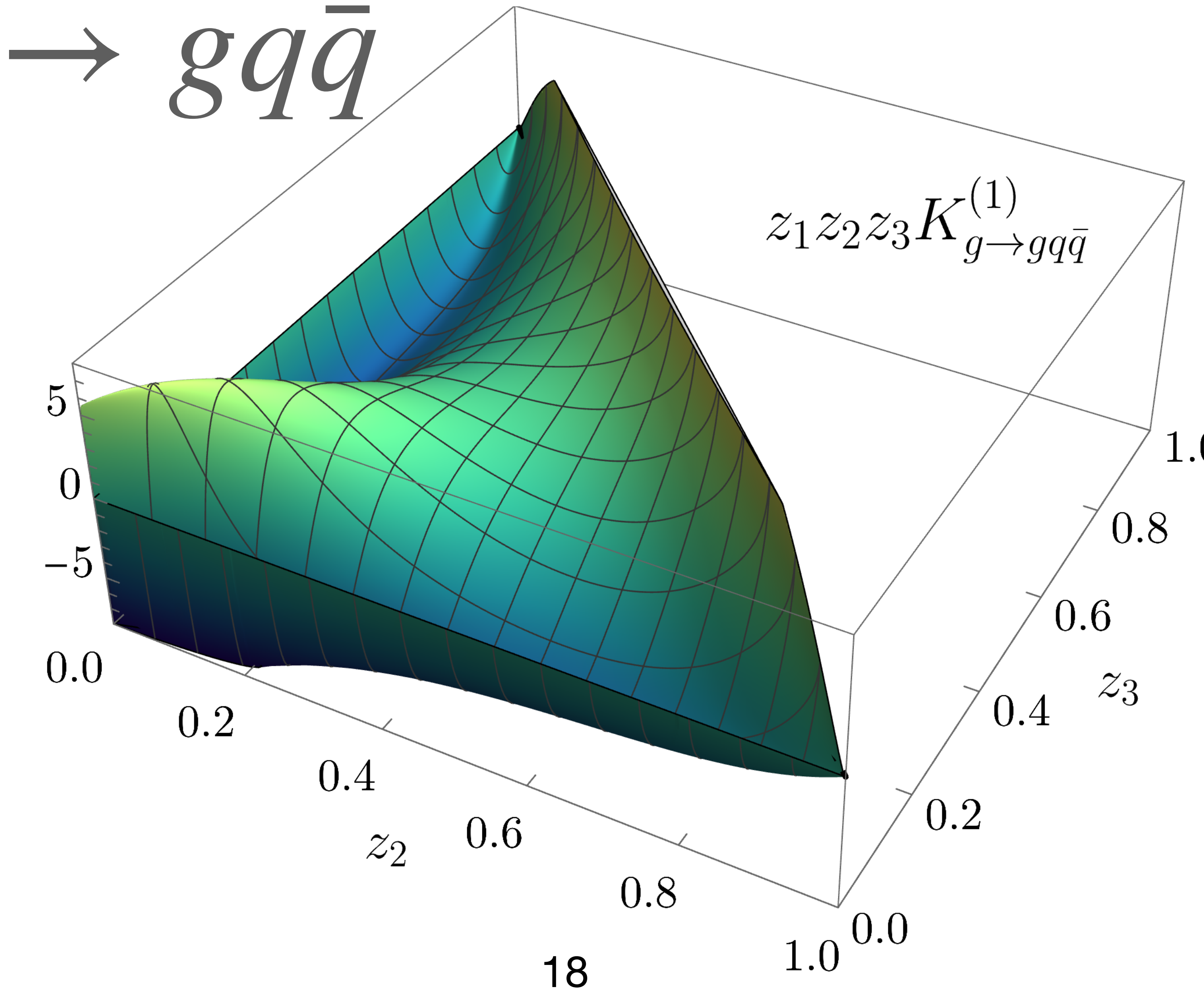
$q \to qQ\bar{Q}$



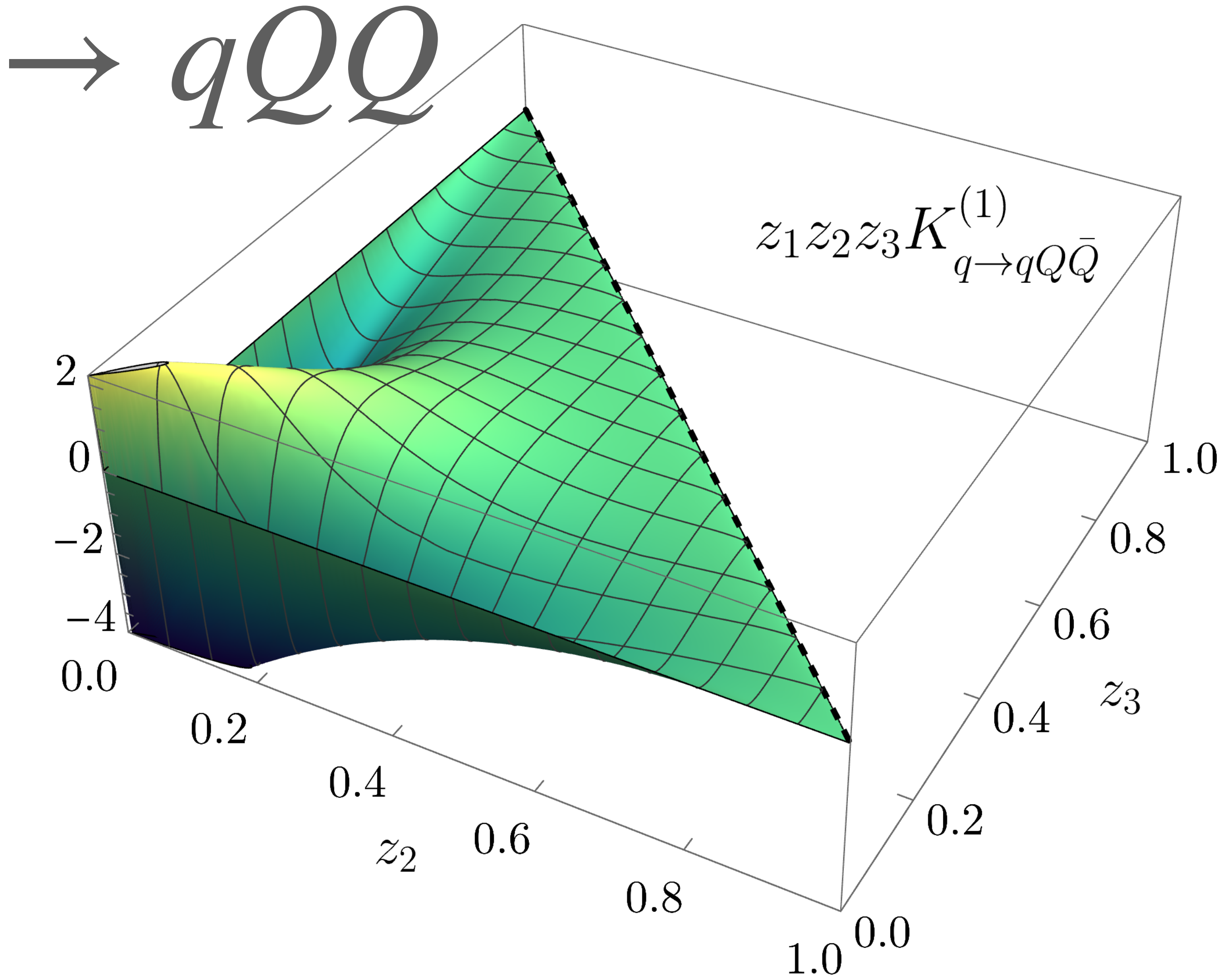
$g \to ggg$



$g \to gq\bar{q}$



$q \to qQ\bar{Q}$

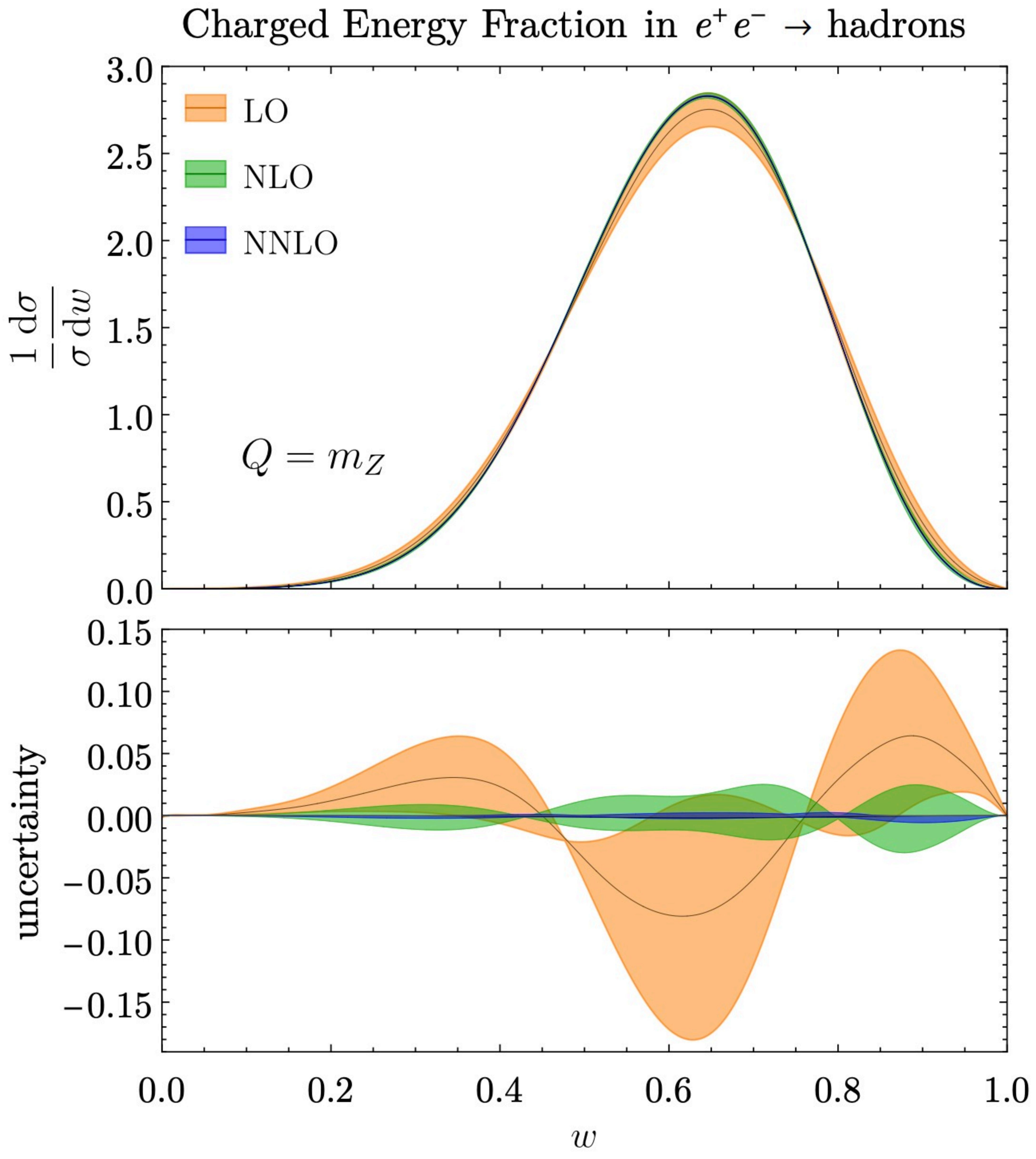


- Numerical methods developed: The moment method, Fourier series, …
  - Initial condition: The NLO track functions extracted from Pythia.

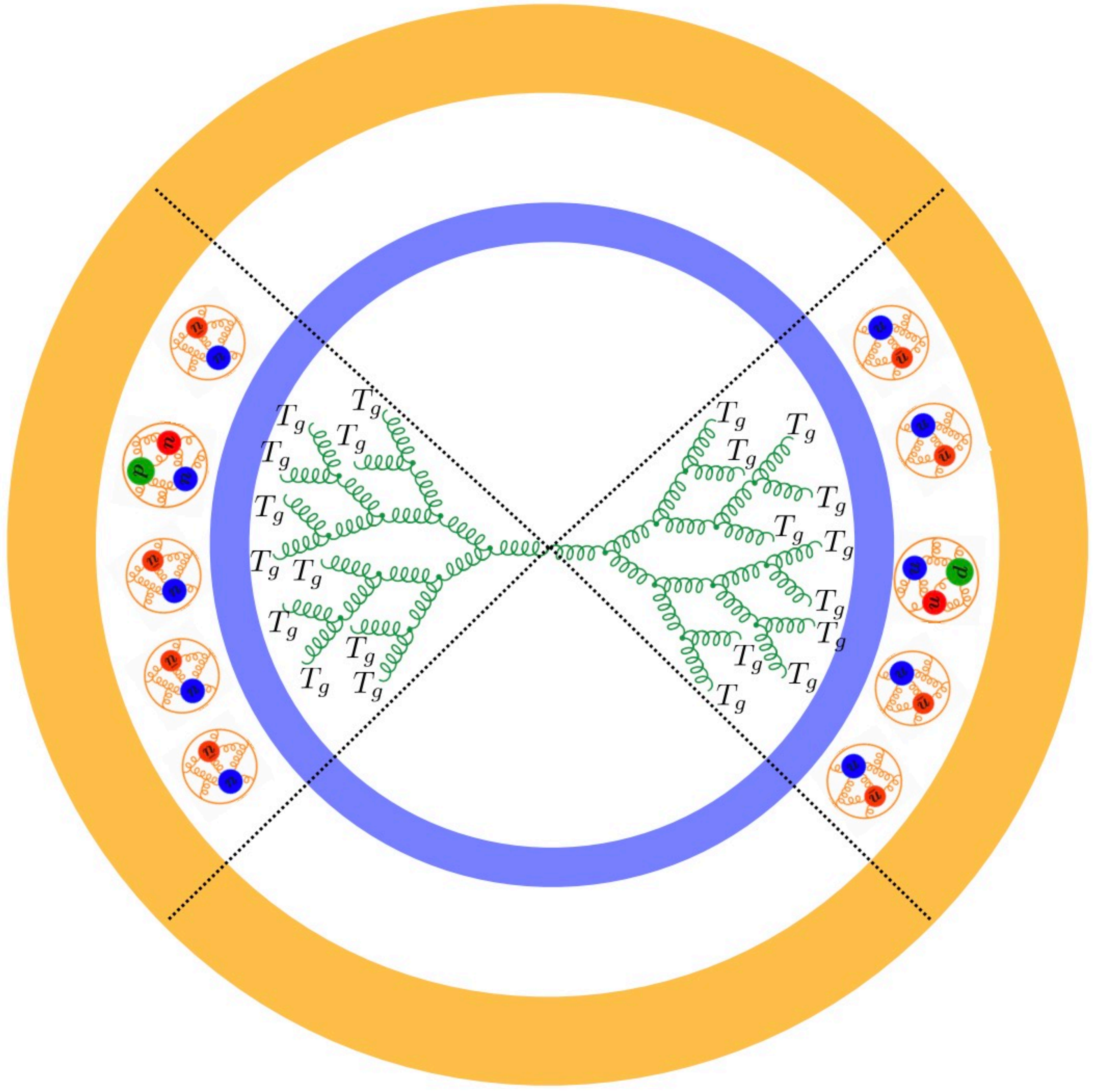
Ready for phenomenology!

Fraction of Charged Hadrons

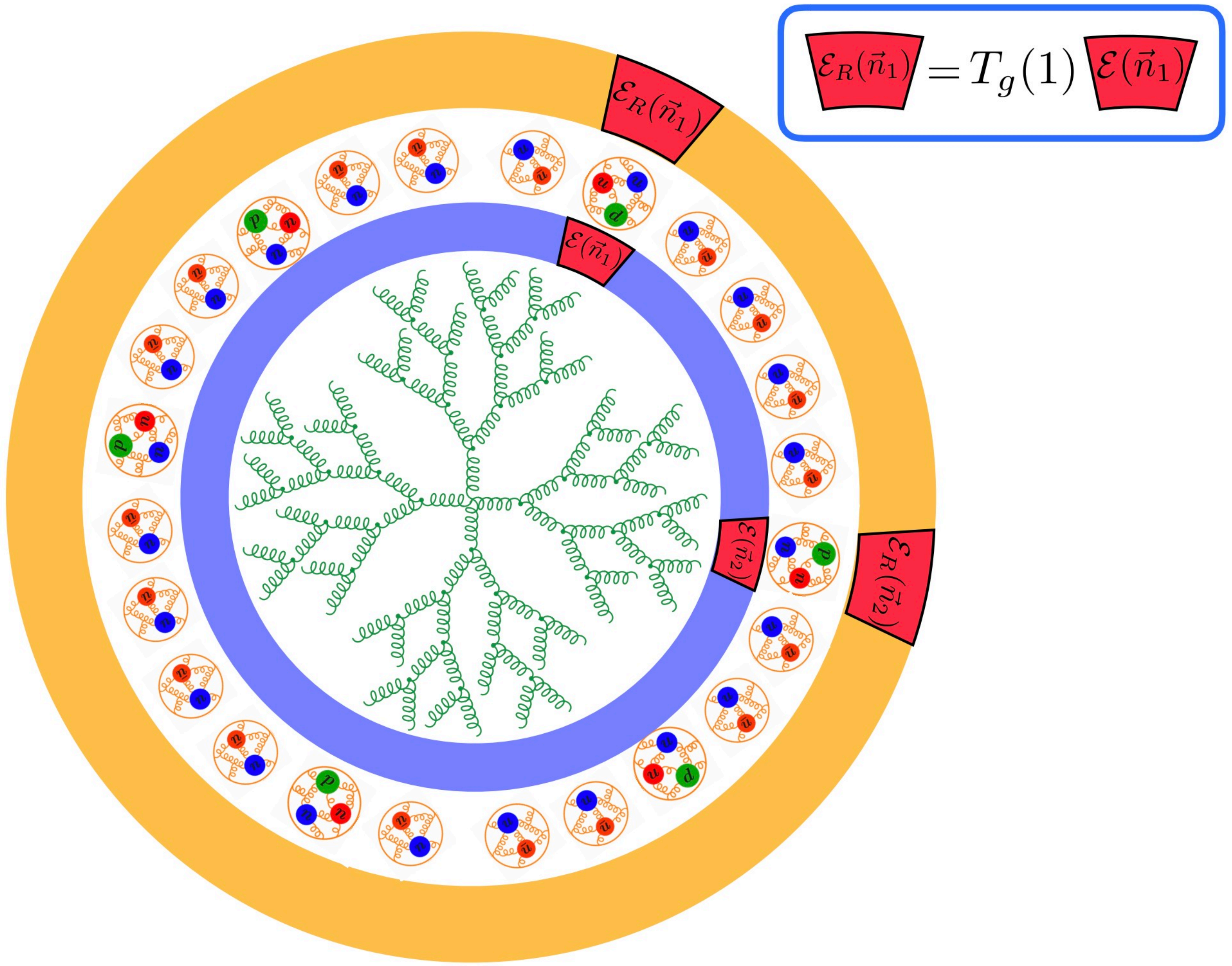
- Significant reduction in scale uncertainties with the NLO evolution of track functions included.



Incorporating Tracks



VS



Two Types of Observables

[1303.6637]

- For a δ-function type observable e measured using partons:

$$\frac{d\sigma}{de} = \sum_N \int d\Pi_N \frac{d\sigma_N}{d\Pi_N} \delta\left[e - \hat{e}(p_i^\mu)\right]$$

↓ tracks

full functional form of T

$$\frac{d\sigma}{d\bar{e}} = \sum_N \int d\Pi_N \frac{d\bar{\sigma}_N}{d\Pi_N} \int \prod_{i=1}^{N} dx_i T_i(x_i) \delta\left[\bar{e} - \hat{e}(x_i p_i^\mu)\right]$$

- **Energy correlators**: tracking easily included, and modern fixed-order techniques applied.

- For correlations of energy flow: k-point correlation functions

$$\langle \mathcal{E}(\vec{n}_1)\mathcal{E}(\vec{n}_2)\cdots\mathcal{E}(\vec{n}_k)\rangle$$

- The energy flow operator that measures energy flow on a restricted set R of final states: $\mathcal{E}_R$ e.g. charged hadrons here
- Then, the k-point correlator on R is

$$\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle$$

- This can be related to the partonic-level correlation functions by a factorization formula:

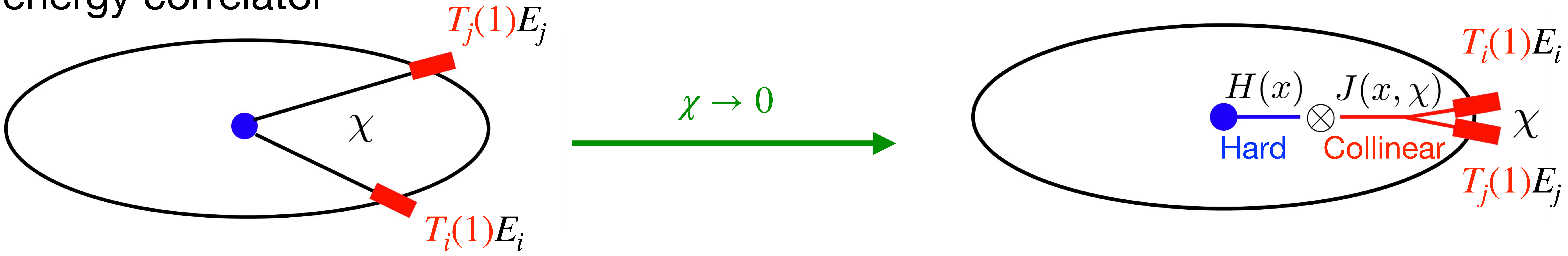
$$\langle \mathcal{E}_R(\vec{n}_1)\mathcal{E}_R(\vec{n}_2)\cdots\mathcal{E}_R(\vec{n}_k)\rangle = \sum_{i_1, i_2, \cdots i_k} T_{i_1}(1)\cdots T_{i_k}(1)\langle \mathcal{E}_{i_1}(\vec{n}_1)\mathcal{E}_{i_2}(\vec{n}_2)\cdots\mathcal{E}_{i_k}(\vec{n}_k)\rangle + \text{contact terms}$$

with dependence on higher moments of T

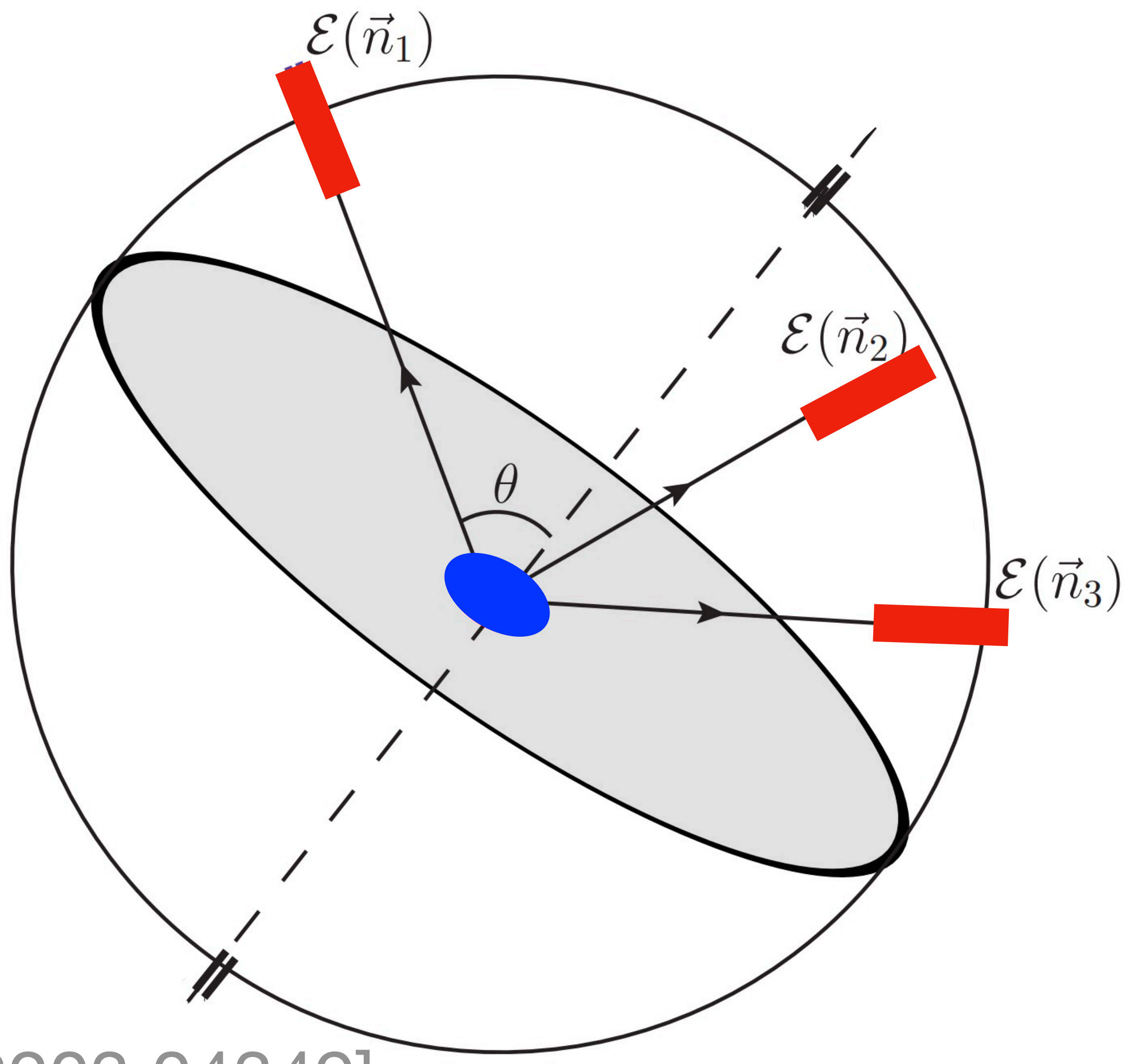
Energy Correlators Within Jets

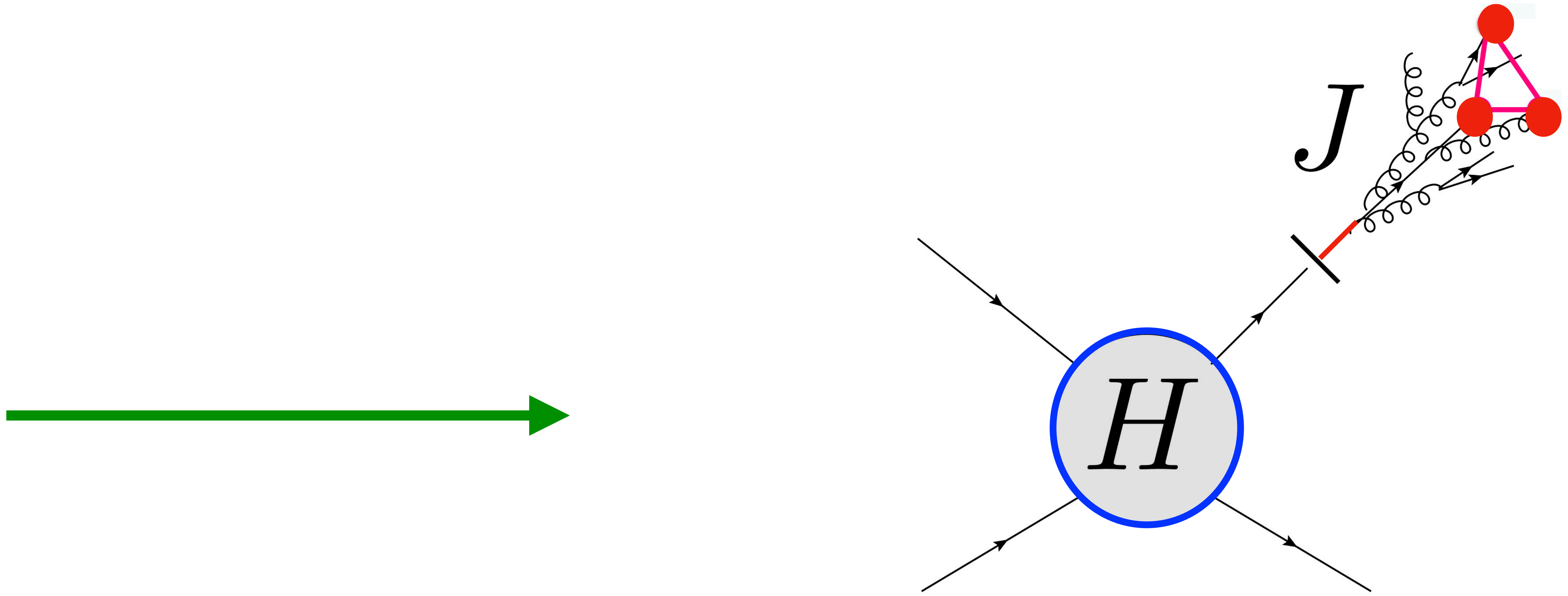
In the collinear limit:

2-point energy correlator



3-point energy correlator





[H. Chen, M.X. Luo, I. Moult, T.Z. Yang, X. Zhang, H.X. Zhu, 1912.11050]

[K. Yan, X. Zhang, 2203.04349]

- The energy correlator is a jet substructure observable: $\Sigma = \vec{J} \otimes \vec{H}$.

Energy Correlators Within Jets

- Projected energy correlators are soft insensitive observables, like the groomed jet mass.

The longest side definition

$$\frac{d\sigma^{[k]}}{dx_L} = \int d\vec{\Omega}\, \delta\left(x_L - \frac{1-\vec{n}_1\cdot\vec{n}_2}{2}\right) \prod_{\substack{1\le i<j\le k\\ i+j>3}} \theta(|\vec{n}_1-\vec{n}_2| - |\vec{n}_i-\vec{n}_j|)\langle \mathcal{E}(\vec{n}_1)\cdots\mathcal{E}(\vec{n}_k)\rangle$$

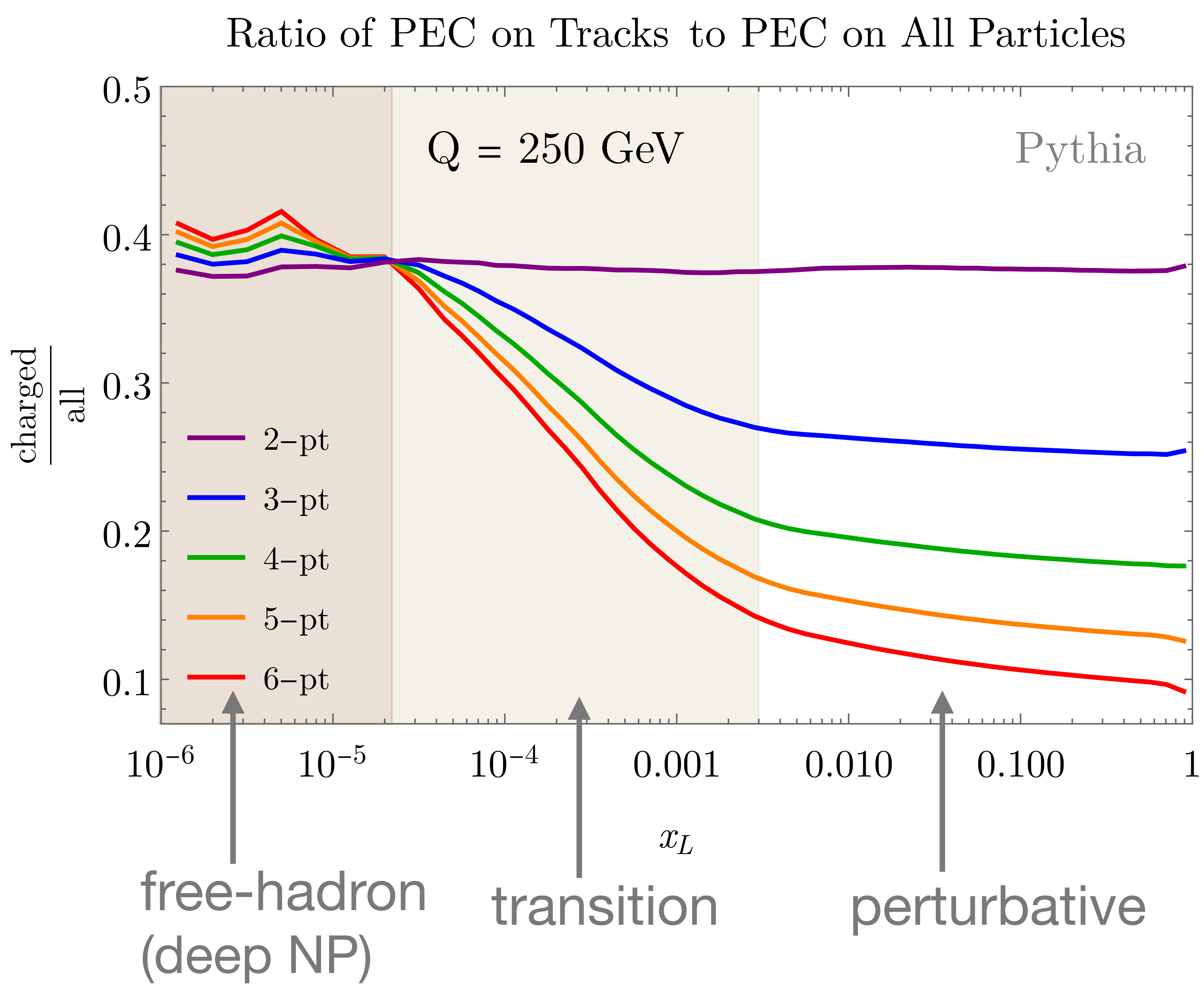
- Jet functions for projected energy correlators on tracks, $\vec{J}_{\text{tr}}\left(\ln\frac{x_L Q^2}{\mu^2}, T_i(n,\mu), a_s(\mu)\right)$: Integer moments $T_i(n,\mu)$ appear as the coefficients. Resummation convenient!
  - The jet function constants (the jet functions with the logarithmic dependence excluded): e.g. for track EECs, up to $\mathcal{O}(\alpha_s^2)$

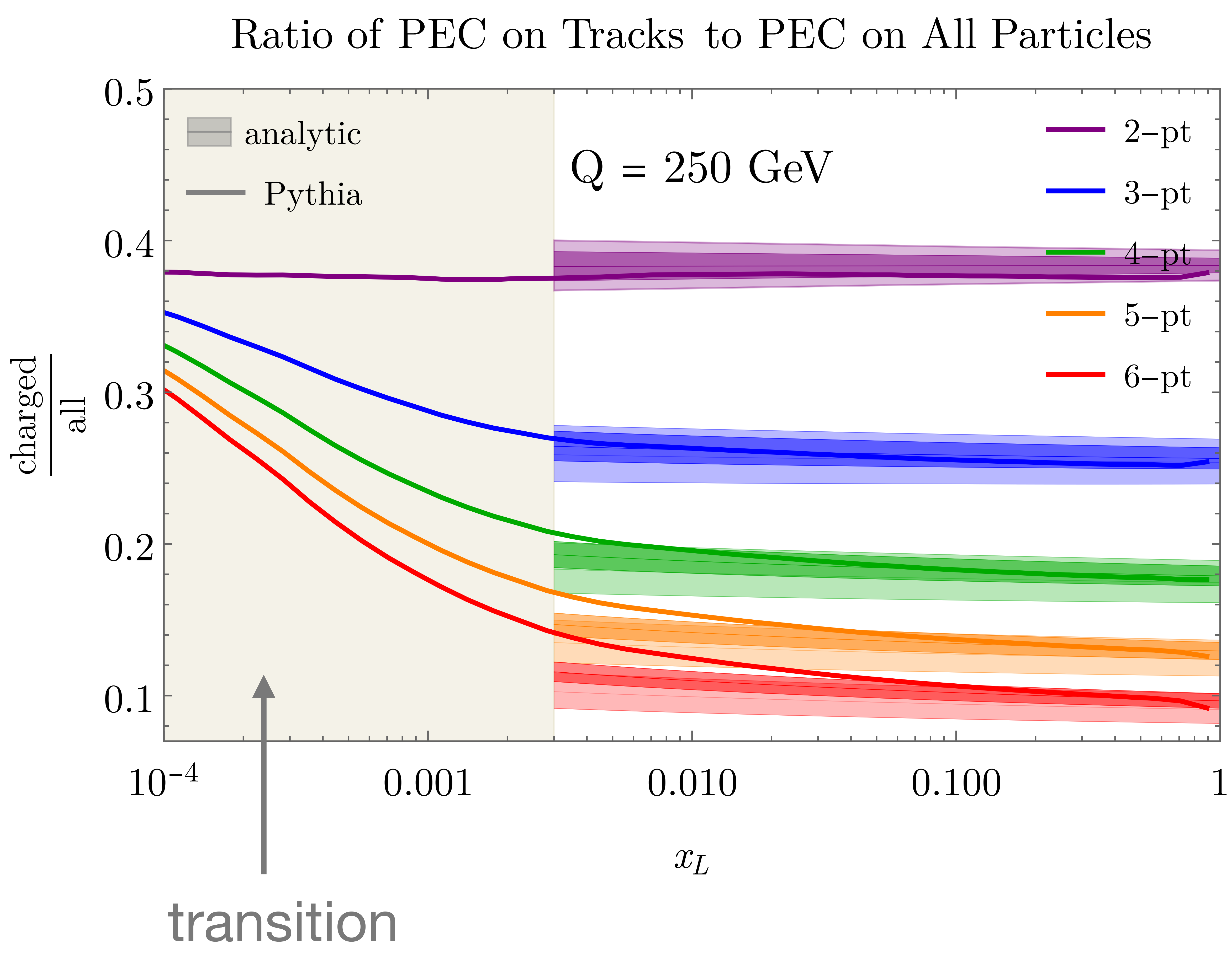
$$j^g = \frac{1}{4}T_g(2) + a_s\left\{T_g(1)T_g(1)C_A\left(-\frac{449}{150}\right) + \sum_q T_q(1)T_{\bar{q}}(1)T_F\left(-\frac{7}{25}\right)\right\}$$

$$+a_s^2\left\{T_g(1)T_g(1)\left\{C_A^2\left(-\frac{527\zeta_3}{10} + \frac{133639871}{3240000} - \frac{2159\pi^2}{1800} + \frac{19\pi^4}{90}\right) + C_A n_f T_F \frac{139}{270}\right\} + \sum_q T_q(1)T_{\bar{q}}(1)\cdots\right\}$$

  - Matches the state-of-the-art calculation for jet substructure, but now on tracks.

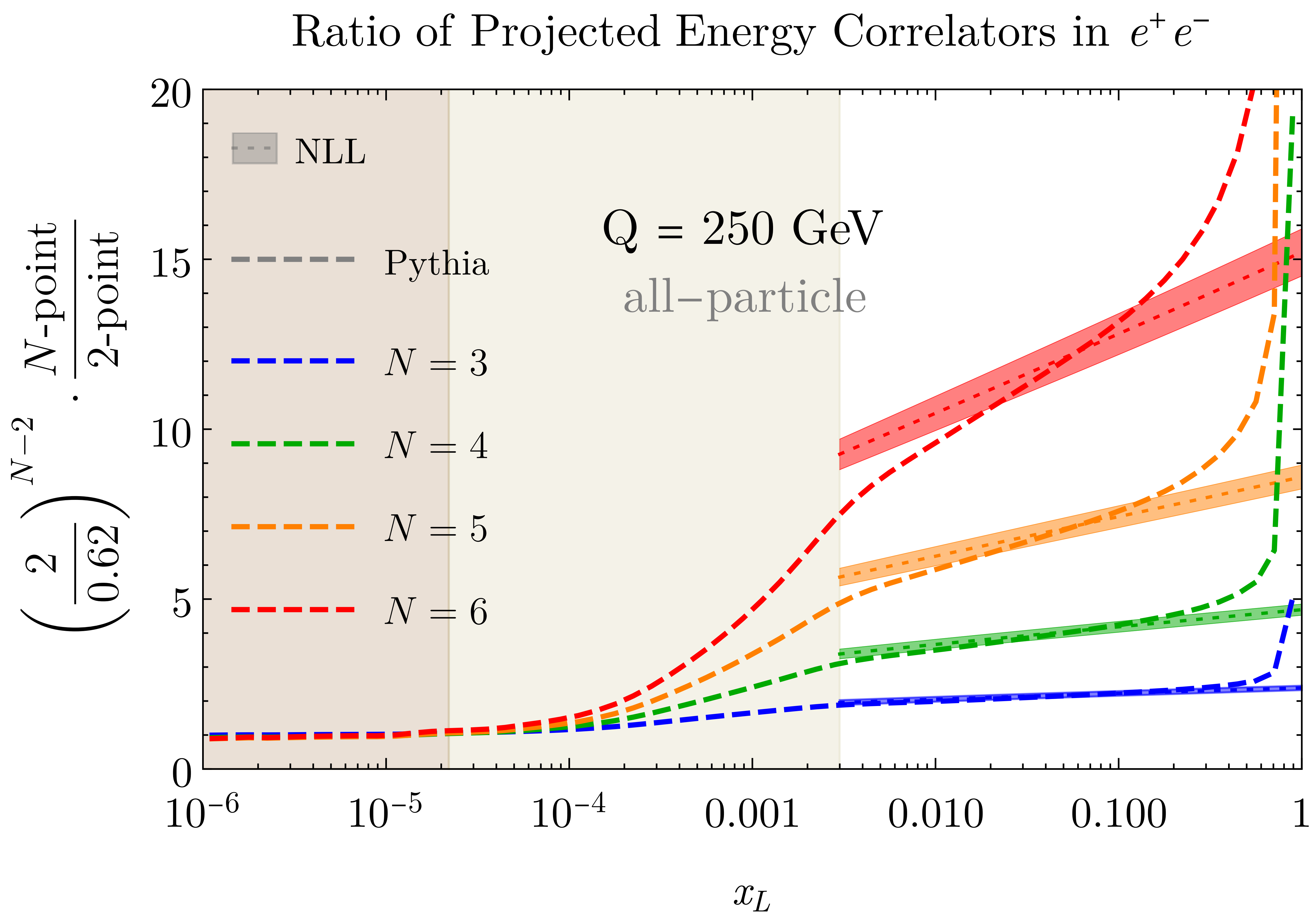
- **NLL for up to 6 point energy correlators computed on tracks!**
- Ratio of projected energy correlator on tracks to that on all particles:

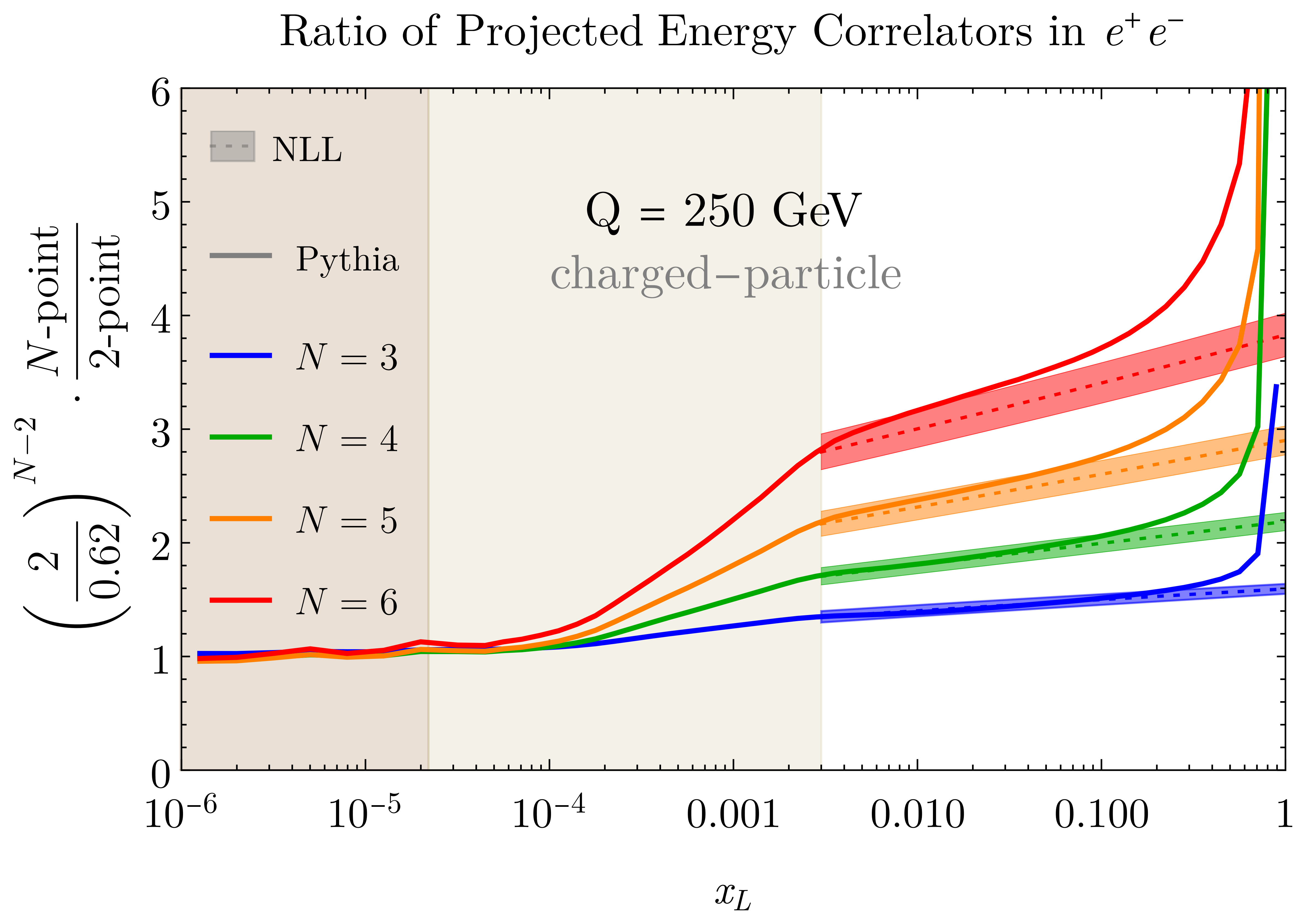




- **NLL for up to 6 point energy correlators computed on tracks!**

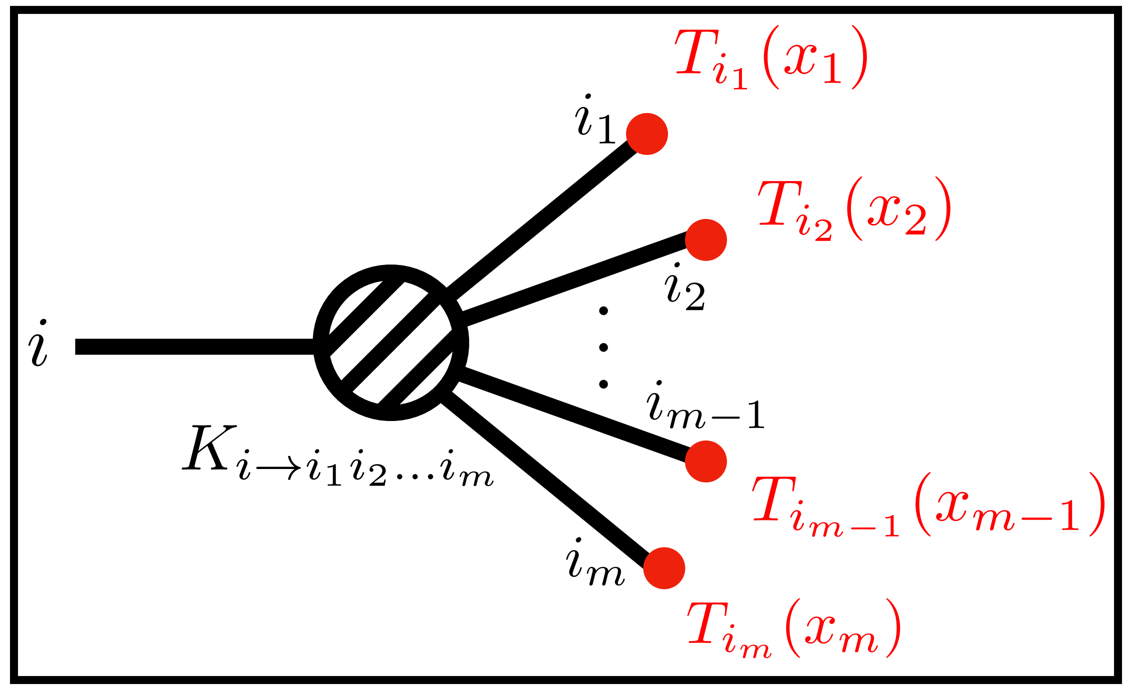
- Ratio of N-point to 2-point projected energy correlators:

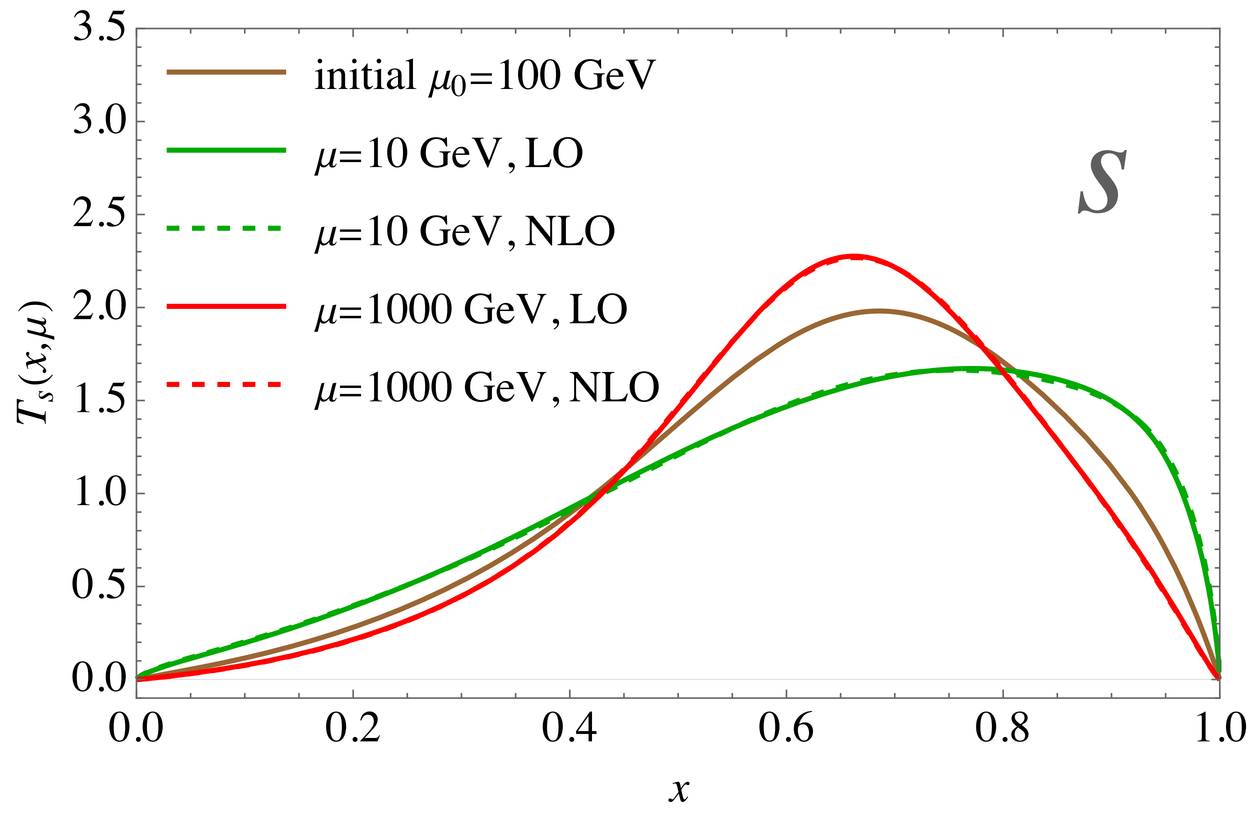


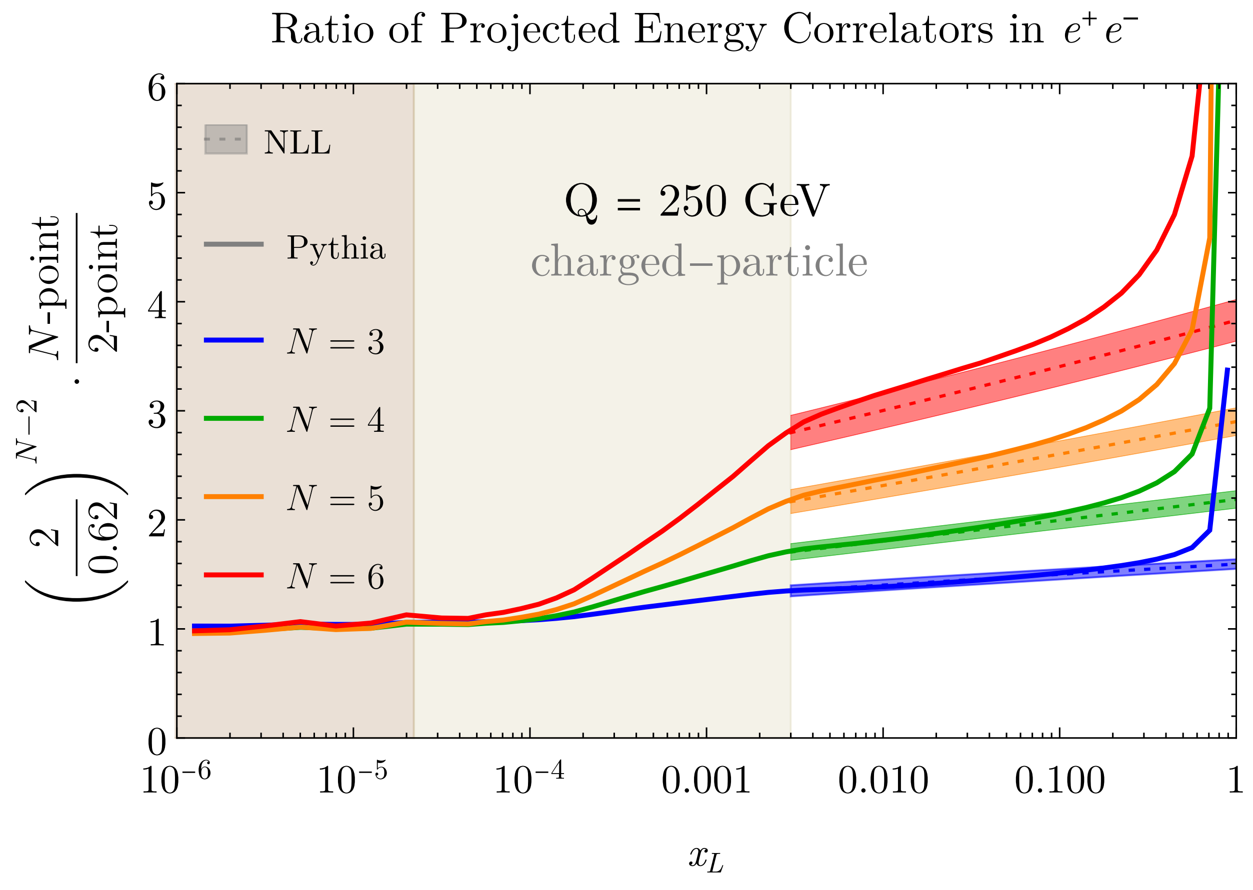


Summary

- Track functions offer a QFT approach to calculating track-based observables and extend the class of systematically computable jet substructure observables
- Full results of the nonlinear x-space evolution at $\mathcal{O}(\alpha_s^2)$.
- Track energy correlators involve the integer moments for which the track function evolution simplifies.
  - NLL resummation for the up to 6-point projected energy correlators achieved!







Others & Outlook

- Extracting track function from jet measurements; a formalism developed! [K. Lee, I. Moult, F. Ringer, W. Waalewijn, arXiv:2308.00028]
- The joint track function formalism to study the energy correlators involving mixed quantum numbers. [K. Lee, I. Moult, arXiv:2308.00746, 2308.01332] → a much broader class of calculable observables.
- A benchmark for triple collinear evolution in parton showers.
- Precision phenomenology with tracks.
- Applications of multi-hadron fragmentation functions.

Backup

Calculational Techniques

$T_i^{(0)}(x) = T_i^{\text{bare}}(x)$

$T_i(x,\mu)$ in a_s-series:

$$T_i(x,\mu) = \underbrace{T_i^{(0)}(x)}_{\text{LO track function}} + a_s \underbrace{T_i^{(1)}(x)}_{\text{UV-renormalized NLO track function}} + a_s^2 \underbrace{T_i^{(2)}(x)}_{\text{UV-renormalized NNLO track function}} + \mathcal{O}(a_s^3)$$

$$T_i^{(1)}(x) = -\frac{1}{\epsilon_{\text{IR}}} K^{(0)}_{i\to jk} \otimes T_j^{(0)} T_k^{(0)}(x)\,,$$

$$T_i^{(2)}(x) = \frac{1}{2}\Bigg\{ -\frac{1}{\epsilon_{\text{IR}}}\Big[K^{(1)}_{i\to jk} \otimes T_j^{(0)} T_k^{(0)}(x) + K^{(1)}_{i\to jmn} \otimes T_j^{(0)} T_m^{(0)} T_n^{(0)}(x)\Big]$$
$$+ \frac{1}{\epsilon_{\text{IR}}^2}\Big\{ K^{(0)}_{i\to jk} \otimes \Big[T_j^{(0)}\Big(K^{(0)}_{k\to mn} \otimes T_m^{(0)} T_n^{(0)}\Big)\Big](x) + \beta_0\, K^{(0)}_{i\to jk} \otimes T_j^{(0)} T_k^{(0)}(x)$$
$$+ K^{(0)}_{i\to jk} \otimes \Big[T_k^{(0)}\Big(K^{(0)}_{j\to mn} \otimes T_m^{(0)} T_n^{(0)}\Big)\Big](x)\Big\}\Bigg\}\,.$$

Track Jet Functions

$$J_{\mathrm{tr},i}(s,x,\mu) = \delta(s)\sum_{\ell=0}^{\infty}\sum_{\{i_f\}}\Big[a_s^\ell \mathcal{J}^{(\ell)}_{i\to\{i_f\}} \otimes \prod_{j\in\{i_f\}} T_j\Big](x,\mu)$$

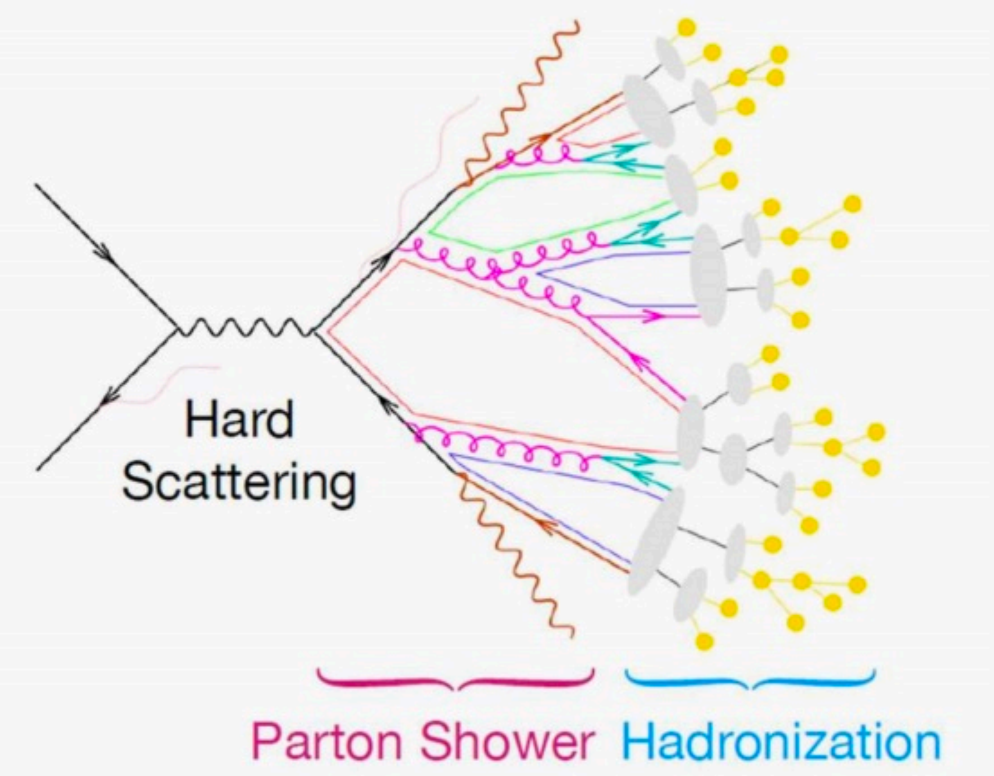
$$\begin{aligned}
&J_{\mathrm{tr},i}(s,x,\mu)\\
&= \delta(s)\,T_i^{(0)}(x) + a_s(\mu)\,\delta(s)\left\{\left(\mathcal{J}^{(1)}_{i\to i} - \frac{1}{\epsilon}K^{(0)}_{i\to i}\right)\otimes T_i^{(0)}(x) + \left(\mathcal{J}^{(1)}_{i\to i_1 i_2} - \frac{1}{\epsilon}K^{(0)}_{i\to i_1 i_2}\right)\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}(x)\right\}\\
&\quad + a_s^2(\mu)\,\delta(s)\left\{\left(\mathcal{J}^{(2)}_{i\to i} - \frac{1}{2\epsilon}K^{(1)}_{i\to i} + \frac{\beta_0}{2\epsilon^2}K^{(0)}_{i\to i}\right)\otimes T_i^{(0)}\right.\\
&\qquad + \left(\mathcal{J}^{(2)}_{i\to i_1 i_2} - \frac{1}{2\epsilon}K^{(1)}_{i\to i_1 i_2} + \frac{\beta_0}{2\epsilon^2}K^{(0)}_{i\to i_1 i_2}\right)\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}\\
&\qquad + \left(\mathcal{J}^{(2)}_{i\to i_1 i_2 i_3} - \frac{1}{2\epsilon}K^{(1)}_{i\to i_1 i_2 i_3}\right)\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}T_{i_3}^{(0)}\\
&\qquad + \left(-\frac{1}{\epsilon}\mathcal{J}^{(1)}_{i\to i} + \frac{1}{2\epsilon^2}K^{(0)}_{i\to i}\right)\otimes\left(K^{(0)}_{i\to i}\otimes T_i^{(0)} + K^{(0)}_{i\to i_1 i_2}\otimes T_{i_1}^{(0)}T_{i_2}^{(0)}\right)\\
&\qquad + \left(-\frac{1}{\epsilon}\mathcal{J}^{(1)}_{i\to i_1 i_2} + \frac{1}{2\epsilon^2}K^{(0)}_{i\to i_1 i_2}\right)\otimes\Big[T_{i_1}^{(0)}\left(K^{(0)}_{i_2\to i_2}\otimes T_{i_2}^{(0)} + K^{(0)}_{i_2\to j_1 j_2}\otimes T_{j_1}^{(0)}T_{j_2}^{(0)}\right)\\
&\qquad\left. + \left(K^{(0)}_{i_1\to i_1}\otimes T_{i_1}^{(0)} + K^{(0)}_{i_1\to k_1 k_2}\otimes T_{k_1}^{(0)}T_{k_2}^{(0)}\right)T_{i_2}^{(0)}\Big]\right\} + \mathcal{O}(a_s^3)\,,
\end{aligned}$$

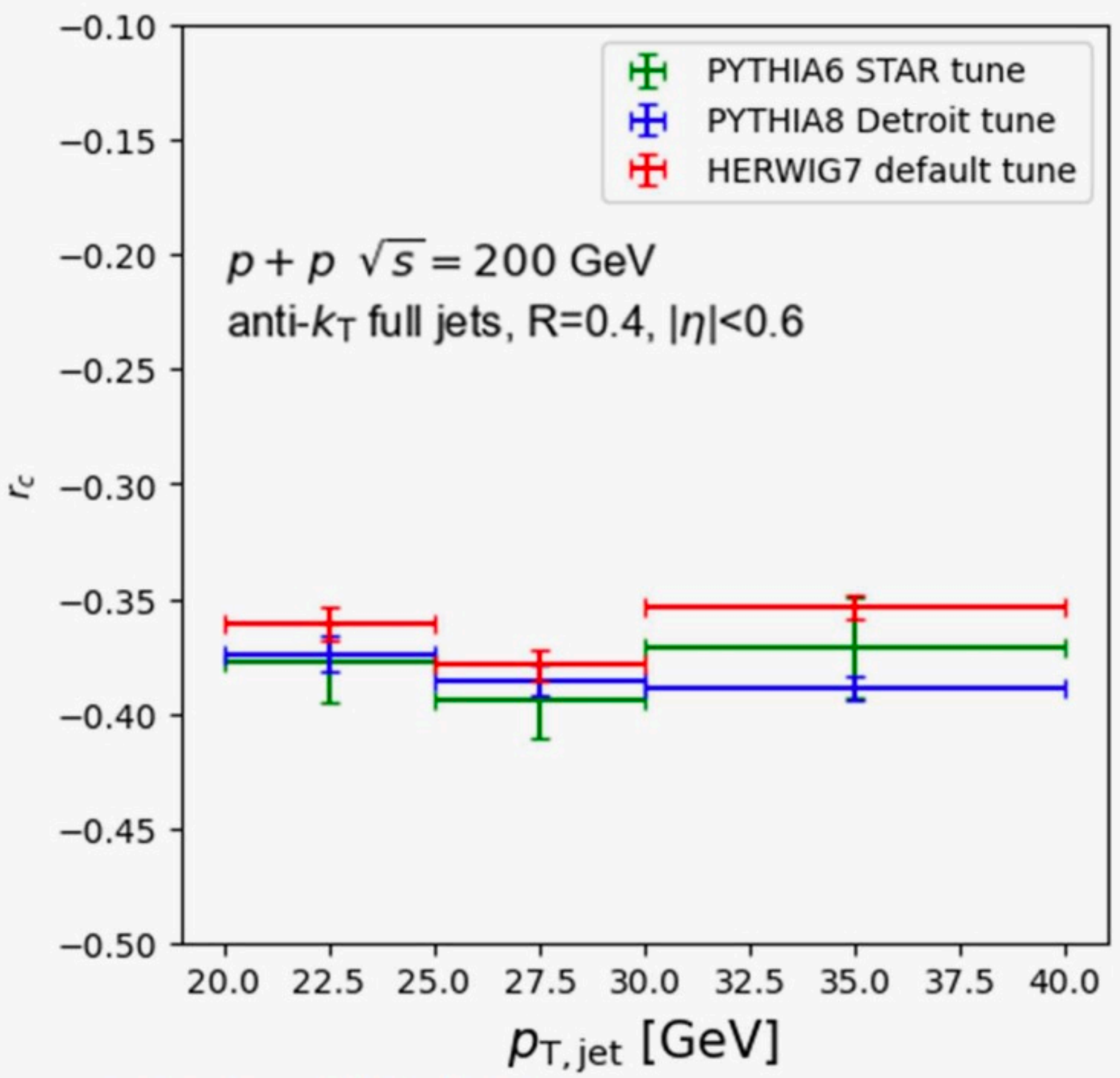
LO evolution kernels

NLO evolution kernels

From Youqi Song's talk at BOOST2023

What's next?





- Study hadronization with jet substructure by measuring r_c

$$r_c(X) = \frac{\mathrm{d}\sigma_{h_1h_2}/\mathrm{d}X - \mathrm{d}\sigma_{h_1\bar{h}_2}/\mathrm{d}X}{\mathrm{d}\sigma_{h_1h_2}/\mathrm{d}X + \mathrm{d}\sigma_{h_1\bar{h}_2}/\mathrm{d}X}$$

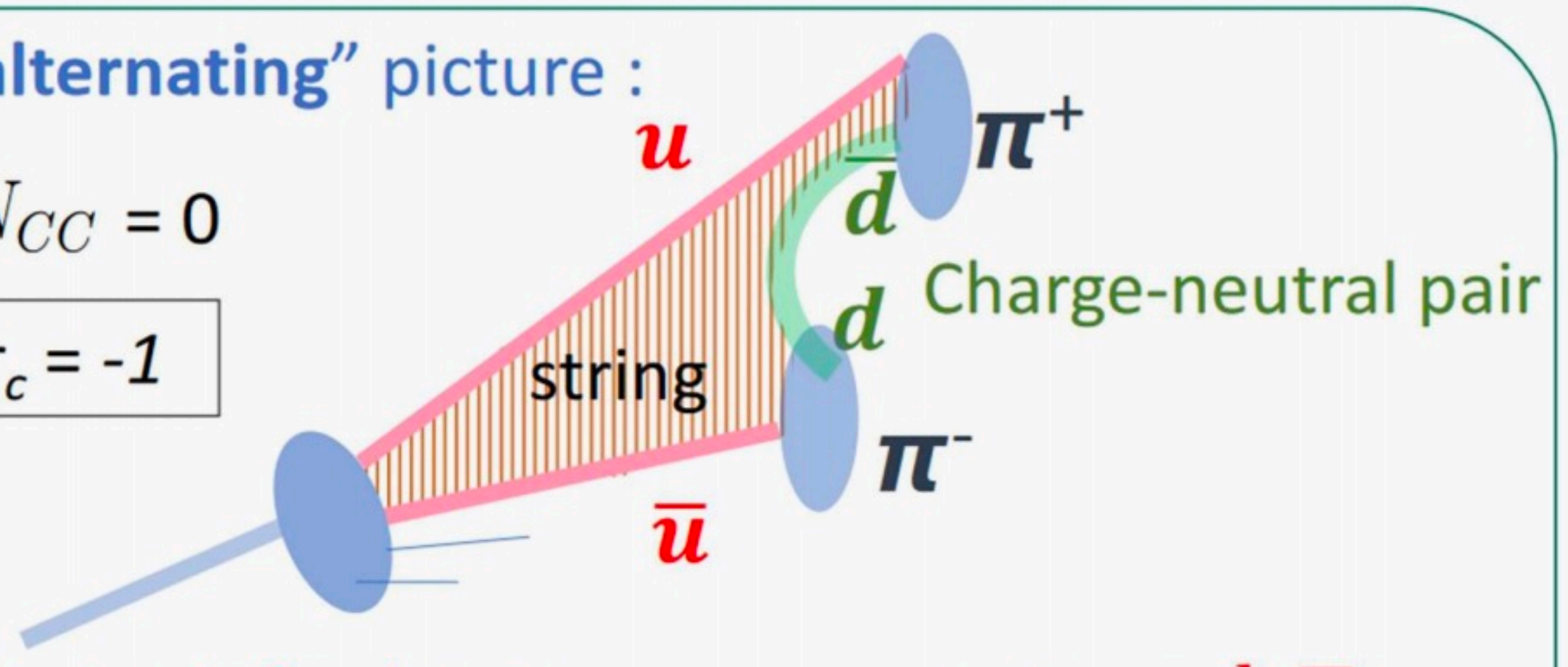
Chien et al. PRD 105 051502 (2022)

- h_1h_2: same charge tracks, $h_1\bar{h}_2$: opposite charge tracks

"alternating" picture :

N_{CC} = 0

r_c = -1



Partonic final state : u and $\bar{u}$

Combine charge-neutral pair : $\bar{d}$ and d

"random" picture :

no charge correlation

N_{CC} = $N_{C\bar{C}}$

r_c = 0

r_c is a measure of the fraction of "string-like hadronization"

Mondal DIS 2022

- To be validated with STAR data!